Unconditional Prime-representing Functions, Following Mills

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Abstract. Mills proved that there exists a real constant A > 1 such that for all $n \in \mathbb{N}$ the values $\lfloor A^{3^n} \rfloor$ are prime numbers. No explicit value of A is known, but assuming the Riemann hypothesis one can choose $A = 1.3063778838 \dots$ Here we give a first *unconditional* variant: $\lfloor A^{10^{10n}} \rfloor$ is prime, where $A = 1.00536773279814724017 \dots$ can be computed to millions of digits. Similarly, $\lfloor A^{3^{13n}} \rfloor$ is prime, with $A = 3.8249998073439146171615551375 \dots$

Mills [9] proved that there exists a real number A > 1 such that for all $n \in \mathbb{N}$ the values $f(n) = \lfloor A^{3^n} \rfloor$ are prime. For some related work see [1, 5, 7, 10, 11, 13, 14] and [3, Exercise 1.23]. Even though such formulae encode existing knowledge of primes, rather than generate new primes, and even though the proof shows that many such values A exist, it is quite astonishing that not a single value A is known. It is not even known whether any $A < 10^{1000000}$ (say) exists such that the statement holds, or not. The reason for this is that Mills made use of a result of Ingham that there is always a prime between n and $n + cn^{5/8}$, for some (nonexplicit) positive constant c, which implies that there always exists a prime between any two *sufficiently large* cubes. With current knowledge this is known only for cubes t^3 of size at least $t^3 \ge e^{e^{33.217}}$, a number which has 115, 809, 481, 360, 809 digits (a result of Dudek, see [2]). This number is very much larger than the largest known primes, which in turn are primes of a very special type $q = 2^n - 1$ (Mersenne primes). Even with an improvement on Dudek's result and with expected progress on primality tests, bridging this huge gap would appear to be several decades away.

Some escape routes out of this dilemma have been studied:

- 1. Caldwell and Cheng [1] observe that assuming the Riemann hypothesis the sequence of primes $b_1 = 2, b_2 = 11, b_3 = 1361, b_4 = 2521008887, b_5 = 16022236204009818131831320183$ can be continued such that Mills' result on $\lfloor A^{3^n} \rfloor$ holds with $A = \lim_{n \to \infty} b_n^{3^{-n}} = 1.3063778838...$ They also write that an *unconditional* value of A is completely out of range of today's methods, due to the issue with "sufficiently large" cubes mentioned above.
- 2. For an unconditional result, Wright [13] introduced a very rapidly increasing tower-type sequence: There exists a constant $\omega = 1.9287800\ldots$ such that with $g_1 = 2^{\omega}, g_{n+1} = 2^{g_n}$, the values of $\lfloor g_n \rfloor$ are prime, for all integers $n \ge 1$, starting with $p_1 = \lfloor 2^{\omega} \rfloor = 3, p_2 = \lfloor 2^{2^{\omega}} \rfloor = 13, p_3 = \lfloor 2^{2^{2^{\omega}}} \rfloor = 16381$.
- 3. Some formulae producing all primes also exist. It follows from Wilson's theorem that the function

$$f(n) = \left\lfloor \frac{n! \mod (n+1)}{n} \right\rfloor (n-1) + 2$$

takes the value $f(n) = p_i$, where p_i is the *i*th prime, when $n = p_i - 1$ and is 2 otherwise. Hence the values of f are prime for all $n \in \mathbb{N}$. Another example is

Gandhi's formula

$$p_n = \left\lfloor 1 - \log_2 \left(-\frac{1}{2} + \sum_{d \mid P_{n-1}} \frac{\mu(d)}{2^d - 1} \right) \right\rfloor,$$

where \log_2 denotes the logarithm to base 2, μ the Möbius function, $P_{n-1} = \prod_{i=1}^{n-1} p_i$, and p_i is the *i*th prime in ascending order.

Very recently a new formula was found [4]: there exists a constant $f_1 = 2.920050977316...$ such that the sequence $f_n = \lfloor f_{n-1} \rfloor (f_{n-1} - \lfloor f_{n-1} \rfloor + 1)$ has the property that $p_n = \lfloor f_n \rfloor$.

A survey on such questions is in Ribenboim's book [12].

In this note we prove the following *unconditional* result on sequences in the spirit of Mills, which grow asymptotically much less rapidly than Wright's sequence.

Theorem. a) Let p be a Mersenne exponent, i.e., $2^p - 1$ is a prime. For every integer $m \ge 1\,438\,989$, there exists a real constant $A_{m,p} > 1$ such that for all $n \in \mathbb{N}$ the values of all functions $f_{m,p}(n) = \lfloor A_{m,p}^{m^n} \rfloor$ are prime. Moreover, the values $A_{m,p}$ can be estimated as follows:

$$\frac{p}{m}\log 2 - \frac{2}{m2^p} < \log A_{m,p} < \frac{p}{m}\log 2.$$

If p is large this gives a very high precision. (The proof gives even more precise estimates.)

- b) Specializing to the Mersenne exponent $p = 77\,232\,917$: There is a constant $A = 1.00536773279814724017\ldots$ such that all values of $\lfloor A^{10^{10n}} \rfloor$, $n \in \mathbb{N}$, are prime. The constant A can be computed to millions of decimal places.
- c) With the same $p: |A^{3^{13n}}|$ is prime with A = 3.82499980734391461716...

The proof makes use of two nontrivial ingredients.

The first ingredient of the proof is an *explicit* variant of the existence of primes in certain intervals. Dudek [2] observed that for $m \ge 4.97117 \cdot 10^9$ there is a prime between n^m and $(n+1)^m$, for *all* values of $n \in \mathbb{N}$. The strongest currently known estimate of this kind is due to Mattner [6]:

Lemma 1. Let $m \ge 1438989$. Then there is a prime with $n^m for all <math>n \ge 1$.

The second ingredient is a quite large prime. We choose the second-largest prime that is currently known, a Mersenne prime, see [8]:

Lemma 2. $2^{77 \, 232 \, 917} - 1$ is a prime number.

Proof of Theorem. For our application we need to reduce the size of the interval in Lemma 1 by one element, namely $(n + 1)^m - 1$ is divisible by n so that the prime satisfies $n^m . From this we can construct a sequence of primes <math>p_1, p_2, \ldots$ with $p_n^m < p_{n+1} < (p_n + 1)^m - 1$. Raising these inequalities (adding 1 where necessary) to the m^{-n-1} th power gives

$$p_n^{m^{-n}} < p_{n+1}^{m^{-n-1}} < (p_{n+1}+1)^{m^{-n-1}} < (p_n+1)^{m^{-n}}.$$

From this we see that the sequence $\alpha_n = (p_n^{m^{-n}})$ is an increasing sequence, whereas the sequence $\beta_n = ((p_n + 1)^{m^{-n}})$ is decreasing with increasing *n*. Hence the sequence α_n is also bounded and therefore the limit $A := \lim_{n \to \infty} p_n^{m^{-n}}$ exists. It follows that $p_n \le A^{m^n} < p_n + 1$ and so $p_n = \lfloor A^{m^n} \rfloor$. We take $p_1 = \lfloor A^{m^1} \rfloor = 2^{77\,232\,917} - 1$.

$$2^{77\,232\,917} - 1 = 2^{77\,232\,917} (1 - \frac{1}{2^{77\,232\,917}}) < A^m < 2^{77\,232\,917}.$$

Taking the natural logarithm and observing that for small x > 0 a simple explicit Taylor estimate gives $-x - x^2 < \log(1 - x) < -x - \frac{x^2}{2}$ we find that

$$\frac{77232917}{m}\log 2 - \frac{1}{m2^{77232917}} - \frac{1}{m4^{77232917}} < \log A < \frac{77232917}{m}\log 2 - \frac{1}{m2^{77232917}} - \frac{1}{2m4^{77232917}}.$$

(This is more precise than stated in the theorem. Higher order Taylor estimates are also possible.)

Note that from this one can evaluate $\log A$ and therefore A with an accuracy of millions of digits. In particular, if $m = 10^{10}$, then A = 1.00536773279814724017..., and similarly for $m = 3^{13} > 1438989$. (To see the implication that knowledge on high precision of $\log A$ implies also high precision for \hat{A} : Let $A_1 < A_2$ be two constants with $A_2 = A_1(1+\varepsilon)$ (say), where $\varepsilon > 0$ is a small constant. Then $\frac{\varepsilon}{2} < \log(1+\varepsilon)$ ε) = log A_2 - log A_1 = log $(1 + \varepsilon) = \varepsilon - \frac{\varepsilon^2}{2} \pm \cdots < \varepsilon$ and

$$A_2 - A_1 = A_1 \varepsilon < 2A_1 (\log A_2 - \log A_1).$$

In other words, when $\log A$ is known with high precision and A is of size, say, 1 < 1A < 10 as in the theorem, then very small deviations of A to A_1 or A_2 would give both $\log A_2 - \log A_1$ and $A_2 - A_1$ with about the same precision. Hence A is also known with high precision.)

Similarly, every reader can produce their own formula by choosing a large number *m* and a quite large prime *q*, for example the largest currently known prime $q = 2^{82,589,933} - 1$. Then $\frac{\log q}{m} - \frac{2}{mq} < \log A < \frac{\log q}{m}$ determines *A* with high precision.

Finally, as the actual distribution of primes might be much better than what can currently be proved, and based on some experiments, we conjecture the following:

Conjecture. There is a constant A (possibly near 1.1966746500705764022) such that $f(n) = |A^{(n+1)^2}|$ is prime for all n > 1.

Note that the exponent $(n + 1)^2$ grows polynomially compared to the exponential growth of m^n (for fixed m) in Mills-type examples. The value above would give $p_1 =$ $2, p_3 = 5, p_4 = 17, p_5 = 89, p_6 = 641, p_7 = 6619, p_8 = 97829, p_9 = 2070443.$

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