# Unconditional Prime-representing Functions, Following Mills 

Christian Elsholtz


#### Abstract

Mills proved that there exists a real constant $A>1$ such that for all $n \in \mathbb{N}$ the values $\left\lfloor A^{3^{n}}\right\rfloor$ are prime numbers. No explicit value of $A$ is known, but assuming the Riemann hypothesis one can choose $A=1.3063778838 \ldots$. Here we give a first unconditional variant: $\left\lfloor A^{10^{10 n}}\right\rfloor$ is prime, where $A=1.00536773279814724017 \ldots$ can be computed to millions of digits. Similarly, $\left\lfloor A^{3^{13 n}}\right\rfloor$ is prime, with $A=3.8249998073439146171615551375 \ldots$


Mills [9] proved that there exists a real number $A>1$ such that for all $n \in \mathbb{N}$ the values $f(n)=\left\lfloor A^{3^{n}}\right\rfloor$ are prime. For some related work see $[\mathbf{1}, 5, \mathbf{7}, 1 \mathbf{1 0}, \mathbf{1 1}, 13,14]$ and [3] Exercise 1.23]. Even though such formulae encode existing knowledge of primes, rather than generate new primes, and even though the proof shows that many such values $A$ exist, it is quite astonishing that not a single value $A$ is known. It is not even known whether any $A<10^{1000} 000$ (say) exists such that the statement holds, or not. The reason for this is that Mills made use of a result of Ingham that there is always a prime between $n$ and $n+c n^{5 / 8}$, for some (nonexplicit) positive constant $c$, which implies that there always exists a prime between any two sufficiently large cubes. With current knowledge this is known only for cubes $t^{3}$ of size at least $t^{3} \geq e^{e^{33.217}}$, a number which has $115,809,481,360,809$ digits (a result of Dudek, see [2]). This number is very much larger than the largest known primes, which in turn are primes of a very special type $q=2^{n}-1$ (Mersenne primes). Even with an improvement on Dudek's result and with expected progress on primality tests, bridging this huge gap would appear to be several decades away.

Some escape routes out of this dilemma have been studied:

1. Caldwell and Cheng [1] observe that assuming the Riemann hypothesis the sequence of primes $b_{1}=2, b_{2}=11, b_{3}=1361, b_{4}=2521008887, b_{5}=$ 16022236204009818131831320183 can be continued such that Mills' result on $\left\lfloor A^{3^{n}}\right\rfloor$ holds with $A=\lim _{n \rightarrow \infty} b_{n}^{3^{-n}}=1.3063778838 \ldots$. They also write that an unconditional value of $A$ is completely out of range of today's methods, due to the issue with "sufficiently large" cubes mentioned above.
2. For an unconditional result, Wright [13] introduced a very rapidly increasing tower-type sequence: There exists a constant $\omega=1.9287800 \ldots$ such that with $g_{1}=2^{\omega}, g_{n+1}=2^{g_{n}}$, the values of $\left\lfloor g_{n}\right\rfloor$ are prime, for all integers $n \geq 1$, starting with $p_{1}=\left\lfloor 2^{\omega}\right\rfloor=3$, $p_{2}=\left\lfloor 2^{2^{\omega}}\right\rfloor=13, p_{3}=\left\lfloor 2^{2^{2^{\omega}}}\right\rfloor=16381$.
3. Some formulae producing all primes also exist. It follows from Wilson's theorem that the function

$$
f(n)=\left\lfloor\frac{n!\bmod (n+1)}{n}\right\rfloor(n-1)+2
$$

takes the value $f(n)=p_{i}$, where $p_{i}$ is the $i$ th prime, when $n=p_{i}-1$ and is 2 otherwise. Hence the values of $f$ are prime for all $n \in \mathbb{N}$. Another example is

Gandhi's formula

$$
p_{n}=\left\lfloor 1-\log _{2}\left(-\frac{1}{2}+\sum_{d \mid P_{n-1}} \frac{\mu(d)}{2^{d}-1}\right)\right\rfloor
$$

where $\log _{2}$ denotes the logarithm to base $2, \mu$ the Möbius function, $P_{n-1}=$ $\prod_{i=1}^{n-1} p_{i}$, and $p_{i}$ is the $i$ th prime in ascending order.

Very recently a new formula was found [4]: there exists a constant $f_{1}=$ $2.920050977316 \ldots$ such that the sequence $f_{n}=\left\lfloor f_{n-1}\right\rfloor\left(f_{n-1}-\left\lfloor f_{n-1}\right\rfloor+1\right)$ has the property that $p_{n}=\left\lfloor f_{n}\right\rfloor$.

A survey on such questions is in Ribenboim's book [12].
In this note we prove the following unconditional result on sequences in the spirit of Mills, which grow asymptotically much less rapidly than Wright's sequence.

Theorem. a) Let p be a Mersenne exponent, i.e., $2^{p}-1$ is a prime. For every integer $m \geq 1438989$, there exists a real constant $A_{m, p}>1$ such that for all $n \in \mathbb{N}$ the values of all functions $f_{m, p}(n)=\left\lfloor A_{m, p}^{m^{n}}\right\rfloor$ are prime. Moreover, the values $A_{m, p}$ can be estimated as follows:

$$
\frac{p}{m} \log 2-\frac{2}{m 2^{p}}<\log A_{m, p}<\frac{p}{m} \log 2 .
$$

If $p$ is large this gives a very high precision. (The proof gives even more precise estimates.)
b) Specializing to the Mersenne exponent $p=77232917$ : There is a constant $A=$ $1.00536773279814724017 \ldots$ such that all values of $\left\lfloor A^{10^{10 n}}\right\rfloor, n \in \mathbb{N}$, are prime. The constant A can be computed to millions of decimal places.
c) With the same $p:\left\lfloor A^{3^{13 n}}\right\rfloor$ is prime with $A=3.82499980734391461716 \ldots$

The proof makes use of two nontrivial ingredients.
The first ingredient of the proof is an explicit variant of the existence of primes in certain intervals. Dudek [2] observed that for $m \geq 4.97117 \cdot 10^{9}$ there is a prime between $n^{m}$ and $(n+1)^{m}$, for all values of $n \in \mathbb{N}$. The strongest currently known estimate of this kind is due to Mattner [6]:

Lemma 1. Let $m \geq 1438$ 989. Then there is a prime with $n^{m}<p<(n+1)^{m}$ for all $n \geq 1$.

The second ingredient is a quite large prime. We choose the second-largest prime that is currently known, a Mersenne prime, see [8]:
Lemma 2. $2^{77232917}-1$ is a prime number.
Proof of Theorem. For our application we need to reduce the size of the interval in Lemma 1 by one element, namely $(n+1)^{m}-1$ is divisible by $n$ so that the prime satisfies $n^{m}<p<(n+1)^{m}-1$. From this we can construct a sequence of primes $p_{1}, p_{2}, \ldots$ with $p_{n}^{m}<p_{n+1}<\left(p_{n}+1\right)^{m}-1$. Raising these inequalities (adding 1 where necessary) to the $m^{-n-1}$ th power gives

$$
p_{n}^{m^{-n}}<p_{n+1}^{m^{-n-1}}<\left(p_{n+1}+1\right)^{m^{-n-1}}<\left(p_{n}+1\right)^{m^{-n}}
$$

From this we see that the sequence $\alpha_{n}=\left(p_{n}^{m^{-n}}\right)$ is an increasing sequence, whereas the sequence $\beta_{n}=\left(\left(p_{n}+1\right)^{m^{-n}}\right)$ is decreasing with increasing $n$. Hence the sequence $\alpha_{n}$ is also bounded and therefore the limit $A:=\lim _{n \rightarrow \infty} p_{n}^{m^{-n}}$ exists. It follows that $p_{n} \leq A^{m^{n}}<p_{n}+1$ and so $p_{n}=\left\lfloor A^{m^{n}}\right\rfloor$.

We take $p_{1}=\left\lfloor A^{m^{1}}\right\rfloor=2^{77232917}-1$.

$$
2^{77232917}-1=2^{77232917}\left(1-\frac{1}{2^{77232917}}\right)<A^{m}<2^{77232917} .
$$

Taking the natural logarithm and observing that for small $x>0$ a simple explicit Taylor estimate gives $-x-x^{2}<\log (1-x)<-x-\frac{x^{2}}{2}$ we find that

$$
\begin{array}{r}
\frac{77232917}{m} \log 2-\frac{1}{m 2^{77232917}}-\frac{1}{m 4^{77232917}} \\
<\log A<\frac{77232917}{m} \log 2-\frac{1}{m 2^{77232917}}-\frac{1}{2 m 4^{77232917}} .
\end{array}
$$

(This is more precise than stated in the theorem. Higher order Taylor estimates are also possible.)

Note that from this one can evaluate $\log A$ and therefore $A$ with an accuracy of millions of digits. In particular, if $m=10^{10}$, then $A=1.00536773279814724017 \ldots$, and similarly for $m=3^{13}>1438989$. (To see the implication that knowledge on high precision of $\log A$ implies also high precision for $A$ : Let $A_{1}<A_{2}$ be two constants with $A_{2}=A_{1}(1+\varepsilon)$ (say), where $\varepsilon>0$ is a small constant. Then $\frac{\varepsilon}{2}<\log (1+$ $\varepsilon)=\log A_{2}-\log A_{1}=\log (1+\varepsilon)=\varepsilon-\frac{\varepsilon^{2}}{2} \pm \cdots<\varepsilon$ and

$$
A_{2}-A_{1}=A_{1} \varepsilon<2 A_{1}\left(\log A_{2}-\log A_{1}\right) .
$$

In other words, when $\log A$ is known with high precision and $A$ is of size, say, $1<$ $A<10$ as in the theorem, then very small deviations of $A$ to $A_{1}$ or $A_{2}$ would give both $\log A_{2}-\log A_{1}$ and $A_{2}-A_{1}$ with about the same precision. Hence $A$ is also known with high precision.)

Similarly, every reader can produce their own formula by choosing a large number $m$ and a quite large prime $q$, for example the largest currently known prime $q=2^{82,589,933}-1$. Then $\frac{\log q}{m}-\frac{2}{m q}<\log A<\frac{\log q}{m}$ determines $A$ with high precision.

Finally, as the actual distribution of primes might be much better than what can currently be proved, and based on some experiments, we conjecture the following:
Conjecture. There is a constant A (possibly near 1.1966746500705764022 ) such that $f(n)=\left\lfloor A^{(n+1)^{2}}\right\rfloor$ is prime for all $n \geq 1$.

Note that the exponent $(n+1)^{2}$ grows polynomially compared to the exponential growth of $m^{n}$ (for fixed $m$ ) in Mills-type examples. The value above would give $p_{1}=$ $2, p_{3}=5, p_{4}=17, p_{5}=89, p_{6}=641, p_{7}=6619, p_{8}=97829, p_{9}=2070443$.

ACKNOWLEDGMENTS. The author would like to thank Timothy Trudgian for drawing the attention to his student Caitlin Mattner's work, and to the referees and editor for useful comments on the manuscript. The author was partially supported by the Austrian Science Fund (FWF): W1230.

1. Caldwell, C. K., Cheng, Y. (2005). Determining Mills' constant and a note on Honaker's problem. J. Integer Seq. 8(4), Article 05.4.1.
2. Dudek, A. (2016). An explicit result for primes between cubes. Funct. Approx. Comment. Math. 55(2): 177-197.
3. Ellison, W., Ellison, F. (1985). Prime Numbers. A Wiley-Interscience Publication. New-York: John Wiley \& Sons; Paris: Hermann.
4. Fridman, D. Garbulsky, J., Glecer, B., Grime, J., Tron Florentin, M. (2019). A prime-representing constant. Amer. Math. Monthly. 126(1): 70-73.
5. Kuipers, L. (1950), Prime-representing functions. Nederl. Akad. Wetensch., Proc. 53: 309-310 = Indagationes Math. 12: 57-58.
6. Mattner, C. (2017). Primes in short intervals. B.Sc. Thesis: Australian National University. Canberra.
7. Matomäki, K. (2010). Prime-representing functions. Acta Math. Hungar. 128(4): 307-314.
8. List of Known Mersenne Prime Numbers www.mersenne.org/primes/ and www.mersenne.org/primes/?press=M77232917, accessed on November 24, 2019.
9. Mills, W. H. (1947). A prime-representing function. Bull. Amer. Math. Soc. 53: 604.
10. Niven, I. (1951). Functions which represent prime numbers. Proc. Amer. Math. Soc. 2: 753-755.
11. Ore, O. (1952). On the selection of subsequences. Proc. Amer. Math. Soc. 3: 706-712.
12. Ribenboim, P. (1996). The New Book of Prime Number Records. New York: Springer-Verlag.
13. Wright, E. M. (1951). A prime-representing function. Amer. Math. Monthly. 58(9): 616-618.
14. Wright, E. M. (1954). A class of representing functions. J. London Math. Soc. 29: 63-71.

Institut für Analysis und Zahlentheorie, Kopernikusgasse 24, 8010 Graz, Austria
elsholtz@math.tugraz.at

