A NOTE ON THE GEOMETRY OF FIGURATE NUMBERS

FRANTIŠEK MARKO

Abstract. We give a short proof of the formula $n^p = \sum_{\ell=0}^{p-1}(-1)^\ell c_{p,\ell} F_n^{p-\ell}$, where $F_n^{p-\ell}$ is the figurate number of dimension $p-\ell$, and $c_{p,\ell}$ is the number of $(p-\ell)$-dimensional facets of p-dimensional simplices obtained by cutting the p-dimensional cube. This formula was formulated as Conjecture 16 in [2].

Recall from [2] that the k-dimensional cube $0 \leq x_1, \dots, x_k \leq n-1$ is cut into $n!$ k-dimensional simplices given by $x_{\sigma_1} \geq x_{\sigma_2} \geq \dots \geq x_{\sigma_k}$ for a permutation σ of the set $\{1, \dots, k\}$. The $(k-\ell)$-dimensional facets of these simplices are given by $0 \leq x_1, \dots, x_k \leq n-1$ and the conditions $x_{\sigma_1} L_1 x_{\sigma_2} L_2 \dots L_{k-1} x_{\sigma_k}$, where exactly ℓ symbols L_i equal " $=$ ", and $k-1-\ell$ symbols L_i equal " $\geq$ ". Denote by $c_{k,\ell}$ the number of all such $(k-\ell)$-dimensional facets.

Also, denote bv $F_n^k = \binom{n+k-1}{n}$ the figure number of dimension k. In Section 5 of [2], we gave a geometric counting argument using the inclusion-exclusion principle in support of Conjecture 16 of [2] stating the following.

Proposition 1. *For all positive integers n and p, $n^p = \sum_{\ell=0}^{p-1}(-1)^\ell c_{p,\ell} F_n^{p-\ell}$.*

Proof: First we observe that $c_{p,\ell}$ counts the number of surjective maps from the set $\{1, \dots, p\}$ to the set $\{1, \dots, p-\ell\}$. Indeed, the $p-1-\ell$ symbols $L_{a_1}, L_{a_2}, \dots, L_{a_{p-1-\ell}}$ that are equal to " $\geq$ " in the expression $x_{\sigma_1} L_1 x_{\sigma_2} L_2 \dots L_{p-1} x_{\sigma_p}$ separate indices $\{1, \dots, p\}$ into $p-\ell$ nonempty groups $\{\sigma_1, \dots, \sigma_{a_1}\}$, $\{\sigma_{a_1+1}, \dots, \sigma_{a_2}\}$, $\{\sigma_{a_{p-\ell-2}+1}, \dots, \sigma_{a_{p-1-\ell}}\}$, and $\{\sigma_{a_{p-1-\ell}+1}, \dots \sigma_p\}$. Therefore the expressions $E = x_{\sigma_1} L_1 x_{\sigma_2} L_2 \dots L_{k-1} x_{\sigma_p}$ are in bijective correspondance to maps f_E given by $f(\sigma_1) = \dots = f(\sigma_{a_1}) = 1$, $f(\sigma_{a_1+1}) = \dots = f(\sigma_{a_2}) = 2, \dots$, $f(\sigma_{a_{p-\ell-2}+1}) = \dots = f(\sigma_{a_{p-\ell-1}}) = p-\ell-1$, and $f(\sigma_{a_{p-\ell-1}+1}) = \dots = f(\sigma_p) = p-\ell$.

According to (6.8) of [3], the number of surjections of a set of m elements onto a set of n elements equals $n!S(m,n)$, where $S(m,n)$ is the Stirling number of the second kind. Thus $c_{p,\ell} = (p-\ell)!S(p,p-\ell)$.

If we substitute $x = -n$ in the formula $x^p = \sum_{j=1}^{r} S(p,j)x(x-1)\dots(x-j+1)$ from Theorem 6.10 of [3], we obtain

$$n^p = \sum_{j=1}^{p}(-1)^{p-j} j! S(p,j) F_n^j = \sum_{\ell=0}^{p-1}(-1)^\ell c_{p,\ell} F_n^{p-\ell}. \qquad \square$$

Note that an earlier (and much longer) proof of this statement was given in [1].

References

[1] Cereceda, J.L., *Figurate numbers and sums of powers of integers.* arXiv:2001.03208

[2] Marko, F., Litvinov, S. (2020). Geometry of figurate numbers and sums of powers of integers. *Amer. Math. Monthly.* 127:1, 4-22.

[3] DeTemple, D., Webb,W., *Combinatorial Reasoning. An Introduction to the Art of Counting.* John Wiley & Sons, 2014.

Pennsylvania State University, 76 University Drive, Hazleton, PA 18202, USA
E-mail address: fxm13@psu.edu