A Gauche Perspective on Row Reduced Echelon Form and Its Uniqueness

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Dedicated to the memory of my father, Ozias "Ozi" Grimberg

Abstract

Using a left-to-right "sweeping" algorithm, we define the *Gauche basis* for the column space of a matrix M. Interpreting the row reduced echelon form (RREF) of M by *Gauche* means gives a direct proof of its uniqueness. A corollary shows that the (right) null space of M determines its row equivalence class, unmasks a sanitized version of the assertion "if two systems are solution equivalent they are row equivalent," and presents the null space as a distinguished graph. We conclude with pedagogical reflections.

1 Introduction.

The row reduced echelon form of a matrix M, RREF(M), is a useful tool when working with linear systems [2, 8, 16, 17]; its uniqueness is an important property. A survey of papers and textbooks yields a variety of uniqueness proofs. Some are simpler [18] and shorter than others. Generally proofs begin with two candidates for RREF(M) and conclude that these are equal. It is deemed desirable to have a direct proof, one that simply identifies every atom and molecule of RREF(M) in terms of properties of M and standard conventions. We use the *Gauche basis* of the column space of M to give such a proof, taking the opportunity to view RREF from a shifted perspective. This context makes it convenient to observe that the (right) null space of M determines its row space, without introducing orthogonality, and yields a near-converse of the familiar assertion "if two systems are row equivalent then they are solution equivalent." In conclusion we offer some reflections on teaching.

2 Conventions and Notations.

We will work mostly in the vector space \mathbb{F}^p , consisting of $p \times 1$ column vectors with entries in the field \mathbb{F} , and sometimes denote these as transposed row vectors, e.g., $(0 \ 1 \cdots 0)^t$. We'll adhere to the ordering conventions of *left to*

right and up to down. Thus the first column of a matrix is the leftmost, and first entry of a column is its top entry. Recall the notation for the "canonical" or "standard" basis of \mathbb{F}^p : $\{\vec{e}_j\}$, where \vec{e}_j stands for the $p \times 1$ column vector $\begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}^t$, with zeros throughout, except for a 1 in the *j*th entry. Recall also that the *span* of a set *S* of vectors in \mathbb{F}^p is the collection of all linear combinations of these vectors. Thus the span of the singelton set $\{\vec{v}\}$ consists of the set of all scalar multiples of \vec{v} , i.e., a line in \mathbb{F}^p , unless $\vec{v} = \vec{0}$, in which case the span of $\{\vec{v}\}$ is $\{\vec{0}\}$. We also have the degenerate case where *S* is the empty set; by convention, the span of the empty set is $\{\vec{0}\}$.

3 The Remembrance of Row Reduced Echelon Form (RREF).

Given a matrix M, viewed as the coefficient portion of a linear system $M\vec{x} = \vec{b}$, we can apply row operations to M, or to the augmented matrix $(M|\vec{b})$, and corresponding equation operations on the system $M\vec{x} = \vec{b}$, to yield a simpler system that is solution equivalent to the original. These operations include scaling a row by a nonzero scalar, interchanging two rows, and subtracting a scalar multiple of one row from another row. This last operation is the most commonly used, and is sometimes called a workhorse row operation.

Starting with a matrix M and applying carefully chosen row operations, one can obtain a matrix E with, arguably, the "best possible" form among all matrices row equivalent to M. This is the row reduced echelon form of M, or RREF(M), or just RREF. We use the definite article the because this form turns out to be be unique, as we'll see.

A matrix E is in RREF if it satisfies the following conditions.

• **Pivots.** Sweeping each row of *E* from the left, the first nonzero scalar encountered, if any, is a 1.

We call this entry, along with its column, a *pivot*.

- **Pivot column insecurity.** In a pivot column, the scalar 1 encountered in the row sweep is the only nonzero entry in its column.
- **Downright conventional.** If a pivot scalar 1 is to the right of another, it is also lower down.
- Bottom zeros. Rows consisting entirely of zeros, if they appear, are at the bottom of the matrix.

Although the RREF conditions may seem labored, a more fluent geometric interpretation will be given below. The term *pivot insecurity* requires explanation. We think of the pivot scalar 1's as insecure: they don't want competition from other nonzero entries along their column. Sorry pivots—row insecurity cannot be accommodated.

4 A Gauche Basis for a Matrix with A Fifth Column.

For the purpose of introduction and illustration we'll begin with a specific matrix [3, SAE]:

$$T \equiv \begin{pmatrix} 2 & 1 & 7 & -7 & 2 \\ -3 & 4 & -5 & -6 & 3 \\ 1 & 1 & 4 & -5 & 2 \end{pmatrix}.$$

We will "sweep" the columns of T from left to right, and designate each column as a *keeper* or as *subordinate*. These are meant to be value-neutral, not value judgments, and we hope that no vectors will take offense. For each column we ask

We will call this the *left-leaning question*, or LLQ for short. Columns for which the answer is *no* will be designated as *keepers* and the rest as *subordinates*.

When focusing on the first column of T, we recall the convention that a linear combination of the empty set is, in the context of a vector space V, the zero vector of V. Thus the LLQ for the first column of T is tantamount to asking:

Is this vector nonzero?

For T the answer is *yes.* Therefore, we adorn column one with the adjective *keeper.* With the aim of responsible accounting, we "journal" our action with the vector $\vec{J_1} \equiv \vec{e_1}$. (Recall that in our context $\vec{e_1}$ is the 3×1 column vector with a 1 in the first entry and zeros elsewhere.)

Next, we focus on column two and the LLQ, which, in the current context, asks:

Is this column a scalar multiple of column one?

The answer is no, so column two is a keeper, and we journal it with $\vec{J}_2 \equiv \vec{e}_2$.

The *LLQ* for third column asks if this column is a linear combination of the first two (keeper) columns of T; by inspection, column three is presentable as a linear combination of columns one and two, with scalings 3, 1, respectively. So column three is *subordinate* and we journal our action with the vector $\vec{J}_3 \equiv 3\vec{e}_1 + 1\vec{e}_2$, which encodes the manifestation of this vector as a linear combination of keeper columns to its left:

$$\begin{pmatrix} 7\\-5\\4 \end{pmatrix} = 3 \cdot \begin{pmatrix} 2\\-3\\1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1\\4\\1 \end{pmatrix}; \quad \vec{J_3} \equiv \begin{pmatrix} 3\\1\\0 \end{pmatrix}.$$

Similarly, the fourth column of T is subordinate, and journaled with

$$\vec{J}_4 \equiv (-2) \cdot \vec{e}_1 + (-3)\vec{e}_2$$

The fifth and final column vector of T is not presentable as a linear combination of previous keepers. The reader is invited to prove this or, alternatively, perform a half-turn on the solution box below.

We declare the fifth column a keeper, at our peril,¹ and journal it with $\vec{J_5} \equiv \vec{e_3}$. Now form a 3 × 5 matrix using the vectors we journaled, in the order we journaled them: $J \equiv \begin{pmatrix} \vec{J_1} & \vec{J_2} & \cdots & \vec{J_5} \end{pmatrix}$, or

$$J \equiv \begin{pmatrix} 1 & 0 & 3 & -2 & 0 \\ 0 & 1 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$
 (1)

This turns out to be the *RREF* of T, perhaps surprisingly. For an independent verification, using Gauss–Jordan elimination on the same matrix T, see Example SAE in [3]. Notice that our procedure does not show that (1) is row equivalent to T, whereas the Gauss–Jordan algorithm, e.g., as in [3], does. It's not difficult to show directly, in this context, that the Gauche procedure yields a matrix that is row equivalent to the original. In case anyone insists, we will prove this later on; the approach is entirely Gauss–Jordan-esque.

5 Beyond the Fifth Column: a General Gauche Algorithm.

Here we detail the procedure for generating the Gauche basis for an arbitrary matrix and use it to produce the corresponding RREF. For student readers, we suggest following the ideas of John H. Hubbard and Bill Thurston in *How To Read Mathematics*, [7]: jump to the illustrative concrete example above, whenever a point in the general procedure below appears sinister.

Let M be a $p \times q$ matrix (over a fixed field \mathbb{F}). We outline a general algorithm that transforms M into row reduced echelon form without invoking row reduction. This exhibits, among other things, the uniqueness of the row reduced

¹Fifth column-a group of secret sympathizers or supporters of an enemy that engage in espionage or sabotage within defense lines or national borders-Merriam-Webster dictionary.

echelon form. Sweeping the columns of M from left to right, we will adorn some of the columns with the title of *keeper*. Initially, the set of keepers is empty. Going from left to right, we take a column of M and ask the LLQ. For the first column of M this is tantamount to asking: Is this column nonzero? If so, we declare it a *keeper* and journal our action with the vector $\vec{J_1} \equiv \vec{e_1} \in \mathbb{F}^p$. If the first column is zero, we do not adorn it with the title of keeper; we call it subordinate and we journal our action with the vector $\vec{J_1} \equiv \vec{0}$.

In general, we examine the *n*th column of M and ask the LLQ. If this column is not in the span of the current keeper set, we adorn this column with the *keeper* designation and journal our action with the vector $\vec{J}_n \equiv \vec{e}_{\ell+1}$, where $\ell + 1$ is the number of keepers adorned up to this step, current column included. If the current column is presentable as a linear combination of (already designated) keepers, say $\alpha_1 \vec{k}_1 + \cdots + \alpha_\ell \vec{k}_\ell$, where the already designated keeper columns are $\{\vec{k}_i\}_{i=1}^{\ell}$, then we call the current column *subordinate* and journal our action with the vector $\vec{J}_n \equiv \alpha_1 \vec{e}_1 + \cdots + \alpha_\ell \vec{e}_\ell$, recalling that we are focusing on column nand we have ℓ keeper columns already designated. The careful (or fussy) reader may object that the current column may be expressible as a linear combination of keepers in more than one way. However, induction readily shows that at each stage the keeper set is linearly independent. At the end of this procedure we obtain a matrix E of the same size as M.

We will call the algorithm above, transforming M into E, the Gauche procedure and the resulting basis for the column space of M the Gauche basis.

Lemma 1. The matrix E is in row reduced echelon form.

Proof. In this discussion we will sometimes tacitly identify columns of E with corresponding columns of M. We take row i of E and "sweep" it from the left. We encounter a first nonzero entry in only one circumstance: where we meet a pivot of E, i.e., a journaled vector $\vec{J_{\ell}}$ corresponding to keeper column of M. (A nonzero entry in a subordinate column is always assigned only *after* a pivot 1 entry has already been assigned earlier in the same row.) In the Gauche algorithm, whenever we introduce a new journal vector $\vec{J_{\ell}}$, corresponding to a pivot, the scalar 1 appears in a lower slot than those of any prior keepers, and prior subordinate columns are linear combinations of prior keepers, so their entries are zero at this row altitude level as well.

What about the *downright* condition? If a pivot 1 is to the right of another, it is also lower down, as it gets adorned with the *keeper* designation at a later stage and is journaled as \vec{e}_k with a larger value of k.

When the pivot journaling stops, no further nonzero entries are journaled in rows lower than the row of the 1 entry in the last pivot column. Hence, in particular, all pure-zero rows are at the bottom of E. Thus we have verified that E is in RREF.

6 Zeroing in on the null space.

We now try to redouble our understanding of the meaning of RREF and its relation to the null space of a matrix.

When solving the linear system $M\vec{x} = \vec{b}$ by row reduction, the matrix M "calls the shots" and the right-hand side \vec{b} "comes along for the ride." That is, the row reduction steps are determined entirely by the coefficient matrix alone, and they are *applied* to the right-hand side vector. This suggests that RREF is not concerned much with the right-hand side \vec{b} , so we focus on the homogeneous system $M\vec{x} = \vec{0}$, i.e., the *null space* null(M).

There is an additional way in which null(M) figures into our discussion. The Gauche algorithm includes steps that may be called *decisional*: we must decide if a column is a keeper or is subordinate. A close re-reading shows that these decisions may be reinterpreted entirely in terms of the null space of M.

We are led to ponder the question: Are RREF and Gauche all about null(M)? Below we will show that this is indeed so: null(M) determines "everything." Moreover, we aim to prove this with "no work at all," somewhat in the spirit of Donald J. Newman's 1990s Thought Less (or thoughtless) Mathematics initiative. Newman sought to systematize a procedure for solving mathematical problems and proving theorems with no ingenuity required at all. The author recalls a colloquium talk delivered by Newman at Temple University, where he gave a thought less proof of the infinitude of the primes. A recently published proof, by I. Mercer (see [11]) is reminiscent of Newman's proof. Alas, not much of Newman's thought less initiative is in the literature. But there is this: [12]. Of course, it goes without saying that setting up a thought less proof is not a thought less undertaking.

To further the re-interpretation of RREF plan, we proceed by setting up a small "dictionary" between linear properties of columns and inclusion properties of the null space. For the benefit of student readers, we point out that small dictionaries are not uncommon in mathematics. (When they get larger, they turn into *categories* [9].) For instance, in [5, p. 11], we find: *one can set up a "dictionary" that translates properties of the matrix into optical properties*.

After these anticipatory remarks and before implementing proofs we need to add to our notational baggage. In working with columns of M (and of E) we used $\{\vec{e}_i\}_{i=1}^p$, the standard basis of \mathbb{F}^p . Now null(M) is a subset of \mathbb{F}^q and we'd like to work with the standard basis of this space as well. To avoid (read: reduce) confusion, we'll use the notation \vec{f}_i for the $q \times 1$ column vector with a 1 in slot i and zeros elsewhere, so that $\{\vec{f}_i\}_{i=1}^q$ is the standard basis of \mathbb{F}^q .

With this notation we observe that the first column of M is $M\vec{f_1}$, so asking if the first column of M is zero is tantamount to asking if the vector $\vec{f_1}$ belongs to the null space of M, i.e., if $M\vec{f_1} = \vec{0}$. Table 1 gives further illustration of this interplay.

7 RREF is unique.

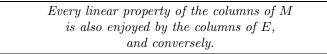
En route to proving the uniqueness of RREF, we state a lemma which, essentially, asserts that the matrix E comprises the columns of the matrix M written in the Gauche basis of the column space of M.

Lemma 2. Let M be a $p \times q$ matrix over a field \mathbb{F} and let E be a matrix in RREF which is row equivalent to M. Let $S \subseteq \{1, \ldots, q\}$ be the index set corresponding to the pivot vectors among the columns of E. Then:

- The columns of M corresponding to the index set S form the Gauche basis for the column space of M.
- Each nonpivot column \vec{c} of E is a linear combination of the pivot columns to its left. This combination exhibits the presentation of the corresponding column of M as a linear combination of Gauche basis vectors to its left. The "top" entries of \vec{c} encode this (unique) linear combination, and the rest of the entries of \vec{c} are "padded" zeros.

Note that if E has no pivots at all, then M = E = 0, which is consistent with the vacuous interpretation of the statement of the lemma. For the matrix J in (1), which is an instance of E, the set S is $\{1, 2, 5\}$.

Proof. It is well known that if M and E are row equivalent then the associated homogeneous linear systems $M\vec{x} = \vec{0}$ and $E\vec{x} = \vec{0}$ have the same solutions [3, REMES], [6, Theorem 3, p. 8]. That is, M and E have the same (right) null space. At the risk of slightly abusing language we state a heuristic principle:



This assertion requires some reflection and interpretation. It is inspired, in part, by a deep principle in the analysis of meromorphic functions [19]. (See [14] for a heuristic principle in the context of linear algebra.) Table 1 provides illustrations of this heuristic for M, and we can do the same for E. (Although we captioned the table as a dictionary, we have taken liberties with the language inside; we hope that this is forgivable.)

Iterating the idea, we can express in this null space way the statement

Columns j_1, \ldots, j_ℓ form the Gauche basis of the column space of (\cdot) .

and others like it. Indeed, in this way, all assertions in the statement of the lemma may be translated into assertions about inclusions in the respective null spaces. Hence these are shared values [13] for E and M.

Theorem 1. Let M be a matrix. Then there is one and only one matrix E in RREF that is row equivalent to M.

Linear Property of Columns	Inclusion Property of Null Space
The first column of M is nonzero.	The vector $\vec{f_1}$ is not in Null(<i>M</i>).
The kth column of M is in the span of columns $j_1, \ldots j_\ell$ of M .	There exist $\alpha_1, \ldots, \alpha_\ell$ so that $\alpha_1 \vec{f}_{j_1} + \cdots + \alpha_\ell \vec{f}_{j_\ell} - \vec{f}_k \in \text{Null}(M).$
Columns j_1, \ldots, j_ℓ of M form a linearly independent set.	For scalars $\alpha_1, \ldots, \alpha_\ell$ the vector $\alpha_1 \vec{f}_{j_1} + \cdots + \alpha_\ell \vec{f}_{j_\ell}$ is in Null (M) \Leftrightarrow the scalars $\alpha_1, \ldots, \alpha_\ell$ all vanish.

Table 1: (column property) \leftrightarrow (null space property) Dictionary.

Proof. The lemma above describes every entry of E in terms of left-down conventions and properties of M, without reference to any process for row reducing M to yield E, e.g., Gauss–Jordan elimination. This proves uniqueness. For existence, one can invoke the Gauss–Jordan algorithm, or prove directly (and, admittedly, with Gauss–Jordan-esque ideas) that E is row equivalent to M, as is done independently, below in Proposition 1.

Corollary 1. The null space of a matrix M determines the RREF and the row space of M. Hence if two matrices of the same size have the same null space, then they are row equivalent.

Proof. The matrix M has a unique RREF and its Gauche construction uses only the null space of M.

The relation between the null space and the row space of a matrix is well known and does not require the concept of orthogonality. This is mentioned repeatedly in [6], at times concretely in examples, at times in generality, but in passing. It is worthy of further promulgation.

8 Geometric User Inferface (GUI)

We take heed of [15] and affirm that, while linear algebra is algebraic, it is geometric as well. Thus the uniqueness of RREF, expressed algebraically above, may be viewed geometrically as well. The Gauche path to the RREF of a matrix e.g., T, presents the null space of the original matrix (T) as a graph over the vector subspace spanned by "axes" corresponding to the subordinate, or nonpivot columns of T. We can read (1) to say that the null space of T is the graph over the span of the third and fourth axes of \mathbb{R}^5 given by the relations



Among all the different ways to present the null space of T as a graph (within the Euclidean space with axes corresponding to columns of T), the RREF way employs as a base the span of the "rightmost" axes available for the task. Why *rightmost*, the reader may ask, given the Gauche perspective? The RREF exercises *leftmost* selection of pivot columns, making nonpivot columns rightmost. The nonpivot columns of T correspond to free variables for solutions of the linear system $T\vec{x} = \vec{0}$ and the pivot columns correspond to dependent variables. This is tantamount to presenting null(T) as a graph.

In the article [8], D. C. Lay points out that vector subspaces of Euclidean space are usually presented as either the locus of solutions to a homogeneous system of linear equations or the span of a collection of vectors, and offers algorithms to link the two presentation types. All the algorithms involve RREF and may be viewed as presenting the vector subspace as a graph over the rightmost span of axes available.

9 The solution determines the problem.

In the television game show *Jeopardy*! contestants are given answers and asked to guess the questions from whence they came. In calculus we introduce antiderivatives as "differentiation Jeopardy." The following linear-algebraic Jeopardy variant may be considered:

> If two linear systems have the same solution set, then they are row equivalent.

Literally, as stated, this assertion is manifestly false. (Please do not invoke it out of context.) For suppose we have two inconsistent linear systems. They both have the empty set of solutions, hence the same set of solutions. But the two systems may not have the same number of equations. They may even involve different variables. Clearly, we need to focus on consistent linear systems of the same size. We will also tacitly assume that they involve the same unknowns.

Corollary 2. If two consistent linear systems of the same size are solution equivalent, then they are row equivalent.

Proof. First assume that the systems are homogeneous. Then the hypothesis says that the corresponding matrices have the same null space. Hence, by the previous corollary, they have the same RREF and are thereby row equivalent. In the general case, simply note that the solution set of a (possibly) inhomogeneous linear system consists of one particular solution added to the solution space of the associated homogeneous system.

10 An Existential Question.

The Gauche procedure takes a matrix M and associates with it a matrix E that is in RREF. But how do we know that there exists a sequence of row operations taking M to E, i.e., why is E row equivalent to M?

We can invoke the Gauss–Jordan elimination algorithm which yields a matrix in RREF that is row equivalent to M and then cite uniqueness considerations to conclude that our Gauche E must be that matrix. But this is unsatisfying—we should be able to show directly that the Gauche-produced matrix E is rowequivalent to M and, if one insists, we can.

Proposition 1. For a matrix M, the Gauche-produced RREF matrix $E \equiv E(M)$ is row equivalent to M.

Proof. We can take M and row reduce it to yield the Gauche-produced matrix E following the algorithm illustrated below:

	$(^{0}$	$\cdot \cdot \cdot 0$	*	*		*)	$(^{0}$	0	$\neq 0$		*)
	0	$\cdot \cdot \cdot 0$	*	*		*	0	0	*		*
	:		:	*		*	1:		:		*
	0	$\cdot \cdot \cdot 0$	$\neq 0$	*		*		0	*		*
	:	 	÷	*		*	:		÷		*
	0	$\cdot \cdot \cdot 0$	*	*		*)	$\langle 0 \rangle$	0	*		*)
\rightarrow	(0)	0	1.		*)	$(^{0}$	0	1	. *)		
	0	0	* .		*	0	0	0	. *		
	:		:		*			:	. *		
	0	0	* .	• •	*	0	0	0.	. *		
	:	::	: .		*	:	 	:	. *		
	(⁰	0	* .	• •	*)	$\langle 0 \rangle$	0	0	. */		

If M is the zero matrix, then E = M and we are done. Otherwise, E has a first pivot column, which corresponds to the first nonzero column of M, say column j_1 . Taking M and permuting rows, we obtain a matrix whose first nonzero column is number j_1 , and which has a nonzero entry in the first slot; after scaling the first row we can assume that this entry is 1. Subtracting scalar multiples of the first row from each of the other rows, i.e., employing workhorse row operations, we obtain a matrix whose first nonzero column is the j_1 st, with entries equal to those of \vec{e}_1 . If E has no other pivot columns, then all later columns are scalar multiples of the j_1 st, and we are done. If E has a second pivot column, say in slot j_2 , then this column must have a nonzero entry below the first pivot. Permuting rows other than the first and then applying workhorse-type operations and rescaling the top nonzero entry in this column, we obtain \vec{e}_2 in the j_2 nd slot while retaining \vec{e}_1 in the first slot. Continuing this way, we produce row operations that place appropriate canonical vectors of the form \vec{e}_{ℓ} in each of the pivot slots. Each of the nonpivot columns is a linear combination of the pivot columns to its left, and requires no additional "processing" by row operations. Thus we have exhibited $E \equiv E(M)$ as the result of a sequence of row operations applied to M.

11 Reflections on Teaching.

The method of elimination via row reduction may be introduced at the very start of a course on linear algebra. Taking the Gauche approach to echelon form, we are led naturally, directly, and concretely to the notions of *linear combination*, span, and linear independence. Definition and application are threaded—no need for a separate introduction with rationale for use. This brings to mind a parallel in a *Math Proof* course. Every such course covers Euclid's proof of the infinitude of primes, and rightly so. But we can also add H. Furstenberg's "topological" proof [1, 4]. Fursternberg's proof leads directly to the basic set operations of intersection, union and complement. Here too, definition and application are threaded and allied; motivation is built in. True, a direct reading of Furstenberg's proof does require some familiarity with topology, possibly turning the motivation upside down. And there is a variant of Furstenberg's proof that does not require topological notions: [10]. Then again, the topological aspect of the proof may be regarded as a teaching feature, not a bug, anticipating notions to come in later courses. Also, this proof requires no theorems in *topol*ogy, but only the definition of the term. The challenge, then, is to introduce the concept of open set in a brief, self contained, pedagogically sound manner, so as to pave the way for Fursternberg's proof early in a proofs course. Here we have tried to address the linear algebraic analogy, which is easier.

We conclude with a question: Is there a *book proof* (see [1]) of the uniqueness of RREF? Is the fact worthy of inclusion in *The Book*?

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