NUMBER CUBES WITH CONSECUTIVE LINE SUMS

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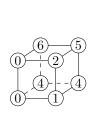
January 9, 2021

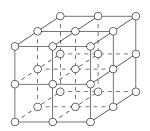
ABSTRACT. We settle the existence of certain 'anti-magic' cubes using combinatorial block designs and graph decompositions to align a handful of small examples.

1. Introduction

We address the following question:

Does there exist, for every integer $n \ge 2$, an $n \times n \times n$ cube of nonnegative integers whose $3n^2$ line sums, in the direction of coordinate axes as shown, are the integers $0, 1, 2, \ldots, 3n^2 - 1$?





After a few moments of thought, the reader might find a solution for n=2 similar to the one shown. The next size, n=3, can be settled with a little more persistence, or perhaps with the help of a computer program. A solution for this (and other small examples) are left for the appendix so as not to spoil the fun of the puzzle. Our main result, stated later as Theorem 4.1, answers the above question in the affirmative.

The two-dimensional version of the problem is fairly easy to settle. Consider integers arranged in an $n \times n$ grid. The sum of all of the row sums is equal to the sum of all of the column sums. So these line sums can exhaust the consecutive integers $0,1,2,\ldots,2n-1$ only if their total, $\frac{1}{2}(2n-1)(2n)=n(2n-1)$, is even. That is, n must be even to admit a number 'square' with consecutive line sums. And indeed, for n even, the array shown below gives a solution. (Blank entries are zero.)

Research of the first author is supported by NSERC grant 312595–2017.

Returning to our (three-dimensional) problem, such number cubes have relationships with combinatorial designs and graph decompositions. We tour several topics in these areas while setting up our constructions. A size-n number cube with line sums $0, 1, 2, \ldots, 3n^2 - 1$ is called a Sarvate-Beam cube, abbreviated SBC(n), for its connection with a combinatorial design variant introduced by D.G. Sarvate and W. Beam, [4].

2. Some related objects

2.1. Latin squares. A Latin square of order n is a $n \times n$ array with entries from an n-element set (often assumed to be $[n] := \{1, 2, \ldots, n\}$), such that every element appears exactly once in each row and each column. A latin square of order n = 3 is given below, and it is not hard to extend the circulant construction to any positive integer n.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Given a Latin square L of order n, we may define a $\{0,1\}$ -valued cube $\widehat{L}:[n]^3 \to \{0,1\}$ such that

$$\widehat{L}(i,j,k) = \begin{cases} 1 & \text{if } L_{ij} = k, \\ 0 & \text{otherwise.} \end{cases}$$

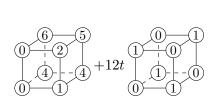
It is simple to check that each of the $3n^2$ line sums of \widehat{L} equals 1. As a result, we may add a multiple of \widehat{L} (entrywise) to an SBC(n) to produce a cube with consecutive line sums starting at any nonnegative integer. We denote an $n \times n \times n$ cube with line sums $a, a+1, a+2, \ldots, a+3n^2-1$ by SBC_a(n).

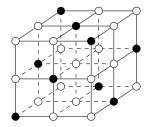
Latin squares also facilitate a construction to "inflate" the size of our number cubes.

Lemma 2.1 (see also [2]). If there exists an SBC(n), then there exists an SBC(mn) for every positive integer m.

PROOF. Let L be a Latin square of order m with entries in $\{0, 1, \ldots, m-1\}$ and, from the remarks above, let C_t be an $SBC_{3tn^2}(n)$ for $t = 0, 1, 2, \ldots, m^2 - 1$. Arrange these m^2 cubes in an $m \times m \times m$ array by putting C_{km+i} in position (i, j, k) whenever $L_{ij} = k$. With all other entries equal to zero, the line sums in the size-mn cube are precisely the line sums in the size-mn cubes, which altogether cover the interval from 0 to $3m^2n^2 - 1$.

We illustrate the method by building an SBC(6). First, we take nine copies of an SBC_{12t}(2) with abutting line sums, as shown at left for $t=0,1,\ldots,8$. These are placed (in any order) at the preimages $(i,j,k) \in \widehat{L}^{-1}(1)$ arising from a latin square L of order 3, shown with solid dots at right. The rest of the $6 \times 6 \times 6$ cube is filled with zeros.





Although Lemma 2.1 reduces our problem of finding SBC(n) to prime values n, a direct construction for primes has eluded our efforts. Later, we introduce a more powerful construction that works equally well for both prime and composite integers.

2.2. **Designs.** Let v be a positive integer and $K \subseteq \{2, 3, 4, ...\}$. A pairwise balanced design PBD(v, K) is a pair (V, \mathcal{B}) , where

- V is a set with |V| = v;
- \mathcal{B} is a family of subsets of V, called *blocks*, where $|B| \in K$ for every $B \in \mathcal{B}$; and
- any two distinct elements of V appear together in exactly one block.

A special class of designs useful for our constructions to follow are the finite planes. From a finite field \mathbb{F}_q of order q, we may construct an affine plane, with elements \mathbb{F}_q^2 and blocks given by the affine lines. This produces a pairwise balanced design $PBD(q^2, \{q\})$. If each of the q+1 parallel classes of lines is projectively extended, the result is a projective plane of order q, which is also a $PBD(q^2 + q + 1, \{q + 1\})$.

A PBD $(v, \{3\})$ is also known as a *Steiner triple system*; these are known to exist for all positive integers $v \equiv 1, 3 \pmod{6}$. Two interesting small examples come from finite planes. A Steiner triple system with v = 7 arises from the projective plane of order q = 2. An explicit construction on $V = \mathbb{Z}/7\mathbb{Z}$ comes from developing the "base block" $\{0,1,3\}$ additively mod 7, producing a family \mathcal{B} of seven 3-subsets covering every pair exactly once. A Steiner triple system with v = 9 arises from the affine plane of order q = 3. Here, a convenient presentation is to take as points the nine elements of a 3×3 grid, and as blocks the rows, columns, diagonals, and broken diagonals, giving twelve 3-subsets in \mathcal{B} . More information on finite planes and Steiner triple systems, including their historical origins, can be found in [1].

In general, Wilson's theory [6] says that pairwise balanced designs PBD(v, K) exist for all v greater than some constant $v_0(K)$, provided certain congruence conditions hold. The congruence conditions disappear if, for instance, K contains three consecutive integers. To illustrate Wilson's theory in this case, and for later reference, we cite the following result.

Lemma 2.2 ([3]). There exists a $PBD(v, \{4, 5, 6\})$ for all $v \ge 24$, and also for $v \in \{4, 5, 6, 13, 16, 17, 20, 21, 22\}$.

In [4], a curious variant of Steiner triple systems was introduced. A Sarvate–Beam triple system of order v, or SBTS(v), is an assignment $f:\binom{[v]}{3}\to\mathbb{Z}_{\geq 0}$ of nonnegative integer weights to the 3-subsets of [v] such that

(2.1)
$$\{\tilde{f}(\{i,j\}): 1 \le i < j \le v\} = \{0,1,2,\ldots,\binom{v}{2}-1\},$$

where $\tilde{f}(\{i,j\}) = \sum_{k \neq i,j} f(\{i,j,k\})$. That is, an SBTS(v) is a multiset of 3-element subsets with the property that the induced frequencies on 2-element subsets cover the range of values from 0 to $\binom{v}{2} - 1$, inclusive. Note that if the right side of (2.1) is changed to $\{1\}$, we recover the definition of a Steiner triple system. For example, an SBTS(5) arises from the following assignment of (positive) block multiplicities.

block	$\{2, 3, 4\}$	$\{1, 3, 5\}$	$\{2, 3, 5\}$	$\{1, 4, 5\}$	$\{2, 4, 5\}$	${\{3,4,5\}}$
multiplicity	2	1	3	2	5	2

See also [5] for a similar example and [2] for a construction for each $v \geq 4$.

2.3. Graph and multigraph decompositions. Taking an alternative viewpoint, a Steiner triple system on v elements is equivalent to an edge-decomposition of the complete graph K_v into triangles. Correspondingly, an SBTS(v) is an edge decomposition into triangles of a certain multigraph on v vertices, namely one whose edge multiplicities are $0, 1, 2, \ldots, \binom{v}{2} - 1$ (in some arrangement).

For a simple graph G and nonnegative integer a, let $\Gamma_a(G)$ denote the set of all multigraphs obtained by assigning multiplicities $a, a+1, \ldots, a+|E(G)|-1$ to the edges of G. Let $\Delta_a(G)$ denote the subset of $\Gamma_a(G)$ consisting of those graphs that admit a triangle decomposition. An SBTS(v) exists if and only if $\Delta_0(K_v) \neq \emptyset$.

Consider now the complete 3-partite graph $K_{n,n,n}$. A triangle decomposition of this graph is equivalent to a Latin square of order n: simply associate each triangle, say on vertices $\{i_1, j_2, k_3\}$ (where subscripts indicate partite sets), with the placement of symbol k in position (i, j). Similar to the above, the existence of a Sarvate-Beam cube of order n is equivalent to showing $\Delta_0(K_{n,n,n}) \neq \emptyset$.

The direct product graph $K_3 \times K_n$ resembles $K_{n,n,n}$, except that the former is missing the edges from a family of n vertex-disjoint triangles. A Latin square L whose diagonal entries satisfy $L_{ii} = i$ is called *idempotent*. It is easy to see that an idempotent Latin square of order n exists for all positive integers $n \neq 2$. Using the off-diagonal entries of an idempotent Latin square of order n, we see that the graph $K_3 \times K_n$ has a triangle decomposition for all $n \neq 2$.

For our construction to follow, it is helpful to have an SBC-like variant based on $K_3 \times K_n$. We define a holey Sarvate-Beam cube, or SBHC $(n, 1^n)$ to be a size-n cube where entry (i, j, k) is filled with a nonnegative integer if and only if i, j, k are pairwise distinct, and such that the 3n(n-1) nonempty line sums are $0, 1, 2, \ldots, 3n(n-1) - 1$. An SBHC $(n, 1^n)$ is equivalent to a triangle decomposition of a multigraph in the class $\Delta_0(K_3 \times K_n)$. The appendix gives an SBHC $(n, 1^n)$ for each $n \in \{4, 5, 6\}$.

Let J_n denote the graph $K_3 \times K_n$ with one additional triangle, as depicted in Figure 1 for $n \in \{3, 4\}$.

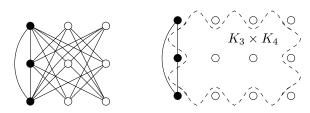


FIGURE 1. The graphs J_3 and J_4 .

Following the notation for holey Sarvate–Beam cubes above, it is natural to use the notation SBHC $(n, 1^{n-1})$ for a triangle decomposition (in cube form) of a multigraph in $\Delta_0(J_n)$, n = 3, 4. An example cube for each of these sizes appears in the appendix.

3. A construction

We illustrate the main idea of our construction to follow by building a solution to our problem for n = 7. In a little more detail, we use a template Steiner triple system on 7 elements and a graph in $\Delta_0(J_3)$ to construct an SBC(7).

Example 3.1. We claim that $K_{7,7,7}$ decomposes into seven copies of J_3 , using a Steiner triple system with blocks

$$\{\underline{0},1,3\},\{\underline{1},2,4\},\{\underline{2},3,5\},\{\underline{3},4,6\},\{\underline{4},5,0\},\{\underline{5},6,1\},\{\underline{6},0,2\}.$$

Each element is replaced by three vertices; each block is to be replaced with a copy of J_3 so that the underlined element takes the role of the three filled vertices of J_3 in Figure 1. Consider an edge (i_1, j_2) of $K_{7,7,7}$. If i = j, this edge appears in the copy of J_3 on the block in which i is underlined. Otherwise, if $i \neq j$, this edge is in the copy of J_3 corresponding to the block of the triple system containing $\{i, j\}$.

Finally, we turn $K_{7,7,7}$ into a triangle-decomposable multigraph as follows: the first copy of J_3 is replaced by a graph in $\Delta_0(J_3)$, the second by a graph in $\Delta_{21}(J_3)$, the third by a graph in $\Delta_{42}(J_3)$, and so on. Note that each graph in $\Delta_a(J_k)$ can be obtained from a corresponding graph in $\Delta_0(J_k)$ by increasing the multiplicity of each edge by a. Because every edge of $K_{7,7,7}$ occurs in some copy of J_3 , and because starting values were chosen as multiples of $|E(J_3)| = 21$, we have constructed a graph in $\Delta_0(K_{7,7,7})$.

We present a general construction that captures the above technique.

Construction 3.2. Suppose $\{G_1, \ldots, G_b\}$ is an edge decomposition of G. Suppose, for each $i = 1, \ldots, b$, the graph G_i has a triangle decomposition and also that $\Delta_0(G_i) \neq \emptyset$. Then $\Delta_0(G) \neq \emptyset$; that is, some multigraph in $\Gamma_0(G)$ has a triangle decomposition.

PROOF. Put $m_i = |E(G_i)|$ for each i, and $a_i = \sum_{j < i} m_j$. Since each G_i is simple and has a triangle decomposition, we may increase all edge multiplicities of a graph in $\Delta_0(G_i)$ to produce a graph in $\Delta_a(G_i)$ for any nonnegative integer a. Let H_i denote a multigraph in $\Delta_{a_i}(G_i)$.

We have that $\{H_1, \ldots, H_b\}$ is an edge decomposition of some graph in $\Gamma_0(G)$, call it H, since the edges of G occur with multiplicities $0, \ldots, a_1 - 1$ in $H_1, a_1, \ldots, a_2 - 1$ in H_2 , and so on, until $a_{b-1}, \ldots, a_b - 1$ in H_b . Since each H_i has a triangle decomposition, it follows that H does as well. \square

4. Solution of the problem

We are now ready for our main result. The proof essentially consists of a few direct constructions and a prescription to apply Construction 3.2 for $G = K_{n,n,n}$. The necessary 'building block' constructions can be found in the appendix.

Theorem 4.1. There exists an SBC(n) for every integer $n \geq 2$.

PROOF. Suppose first that there exists a $PBD(n + 1, \{4, 5, 6\})$. We claim there exists an SBC(n).

If we delete one element from the hypothesized PBD, the result is a pairwise balanced design on n elements with a 'parallel class' \mathcal{A} of blocks with sizes in $\{3,4,5\}$, and other block sizes in $\{4,5,6\}$. Replace every element x with three vertices x_1,x_2,x_3 . Replace every block B, where $|B|=m\in\{3,4,5,6\}$, either by the graph $K_{m,m,m}$ if $B\in\mathcal{A}$, or by $K_3\times K_m$ otherwise. The result is an edge decomposition of $K_{n,n,n}$ into graphs isomorphic to $K_{m,m,m}$ for $m\in\{3,4,5\}$, and $K_3\times K_m$ for $m\in\{4,5,6\}$. From Construction 3.2 and our examples given in the appendix, it follows that $\Delta_0(K_{n,n,n})\neq\emptyset$; that is, there exists an SBC(n).

In view of Lemma 2.2, the preceding construction leaves as exceptions $n \in \{7, ..., 11, 13, 14, 17, 18, 22\}$. Except for n = 7, 11, 13, 17, each of these values has a prime divisor less than or equal to 5, and so an SBC(n) exists by Lemma 2.1 and the Sarvate-Beam cubes of order 2, 3, and 5 given in the appendix. An SBC(7) was built in Example 3.1. We turn to the remaining three values.

For n = 11, we start with an affine plane of order three, or PBD(9, $\{3\}$), and extend two parallel classes using one new element for each to produce a PBD(11, $\{2, 3, 4\}$). The block set is, say,

```
\begin{array}{lll} \{\underline{1},2,3,\infty_1\}, & \{1,\underline{4},7,\infty_2\}, & \{1,\underline{5},9\}, & \{1,6,8\}, \\ \{4,5,\underline{6},\infty_1\}, & \{\underline{2},5,8,\infty_2\}, & \{2,6,\underline{7}\}, & \{2,4,9\}, \\ \{7,\underline{8},9,\infty_1\}, & \{3,6,\underline{9},\infty_2\}, & \{\underline{3},4,8\}, & \{3,5,7\}, & \{\underline{\infty_1},\underline{\infty_2}\}. \end{array}
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Blocks with one underlined element are replaced with J_3 or J_4 according to the block size. The underlined element takes the role of filled vertices as in Example 3.1. The remaining blocks of size three are replaced with $K_3 \times K_3$. The block of size two is replaced with $K_{2,2,2}$. The above set of graphs decompose $K_{11,11,11}$, and each satisfies the assumptions of Construction 3.2. This gives an SBC(11).

For n = 13, we take a projective plane of order three, or PBD(13, {4}), with blocks

$$x + \{0, 1, 3, 9\}, x = 0, 1, 2, \dots, 12,$$

where addition, mod 13, distributes into the block. Using this cyclic structure, it is possible to underline exactly one element from each block, as shown. Accordingly, replace each block with a copy of J_4 . This gives a covering of the edges of $K_{13,13,13}$ as needed for Construction 3.2, and hence an SBC(13).

For n = 17, we start with an affine plane of order four and extend one parallel class to produce a PBD(17, $\{4,5\}$). Explicitly, the blocks can be taken as

```
\{4, 5, 8, 14\},\
\{\underline{1},\underline{2},\underline{3},\underline{4},\underline{\infty}\},\
                                          \{1, \underline{5}, 9, 16\},\
                                                                             \{2, 5, 11, \underline{15}\},\
                                                                                                             \{3, 5, 7, \underline{12}\},\
                                         \{1, \underline{6}, 7, 8\},\
                                                                             \{2, 6, 12, \underline{14}\},\
\{5, 6, 10, 13, \infty\},\
                                                                                                             \{3, 6, \underline{9}, 15\},\
                                                                                                                                                 \{4, 6, 11, 16\},\
                                         \{1, \underline{10}, 14, 15\},\
                                                                          \{2, \underline{7}, 10, 16\},\
\{7, 9, 11, 14, \infty\},\
                                                                                                             \{3, 8, 10, \underline{11}\}.
                                                                                                                                                 \{4, 9, 10, 12\},\
\{8, 12, 15, 16, \infty\}, \{1, 11, 12, \underline{13}\}, \{2, \underline{8}, 9, 13\},\
                                                                                                             {3, 13, 14, 16},
```

We have again underlined each element exactly once. The first block is to be replaced by $K_{5,5,5}$. Blocks with exactly one underlined element are to be replaced by J_4 . Other blocks are replaced by $K_3 \times K_4$ and $K_3 \times K_5$ according to the size. Similar to before, we have covered the edges of $K_{17,17,17}$ so as to satisfy the hypotheses of Construction 3.2. The result is an SBC(17).

Appendix

Apart from the simple case of SBC(2), our proof requires only seven explicit cubes or cube-variants. Layers of the $n \times n \times n$ cube are given in separate $n \times n$ grids and line sums appear in bold text.

SBC(3) (underlying graph $K_{3,3,3}$; see also [2])

0 1	12 13	0	14 8	22			1			0	16	20
2 3	1 6	0	5 0	5		7	11	3	21	9	19	4
0 10	5 15	3	7 0	10		4	0	20	24	7	17	25
2 14	18	3	26 8		_	11	12	23				

SBC(5) (underlying graph $K_{5,5,5}$)

```
16 9 6 9 1 41
                            0 0 10 2 0 12
                                                       15 34 2 6 15 72
14 0 0 6 27 47
                            1 2 2 1 0 6
                                                       35 0 3 0 2 40

    0
    0
    0
    2
    0
    2

    0
    60
    14
    0
    0
    74

    0
    7
    8
    49
    3
    67

    1
    69
    34
    54
    3

4 13 5 14 0 36
                                                      13 13 16 1 2 45
20 2 0 2
7 8 41 0
                5 29
                                                       0 0 0 0 0
                                                       8 2 7 0 1 18
                   60
               4
                                                      71 49 28 7 20
61 32 52 31 37
                            5 7 6 25 0 43
10 0 0 0 5 15
                                                      46\ 50\ 24\ 42\ 21
                            2 1 1 15 4 23
29 0 4 2 0 35
37 0 20 0 0 57
1 6 5 5 0 17
                                                      53 9 11 27 33
2 0 0 39 14 55
                                                      48\ 26\ 25\ 58\ 16
5 2 32 12 0 51
                                                       62\ 64\ 66\ 14\ 5
                             0 0 8 2 0 10
41 5 1 12 11 70
                                                      56 22 65 63 19
59 13 38 68 30
                            73 8 39 44 4
```

SBHC(4, 1^4) (underlying graph $K_3 \times K_4$)

			14		26	8	34				29								
	14	0	14					1			1							35	0
			3	22		9	31					0	12		12	22	13		11
	9		27	2	6		8	5	0		5					7	18	15	
19	23	2		24	32	17		6	4	25		9	28	21					

 ${\rm SBHC}(5,1^5) \quad \ ({\rm underlying \ graph} \ K_3 \times K_5)$

```
. . 49 9 0 58
. . . . . . .
1 . . 9 2 12
. 8 . 27 1 36
                                        3 . . 10 27 40
. 0 . 10 1 11
                                       . 3 0 . 3 6
                    6 . 2 . 2 10

    . 24 0 4 . 28

    27 0 59 7

                   19 . 0 25 . 44
26 51 43 4
                                       18 30 37 47
. 9 1 . 15 25
                    . 22 0 19 . 41
                                          39 50 55 16
42 . 2 . 2 46
                    0 . 0 1 . 1
                                              2 56 32
                                        45
4 4 . . 0 8
                    0 9 . 0 . 9
                                        5 13 19 3
                               . 54
                    33 0 21 .
                                        42\ 15\ 23
                                                    24
7 1 49 . . 57
                                        38 35 49 29
53 14 52 17
                    33 31 21 20
```

SBHC(6, 1⁶) (underlying graph $K_3 \times K_6$)

							1	0	1	1	3		7	1	24	16	48
	6	12	1	9	28							29		7	5	1	42
5		4	12	16	37	5		0	0	0	5						
2	3		0	2	7	8	1		14	16	39	3	52		10	1	66
3	11	1		36	51	70	0	0		14	84	0	11	0		1	12
19	2	4	2		27	5	43	0	32		80	3	0	9	4		16
29	22	21	15	63		88	45	0	47	31		35	70	17	43	19	

```
. 3 0 . 13 16 32
                     . 21 56 0 . 1 78
                                          . 38 7 0 37 . 82
2 . 0 . 43 36 81
                     0 . 0 9 . 0 9
                                         48 . 5 2 4 . 59
40 10 . . 6 11 67
                     0 12 . 5 . 9 26
                                         15 50 . 1 2 . 68
                     2 4 0 . . 66 72
                                          1 0 4 . 1 . 6
                    38 1 0 10 . .
7 18 42 . . 20 87
                                                      . 13
                                          9 1 2 1 .
8 21 20 . 12 . 61
57 52 62 74 83
                     40 38 56 24
                                         73 89 18 4 44
```

```
69 64 1 75 34
79 11 30 53 46
60 77 10 20 36
14 58 8 25 85
86 33 55 2 71
54 41 65 23 50
```

SBHC $(3, 1^2)$ (underlying graph J_3)

SBHC(4, 1^3) (underlying graph J_4)

1	13	4	2	20	3	. 0	1	4	1	4	. 31	36	1	0	18	. 1	9	6	17	22	34
1		19	10	30					22		. 4	26	0		9		9	23		2 8	14
25	7		0	32	3		15	18					10	3		. 1	.3	38	10		15
6	1	1			25				6									37	2	1	
33	21	24	12	_	31	0	16	_	29	5	35	_	11	3	27						

References

- [1] C.J. Colbourn and J.H. Dinitz, Handbook of Combinatorial Designs, 2nd ed., CRC Press, 2007.
- [2] P.J. Dukes and J.A. Short-Gershman, A complete existence theory for Sarvate-Beam triple systems. *Australas. J. Combin.* 54 (2012), 261–272.
- [3] H. Lenz, Some remarks on pairwise balanced designs. Mitt. Math. Sem. Giessen 165 (1984), 49-62.
- [4] D. Sarvate and W. Beam, A new type of block design. Bull. Inst. Combin. Appl. 50 (2007), 26–28.
- [5] R.G. Stanton, Sarvate-Beam Triple Systems for $v \equiv 2 \pmod{3}$. J. Combin. Math. Combin. Comput. 61 (2007), 129–134.
- [6] R.M. Wilson, An existence theory for pairwise balanced designs: II, The structure of PBD-closed sets and the existence conjectures. J. Combin. Theory Ser. A 13 (1972), 246–273.

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