

A Fractal Eigenvector

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Abstract. The recursively-constructed family of Mandelbrot matrices M_n for $n = 1, 2, \dots$ have nonnegative entries (indeed just 0 and 1, so each M_n can be called a *binary* matrix) and have eigenvalues whose negatives $-\lambda = c$ give periodic orbits under the Mandelbrot iteration, namely $z_k = z_{k-1}^2 + c$ with $z_0 = 0$, and are thus contained in the Mandelbrot set. By the Perron–Frobenius theorem, the matrices M_n have a dominant real positive eigenvalue, which we call ρ_n . This article examines the eigenvector belonging to that dominant eigenvalue and its fractal-like structure, and similarly examines (with less success) the dominant singular vectors of M_n from the singular value decomposition.

1. PLOTS, DIRECTED GRAPHS, AND AN EPIGRAPH.

For a construction to be useful and not mere waste of mental effort, for it to serve as a stepping-stone to higher things, it must first of all possess a kind of unity enabling us to see something more than the juxtaposition of its elements. —Henri Poincaré, *Science and Hypothesis* [15]

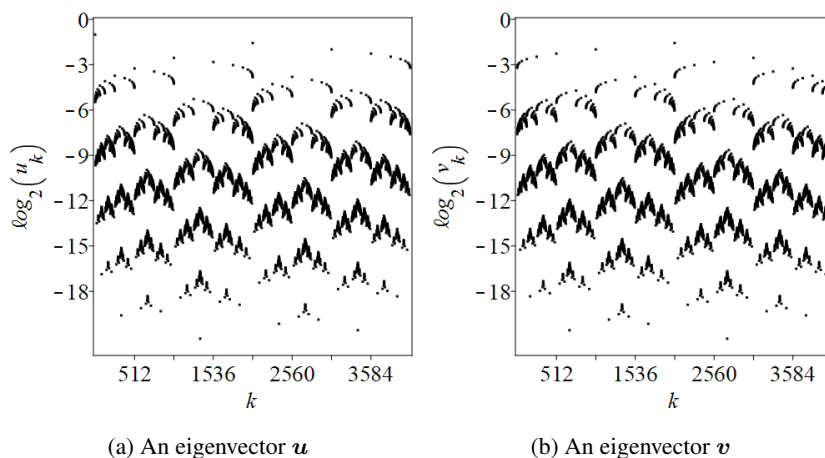


Figure 1. A discrete plot of the components of eigenvectors corresponding to the (same) dominant eigenvalue of two particular nonnegative integer 4095-by-4095 matrices, graphed on a base-2 logarithmic scale. At this dimension, there are enough components to give the illusion of connected structures, which we seek to understand. A higher-dimensional plot is shown later, in Figure 11.

This article seeks to explain a visual curiosity, namely that of Figure 1, where visible structures seem to repeat, slightly transformed, at smaller scales. But what do we mean by an “explanation?” What constitutes a *mathematical* explanation? By the way, those structures are *not* the result of rounding errors, in spite of our doing the computation only in standard hardware precision floating-point arithmetic.

We will see a connection with the Mandelbrot set. After seeing the name Mandelbrot get involved, the reader might no longer be surprised that repeating transformed

structures occur, because nowadays self-similarity and fractals are familiar features of the mathematical landscape.¹ But can we say more than that, and can we satisfy Poincaré’s dictum quoted at the start of this section? We think so.

Let us begin with a recursive construction of a family of directed graphs (digraphs). Consider the following digraphs G_n on $d_n = 2^n - 1$ vertices labeled $1, 2, 3, \dots, d_n$. For $n = 1$ we define the digraph G_1 to be just one vertex with one loop; that is, an edge connecting the vertex to itself. See Figure 2.

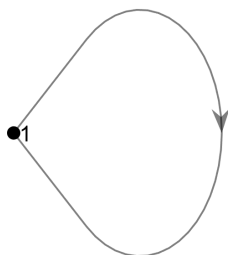


Figure 2. The directed graph G_1 : just a single loop.

For $n = 2$ we define the digraph G_2 to consist of two copies of G_1 , together with a new vertex between the two copies, and three new edges: two connecting the new vertex to each copy and the third directly connecting the first copy to the second. That is, we make two copies, add a vertex, and connect them all with three new edges. See Figure 3.

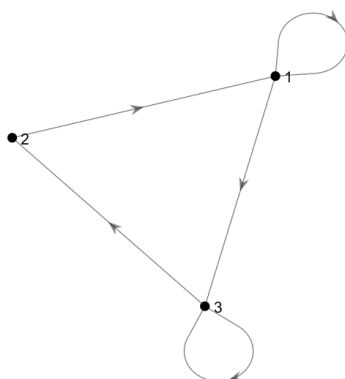


Figure 3. The directed graph G_2 , drawn with the “force” model in Matlab: imagine an electrical charge placed at each vertex, repelling all other vertices; and imagine a mechanical spring replacing each edge, pulling connected vertices together. The pictured configuration is an approximate minimization of the potential energy of this model. At equilibrium in G_2 , the vertices are equidistant by symmetry. The “springs” of the two loops have no effect, of course.

For $n = 3$ we repeat the process. The digraph G_3 is defined to consist of two copies of G_2 , with a new vertex between the two copies and edges connecting the new vertex to each copy and a new edge from the first copy to the second. Again we have made two copies, added a vertex, and added three edges. See Figure 4.

¹We won’t formally define *fractal* here, or pursue the many known facts about the Mandelbrot set. We’re going to stick to the finite, and say only that some things that we draw *look like* they might become fractals in the limit as the dimension goes to infinity.

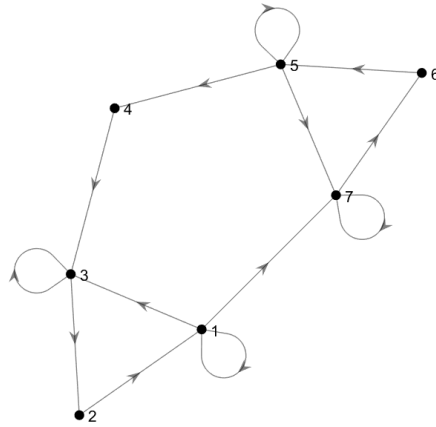


Figure 4. The directed graph G_3 , containing two copies of G_2 , drawn with the “force” model in Matlab. We see the two copies of G_2 with their loops, and the new vertex 4 between the two copies, and the new edge between vertices 1 and 7. We can imagine the effect of the repulsion between vertices owing to the electrical “charge” being balanced by the pull of the “springs.”

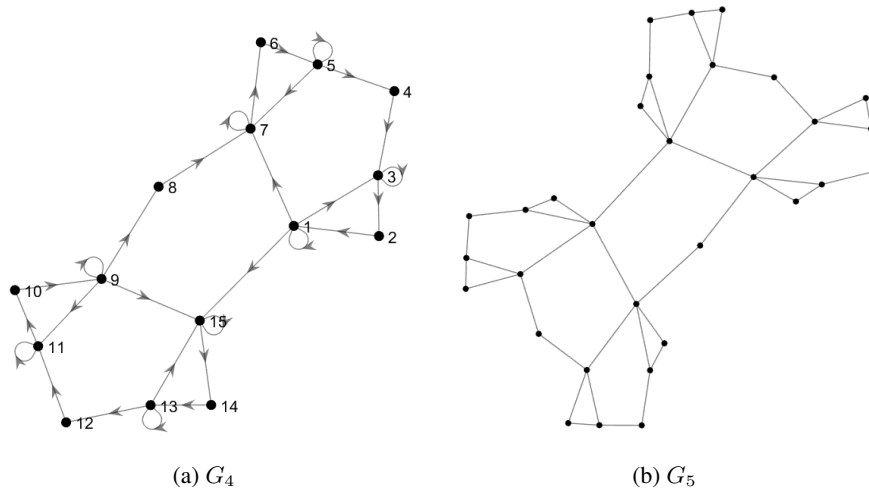


Figure 5. Force digraphs G_4 and G_5 . To remove clutter in the larger digraphs, we don’t print the arrows or the loops in G_n for $n \geq 5$. The recursive construction should now be clear.

By now the recursive construction is clear (we will formalize it in Definition 1 below), but for thoroughness G_4 and G_5 are shown in Figure 5. All of these digraphs are *strongly connected*: that is, there is a *closed walk* along the directed edges that includes all the vertices. Finally, just because we think that the digraphs with lots of vertices are beautiful, we show G_{12} and G_{13} in Figure 6, with $2^{12} - 1 = 4095$ vertices and $2^{13} - 1 = 8191$ vertices, respectively.

Remark. We use the graph visualization methods which we learned of first from the SuiteSparse collection of sparse matrices [7]; that is, take the graph associated with the matrix and put an attracting spring on every edge, put a repelling charge on every vertex, and look for a minimum energy configuration. We used the implementation in Matlab: for a digraph G issue the command `plot(G, 'Layout', 'force')`.

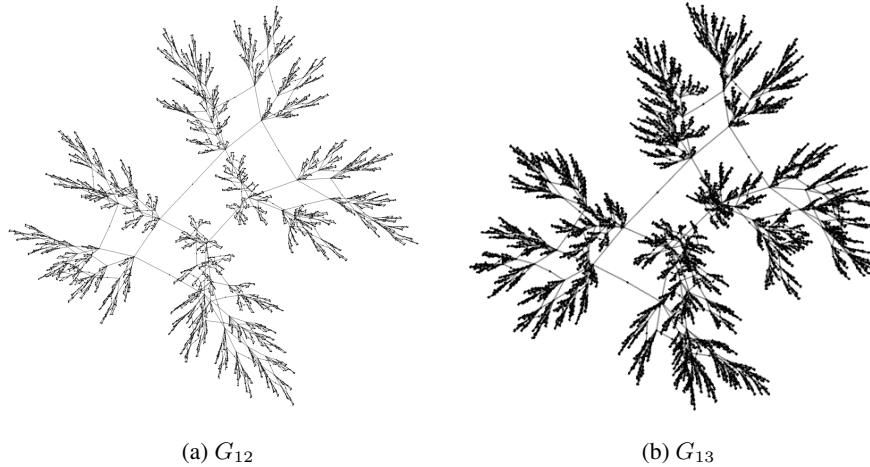


Figure 6. Force digraphs of G_{12} and G_{13} . The digraph on the right has nearly twice as many vertices and appears darker because of that, but the likeness in shape is evident.

Definition 1. G_1 is defined as in Figure 2. To construct G_n for $n > 1$, take two copies of G_{n-1} and one new vertex. Give the number 2^{n-1} to the new vertex. Keep the numbering on one copy of G_{n-1} the same as it was: 1 through $2^{n-1} - 1$. Renumber the vertices on the other copy to be $2^{n-1} + 1$ through $2^n - 1$: that is, add 2^{n-1} to each vertex number in this copy and renumber all edges (i, j) in this copy of G_{n-1} to become edges $(2^{n-1} + i, 2^{n-1} + j)$. Now add three new edges: the first directed from vertex 1 to the newly-renumbered vertex $2^n - 1$, the second from newly-renumbered vertex $2^{n-1} + 1$ to the new vertex 2^{n-1} , and the third from the new vertex 2^{n-1} to vertex $2^{n-1} - 1$. Call the resulting graph G_n .

Proposition 1. G_n is strongly connected.

Proof. By construction, from vertex 1 we may travel directly to vertex $2^n - 1$. Inductively we may travel from vertex $2^n - 1$ to $2^{n-1} + 1$; by construction from there through vertex 2^{n-1} to $2^{n-1} - 1$; and inductively from there to vertex 1. \square

2. THE ADJACENCY MATRICES. The *adjacency matrix* M of a directed graph G with d nodes is a d -by- d matrix with entry $M_{i,j} = 1$ if there is an edge from vertex i to vertex j , and is zero otherwise. Define M_n to be the adjacency matrix for G_n . For reasons that we will explain soon, we will call them *Mandelbrot matrices*. Explicitly, put

$$M_1 = [1] . \quad (1)$$

This is the adjacency matrix for the digraph G_1 : the matrix contains a 1 in its $(1, 1)$ entry because there is an edge connecting vertex 1 to itself, i.e., a loop.

We then put

$$M_2 = \begin{bmatrix} M_1 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & M_1 \end{bmatrix} . \quad (2)$$

This is the adjacency matrix for G_2 : we have a copy of G_1 situated in the upper left corner and another in the lower right corner; we have a new vertex (numbered 2, in

between the copies at 1 and at 3) and three new edges (red entries in the matrix) connecting vertex 1 to vertex 3, vertex 2 to vertex 1, and vertex 3 to vertex 2.

Proceeding in a similar fashion to construct M_3 , but this time explicitly showing the copies of M_2 in the outlined blocks:

$$M_3 = \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 & 1 \end{matrix}} & \begin{matrix} 0 & 0 & 0 \end{matrix} & \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 1 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 1 & 1 \end{matrix} & \begin{matrix} 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \boxed{\begin{matrix} 0 & 0 & 1 \end{matrix}} & \begin{matrix} 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & \boxed{\begin{matrix} 1 & 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 1 & 0 \end{matrix} & \boxed{\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 1 \end{matrix} & \boxed{\begin{matrix} 0 \\ 1 & 1 \end{matrix}} \end{bmatrix}. \quad (3)$$

In general, if $e_1 = [1, 0, 0, \dots, 0]^T$ is the leading elementary column vector of dimension $d_n = 2^n - 1$ and e_{d_n} is the final elementary column vector of the same dimension, then we may construct M_{n+1} from two copies of M_n in the following way.

Definition 2. Mandelbrot matrices. The Mandelbrot matrix M_1 is defined as above, namely the 1-by-1 matrix $M_1 = [1]$. For $n \geq 1$,

$$M_{n+1} = \begin{bmatrix} M_n & 0 & e_1 e_{d_n}^T \\ e_{d_n}^T & 0 & 0 \\ 0 & e_1 & M_n \end{bmatrix}. \quad (4)$$

We have the following facts, which we present without proof:

1. The matrix M_{n+1} has dimension $d_{n+1} = 2d_n + 1 = 2^{n+1} - 1$.
2. $\det M_n = 1$ for all $n \geq 1$.
3. M_n is the adjacency matrix for G_n for $n \geq 1$.
4. The matrices M_n are all “unit upper Hessenberg”: that is, they are upper triangular, except that the principal subdiagonal is also nonzero and contains only 1s.
5. Since there is a *walk* or directed path in G_n that contains all vertices, i.e., a complete circuit, the graph is *strongly connected* and the adjacency matrices M_n are *irreducible* [11, Chapter 40].
6. The *period* h of M_n is defined to be the greatest common divisor (GCD) of the length of all cycles in G_n ; here this is $h = 1$.
7. $\|M_n\|_1 = \|M_n\|_\infty = n$.
8. $\|M_n^{-1}\|_1 = \|M_n^{-1}\|_\infty = 2n - 1$.
9. The number of nonzero entries in M_n is $2d_n - 1$.

The Mandelbrot matrices are defined differently in some works, e.g., in [1, 2, 5], so that their characteristic polynomials $p_n(\lambda)$ satisfy $p_0(\lambda) = 0$ and the recurrence relation

$$p_{n+1}(\lambda) = \lambda p_n^2(\lambda) + 1. \quad (5)$$

This recurrence relation is a transformation of Mandelbrot’s fundamental recurrence $z_{n+1} = z_n^2 + c$; divide that fundamental recurrence by c and put $p_n = z_n/c$ and rename c to be λ . Zeros of these polynomials (and therefore eigenvalues of the Mandelbrot matrices) give *periodic* points in—centers of hyperbolic components of—the Mandelbrot set. See Figure 7. These eigenvalues are known to be all *simple*: see [13, 16, 19].

One inconvenience of that alternative definition of the Mandelbrot matrices is that it entails that the entries of each so-defined Mandelbrot matrix are either 0 or -1 . In order to minimize minus signs, we changed the definition \mathbf{M}_n so that its entries are either 0 or 1—that is, so that \mathbf{M}_n is a *binary matrix*. This has the consequence that the Mandelbrot polynomials as defined above are related to $\det(\lambda\mathbf{I} + \mathbf{M}_n)$. This *matrix pencil*² has the opposite sign to the usual definition of a characteristic polynomial of a matrix \mathbf{A} , namely $\det(\lambda\mathbf{I} - \mathbf{A})$.

Another difference in our definition here is that the prior definition indexes from 0. This makes $p_1 = 1$, which has no zeros; this would correspond to the empty matrix, which has no eigenvalues. Instead, we index from 1, here. In [6] the elements in the matrices are all nonpositive, but the indexing is as here. The reason that the other indexing convention is used is so that zeros of $p_n(z)$ give rise to points of period n in the Mandelbrot iteration. Here, we do not need this, and there are several favorable consequences: for instance, the maximum degree of G_n is n , and this means the maximum row sum of \mathbf{M}_n is n . Since the notational ambiguity is already in the literature, we feel required to warn the reader; and we feel entitled to use the most convenient notation here.

Proposition 2. *The matrices \mathbf{M}_n as defined above satisfy $\det(\lambda\mathbf{I} + \mathbf{M}_n) = p_{n+1}(\lambda)$, where $p_n(\lambda)$ are the Mandelbrot polynomials defined in equation (5).*

Proof. The determinant function is linear in the entries of the first row. Therefore the determinant of $\lambda\mathbf{I} + \mathbf{M}_{n+1}$ is the sum of the determinant of a block lower-triangular matrix with three blocks, namely $\lambda\mathbf{I} + \mathbf{M}_n$, λ , and $\lambda\mathbf{I} + \mathbf{M}_n$ again, and $(-1)^{d_{n+1}-1} = 1$ times the determinant of an upper-triangular matrix of dimension $d_{n+1} - 1$ with ones on the diagonal (because \mathbf{M}_{n+1} is upper Hessenberg with unit subdiagonal). Since the case $n = 1$ gives $p_2(\lambda) = \lambda + 1$, the theorem follows by induction. \square

Remark. To ease reading about and working with these matrices, we define the *characteristic polynomials* $C_n(\lambda) = \det(\lambda\mathbf{I} - \mathbf{M}_n)$ with the proper signs. By inspection, we have $C_n(\lambda) = -p_{n+1}(-\lambda)$ because the degrees are always odd: $d_n = 2^n - 1$. The recurrence relation that C_n satisfies is $C_{n+1} = \lambda C_n^2 - 1$ with $C_0 = 1$.

3. THE DOMINANT EIGENVALUES.

The eigenvalue of largest absolute value of a positive (square) matrix A is both simple and positive and belongs to a positive eigenvector. All other eigenvalues are smaller in absolute value.

—O. Perron, as quoted in [12]

A matrix or vector is called *positive* if all its entries are positive. A matrix is called *nonnegative* if all its nonzero entries are positive. Positive matrices have a dominant eigenvalue ρ which is also positive, by the Perron–Frobenius theorem. A nonnegative matrix can be taken as a limit of a set of positive matrices and therefore its eigenvalue of largest absolute value, ρ , is also nonnegative and the eigenvector belonging to it is nonnegative; however, there may be other eigenvalues of equal (largest) magnitude.

In the case in which the matrices are *irreducible*, more can be said. In such a case, the *period* h of the matrix is defined to be the GCD of the lengths of all the circuits in the digraph associated with the matrix. Then those other equally largest magnitude eigenvalues must be of the form $\exp(2\pi i/h)\rho$.

²A *matrix pencil* involving the pair of (usually square) matrices (\mathbf{A}, \mathbf{B}) is the linear matrix polynomial $\lambda\mathbf{B} + \mathbf{A}$.

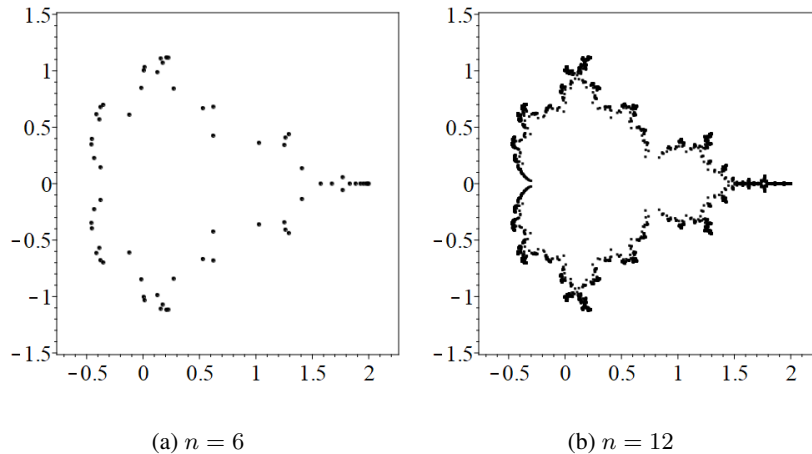


Figure 7. The eigenvalues of M_6 , which are the roots of $C_6(\lambda)$ (Figure 7a, degree 63) and the eigenvalues of M_{12} , which are the roots of $C_{12}(\lambda)$ (Figure 7b, degree 4095). The negatives of these points are periodic points in the Mandelbrot set. This is why the name “Mandelbrot polynomials” is given to $p_{n+1}(\lambda) = -C_n(-\lambda)$ and why we call the matrices “Mandelbrot matrices.”

Since our matrices M_n are nonnegative, and since the digraphs G_n associated with the matrices are strongly connected (which implies the matrices M_n are irreducible) the Perron–Frobenius theory applies. Since there are cycles of length 1, we see that the period h as defined in Definition 2 is just 1 and therefore the largest eigenvalue is in fact unique. See [12] for several proofs of Perron’s theorem, which states that an irreducible nonnegative matrix has a single, simple, positive, largest real eigenvalue, denoted ρ . The dominant eigenvalue of M_n , which we will call ρ_n , has been found in [6] to have the asymptotic expansion, valid as $n \rightarrow \infty$,

$$\rho_n = 2 - \frac{3}{8}\pi^2 4^{-n} + \tilde{O}(4^{-2n}). \quad (6)$$

Here the “soft-Oh” notation $\tilde{O}(g(n))$ is shorthand for $O(g(n) \log^m g(n))$ for some fixed m . For instance, $n \cdot 4^{-2n}$ and $n^{10} 4^{-2n}$ are both $\tilde{O}(4^{-2n})$. In this article we are concerned not with the eigenvalue, but rather with the eigenvector belonging to it. For our purposes, a more accurate ρ_n can be found by simple Newton iteration on the recurrence relation³ $C_{k+1}(z) = zC_k^2(z) - 1$ with $C_0(z) = 1$, so of course $C'_{k+1}(z) = C_k^2(z) + 2zC_k(z)C'_k(z)$ can be computed simultaneously. It is interesting to note as the authors of [6] do that, because the derivatives of $C_k(z)$ are so large, starting with just $\rho_n \approx 2$ is *not good enough for convergence*, and one must use the starting estimate from equation (6). For instance, for $n = 7$ the asymptotic estimate gives $\rho_7 \doteq 1.99977410268247$, and two Newton iterations achieve full double-precision accuracy at $\rho_7 = 1.99977404869373$; comparison shows the red digits were wrong.

This treatment works perfectly, though, and *we may regard the dominant eigenvalue as known to full accuracy*. Computing the eigenvalue is usually the hard part, but not here. We may now continue with the *eigenvector*.

The Perron–Frobenius theory states that each entry of the eigenvector belonging to

³As detailed in [4] the coefficients of the expanded polynomial grow *doubly exponentially* with n , and it is a bad idea numerically to do that expansion before trying to find roots. As a further benefit, using the recurrence relation instead takes only $O(n)$ operations, whereas evaluating the polynomial any standard way would require $O(d_n)$ operations, i.e., exponentially greater cost. There are fewer rounding errors, too.

ρ_n can also be taken to be nonnegative. We will see that, in practice, all entries are in fact positive.

To compute the eigenvector once the eigenvalue ρ_n is known, we do the simplest thing imaginable: we put $x_{d_n} = 1$ and $\hat{x} = [x_1, x_2, \dots, x_{d_n-1}]^T$ and solve the (very sparse) triangular system $T_n \hat{x} = -x_{d_n} \mathbf{b}$ that arises from looking for the null vector of $R(\rho_n) = (M_n - \rho_n I)$. Delete the first row of R and call the result \tilde{R} . The vector \mathbf{b} is the last column of \tilde{R} and the upper triangular matrix T_n comprises the first $d_n - 1$ columns of \tilde{R} .

For the *eigenvector* problem, the accurate computation of the dominant eigenvector of M_n therefore costs only $O(d_n)$ arithmetic operations.⁴ For a dense upper Hessenberg matrix the cost would instead be $O(d_n^2)$. Why is the computation so cheap? Basically, because the matrix M_n is so sparse (it has only $2d_n - 1$ nonzero entries).

Because the recurrence relation for the characteristic polynomial is so economical, this technique is cheaper than the more general technique for *quasiseparable* matrices discussed in [9], which also uses Newton's method on the characteristic polynomial.

4. “FRACTAL” EIGENVECTORS. We begin with pictorial representations of these eigenvectors, by plotting the components x_k against their index, k . The eigenvector of M_1 is trivial, being just a single dot: when $k = 1$, $x_k = 1$. This needs little comment. So let us consider instead M_2 . We choose to normalize the eigenvector by taking $x_d = 1$, and denote it as $[x_1, x_2, 1]^T$. We then have

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \rho_2 \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}. \quad (7)$$

This gives $x_2 = \rho_2 - 1$ from the third equation and $x_1 = \rho_2(\rho_2 - 1)$ from the second; the remaining equation simply gives (of course) the characteristic polynomial that $\lambda = \rho_2$ has to satisfy, namely $C_2(\lambda) = \lambda^3 - 2\lambda^2 + \lambda - 1 = 0$. We plot this eigenvector in Figure 8.

It will turn out to be convenient to normalize these eigenvectors by $x_d = 1$ in analytic computation; however, for visual presentation when there are many components, the plots turn out to be more intelligible if instead we choose $x_1 = 1$. We do this in all of Figures 8–10. This means there is always a component plotted in the upper left corner.

Maybe the simplest explanation. When we look at Figures 9–10 we see that each eigenvector can be split into (nearly) two halves: $d_n = 2^n - 1$ is odd and so the middle component can be taken to be special. The 2^{n-1} -dimensional subvectors consisting of the two halves of the remaining elements have a symmetry that, once seen, is striking: the two halves are visually identical, apart from scaling. Indeed we will prove that there is a single common scaling factor relating the two halves: $x_j = K x_{j+2^{n-2}+1}$ for $1 \leq j \leq 2^{n-2}$.

Also, there seems to be a significant likeness of the second half of the eigenvector to the full eigenvector of the previous case ($n - 1$). What could explain that?

An important element of the explanation comes from the following observation. Suppose $x(\rho_n)$ (normalized so its *final* entry is 1) is the eigenvector of M_n belonging

⁴At precision higher than offered by hardware, the bit complexity becomes relevant—not just the number of operations themselves—because the cost of each arithmetic operation increases if the precision is increased. Since the numerical condition of a generic eigenvalue problem is expected to grow like $O(d_n^2)$, and does so in this case, one expects to have to use greater than double precision if $d_n > 10^8$, which occurs if $n > 26$.

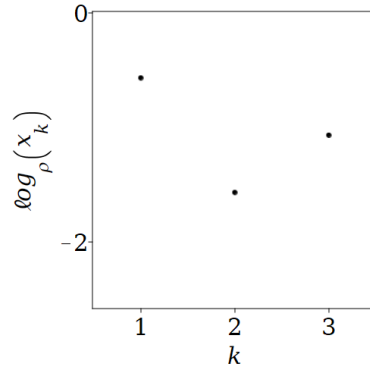


Figure 8. A semilog plot of the eigenvector components in the case $n = 2$ (dimension $d_n = 2^n - 1 = 3$) of the dominant eigenvector of M_n (normalized to have $x_1 = 1$). At this dimension it does not seem useful to make a discrete plot of the components of the eigenvector. Such plots *are* made in vibration studies, where the eigenvector gives the so-called “mode shape.” Here the purpose will only become clear as we increase the dimension.

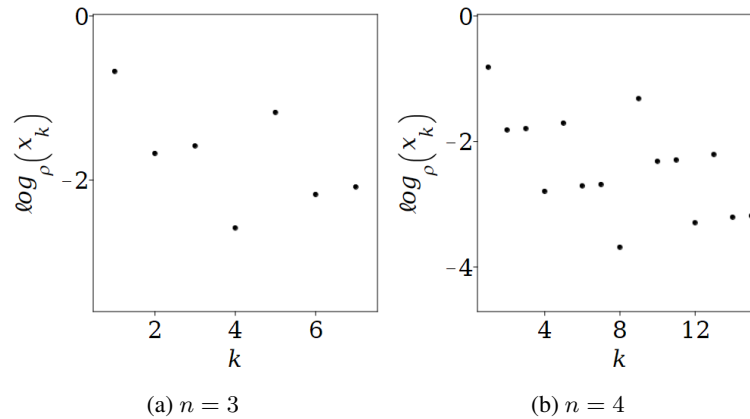


Figure 9. A semilog plot of the eigenvector components in the cases $n = 3$ (dimension $d_n = 2^n - 1 = 7$) and $n = 4$ (dimension $d_n = 2^n - 1 = 15$) of the dominant eigenvector of M_n . The largest component is $x_1 = 1$ (upper left corner). Notice the curious symmetry of the final half of the eigenvector components compared to the first half: ignoring the component exactly in the middle, the first half is a copy of the second, but scaled upwards slightly.

to ρ_n . Each component of $x(\rho_n)$ is a polynomial in ρ_n . For instance, when $n = 2$ we have

$$x = \begin{bmatrix} \rho_2(\rho_2 - 1) \\ \rho_2 - 1 \\ 1 \end{bmatrix}, \quad (8)$$

where $C_2(\rho_2) = \rho_2^3 - 2\rho_2^2 + \rho_2 - 1 = 0$. For $n = 3$ we have instead (after expanding

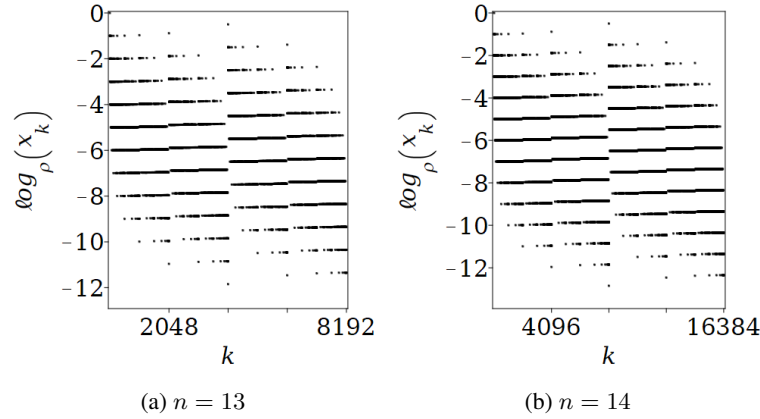


Figure 10. A semilog plot of the eigenvector components in the cases $n = 13$ (dimension $d_n = 2^n - 1 = 8191$) and $n = 14$ (dimension $d_n = 2^n - 1 = 16383$) of the dominant eigenvector of M_n . In these figures we normalized so that the largest component is $x_1 = 1$.

and factoring the results symbolically⁵)

$$x = \begin{bmatrix} \rho_3^2(\rho_3 - 1)(\rho_3^3 - 2\rho_3^2 + \rho_3 - 1) \\ \rho_3(\rho_3 - 1)(\rho_3^3 - 2\rho_3^2 + \rho_3 - 1) \\ \rho_3(\rho_3^3 - 2\rho_3^2 + \rho_3 - 1) \\ \rho_3^3 - 2\rho_3^2 + \rho_3 - 1 \\ \rho_3(\rho_3 - 1) \\ \rho_3 - 1 \\ 1 \end{bmatrix}. \quad (9)$$

Notice the occurrence of $C_2(\rho_3)$ in this vector. Because $C_3(\rho_3) = \rho_3 C_2^2(\rho_3) - 1 = 0$, we may write this as $C_2(\rho_3) = 1/\sqrt{\rho_3}$. Notice also that the final three components are the *same* polynomials as occurred for $n = 2$, only now evaluated at ρ_3 , not ρ_2 .

This is because M_{n+1} has two copies of M_n in it, and is upper Hessenberg so that we may find the eigenvector by solving a unit upper triangular system. In block form, we have

$$\begin{bmatrix} M_n & \mathbf{0} & e_1 e_{d_n}^T \\ e_{d_n}^T & 0 & \mathbf{0} \\ \mathbf{0} & e_1 & M_n \end{bmatrix} \begin{bmatrix} \tilde{x} \\ u \\ x \end{bmatrix} = \rho_{n+1} \begin{bmatrix} \tilde{x} \\ u \\ x \end{bmatrix}. \quad (10)$$

Theorem 1. *The solution to equation (10) can be constructed recursively as follows. Put $x_1(\rho) = [1]$, a one-vector containing a trivial polynomial in ρ . Subsequent vectors of dimension $2^{n+1} - 1$ are defined by the following polynomial vector recurrence relation:*

$$x_{n+1}(\rho_{n+1}) = \begin{bmatrix} \rho_{n+1} C_n(\rho_{n+1}) x_n(\rho_{n+1}) \\ C_n(\rho_{n+1}) \\ x_n(\rho_{n+1}) \end{bmatrix}. \quad (11)$$

⁵This is not a sensible thing to do numerically; these explicit expressions for eigenvector components rapidly become numerically unstable, owing to the *doubly* exponential growth of the monomial basis coefficients [6]. We do not use these symbolic expressions for numerical computation.

Proof. Notice first that the final component of each $\mathbf{x}_n(\rho_{n+1})$ is 1, as intended. Substitution of equation (11) into equation (10) gives two matrix equations, (12) and (13) and a scalar equation, (14):

$$\rho_{n+1}C_n(\rho_{n+1})\mathbf{M}_n\mathbf{x}_n(\rho_{n+1}) + \mathbf{e}_1 = \rho_{n+1}^2C_n(\rho_{n+1})\mathbf{x}_n(\rho_{n+1}), \quad (12)$$

$$C_n(\rho_{n+1})\mathbf{e}_1 + \mathbf{M}_n\mathbf{x}_n = \rho_{n+1}\mathbf{x}_n(\rho_{n+1}), \quad (13)$$

and, because the final component of $\mathbf{x}_n(\rho_{n+1})$ is 1, the scalar equation simply becomes the identity

$$\rho_{n+1}C_n(\rho_{n+1}) = \rho_{n+1}C_n(\rho_{n+1}). \quad (14)$$

Next, using $C_n(\rho_{n+1}) = 1/\sqrt{\rho_{n+1}}$, we see that the matrix equation (12) is a simple scalar multiple of the matrix equation (13). Thus, we only need to solve this final matrix equation. But this has already been done, recursively: \mathbf{M}_n is upper Hessenberg, so the eigenvector $\mathbf{x}_n(\rho_{n+1})$ is completely determined by solving rows 2 through d_n by back substitution given that the final component is 1. \square

Remark. The unused row in the matrix equation, namely

$$C_n(\rho_{n+1}) + \sum_{j \geq 1} M_{1,j}x_j(\rho_{n+1}) = \rho_{n+1}x_1(\rho_{n+1}),$$

must simply be a restatement of the characteristic polynomial. In some sense we don't need to explicitly solve it: we know how it will work out because by definition \mathbf{x}_{n+1} is an eigenvector and the only variable left free is ρ_{n+1} . Nonetheless, it is an interesting equation to solve: it involves the $1, 3, 7, \dots, 2^n - 1$ components of $\mathbf{x}_n(\rho_{n+1})$ (these are the only entries of the first row of \mathbf{M}_n that are nonzero) and does not lead directly to the recurrence relation $C_{n+1}(\rho) = \rho C_n^2(\rho) - 1$ but rather needs to use it and the recursive construction of the vector \mathbf{x} itself. We leave this as fun for the reader, but note that it gives a sparse representation for $C_{n+1}(\rho)$ that may have other uses.

Remark. The details of that proof also identify both the smallest element of that vector, namely $C_n(\rho_{n+1}) = 1/\sqrt{\rho_{n+1}}$ in the middle, and the largest entry x_1 , which is $\sqrt{\rho_{n+1}}$ times an approximation for the largest entry of the previous vector.

Since equation (11) shows that the lower entries of $x_k(\rho_{k+1})$ are *fixed* polynomials in ρ_{k+1} , and we know $\rho_k \rightarrow 2$ as $k \rightarrow \infty$, those lower entries actually converge to $[1, 1, 2, 1, 2, 2, 4, 1, 2, 2, 4, 2, 4, 4, 8, 1, \dots]$. This shows up as Sequence A048896 in the On-line Encyclopedia of Integer Sequences and is connected to Catalan numbers and to the number of 1s in the binary expansion of n , apparently [14]. We can see in retrospect that this is natural: each entry of the eigenvector is either a power of λ at ρ_{n+1} or an evaluation of some $C_k(\lambda)$ at ρ_{n+1} , and these go to 2 or 1, respectively, as $k \rightarrow \infty$. We do not pursue this further here, although it is extremely tempting.

The *upper* part of the vector is somehow more surprising: the leading entry is

$$x_{n+1,1}(\rho_{n+1}) = \rho_{n+1}^n \prod_{k=1}^n C_k(\rho_{n+1}) = \frac{2^{n+1}}{\pi} \left(1 + \tilde{O}(4^{-n})\right). \quad (15)$$

Note that $\rho_{n+1}C_k(\rho_{n+1})$ for $k = 1, \dots, n$ are the nonzero elements of the generated periodic orbit of the Mandelbrot set. We have established that last asymptotic equality only by high-precision numerical experiments, up to $n = 15$ where

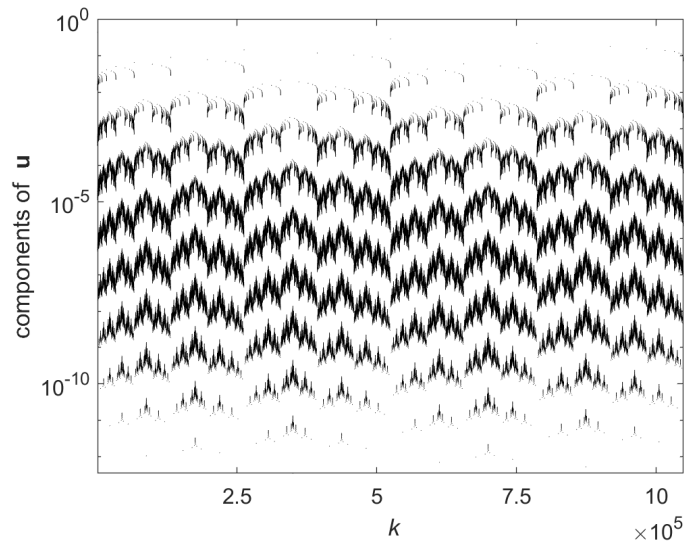


Figure 11. A discrete plot of the components of the singular vector (dimension $2^{20} - 1 = 1,048,575$) corresponding to the dominant singular value of M_{20} as computed by Matlab with its sparse SVD command `svds`, plotted on a logarithmic scale. We see many complex structures.

$d_{n+1} = 65,535$. We are quite convinced it's true, but have no proof. We do not pursue this further here either, although it is also extremely tempting. Another interesting and unexplained experimental fact is that the top of the vector, once the factor of π has been removed, appears to be in a scaled *Gould's sequence* oeis.org/A001316: if we compute $x_{16}(\rho_{16})$ and scale the topmost 16 entries (say), we get $2^{-11}\pi x_{16,16:1} = [1, 2, 2, 4, 2, 4, 4, 8, 2, 4, 4, 8, 8, 16]$. Gould's sequence is visible at least up to the topmost 128 entries.

The recursive application of powers of ρ_j , all nearly equal to 2, explains the bands visible on a \log_2 scale. Since the upper half of the eigenvector is a scaled version of the lower half, with the same scaling factor $\sqrt{\rho_{n+1}}$ applied to each component, this explains the rest. For *this* question, we believe that this answer satisfies Poincaré's dictum because in order to reach our explanation, we had to use several powerful mathematical ideas. We now turn to a harder problem.

5. SINGULAR VALUES AND VECTORS. Matlab's sparse singular value decomposition (SVD) routine for $A = U\Sigma V^T$ can compute the singular vectors belonging to the largest singular value of M_n quite rapidly—seemingly also of cost $O(d_n)$ —and moreover to do so accurately. As an instance of timing, Matlab 2019b can compute and plot each of the dominant left and right singular vectors for $n = 20$, which means $d_{20} = 2^{20} - 1 = 1,048,575$, in under 11 seconds on a 2017 Microsoft Surface Pro; that is, it can work with a (very sparse, true) million-plus by million-plus matrix and compute two million-plus vectors in a ludicrously short time, on a tablet computer.

In order to *explain* the features of Figure 11 we are going to have to use some facts about the singular value decomposition. The following section summarizes some things we need.

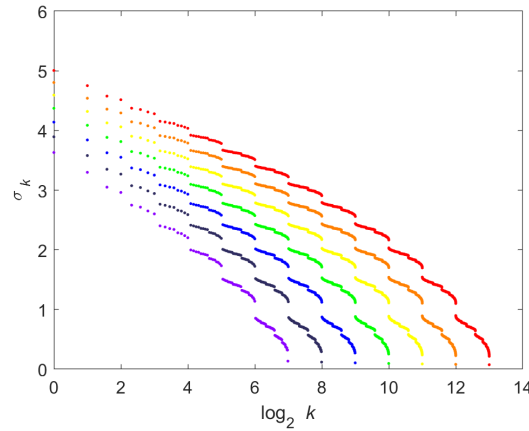


Figure 12. All singular values $\sigma_{n,k}$ for $1 \leq n \leq 2^k - 1$ of M_k for various k ; specifically, M_7 (in violet), M_8 (in indigo), M_9 (in blue), M_{10} (in green), M_{11} (in yellow), M_{12} (in orange), and M_{13} (in red), plotted against $\log_2 n$. In this plot we see apparent self-similarity at different scales: for each increment in k , the singular values move up and a likeness of the old ones seems to be inserted at the bottom.

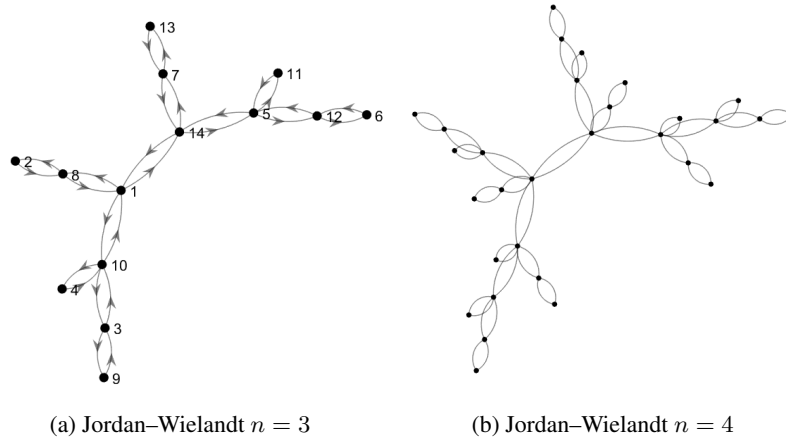


Figure 13. Force digraphs associated with the Jordan–Wielandt matrix of equation (16). The period (2) can be seen in these bipartite digraphs. As before, we declutter the digraph if the number of vertices is too high to show labels and arrows effectively.

The Jordan–Wielandt matrix. The singular values of M_n can be found from the eigenvalues of the well-known Jordan–Wielandt matrix corresponding to M_n :

$$\begin{bmatrix} 0 & M_n \\ M_n^T & 0 \end{bmatrix}. \quad (16)$$

See Figure 13 for some digraphs associated with these matrices. It is important to note that these graphs are *bipartite*: we can divide the vertices into two groups (say “red” and “green”) and each vertex is connected only to vertices of the other color.

Alternatively, we could use the eigenvalues of $M_n^T M_n$ and $M_n M_n^T$ which give the *squares* of the singular values of M_n . But let us continue with the Jordan–Wielandt matrix.

If the singular value decomposition of M_n is given by $M_n = U \Sigma V^T$, with or-

thogonal matrices U and V and diagonal matrix Σ with its entries ordered⁶ so that $\sigma_1 \geq \sigma_2 \geq \cdots$, then we can form an invertible matrix

$$X = \begin{bmatrix} U & -U \\ V & V \end{bmatrix}, \quad X^{-1} = \frac{1}{2} \begin{bmatrix} U^T & V^T \\ -U^T & V^T \end{bmatrix}, \quad (17)$$

and when we apply this as a similarity transform to the Jordan–Wielandt matrix we get (writing without inverses)

$$\begin{bmatrix} 0 & M_n \\ M_n^T & 0 \end{bmatrix} \begin{bmatrix} U & -U \\ V & V \end{bmatrix} = \begin{bmatrix} U & -U \\ V & V \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}. \quad (18)$$

This reveals the well-known fact that the eigenvalues of the Jordan–Wielandt matrix, which is a nonnegative matrix, are $\pm\sigma_k$ for $1 \leq k \leq d$, where d is the dimension of the square matrix M_n . This is a characteristic of adjacency matrices for bipartite graphs: they can always be reordered so that their adjacency matrix is in the above form (not necessarily with square matrix blocks), and the eigenvalues occur in \pm pairs (possibly including 0).

In particular, here, one largest magnitude eigenvalue of the Jordan–Wielandt matrix is σ_1 , but there is another eigenvalue equally large in magnitude, namely $-\sigma_1$.

There is more. We have already shown the computed singular vectors in the case $n = 12$ (dimension $d_n = 4095$) in Figure 1, although we called them eigenvectors, there (they are: of $M_n^T M_n$ or of $M_n M_n^T$). The symmetry shown there—namely that the u vector and v vector look to be mirror images of each other—reflects the fact that M_n is symmetric about the anti-diagonal; this means that M_n^T is also symmetric about the anti-diagonal, and hence the Jordan–Wielandt matrix must have eigenvectors symmetric about the half-way point. Indeed, the vectors U and V are the same, but in reverse order. This suggests that there is an economy that might be useful. We look for such, in the next section.

Smaller matrices. After some thought, we notice that we may use the involutory symmetry of M_n as follows. As previously noted, the left and right singular vectors are the same, except in reverse order. This is because the matrix $M_n J_n$ is *symmetric*, where J_n is the involutory “anti-identity”: for instance, when $n = 2$ and the dimension $d_n = 2^n - 1$ is 3, we have

$$J_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \quad (19)$$

From the Jordan–Wielandt matrix, we have that $M_n v = \sigma u$ and $M_n^T u = \sigma v$; because $u = J_n v$ and $v = J_n u$, we see that

$$M_n J_n u = \sigma u. \quad (20)$$

That is, a singular vector u of M_n is an *eigenvector* of the sparse, symmetric matrix $M_n J_n$. Indeed, we have the following propositions.

⁶Here we have a notational conflict. We would like to use the notation σ_k to refer to the *largest* singular value of the matrix M_k , but this is confusing; ordinarily the largest singular value of a matrix is σ_1 . We will use $\sigma_{1,k}$ to mean the largest singular value of M_k .

Proposition 3. *The matrices $S_n = M_n J_n$ can be constructed recursively as follows: as a base, $S_1 = [1]$; then*

$$S_{n+1} = \begin{bmatrix} e_1 e_1^T & 0 & S_n \\ 0 & 0 & e_1^T \\ S_n & e_1 & 0 \end{bmatrix}. \quad (21)$$

Proof. From equation (4) we find

$$M_{n+1} J_{n+1} = \begin{bmatrix} M_n & 0 & e_1 e_{d_n}^T \\ e_{d_n}^T & 0 & 0 \\ 0 & e_1 & M_n \end{bmatrix} \begin{bmatrix} & & J_n \\ & 1 & \\ J_n & & \end{bmatrix} \quad (22)$$

$$= \begin{bmatrix} e_1 e_1^T & 0 & M_n J_n \\ 0 & 0 & e_1^T \\ M_n J_n & e_1 & 0 \end{bmatrix} \quad (23)$$

because $e_1 e_{d_n}^T J_n = e_1 e_1^T$. Equation (21) follows. \square

Proposition 4. *The matrices $S_n = M_n J_n$ have eigenvalues $\pm\sigma$ where σ is a singular value of M_n . Moreover, each singular value of M_n occurs as the absolute value of some eigenvalue of S_n .*

Proof. Because

$$\begin{bmatrix} 0 & M_n \\ M_n^T & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \pm\sigma \begin{bmatrix} u \\ v \end{bmatrix} \quad (24)$$

(we do not know which sign of the eigenvalue σ belongs to the eigenvector), and because $u = J_n v$, we see that

$$M_n v = M_n J_n u = \pm\sigma u. \quad (25)$$

This establishes that either σ or $-\sigma$ is an eigenvalue of $S_n = M_n J_n$. Since J_n cannot change the magnitude of the singular values because it is orthogonal, all singular values of M_n appear as absolute values of eigenvalues of S_n . \square

Let $D_n(\lambda) = \det(\lambda I - S_n)$ be the characteristic polynomial of S_n . Here are some facts about S_n , without proof.

1. S_n is symmetric.
2. $\det S_n = -1$ for $n > 1$.
3. $\text{trace} S_n = 1$.
4. If $N_n = \text{trace} S_n^2$, then $N_{n+1} = 2N_n + 3$, so $N_n = 2^{n+1} - 3$. Apparently coincidentally, $N_n = 2d_n - 1$ is the number of nonzero entries in M_n .
5. $S_n = M_n J$ is a matrix square root of $M_n M_n^T$.
6. The eigenvalues of S_n are distinct.

Something that isn't quite a "fact" is that the digraph of S_n looks rather like the directed graph of the Jordan–Wielandt matrix for M_{n-1} , except it has one simple loop on the first vertex. This is because, apart from that simple loop, the digraph for S_n is

also bipartite! We will use this in what follows. Let \tilde{S}_{n+1} be the matrix equal to S_{n+1} apart from the $(1, 1)$ entry, which is zeroed:

$$S_{n+1} \approx \tilde{S}_{n+1} = \begin{bmatrix} 0 & 0 & S_n \\ 0 & 0 & e_1^T \\ S_n & e_1 & 0 \end{bmatrix}. \quad (26)$$

Theorem 2. *If $n > 1$, then the d_n eigenvalues of S_n arranged in descending magnitude have $d_n - 1$ alternations in sign, and the largest magnitude eigenvalue is positive. That is, the eigenvalues of S_n are $(-1)^{i-1} \sigma_{i,n}$ for $1 \leq i \leq d_n$ where $\sigma_{i,n}$ are the singular values of M_n .*

Remark. This makes the characteristic polynomial of S_n self-interlacing in the sense of [18].

Proof. We assume $n \geq 1$ and work with S_{n+1} . Consider the graph one gets by deleting vertex 1, i.e., \tilde{S}_{n+1} . Its eigenvalues, which we know are distinct by an additional induction, interlace with those of S_{n+1} by Fact 1 of [11, Section 47.4]. Now consider the characteristic polynomial of S_{n+1} , namely $D_{n+1}(\lambda) = \det(\lambda I - S_{n+1})$. Because the determinant is linear in the first row, and the $(1, 1)$ entry of the matrix is $\lambda - 1$, this is

$$\begin{aligned} D_{n+1}(\lambda) &= \det \begin{bmatrix} \lambda I - e_1 e_1^T & 0 & -S_n \\ 0 & \lambda & -e_1^T \\ -S_n & -e_1 & \lambda I \end{bmatrix} \\ &= \det \begin{bmatrix} \lambda I & 0 & -S_n \\ 0 & \lambda & -e_1^T \\ -S_n & -e_1 & \lambda I \end{bmatrix} - \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda I_{d_n-1} & 0 & -L_n \\ 0 & 0 & \lambda & -e_1^T \\ -s_1 & -L_n^T & -e_1 & \lambda I \end{bmatrix}. \end{aligned} \quad (27)$$

Here $-s_1$ is the first column of $-S_n^T$, and L_n is defined by the partition $S_n^T = [s_1 | L_n^T]$; that is, L_n is the matrix that remains after we have removed the first row of S_n . The first determinant is the characteristic polynomial of the adjacency matrix of a bipartite graph, namely the one that is obtained by deleting the loop in the graph for S_{n+1} . Its characteristic polynomial, which also can be established by a separate induction to have distinct roots, may therefore be written as $\lambda p(\lambda) p(-\lambda)$, because it is of odd dimension and its nonzero eigenvalues (call them s_i , say, for $1 \leq i \leq 2d_n$) must occur in pairs of positive and negative elements. The second determinant can be written by Laplace expansion about the first row as 1 times the characteristic polynomial of the adjacency matrix of another bipartite graph, but now of even dimension; its characteristic polynomial may be written as $q(\lambda) q(-\lambda)$ where the (distinct, by separate induction) roots of this product (call them t_i , say, for $1 \leq i \leq 2d_n$) must interlace the eigenvalues of S_{n+1} by Fact 1 of [11, Section 47.4]. This means that the signs of $D_{n+1}(t_i) = t_i p(t_i) p(-t_i)$, $i = 1, 2, \dots, 2d_n$ must alternate. This entails that the s_i also interlace the roots of $\lambda p(\lambda) p(-\lambda)$. Since the nonzero s_i occur in positive and negative pairs, this establishes that there will be $2d_n + 1$ sign alternations; and since for a connected graph such as S_{n+1} the largest eigenvalue is always positive, the only way to have this interlacing is for the next-largest magnitude eigenvalue be negative. Similar reasoning establishes that the alternation continues until the requisite intervals

are exhausted. The sign in the smallest interval, which includes 0, is settled by appealing to the sign of the determinant, which is -1 . Because the number of eigenvalues is odd, the smallest eigenvalue is also positive. \square

Patterns versus pareidolia. Now we come nearer to our original goal, namely understanding the pictures of the singular vectors. When we examine the singular vector belonging to $\sigma_{1,k}$ for each of M_k up to M_6 we begin to see patterns: repeated groups of points shaped vaguely like daggers, or arrowheads, or perhaps boomerangs. See Figure 14.

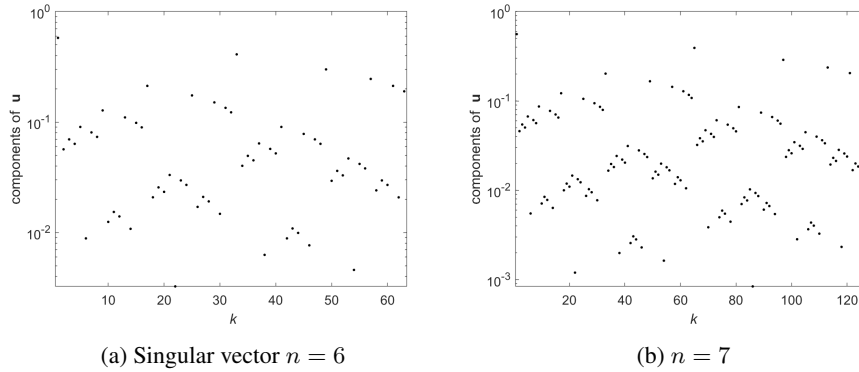


Figure 14. A discrete plot of the components of the singular vectors corresponding to the dominant singular values of M_6 and M_7 drawn on a logarithmic scale. At this density, we begin to see the emergence of complex structures.

Moreover, there are the *correct number* of copies of these “daggers”: twice as many for M_{n+1} as there were for M_n . Humans, however, are perhaps overly adroit at seeing patterns—when people see patterns that aren’t really there it is called “pareidolia”—and so we would like to have proof, just as we had for the eigenvector case.

Let us try do so by a *homotopy continuation*:

$$T(\varepsilon) = \begin{bmatrix} \varepsilon e_1 e_1^T & 0 & S_n \\ 0 & 0 & \varepsilon e_1^T \\ S_n & \varepsilon e_1 & 0 \end{bmatrix}. \quad (28)$$

When $\varepsilon = 1$ the eigenvalues of T are the same as the singular values of M_{n+1} , in absolute value. When $\varepsilon = 0$, the eigenvalues are 0 and $\{\pm\lambda\}$ where $\{\lambda\}$ are the eigenvalues of S_n . We just proved in Theorem 2 that the eigenvalues of S_n self-interlace, so we may conclude that the set $\{\pm\lambda\}$ contains $2d_n$ distinct values; adding 0 gives $2d_n + 1$ distinct values (no M_n or S_n is singular). This suggests that we may link the singular values of M_{n+1} directly to those of M_n by following the path of each eigenvalue of T as ε varies from 0 to 1.

One expects that the largest singular value of S_n would be transformed by this process to be the largest singular value of S_{n+1} . This seems to work: for every ε the matrix is symmetric and hence the roots are real, and they do not seem to cross (we conjecture that they do not).

Conjecture 1. *The eigenvalues of $T(\varepsilon)$ are simple on $0 \leq \varepsilon$.*

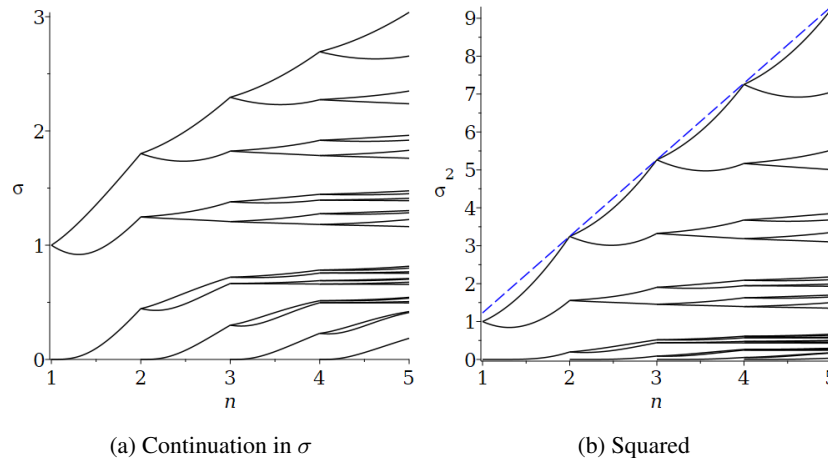


Figure 15. The homotopies for $n = 2, 3, 4$, and 5 plotted together. The role of ε is played by $t - 1$ in $1 \leq t \leq 2$, by $t - 2$ in $2 \leq t \leq 3$, and so on. These figures were drawn by solving the so-called Davidenko equations numerically, where the *initial conditions* for the solution on each interval were provided by the endpoints of the previous solutions, *their negatives*, and the new point 0 . We plot only the absolute values of the eigenvalues on the left, which shows the connections of singular values. On the right, plotting the squares shows our conjectured bound (blue dashed line) from equation (29)

The evidence we have for this conjecture is that the discriminants we have calculated, of the characteristic polynomials of $T(\varepsilon)$ with respect to λ , have strictly positive coefficients as monomial basis polynomials in ε . We do not have a general proof.

The largest singular value of the previous matrix becomes the largest singular value of the current one (the new root that starts from the negative of the largest singular value becomes the second largest of the next one). The plots, put together, are interesting. See Figure 15. To compute these, we first computed the characteristic polynomials $F_k(\lambda, \varepsilon)$ for $1 \leq k \leq 5$. We then differentiated $F_k(\lambda(\varepsilon), \varepsilon) = 0$ with respect to ε to get a differential equation for $\lambda(\varepsilon)$; this equation is called the *Davidenko equation* [3]. For $k = 1$, the initial conditions were $\lambda(0) = 0, 1$, and -1 . We used Maple's `dsolve/numeric` [17] to solve the Davidenko equation up to $\varepsilon = 1$. We then used the solutions at $\varepsilon = 1$, together with their negatives, and zero, as the initial conditions for a similar problem with k replaced by $k + 1$. We plot the *absolute values* of the eigenvalue paths in Figure 15. This gives the appearance of each old singular value giving birth to two new ones; which, in a sense, is true.

Another conjecture that comes from this experiment is

Conjecture 2.

$$\sigma_{1,n} \leq \sqrt{2.0193n - 0.7914}. \quad (29)$$

The numbers come from fitting a straight line to the largest singular values at $n = 2$ and at $n = 3$. At $n = 4$ the conjectured bound is 0.43% larger than the true value. By $n = 20$, the conjectured bound is larger than the true value by 0.85%.

We infer from our numerical experiments that there are 2^{n-1} double roots larger than 1 in magnitude at $\varepsilon = 0$, one root exactly 0, and $2^{n-1} - 1$ double roots smaller than 1 in magnitude at $\varepsilon = 0$. As we move to $\varepsilon = 1$, this changes to 2^n roots larger than 1 and $2^n - 1$ roots smaller than 1. We add a new root at $\sigma = 0$ and start again. This homotopy, if we could prove that it behaves as we think it does, would actually explain Figure 12 (in particular, it explains the gap near $\sigma = 1$ because the new eigenvalue starting at 0 never seems to cross the line).

But in fact we stop here: we now have a partial explanation for Figure 12, assuming that our conjecture (that the discriminant has only positive coefficients) is true.

6. CONCLUDING REMARKS.

A construction only becomes interesting when it can be placed side by side with other analogous constructions for forming species of the same genus. —Henri Poincaré [15]

We showed in Theorem 1 that the dominant eigenvector of M_n appears to acquire a fractal structure in the limit as $n \rightarrow \infty$: the eigenvector has two halves, the bottom half being something related to the previous eigenvector and the top half being a scaled copy of that. This theorem “explains” the visual appearance of the numerically computed eigenvectors; or, at least, one aspect of that appearance. The presence of powers of ρ_n (for $\rho_n = 2 - O(4^{-n})$) explains the discrete levels of values in the eigenvectors. The appearance of the OEIS sequence oeis.org/A048896 at the bottom of the bottom half is explained by the asymptotics of the dominant root, equation (6). The (conjectured) appearance of π in the upper half of the eigenvector, according to equation (15), will need future work to explain.

We claimed at the beginning of the article that the visible features that we tried to explain were not numerical artifacts, but were in fact faithful to the mathematics; we have not proved that fidelity here. The numerical analysis is almost, but not quite, straightforward. For a clear treatment of the standard eigenvalue and eigenvector perturbation theory, see [10]; for a treatment specialized to perturbation of Perron vectors, see [8]. The key fact needed is that the *next* largest eigenvalue is, by the results of [6], $O(4^{-n})$ away, giving an estimate of $O(d_n^2)$ for the condition number for the dominant eigenvector. We detailed the argument to our own satisfaction, but it is frankly simpler to do the computation again in high precision (we used Maple’s variable precision, with 30 and again with 60 Digits; 30 was more than enough) to estimate the largest relative error in the eigenvector. Our experiments showed (in agreement with our analysis) that this largest relative error grew like $8 \times 10^{-18} d_n^2$, resulting in an error of about 2×10^{-9} by $n = 14$ when $d_n = 16383$. Thus, all our eigenvector plots are correct to better than visual accuracy.

The patterns in the singular values and the singular vectors, on the other hand, were not confirmed by any theorems. Nonetheless, the results are accurate, because the two largest singular values are not that close to one another, and the gap between them determines the sensitivity of the dominant singular vector [10].

The Mandelbrot set features prominently in the theory of dynamical systems, and much is known about it. We suspect that our equation (15), which gives an expression for the geometric mean of the nonzero elements of the periodic orbit in the Mandelbrot set corresponding to the largest c , must be connected in some way with this vast theory, but at this moment we do not know just in what way.

Mandelbrot matrices and polynomials also have many connections to combinatorial problems. For instance, the fixed point for equation (5) is a generating function for the Catalan numbers, so the trailing coefficients of Mandelbrot polynomials are Catalan numbers. The polynomials appear to be *unimodal*, meaning that the coefficients increase in size to a maximum, then decay monotonically. We know of no proof.

The matrix family $\{M_n\}$ has been generalized in at least two separate ways. For instance, we mention the recurrence $q_{n+1} = \lambda q_n q_{n-1} + 1$ which generates the *Fibonacci–Mandelbrot* polynomials and their analogous companion matrices which contain only elements $\{-1, 0, 1\}$; there are puzzles here, too.

The matrices M_n themselves have yet more to tell us. The inverse of M_n is sparse

and contains only elements from the population $\{-1, 0, 1\}$; its largest magnitude eigenvalues correspond to the *smallest* magnitude eigenvalues of M_n . What more can be said about those eigenvalues and eigenvectors? We look forward to finding out.

Dedicated to the memory of Peter B. Borwein, May 10, 1953–August 23, 2020. We remember him by using the “Peter Borwein end-of-proof symbol”, \natural (“naturally”).

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REFERENCES

1. Bini, D. A., Fiorentino, G. (2000). Design, analysis, and implementation of a multiprecision polynomial rootfinder. *Numer. Algorithms*, 23(2): 127–173. doi.org/10.1023/a:1019199917103.
2. Bini, D. A., Robol, L. (2014). Solving secular and polynomial equations: A multiprecision algorithm. *J. Comput. Appl. Math.*, 272: 276–292. doi.org/10.1016/j.cam.2013.04.037.
3. Boyd, J. P. (2014). *Solving Transcendental Equations: The Chebyshev Polynomial Proxy and Other Numerical Rootfinders, Perturbation Series, and Oracles*. Philadelphia, PA: SIAM.
4. Chan, E. Y. S. (2016). A comparison of solution methods for Mandelbrot-like polynomials. Master’s thesis, Western University, London, Canada.
5. Chan, E. Y. S., Corless, R. M., Gonzalez-Vega, L., Sendra, J. R., Sendra, J. (2019). Algebraic linearizations of matrix polynomials. *Linear Algebra Appl.*, 563: 373–399. doi.org/doi.org/10.1016/j.laa.2018.10.028.
6. Corless, R. M., Lawrence, P. W. (2013). The largest roots of the Mandelbrot polynomials. In: Bailey, D. H., Bauschke, H. H., Borwein, P., Garvan, F., Théra, M., Vanderwerff, J. D., Wolkowicz, H., eds., *Computational and Analytical Mathematics*. New York, NY: Springer, pp. 305–324.
7. Davis, T. A. (2019). Algorithm 1000: Suitesparse:graphblas: Graph algorithms in the language of sparse linear algebra. *ACM Trans. Math. Software*, 45(4): 1–25. doi.org/10.1145/3322125.
8. Dietzenbacher, E. (1988). Perturbations of matrices: A theorem on the Perron vector and its applications to input-output models. *J. Economics*, 48(4): 389–412. doi.org/10.1007/bf01227544.
9. Eidelman, Y., Gohberg, I., Olshevsky, V. (2005). Eigenstructure of order-one-quasiseparable matrices. three-term and two-term recurrence relations. *Linear Algebra Appl.*, 405: 1–40. doi.org/10.1016/j.laa.2005.02.039.
10. Greenbaum, A., Li, R.-C., Overton, M. L. (2020). First-order perturbation theory for eigenvalues and eigenvectors. *SIAM Rev.*, 62(2): 463–482. doi.org/10.1137/19m124784x.
11. Hogben, L., ed. (2014). *Handbook of Linear Algebra*, 2nd ed. Boca Raton, FL: Taylor & Francis Group.
12. MacCluer, C. R. (2000). The many proofs and applications of Perron’s theorem. *SIAM Rev.*, 42(3): 487–498. doi.org/10.1137/s0036144599359449.
13. Morton, P., Patel, P. (1994). The Galois theory of periodic points of polynomial maps. *Proc. London Math. Soc.* (3), 68(2): 225–263. doi.org/10.1112/plms/s3-68.2.225.
14. OEIS Foundation Inc. (2021). The On-Line Encyclopedia of Integer Sequences. Published electronically at oeis.org.
15. Poincaré, H. (1905). *Science and Hypothesis*. New York, NY: The Walter Scott Publishing Co., Ltd. en.wikisource.org/wiki/Science_and_Hypothesis.
16. Schleicher, D. (2017). Internal addresses of the Mandelbrot set and Galois groups of polynomials. *Arnold Math. J.*, 3(1): 1–35. doi.org/10.1007/s40598-016-0042-x.
17. Shampine, L. F., Corless, R. M. (2000). Initial value problems for ODEs in problem solving environments. *J. Comput. Appl. Math.*, 125(1–2): 31–40. doi.org/10.1016/s0377-0427(00)00456-8.
18. Tyaglov, M. (2017). Self-interlacing polynomials. *Linear Algebra Appl.*, 535: 12–34. doi.org/10.1016/j.laa.2017.08.020.
19. Vivaldi, F., Hatjispyros, S. (1992). Galois theory of periodic orbits of rational maps. *Nonlinearity*, 5(4): 961–978. doi.org/doi:10.1088/0951-7715/5/4/007.