Large Deviations Of Sums Mainly Due To Just One Summand

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Abstract. We present a formalization of the well-known thesis that, in the case of independent identically distributed random variables $X_1, \dots, X_n$ with power-like tails of index $\alpha \in (0, 2)$, large deviations of the sum $X_1 + \dots + X_n$ are primarily due to just one of the summands.

1. INTRODUCTION, SUMMARY, AND DISCUSSION. Let $X_1, X_2, \dots$ be independent identically distributed random variables. For each natural n, let $S_n := \sum_1^n X_i$.

Heyde [**3**] showed the following: Suppose that, for some sequence (B_n) of positive real numbers, S_n/B_n converges in distribution to a stable law of index $\alpha \in (0, 2) \setminus \{1\}$, whose support is the entire real line $\mathbb{R}$. (For a definition and basic properties of stable laws, see e.g. [**6**, §IV.3].) Then, for any sequence (x_n) going to ∞,

$$\mathsf{P}(|S_n| > x_n B_n) \sim \mathsf{P}(\max_{1\le i\le n} |X_i| > x_n B_n). \tag{1}$$

As indicated in [**3**], one-sided analogs of (1) could also be obtained, even in the case $\alpha = 1$. However, such a task would involve additional technical difficulties.

The conditions in [**3**] for (1) imply that the tail of the distribution of each X_i is power-like – more specifically,

$$\mathsf{P}(|X_1| > u) = u^{-\alpha+o(1)} \quad \text{as} \quad u \to \infty. \tag{2}$$

This work by Heyde was followed by a large number of publications, including [**4, 5, 7, 1**].

The asymptotic equivalence (1) and, especially, its proof suggest the well-known interpretation that, in the cases of power-like tails as in (2), large deviations of the sum S_n are mainly due to just one of the summands $X_1, \dots, X_n$.

In this note, we present a formal version of this interpretation:

Theorem 1. *Take any $\alpha \in (0, 2)$. Let $X_1, X_2, \dots$ and S_n be as in the first paragraph of this note. To avoid technicalities, suppose that the distribution of X_1 is symmetric about 0 and has a probability density function f such that*

$$f(u) \asymp u^{-1-\alpha} \quad \textit{as} \quad u \to \infty \tag{3}$$

(cf. (2)). Then

$$\mathsf{P}(S_n > x) \sim \mathsf{P}\Big(S_n > x, \bigcup_{i\in[n]} \{X_i > x, |S_n - X_i| \le bx, \max_{j\in[n]\setminus\{i\}} |X_j| \le cx\}\Big) \tag{4}$$

whenever $n \in \mathbb{N}$, $x \in (0, \infty)$, $c \in (0, 1)$, and $b \in (0, 1)$ vary in such a way that

$$n << x^\alpha, \tag{5}$$

$$nx^{-\alpha} << c^{2\alpha}, \tag{6}$$

$$nx^{-\alpha} << b^2 c^{\alpha-2}. \tag{7}$$

Here, as usual, $\mathbb{N} := \{1, 2, \dots\}$ and $[n] := \{1, \dots, n\}$ for $n \in \mathbb{N}$. For positive expressions E and F (in terms of x, n, c, b), we write (i) $E \sim F$ if $E/F \to 1$; (ii) $E << F$ or, equivalently, $F >> E$ if $E = o(F)$—that is, if $E/F \to 0$; (iii) $E \lesssim F$ or, equivalently, $F \gtrsim E$ if $\limsup E/F < \infty$; and (iv) $E \asymp F$ if $E \lesssim F \lesssim E$. The "much smaller than" sign $<<$ should not be confused with Vinogradov's symbol $\ll$ (the latter is usually taken to mean the same as $\lesssim$).

Proposition 2. *For S_n as in Theorem 1 and for all $n \in \mathbb{N}$ and $x \in (0, \infty)$, we have $\mathsf{P}(S_n > x) \to 0$ if and only if condition* (5) *holds. Moreover, if either* (5) *holds or $\mathsf{P}(S_n > x) \to 0$, then $\mathsf{P}(S_n > x) \asymp nx^{-\alpha}$.*

Remark 3. Condition $\mathsf{P}(S_n > x) \to 0$ means precisely that $\mathsf{P}(S_n > x)$ is a large-deviation probability for S_n. So, in view of Proposition 2, Theorem 1 concerns all the large deviations of S_n. ■

Remark 4. Given (6), for (7) to hold it is enough that $b \asymp c$ or even $b \gtrsim c^{1+\alpha/2}$. Therefore and because the probability on the right-hand side of (4) is non-decreasing in c and in b, without loss of generality

$$c << 1 \quad \text{and} \quad b << 1. \tag{8}$$

So, (4) shows that the large deviation event $\{S_n > x\}$ is mainly due to just one of the summands $X_1, \dots, X_n$. More specifically, (4) tells us that, given $S_n > x$, the conditional probability that exactly one of the X_i's is $> x$ while the absolute values of the other X_i's and of the sum of the other X_i's are all $o(x)$ is close to 1. ■

Remark 5. In contrast with (1), the condition $n \to \infty$ is not required in Theorem 1; in particular, n may be fixed there. However, it is clear that condition (5) in Theorem 1 necessarily implies that $x \to \infty$. In another distinction from (1), in Theorem 1 the common distribution of the X_i's is not required to be in the domain of attraction of a stable law. ■

2. PROOFS.

Proof of Theorem 1. This proof is based on two lemmas. To state the lemmas, let us introduce the following notations:

$$p_0(n, x) := \mathsf{P}\Big(S_n > x,\ \max_{j\in[n]} |X_j| \le cx\Big), \tag{9}$$

$$p_{\ge 2}(n, x) := \mathsf{P}\Big(S_n > x,\ \bigcup_{i\in[n]} \bigcup_{j\in[n]\setminus\{i\}} \{|X_i| > cx, |X_j| > cx\}\Big), \tag{10}$$

$$p_{1,0}(n, x) := \mathsf{P}\Big(S_n > x,\ \bigcup_{i\in[n]} \{cx < |X_i| \le x,\ \max_{j\in[n]\setminus\{i\}} |X_j| \le cx\}\Big), \tag{11}$$

$$p_{1,1,-}(n, x) := \mathsf{P}\Big(S_n > x,\ \bigcup_{i\in[n]} \{X_i < -x,\ \max_{j\in[n]\setminus\{i\}} |X_j| \le cx\}\Big), \tag{12}$$

$$p_{1,1,+}(n, x) := \mathsf{P}\Big(S_n > x,\ \bigcup_{i\in[n]} \{X_i > x,\ \max_{j\in[n]\setminus\{i\}} |X_j| \le cx\}\Big). \tag{13}$$

Lemma 6. *For n and x as in the conditions of Theorem 1 (that is, for $n \in \mathbb{N}$ and $x \in (0, \infty)$ such that* (5) *holds), we have*

$$\mathsf{P}(S_n > x) \gtrsim n\,\mathsf{P}(X_1 > x) \asymp nx^{-\alpha}. \tag{14}$$

Proof. By (3),

$$\mathsf{P}(X_1 > u) \asymp u^{-\alpha} \quad \text{as} \quad u \to \infty. \tag{15}$$

So, in view of (5), $n\,\mathsf{P}(X_1 > x) \asymp nx^{-\alpha} << 1$. Now (14) follows from [**2**, inequality V, (5.10)], which immediately implies $\mathsf{P}(S_n > x) \geq \frac{1}{4}\big(1 - e^{-2n\,\mathsf{P}(X_1 > x)}\big)$ (since the distribution of X_1 is symmetric and absolutely continuous). ∎

Lemma 7. *For n, x, and c as in the conditions of Theorem 1,*

$$p_0(n, x) << nx^{-\alpha}, \tag{16}$$

$$p_{\geq 2}(n, x) << nx^{-\alpha}, \tag{17}$$

$$p_{1,0}(n, x) << nx^{-\alpha}, \tag{18}$$

$$p_{1,1,-}(n, x) << nx^{-\alpha}. \tag{19}$$

Proof. For all natural i, let

$$Y_i := X_i\ 1(|X_i| \leq cx),$$

where $1(A)$ denotes the indicator of an assertion A, so that $1(A) = 1$ if A is true, and $1(A) = 0$ if A is false. Then the Y_i's are independent identically distributed symmetric random variables. Also, by (6) and (5), $(cx)^{2\alpha} >> nx^{\alpha} >> 1$, so that $cx >> 1$. Therefore, in view of (3), for some real $A > 0$ we have

$$\mathsf{E}\,Y_1^2 \lesssim \int_0^A u^2 f(u)\,du + \int_A^{cx} u^2 u^{-1-\alpha}\,du \asymp (cx)^{2-\alpha}.$$

Therefore, with

$$T_n := \sum_1^n Y_i,$$

by (9), Markov's inequality, and (8),

$$p_0(n, x) \leq \mathsf{P}(T_n > x) \leq \frac{\mathsf{E}\,T_n^2}{x^2} = \frac{n\,\mathsf{E}\,Y_1^2}{x^2} \lesssim c^{2-\alpha}\frac{n}{x^{\alpha}} << nx^{-\alpha}. \tag{20}$$

So, (16) is proved.

Next, by (10), (15), and (6),

$$p_{\geq 2}(n, x) \leq \binom{n}{2}\mathsf{P}(|X_1| > cx, |X_2| > cx) \lesssim n^2(cx)^{-2\alpha} << nx^{-\alpha},$$

which proves (17).

Further, using (11), (3), and Markov's inequality as in (20), we have

$$\begin{aligned} p_{1,0}(n,x) &= n\,\mathsf{P}(S_n > x,\ cx < |X_1| \le x, |X_2| \le cx, \dots, |X_n| \le cx) \\ &\le n\,\mathsf{P}(cx < |X_1| \le x, Y_2 + \cdots + Y_n > x - X_1) \\ &\asymp n \int_{cx}^{x} u^{-1-\alpha}\,\mathsf{P}(Y_2 + \cdots + Y_n > x - u)\,du \lesssim I, \end{aligned} \tag{21}$$

where

$$I := \int_{cx}^{x} g(u)\,du, \quad g(u) := nu^{-1-\alpha} \min\Big(1, \frac{n(cx)^{2-\alpha}}{(x-u)^2}\Big).$$

Next,

$$u_x := x - n^{1/2}(cx)^{1-\alpha/2} \sim x, \tag{22}$$

by conditions (6) and (8) on c. It follows that

$$I = I_1 + I_2 + I_3,$$

where

$$I_1 := \int_{cx}^{x/2} g(u)\,du \le \int_{cx}^{\infty} nu^{-1-\alpha} \frac{n(cx)^{2-\alpha}}{(x/2)^2}\,du \asymp \Big(\frac{n}{x^\alpha}\Big)^2 c^{2-2\alpha} << nx^{-\alpha},$$

again by the mentioned conditions on c;

$$\begin{aligned} I_2 := \int_{x/2}^{u_x} g(u)\,du \le \int_{-\infty}^{u_x} n(x/2)^{-1-\alpha} \frac{n(cx)^{2-\alpha}}{(x-u)^2}\,du \\ \asymp \Big(\frac{n}{x^\alpha}\Big)^{3/2} c^{1-\alpha/2} << nx^{-\alpha}, \end{aligned}$$

once again by the conditions on c; and, in view of the definition of u_x in (22),

$$I_3 \ := \ \int_{u_x}^{x} g(u)\,du \ \le \ (x - u_x) n u_x^{-1-\alpha} \ \asymp \ \Big(\frac{n}{x^\alpha}\Big)^{3/2} c^{1-\alpha/2} \ << \ nx^{-\alpha},$$

as in the bounding of I_2. So, the bound on $p_{1,0}(n,x)$ in (18) follows immediately from (21) and the bounds on the integrals I_1, I_2, I_3.

Finally, in view of the definition of $p_{1,1,-}(n,x)$ in (12),

$$\begin{aligned} p_{1,1,-}(n,x) &= n\,\mathsf{P}\Big(S_n > x,\ X_1 < -x,\ \max_{j\in[n]\setminus\{1\}} |X_j| \le cx\}\Big) \\ &\le n\,\mathsf{P}\Big(S_n - X_1 > x,\ X_1 < -x,\ \max_{j\in[n]\setminus\{1\}} |X_j| \le cx\}\Big) \\ &\le n\,\mathsf{P}(T_n - Y_1 > x,\ X_1 < -x) \\ &= \mathsf{P}(T_n - Y_1 > x)\,n\,\mathsf{P}(X_1 < -x) \lesssim \frac{nc^{2-\alpha}}{x^\alpha}\frac{n}{x^\alpha} << nx^{-\alpha}. \end{aligned}$$

The $\lesssim$ comparison here is obtained by bounding $\mathsf{P}(T_n - Y_1 > x)$ similarly to the bounding of $\mathsf{P}(T_n > x)$ in (20) and using the symmetry of the distribution of X_1, the condition $x \to \infty$, and the relation (15); the $<<$ comparison in the above multiline display follows, yet again, by the conditions on c. So, (19) is proved as well.

This completes the proof of Lemma 7. ■

Now we can complete the proof of Theorem 1. Note that

$$\mathsf{P}(S_n > x) = p_0(n,x) + p_{\ge 2}(n,x) + p_{1,0}(n,x) + p_{1,1,-}(n,x) + p_{1,1,+}(n,x).$$

So, by Lemmas 7 and 6,

$$\mathsf{P}(S_n > x) \sim p_{1,1,+}(n,x). \tag{23}$$

Finally, the difference between $p_{1,1,+}(n,x)$ and the probability on the right-hand side of (4) is

$$\begin{aligned}
&\le n\,\mathsf{P}\Big(X_1 > x, |S_n - X_1| > bx,\ \max_{j\in[n]\setminus\{1\}} |X_j| \le cx\}\Big) \\
&\le n\,\mathsf{P}(X_1 > x)\,\mathsf{P}(|T_n - Y_1| > bx) \\
&\lesssim \mathsf{P}(S_n > x)\frac{n(cx)^{2-\alpha}}{(bx)^2} << \mathsf{P}(S_n > x);
\end{aligned}$$

the $\lesssim$ comparison here is obtained using the $\gtrsim$ comparison in (14) and bounding $\mathsf{P}(T_n - Y_1 > bx)$ similarly to the bounding of $\mathsf{P}(T_n > x)$ in (20); and the latter $<<$ comparison follows by (7). Now (4) follows from (23).

The proof of Theorem 1 is complete. ■

Proof of Proposition 2. Suppose first that condition (5) holds. Then, by (23), (13), and (14), $\mathsf{P}(S_n > x) \sim p_{1,1,+}(n,x) \le n\,\mathsf{P}(X_1 > x) \asymp nx^{-\alpha} \lesssim \mathsf{P}(S_n > x)$, so that $\mathsf{P}(S_n > x) \asymp nx^{-\alpha} \to 0$.

On the other hand, if $\mathsf{P}(S_n > x) \to 0$, then, by the inequality $\mathsf{P}(S_n > x) \ge \frac{1}{4}\left(1 - e^{-2n\,\mathsf{P}(X_1 > x)}\right)$ in the proof of Lemma 6, we have $n\,\mathsf{P}(X_1 > x) \to 0$, and hence $\mathsf{P}(X_1 > x) \to 0$ and $x \to \infty$. So, $\mathsf{P}(S_n > x) \gtrsim n\,\mathsf{P}(X_1 > x) \asymp nx^{-\alpha}$, by (15). Thus, $\mathsf{P}(S_n > x) \to 0$ implies (5), which in turn implies $\mathsf{P}(S_n > x) \asymp nx^{-\alpha} \to 0$. ■

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