

# Lagrange Inversion Formula by Induction

Erlang Surya and Lutz Warnke

**Abstract.** We present a simple inductive proof of the Lagrange Inversion Formula.

**1. INTRODUCTION.** The Lagrange inversion formula is a fundamental result in combinatorics. In its most basic form (see Theorem 1 with  $H(z) = z$  and  $H'(z) = 1$ ), it solves the functional equation  $A(x) = x\Phi(A(x))$  for  $A(x)$ , by expressing the coefficients of the formal power series  $A(x)$  in terms of the coefficients of the formal power series  $\Phi(z)$ . Functional equations of this form frequently arise in enumerative combinatorics, and in many applications the Lagrange inversion formula thus yields explicit counting formulas (e.g., for trees, permutations, and planar maps); see [4, 6, 7, 19, 20].

As is so often the case for fundamental results, there are many different proofs of the Lagrange inversion formula, including ones based on Cauchy’s coefficient formula for holomorphic functions, residues of formal Laurent Series, and tree-counting arguments, just to name a few (see [20, Section 5.1], [9, Section 4], [19, Section 5.4] and [2, 3, 5, 11, 12, 13] for more details and additional proofs). Furthermore, as one might expect, there are many different generalizations of the Lagrange inversion formula, including multivariate forms (see [8, 9, 10, 14, 17] and the references therein).

In this expository note we present a simple and elementary ‘just-do-it’ inductive proof of the Lagrange inversion formula (where all proof-steps emerge naturally).

**Theorem 1 (Lagrange Inversion Formula).** *Assume that  $A(x) = \sum_{n \geq 0} a_n x^n$  and  $\Phi(z) = \sum_{r \geq 0} c_r z^r$  are formal power series satisfying*

$$A(x) = x\Phi(A(x)). \tag{1}$$

*Then, for any integer  $n \geq 0$  and any formal power series  $H(z) = \sum_{r \geq 0} h_r z^r$ ,*

$$n[x^n]H(A(x)) = [z^{n-1}]H'(z)\Phi^n(z). \tag{2}$$

To fully understand and appreciate the statement and conclusion of Theorem 1, it might be useful to study the frequently asked questions discussed in the next section.

## 2. FREQUENTLY ASKED QUESTIONS.

**What does the notation  $[x^n]F(x)$  mean?** This is a widely-used [1, 4, 6, 7, 18, 20] short-hand notation for the coefficient of  $x^n$  in the formal power series  $F(x)$ , i.e.,

$$[x^n]F(x) := f_n \quad \text{when} \quad F(x) = \sum_{r \geq 0} f_r x^r. \tag{3}$$

**What are formal power series?** In brief: given a commutative coefficient ring  $R$ , the ring  $R[[x]]$  of formal power series is the set of all ‘formal sums’ of the form  $\sum_{r \geq 0} c_r x^r$  with  $c_r \in R$ , where addition, multiplication and differentiation are defined naturally:

$$\sum_{r \geq 0} a_r x^r + \sum_{r \geq 0} b_r x^r := \sum_{r \geq 0} (a_r + b_r) x^r, \tag{4}$$

$$\sum_{r \geq 0} a_r x^r \cdot \sum_{r \geq 0} b_r x^r := \sum_{r \geq 0} \left( \sum_{0 \leq s \leq r} a_s b_{r-s} \right) x^r, \tag{5}$$

$$\left( \sum_{r \geq 0} a_r x^r \right)' := \sum_{r \geq 1} (r a_r) x^{r-1} = \sum_{r \geq 0} a_r (x^r)'. \tag{6}$$

The ring  $R[[x]]$  of formal power series satisfies (more or less) all the properties one would expect, including the following well-known derivative formula:

$$(\Phi^m(z))' = m\Phi'(z)\Phi^{m-1}(z) \quad \text{for any integer } m \geq 1, \tag{7}$$

see Section 5 for a short proof. For more details about formal power series we refer to the textbooks [10, 18] or the *American Mathematical Monthly* expository paper [16] (which won the Lester R. Ford Award for expository excellence); see also [15].

**Why is the conclusion (2) useful in enumerative combinatorics?** In many enumeration problems, the formal power series  $A(x)$  is used as follows [1, 4, 6, 7, 18, 20]: the coefficients  $a_n$  encode the number of objects of size  $n$ , such as the number of certain  $n$ -vertex trees. Exploiting combinatorial properties of the objects of interest, one then infers a functional equation for  $A(x)$ : for example, given an integer  $t \geq 1$ , when  $a_n$  denotes the number of unlabelled rooted plane  $t$ -ary trees with  $n$  vertices (both external and internal) as in [6, Example I.14 (p. 68)], then one can obtain that

$$A(x) = x(1 + A^t(x)). \tag{8}$$

The Lagrange inversion formula shines when these functional equations cannot be explicitly solved for  $A(x)$ , which in example (8) is the case when  $t \geq 5$ . In these situations the crux is that (2) still allows us to determine the coefficients of  $A(x)$ : setting  $\Phi(z) := 1 + z^t$  and  $H(z) := z$  (so that  $H'(z) = 1$ ), for  $n \geq 1$  it directly gives

$$a_n = [x^n]A(x) \stackrel{(2)}{=} \frac{1}{n} [z^{n-1}] \underbrace{(1 + z^t)^n}_{=\sum_{r=0}^n \binom{n}{r} z^{tr}} = \frac{1}{n} \binom{n}{\frac{n-1}{t}} \quad \text{provided } t \mid n-1. \tag{9}$$

This illustrates the conceptual upshot of (2) in applications: it can specify the coefficients of an unknown formal power series  $A(x)$  that is defined by the functional equation (1) in terms of the known formal power series  $\Phi(z)$ ; see [4, 6, 7, 19, 20].

**What is the point of arbitrary  $H(z)$  in (2): is  $H(z) = z$  not enough?** Building upon the previous question, in many enumeration problems the following idea is used: when a formal power series counts certain objects, then suitable functions of it count other objects of interest (see [1, Chapter 3], [6, Sections I.2 and II.2], or [7, Sections 5.2 and 5.3]). For example, if  $A(x)$  counts  $t$ -ary trees as in (8), then

$$B(x) := A^k(x) \tag{10}$$

counts so-called ordered  $k$ -forests of  $t$ -ary trees, which are simply  $k$ -sequences of  $t$ -ary trees; see [6, below (69) on p. 66]. Having  $H(z)$  in (2) crucially allows us to directly determine the coefficients of  $B(x)$ : using  $H(z) := z^k$  and  $\Phi(z) := 1 + z^t$  gives

$$b_n = [x^n]B(x) \stackrel{(2)}{=} \frac{1}{n} [z^{n-1}] k z^{k-1} \Phi^n(z) = \frac{k}{n} [z^{n-k}] (1 + z^t)^n$$

for  $n \geq 1$ , which can then be computed analogously to (9). This illustrates why for applications it is useful to allow for arbitrary formal power series  $H(z)$  in (2).

**Doesn't the Lagrange Inversion Formula require  $a_0 = 0$  and  $c_0 \neq 0$ ?** The two assumptions  $a_0 = 0$  and  $c_0 \neq 0$  arise naturally in applications of the Lagrange Inversion Formula. Indeed, it is well-known (and not difficult to check) that the assumed functional equation (1) requires  $a_0 = 0$ , and, furthermore, that the special case  $c_0 = 0$  corresponds to the degenerate power series  $A(x) = 0$ . Nevertheless, it turns out that Theorem 1 is true without these two assumptions, i.e., they are formally redundant.

**Isn't the Lagrange Inversion Formula about the compositional inverse?** There are indeed formulations of the Lagrange Inversion Formula that concern the compositional inverse  $F^{(-1)}(x)$  of a given formal power series  $F(x) = \sum_{r \geq 1} f_r x^r$ , which satisfies  $F^{(-1)}(F(x)) = x = F(F^{(-1)}(x))$ . To this end we need to assume the existence of  $F^{(-1)}(x)$ , which turns out to be equivalent to  $f_1$  being invertible in the coefficient ring  $R$  (in which case  $F^{(-1)}(x) = \sum_{r \geq 1} g_r x^r$  with  $g_1 = 1/f_1$ ; see [19, Section 5.4] and [9, Section 1.1]). To relate this setup to Theorem 1, we set  $\phi(x) := x/F(x)$  and observe that  $x\phi(F^{(-1)}(x)) = F^{(-1)}(x)$ , so by invoking (2) with  $A(x) := F^{(-1)}(x)$  it follows that, for any integer  $n \geq 0$  and any formal power series  $H(z) = \sum_{r \geq 0} h_r z^r$ ,

$$n[x^n]H(F^{(-1)}(x)) = [x^{n-1}]H'(x) \left( \frac{x}{F(x)} \right)^n. \quad (11)$$

The conceptual crux is that (11) relates the coefficients of the formal power series  $F(x)$  and its compositional inverse  $F^{(-1)}(x)$ ; for more details we refer to [3, Section 3.8], [7, Section 6.12], [19, Section 5.4] and the references therein.

### 3. PROOF BY INDUCTION.

*Proof of Theorem 1.* Using induction on  $n \geq 0$ , for each  $n$  we shall prove that (2) holds for any formal power series  $H(z)$ . The base case  $n = 0$  is trivial, since both sides of (2) are zero (for the right-hand side the crux is that the power  $n - 1$  of  $z$  is negative).

We now turn to the induction step  $n \geq 1$ , where we first exploit that the derivative is a linear operator: indeed, for the induction step it suffices to establish that

$$n[x^n]A^k(x) = [z^{n-1}](z^k)' \Phi^n(z) \quad (12)$$

for all integers  $k \geq 0$ , as the desired identity (2) then follows for any formal power series  $H(z) = \sum_{k \geq 0} h_k z^k$  using linearity of the  $[x^n]$  operator and (6):

$$\begin{aligned} n[x^n]H(A(x)) &= \sum_{k \geq 0} h_k n[x^n]A^k(x) \stackrel{(12)}{=} \sum_{k \geq 0} h_k [z^{n-1}](z^k)' \Phi^n(z) \\ &= [z^{n-1}] \left( \sum_{k \geq 0} h_k (z^k)' \right) \Phi^n(z) \stackrel{(6)}{=} [z^{n-1}]H'(z) \Phi^n(z). \end{aligned}$$

To complete the induction step, in view of  $(z^k)' = kz^{k-1}$  it thus suffices to prove that, for all integers  $k \geq 0$ ,

$$n[x^n]A^k(x) = k[z^{n-k}] \Phi^n(z). \quad (13)$$

It will be convenient (and instructive) to first verify (13) in a few degenerate cases:

- *Case  $k = 0$ :* here both sides of (13) are zero: for the left-hand side the crux is that  $A^k(x) = A^0(x)$  contains no powers of  $x$  of form  $x^n$  with  $n \geq 1$ .
- *Case  $k > n$ :* here both sides of (13) are again zero: for the right-hand side the crux is that the power  $n - k$  of  $z$  is negative, and for the left-hand side the crux is that the assumption (1) implies that in  $A(x)^k$  all occurring powers of  $x$  are higher than  $n$ .
- *Case  $k = n$ :* here (13) is true since, using the assumption (1) and  $n = k$ , it readily follows that  $n[x^n]A^k(x) = n[x^0]\Phi^n(A(x)) = n(c_0)^n = k[z^{n-k}]\Phi^n(z)$ .

It thus remains to verify (13) in the case  $1 \leq k < n$ . Assumption (1) implies that

$$n[x^n]A^k(x) \stackrel{(1)}{=} n[x^n]x^k\Phi^k(A(x)) = n[x^{n-k}]\Phi^k(A(x)). \tag{14}$$

By the induction hypothesis we may apply (2) with  $H(z) = \Phi^k(z)$  and  $n$  replaced by  $n - k$ , and so by the derivative identity (7) with  $m = k$  and  $m = n$  it follows that

$$\begin{aligned} n[x^{n-k}]\Phi^k(A(x)) &\stackrel{(2)}{=} \frac{n}{n-k}[z^{n-k-1}](\Phi^k(z))'\Phi^{n-k}(z) \\ &\stackrel{(7)}{=} \frac{k}{n-k}[z^{n-k-1}]n\Phi'(z)\Phi^{n-1}(z) \\ &\stackrel{(7)}{=} \frac{k}{n-k}[z^{n-k-1}](\Phi^n(z))'. \end{aligned} \tag{15}$$

Finally, observe that by definition (see (3) and (6) above) we have

$$[z^{n-k-1}](\Phi^n(z))' \stackrel{(6)}{=} (n-k)[z^{n-k}]\Phi^n(z), \tag{16}$$

which together with (14) and (15) establishes the desired identity (13), completing the proof of the induction step (and thus Theorem 1). ■

**4. DISCUSSION.** Let us now take a step back, and discuss the structure of our inductive proof of the Lagrange inversion formula. The first reduction step is standard (and intuitive): exploiting the linearity of the derivative, it suffices to prove the desired identity (2) for monomials  $H(z) = z^k$ , i.e., it suffices to prove (12), which directly reduces to (13). In the induction step, it is natural to insert assumption (1) to arrive at (14), which is directly amenable to the induction hypothesis for a suitable formal power series  $H$  (exploiting that the induction hypothesis applies to arbitrary  $H$  instead of just monomials). The remaining steps from (15) onwards are again natural, and simply use the well-known derivative identities (7) and (16). To sum up: all steps of the proof emerged naturally (since none of them required any non-trivial ideas or insights), so we arguably presented a ‘just-do-it’ proof of the Lagrange Inversion Formula.

We remark that [9, Section 4.2] also contains an inductive proof, which has some similarities to the one given above. However, that inductive proof is used to prove a somewhat indirect and less natural variant of the Lagrange inversion formula, so an additional argument is needed to deduce (1). For combinatorial applications (1) is perhaps the most useful form of the Lagrange inversion formula, so it seems adequate to record our more direct (and simpler) inductive proof for expository reasons.

**5. APPENDIX.** We close by outlining, for completeness, a short proof of the well-known derivative identity (7) for formal power series. First note that (6) gives

$$(z^r z^s)' \stackrel{(6)}{=} (r+s)z^{r+s-1} \stackrel{(6)}{=} (z^r)'z^s + z^r(z^s)', \tag{17}$$

which by linearity implies (similar to the induction step in Section 3) that

$$(F(z)G(z))' = F'(z)G(z) + F(z)G'(z) \quad (18)$$

for any two formal power series  $F(z) = \sum_{r \geq 0} f_r z^r$  and  $G(z) = \sum_{s \geq 0} g_s z^s$ . Using the product rule (18), it then is easy to prove the desired derivative identity (7) by induction on  $m \geq 1$  (the base case  $m = 1$  being trivial).

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**ERLANG SURYA** *Department of Mathematics, University of California San Diego, La Jolla CA 92093, USA*  
*esurya@ucsd.edu*

**LUTZ WARNKE** *Department of Mathematics, University of California San Diego, La Jolla CA 92093, USA*  
*lwarnke@ucsd.edu*