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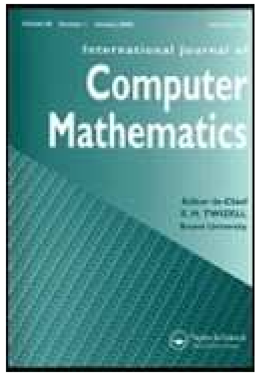

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A note on the first order theories of equilibrium figures of celestial bodies

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One of main problems in celestial mechanics is the determination of the shape of the equilibrium configuration of celestial bodies. In this paper a model of a fluid mass rotating in space like a rigid body will be developed.
To this aim, the equipotential surfaces are developed by using the Neumann series with respect to the Clairaut coordinates, and from these developments, the equilibrium equations and the boundary conditions can be obtained. Classical methods involve convergence problems, and in this paper two methods are developed to solve this problem, one based on numerical quadrature methods and the other one based on an analytical development.

Keywords: Figures of celestial bodies, Potential theory, Spherical harmonics, Celestial mechanics.

AMS Subject Classifications: 33E99,70F15,85A30.

1. Introduction

The study of the equilibrium configurations of celestial bodies is a classic problem in celestial mechanics, and they have been studied by classical authors. This paper focuses on the particular case of the study of the figures of equilibrium of rotating deformable bodies based on the use of the developments in Clairaut. Let us consider M as an isolated deformable mass with a uniform rotation $\overrightarrow{\omega}$. Let the rotating system of coordinates be defined by $OXYX$ axes where O is placed in the centre of masses of the body, OZ axe is parallel to $\overrightarrow{\omega}$ and OX, OY defines a direct trihedron with OZ. The potential in an internal point $\overrightarrow{r}$ of coordinate (x, y, z) is given by

$$\Psi = \Omega + V_c = G \int_M \frac{dm'}{\Delta} + \frac{\omega^2}{2}(x^2 + y^2) \tag{1}$$

where the first term is the so-called self-ravitational potential, and the second term is the centrifugal potential due to the rotation of the coordinate system. In this equation M denotes the mass of the body, dm' is the element of mass of an arbitrary internal point $\overrightarrow{r}'$ with coordinates (x', y', z'), and Δ is the distance between the point of vector radii $\overrightarrow{r}$ and $\overrightarrow{r}'$.

The condition of rigid rotation implies hydrostatic equilibrium $dP = \rho d\Psi$ and from this condition, and according to Kopal [7], [8] and Faulkner [3], this state implies the identification of the equipotential, isobaric, isothermal and isopycnic surfaces. To integrate this problem in a general case of mass distribution, a state equation is needed to connect the pressure and the density.

In Section 2 the coordinate system of Clairaut is defined and the classical potential development according to this coordinate system is given. From this development a set of integral equations for the amplitudes is obtained. Classical methods assumes that $U_n = K_n$ and $V_n = W_n$.

In section 3 we develop two main results; firstly, we show that the assumptions made in classical methods are not true to first order in the amplitudes and, secondly, we prove that, despite the above not be true,

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the development of the globlal potential for the first order in amplitudes is coincident with the classical theories.

In section 4 a new analytical method to arrange the potential is developed.

2. Development of the potential in Clairaut coordinates

To study the potential at a point in the primary component, classical methods use the Clairaut coordinate system (a, θ, λ) where a is a constant parameter on each equipotential surface. In this paper the parameter a was taken as the radius of the sphere with the same mass as the equipotential surface. The spherical coordinates (r, θ, λ) are connected to Clairaut ones by $r = r(a, \theta, \lambda)$. The equipotential surfaces are determined by a constant value of the parameter a. Since the Jacobian J of the transformation from spherical coordinates to Clairaut ones is of the form $J = \frac{\partial r'}{\partial a'}$, the element of mass dm' can be written, according to Clairaut coordinates, as

$$dm' = \rho(a')r(a', \theta', \lambda')^2 \left| \frac{\partial r}{\partial a'} \right| \cos\theta' d\theta' d\lambda' da'$$

To evaluate the self-gravitational potential Ω it is necessary to develop the inverse of the distance. Classical methods (Finlay [4], Kopal [7], Jardetzky [6], López [9]) are based on the development of the distance between two mass elements dm, dm' given by

$$\frac{1}{\Delta} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} = \begin{cases} \frac{1}{r}\sum\limits_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\gamma) & r > r' \\ \frac{1}{r'}\sum\limits_{n=0}^{\infty}\left(\frac{r}{r'}\right)^n P_n(\cos\gamma) & r < r' \end{cases} \tag{2}$$

where $P_n(\cos\gamma)$ are the Legendre polynomials.
Let (r, θ, λ) and (r', θ', λ') be the spherical coordinates of mass elements dm and dm'. The self-gravitational potential in a point of spherical coordinates (r, θ, λ) can be evaluated as

$$\Omega = U + V \tag{3}$$

where

$$U = G\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_r^{r_1} \frac{\rho(a')r'^2}{\Delta}\cos\theta' dr' d\theta' d\lambda'$$
$$V = G\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{r} \frac{\rho(a')r'^2}{\Delta}\cos\theta' da' d\theta' d\lambda' \tag{4}$$

and where r_1 is the minor radius of a sphere centred at 0 containing the mass distribution.
To evaluate these integrals it is convenient to replace $\frac{1}{\Delta}$ by its development in Legendre polynomial series.

$$U = \sum_{n=0}^{\infty} U_n r^n, \quad V = \sum_{n=0}^{\infty} V_n r^{-n-1} \tag{5}$$

where

$$U_n = G\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_r^{r_1} r'^{1-n}P_n(\cos\gamma)\rho(r',\theta',\lambda')\cos\theta' dr'd\theta'd\lambda'$$

$$V_n = G\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{r} r'^{2+n}P_n(\cos\gamma)\rho(r',\theta',\lambda')\cos\theta' dr'd\theta'd\lambda' \tag{6}$$

Let us define K_n, W_n as

$$K_n = G\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_a^{a_1} r'^{1-n}P_n(\cos\gamma)\rho(r',\theta',\lambda')\cos\theta'\frac{\partial r'}{\partial a'}da'd\theta'd\lambda'$$

$$W_n = G\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a} r'^{2+n}P_n(\cos\gamma)\rho(r',\theta',\lambda')\cos\theta'\frac{\partial r'}{\partial a'}da'd\theta'd\lambda' \tag{7}$$

where a_1 the first root of the equation $\rho(a) = 0$.
Classical methods assumes that $U_n = K_n$ and $V_n = W_n$ and consequently,

$$U = \sum_{n=0}^{\infty} K_n r^n, \qquad V = \sum_{n=0}^{\infty} W_n r^{-n-1} \tag{8}$$

If so, however, then evidently

$$K_n = \frac{G}{2-n}\int_a^{a_1}\rho(a')\frac{\partial}{\partial a'}\left[\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r'^{2-n}P_n(\cos\gamma)\cos\theta' d\theta' d\lambda'\right]da', \quad n \neq 2$$

$$K_2 = G\int_a^{a_1}\rho(a')\frac{\partial}{\partial a'}\left[\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log r' P_n(\cos\gamma)\cos\theta' d\theta' d\lambda'\right]da'$$

$$W_n = \frac{G}{n+3}\int_0^{a}\rho(a')\frac{\partial}{\partial a'}\left[\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r'^{n+3}P_n(\cos\gamma)\cos\theta' d\theta' d\lambda'\right]da' \tag{9}$$

Let us assume that the radius vector r' of an equipotential surface can be developed as

$$r' = a'\left[1 + \sum_{n=0}^{\infty}\sum_{m=-n}^{m} f_{n,m}(a')Y_{n,m}(\theta',\lambda')\right] \tag{10}$$

where $f_{n,m}(a')$ are the so-called amplitudes, and $Y_{n,m}(\theta',\lambda')$ the spherical functions [1].
Spherical functions satisfy the orthogonality condition

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Y_{n,m}(\theta,\lambda)Y_{r,s}(\theta,\lambda)\cos\theta\, d\theta\, d\lambda = \delta_{n,r}\delta_{r,s} \tag{11}$$

where $\delta_{i,j}$ is the delta of Kronecher. On the other hand $P_n(\cos\gamma)$ satisfies [5].

$$P_n(\cos\gamma) = \frac{4\pi}{2n+1}\sum_{m=-n}^{n} Y_{n,m}(\theta,\lambda)Y_{n,m}(\theta',\lambda') \tag{12}$$

Due to reasons concerned with symmetry, in the particular case of only rotation, vector radius r' can be developed as $r' = a'\left(1 + \sum_{k=0}^{\infty} f_{2k}(a')P_{2k}(\sin(\theta'))\right)$ where P_n are the Legendre polynomials, or in abbreviated form $r' = a'(1 + \Sigma')$.
In order to evaluate the last integrals it is convenient to approach r'^p and $\log r'$ by

$$r'^p = a'^p(1 + p\Sigma' + \frac{1}{2}p(p-1)\Sigma'^2 + \frac{1}{6}p(p-1)(p-2)\Sigma'^3 + ..)$$
$$\log r' = \log a + \Sigma' - \frac{1}{2}\Sigma'^2 + \frac{1}{3}\Sigma'^3 + ... \tag{13}$$

The product of the Legendre polynomials for $m \leq n$ is given by the Adams-Neumann formulae [2]

$$P_n(x)P_m(x) = \sum_{j=0}^{m} \frac{A_{m-j}A_jA_{n-j}}{A_{n+m-j}} \left\{\frac{2n+2m+1-4j}{2n+2m+1-2j}\right\} P_{n+m-2j}(x), \quad A_j = \frac{(2j-1)!!}{j!} \tag{14}$$

Replacing (2), (13) in (8), (9), and approximating r'^{2-n}, $\log r'$ and r'^{n+3} to an appropriate order in amplitudes, the self-gravitational potential can be written as

$$\Omega = 4\pi G \sum_{n=0}^{\infty} \frac{E_n(a)r^n + F_n(a)r^{-n-1}}{2n+1} P_n(\sin\theta) \tag{15}$$

Note that from the last equation $4\pi F_0(a) = M(a)$, where $M(a)$ is the mass contained in the equipotential surface of parameter a, and following Kopal [7] from this condition, to third order in amplitudes we have

$$f_0(a) = -\frac{1}{5}f_2^2(a) - \frac{2}{105}f_2^3(a) + ... \tag{16}$$

In the first order in aplitudes, functions $E_n(a)$, $F_n(a)$ can be written as.

$$E_0(a) = \int_a^{a_1} \rho(a')a'\,da' \qquad E_n(a) = \int_a^{a_1} \rho(a')\frac{\partial}{\partial a'}\left[a'^{2-n}f_n(a')\right] da'$$
$$F_0(a) = \int_0^{a} \rho(a')a'^2\,da' \qquad F_n(a) = \int_0^{a} \rho(a')\frac{\partial}{\partial a'}\left[a'^{n+3}f_n(a')\right] da' \tag{17}$$

The centrifugal potential V_c is given by

$$V_c = \frac{1}{2}r^2\left[1 - P_2(\sin\theta)\right] \tag{18}$$

Replacing r^n and r^{-n-1} in (22), (18) by their developments, the total potential (1) can be writen as

$$\Psi(a) = \sum_{n=0}^{\infty} \Psi_n(a)P_n(\sin\theta) \tag{19}$$

and consequently

$$\Psi(a) = \Psi_0(a), \qquad \Psi_n(a) = 0 \qquad n \neq 0 \tag{20}$$

In the firts order in ω^2 we get [7] [8] [9]

$$\frac{a^2E_2(a)}{5}+\frac{a^{-3}F_2(a)}{5}-a^{-1}f_2(a)F_0(a)=\frac{\omega^2a^2}{12\pi G}$$
$$\frac{a^nE_n(a)}{2n+1}+\frac{a^{-n-1}F_n(a)}{2n+1}-a^{-1}f_n(a)F_0(a)=0,\quad n=2,4,... \tag{21}$$

and from these integral equations, only $f_2(a)$ is not zero in the first order.
To get developments of $E_n(a)$ and $F_n(a)$ of order greather than one see [8].

3. First order theory: Numerical quadrature method

Unfortunately, the right-hand series do not converge in the layer defined by $r\in[r_{min}(a),r_{max}(a)]$ where $r_{min}(a)=\min\left\{r(a,\theta,\lambda)|\theta\in[-\frac{\pi}{2},\frac{\pi}{2}],\lambda\in[0,\pi]\right\}$,$r_{max}(a)=\max\left\{r(a,\theta,\lambda)|\theta\in[-\frac{\pi}{2},\frac{\pi}{2}],\lambda\in[0,\pi]\right\}$.
To solve this problem we can proceed as follows.
The potential Ω can be evaluated as

$$\Omega=\sum_{n=0}^{\infty}\left[U_nr^n+V_nr^{-n-1}\right] \tag{22}$$

To evaluate U_n and V_n it is more convenient to use of the Clairaut coordinates. Let us define $\Sigma=\sum\limits_{n=0}^{\infty}f_n(a)P_n(\sin\theta)$ and $\Sigma'=\sum\limits_{n=0}^{\infty}f_n(a')P_n(\sin\theta')$. The value of the vector radii r and r' of the equipotential surfaces that contain dm and dm', are given by $r=a(1+\Sigma)$, $r'=a'(1+\Sigma')$ and let (a,θ,λ) be the Clairaut coordinates of this surface in the (θ',λ') direction given by $(a(1+\Sigma)(1+\Sigma')^{-1},\theta',\lambda')$.

$$U_n=\frac{G}{2-n}\left[\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{a(1+\Sigma)(1+\Sigma')^{-1}}^{a_1}\frac{\partial r'^{2-n}}{\partial a'}P_n(\cos\gamma)\rho(r',\theta',\lambda')\cos\theta'da'd\theta'd\lambda'\right] \tag{23}$$

if $n\neq 2$. For $n=2$

$$U_2=G\left[\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{a(1+\Sigma)(1+\Sigma')^{-1}}^{a_1}\frac{\partial\log r'}{\partial a'}P_n(\cos\gamma)\rho(r',\theta',\lambda')\cos\theta'dr'd\theta'd\lambda'\right] \tag{24}$$

$$V_n=\frac{G}{n+3}\left[\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a(1+\Sigma)(1+\Sigma')^{-1}}\frac{\partial r'^{n+3}}{\partial a'}P_n(\cos\gamma)\rho(r',\theta',\lambda')\cos\theta'da'd\theta'd\lambda'\right] \tag{25}$$

To evaluate V_n it is convenient to compute the integral

$$\int_0^{a(1+\Sigma)(1+\Sigma')^{-1}}\rho(a')\frac{\partial r'^{n+3}}{\partial a'}da'=\int_0^a\rho(a')\frac{\partial r'^{n+3}}{\partial a'}da'+\int_a^{a(1+\Sigma)(1+\Sigma')^{-1}}\rho(a')\frac{\partial r'^{n+3}}{\partial a'}da' \tag{26}$$

To evaluate the second integral, a numerical quadrature formula of an appropriate order can be used. To build up a first order theory in the amplitudes $f_n(a)$, the approach $a(1+\Sigma)(1+\Sigma')^{-1}=a(1+\Sigma-\Sigma')$ and the rectangle quadrature formula can be used

$$\int_a^{a(1+\Sigma)(1+\Sigma')^{-1}}\rho(a')\frac{\partial r'^{n+3}}{\partial a'}da'\approx\int_a^{a(1+\Sigma-\Sigma')}\rho(a')\frac{\partial r'^{n+3}}{\partial a'}da'\approx\rho(a')\frac{\partial r'^{n+3}}{\partial a'}(\Sigma-\Sigma') \tag{27}$$

In zero order in amplitudes $r'^{n+3} = a'^{n+3}$, and from them we have in first order

$$\int_a^{a(1+\Sigma)(1+\Sigma')^{-1}} \rho(a')\frac{\partial r'^{n+3}}{\partial a'}da' = (n+3)a^{n+2}\rho(a)\left[\sum_{n=0}^{\infty} f_n(a)(P_n(\sin\theta) - P_n(sin\theta'))\right] \tag{28}$$

From this result we have

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a(1+\Sigma)(1+\Sigma')^{-1}} \frac{\partial r'^{n+3}}{\partial a'}P_n(\cos\gamma)\rho(r',\theta',\lambda')\cos\theta' dr' d\theta' d\lambda' =$$

$$= \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a} \rho(a')\frac{\partial r'^{n+3}}{\partial a'}P_n(\cos\gamma)\cos\theta' da' d\theta' d\lambda' +$$

$$+\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(n+3)a^{n+2}\rho(a)\left[\sum_{n=0}^{\infty} f_n(a)(P_n(\sin\theta) - P_n(sin\theta'))\right]P_n(\cos\gamma)\cos\theta' d\theta' d\lambda' \tag{29}$$

If $n \neq 0$, the first integral can be approached in the first order in amplitudes by

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a} \rho(a')\frac{\partial r'^{n+3}}{\partial a'}P_n(\cos\gamma)\cos\theta' da' d\theta' d\lambda' =$$

$$= \int_0^{a}\rho(a')\frac{\partial}{\partial a'}\left[\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r'^{n+3}P_n(\cos\gamma)\cos\theta' d\theta' d\lambda'\right]da' =$$

$$= \int_0^{a}\rho(a')\frac{\partial}{\partial a'}\left[\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a'^{n+3}(1+(n+3)\sum_{k=0}^{\infty} f_k(a')P_k(\sin\theta))P_n(\cos\gamma)\cos\theta' d\theta' d\lambda'\right]da' =$$

$$= \frac{4\pi}{2n+1}(n+3)\int_0^{a}\rho(a')\frac{\partial}{\partial a'}\left[a'^{n+3}f_n(a')\right]da' P_n(\sin\theta) \tag{30}$$

For $n = 0$ we have

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a} \rho(a')\frac{\partial r'^3}{\partial a'}\cos\theta' da' d\theta' d\lambda' = 12\pi\int_0^{a}\rho(a')a'^2 da' \tag{31}$$

The value of the second integral is given by

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(n+3)a^{n+2}\rho(a)\left[\sum_{n=0}^{\infty} f_n(a)(P_n(\sin\theta) - P_n(\sin\theta'))\right]P_n(\cos\gamma)\cos\theta' d\theta' d\lambda' =$$

$$= -4\pi\frac{n+3}{2n+1}a^{n+2}f_n(a)\rho(a)P_n(\sin\theta), \quad n \neq 0 \tag{32}$$

if $n = 0$ the integral is null.
Replacing (30), (32), in (29) we get

$$V_n = \frac{4\pi}{2n+1}G\int_0^{a}\rho(a')\frac{\partial}{\partial a'}\left[a'^{n+3}f_n(a')\right]da' - \frac{4\pi}{2n+1}Ga^{n+2}f_n(a)\rho(a)P_n(\sin\theta) \tag{33}$$

if $n \neq 0$. If $n = 0$, its value is found by

$$V_0 = 4\pi G \int_0^a \rho(a')a'^2 da' \tag{34}$$

The first integral of (33) coincides with the value of the classical theory.
The analyses of the corresponding U_n terms are similar. For $n \neq 0$ we have

$$U_n = \frac{4\pi}{2n+1} G \int_a^{a_1} \rho(a') \frac{\partial}{\partial a'} \left[a'^{2-n} f_n(a') \right] da' + \frac{4\pi}{2n+1} G a^{1-n} f_n(a) \rho(a) P_n(\sin\theta) \tag{35}$$

$$U_0 = 4\pi G \int_a^{a_1} \rho(a')a' da' \tag{36}$$

Replacing r^p by $r^p = a^p(1 + p \sum_{k=0}^{\infty} f_n(a)P_n(\sin\theta))$ in (22) we have in the first order in the amplitudes

$$\begin{aligned}
\Omega = 4\pi G &\left[a^{-1} \int_0^a \rho(a')a'^2 da' - a^{-1} \sum_{n=0}^{\infty} \left[\int_0^a \rho(a')a'^2 da' \right] f_n(a) P_n(sin\theta) \right] + \\
&+ 4\pi G \int_a^{a_1} \rho(a')a' da' + \sum_{n=1}^{\infty} \frac{4\pi G}{2n+1} \left\{ a^{-n-1} \int_0^a \rho(a') \frac{\partial}{\partial a'} \left[a'^{2-n} f_n(a') \right] da' + \right. \\
&\left. + a^n \int_a^{a_1} \rho(a') \frac{\partial}{\partial a'} \left[a'^{2-n} f_n(a') \right] da' \right\} P_n(\sin\theta) + \\
&+ \sum_{n=1}^{\infty} \left[-a^{-n-1} \frac{2\pi}{2n+1} G a^{n+2} f_n(a) + a^n \frac{2\pi}{2n+1} G a^{1-n} f_n(a) \right] P_n(\sin\theta)
\end{aligned} \tag{37}$$

Note that the last sum in equation (37) is null. Replacing (17) in (37) we get

$$\begin{aligned}
\Omega = 4\pi G &\left[E_0(a) + a^{-1} F_0(a) \right] + \\
&+ \sum_{n=1}^{\infty} 4\pi G \left\{ \frac{1}{2n+1} \left[a^n E_n(a) + a^{-n-1} F_n(a) \right] - a^{-1} F_0(a) f_n(a) \right\} P_n(\sin\theta)
\end{aligned} \tag{38}$$

This result coincides with the classical expression of the potential [8].

4. First order theory: Analytical method

A second way, based on the analytical development of the inverse of the distance, can be formulated as follows:

$$\Omega = K + W \tag{39}$$

where

$$K = G\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_a^{a_1}\frac{\rho(a')r'^2}{\Delta}\frac{\partial r'}{\partial a'}\cos\theta' da' d\theta' d\lambda'$$
$$W = G\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a}\frac{\rho(a')r'^2}{\Delta}\frac{\partial r'}{\partial a'}\cos\theta' da' d\theta' d\lambda' \quad (40)$$

To evaluate K and W we cannot use $W = \sum_{n=0}^{\infty} W_n r^{-n-1}$ and $K = \sum_{n=0}^{\infty} K_n r^n$, where W_n and K_n are defined by (7) because the developments of $\frac{1}{\Delta}$ given by (2) do not converge in the layer defined by $r \in [r_{min}(a), r_{max}(a)]$ where $r_{min}(a) = \min\left\{r(a,\theta,\lambda)| \quad \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], \lambda \in [0,\pi]\right\}$, $r_{max}(a) = \max\left\{r(a,\theta,\lambda)|\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], \lambda \in [0,\pi]\right\}$.
To solve this problem, let us define

$$D(a,a') = \frac{1}{\sqrt{a^2 + a'^2 - 2aa'\cos\gamma}} \quad (41)$$

The inverse of the distance between dm and dm' can be developed to the second order in Σ, Σ'

$$\frac{1}{\Delta} = D(a,a') + D_a(a,a')a\Sigma + D_a(a,a')a'\Sigma' + \frac{1}{2}D_{aa}a^2\Sigma^2 + D_{aa'}aa'\Sigma\Sigma' + \frac{1}{2}D_{a'a'}a'^2\Sigma'^2 + .. \quad (42)$$

where subcript x denotes the partial derivative with respect to x.
On the other hand, we have

$$dm' = \rho(a')a'^2(1 + 3\Sigma' + a'\Sigma'_{a'} + 3\Sigma'^2 + 2a'\Sigma'\Sigma'_{a'} + ..)\cos\theta' da' d\theta' d\lambda' \quad (43)$$

To evaluate the potential integral inside the equipotential surface of dm, $D(a,a')$ can be evaluated by

$$D(a,a') = \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{a'}{a}\right)^n P_n(\cos\gamma) \quad (44)$$

while, for outside this surface, it can be evaluated by

$$D(a,a') = \frac{1}{a'}\sum_{n=0}^{\infty}\left(\frac{a}{a'}\right)^n P_n(\cos\gamma) \quad (45)$$

In order to evaluate W we have

$$\frac{1}{\Delta} = \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{a'}{a}\right)^n\left\{1 - (n+1)\Sigma + n\Sigma'\right\}P_n(\cos\gamma) \quad (46)$$

Replacing (43) and (44) in (40) we get

$$W = G\int_0^{a}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sum_{n=0}^{\infty}\frac{a'^{n+2}}{a^{n+1}}\left\{1 - (n+1)\Sigma + (n+3)\Sigma' + a'\Sigma'_{a'}\right\}P_n(\sin\theta')\rho'(a)\cos\theta' d\theta' d\lambda' da' \quad (47)$$

To evaluate K we can procced by a similar way

$$\frac{1}{\Delta} = \frac{1}{a'}\sum_{n=0}^{\infty}\left(\frac{a}{a'}\right)^n \left\{1 + n\Sigma - (n+1)\Sigma'\right\} P_n(\cos\gamma) \tag{48}$$

Repalcing(43) and (45) in (40) we get

$$K = G\int_a^{a_1}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sum_{n=0}^{\infty}\frac{a^n}{a'^{n-1}}\left\{1 + n\Sigma + (2-n)\Sigma' + a'\Sigma'_{a'}\right\} P_n(\sin\theta')\rho'(a)\cos\theta' d\theta' d\lambda' da' \tag{49}$$

To evaluate (47) and (49), we get if $n \neq 0$

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Sigma P_n(\cos\gamma)d\theta' d\lambda' = 0, \qquad \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Sigma P_0(\cos\gamma)d\theta' d\lambda' = 4\pi\Sigma \tag{50}$$

and

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Sigma' P_n(\cos\gamma)d\theta' d\lambda' = \frac{4\pi}{2n+1}f_n(a')P_n(\sin\theta')$$

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}a'\Sigma'_{a'}P_n(\cos\gamma)d\theta' d\lambda' = \frac{4\pi}{2n+1}a' f'_n(a')P_n(\sin\theta') \tag{51}$$

and consequently

$$W = 4\pi G\int_0^a \rho(a')a'^2 da' - 4\pi G\sum_{n=0}^{\infty} f_n(a)\int_0^a \rho(a')a'^2 da' P_n(\sin\theta) + \\ + \sum_{n=0}^{\infty}\frac{4\pi G}{2n+1}\int_0^a \rho(a')\frac{\partial}{\partial a'}\left[a'^{n+3}f_n(a')\right] da' \tag{52}$$

$$K = 4\pi G\int_a^{a_1}\rho(a')a' da' + \sum_{n=0}^{\infty}\frac{4\pi G}{2n+1}\int_a^{a_1}\rho(a')\frac{\partial}{\partial a'}\left[a'^{2-n}f_n(a')\right] da' \tag{53}$$

Replacing (52), (53), (17) in (39) we get

$$\Omega = K + W = 4\pi G\left[E_0(a) + a^{-1}F_0(a)\right] + \\ + \sum_{n=1}^{\infty} 4\pi G\left\{\frac{1}{2n+1}\left[a^n E_n(a) + a^{-n-1}F_n(a)\right] - a^{-1}F_0(a)f_n(a)\right\} P_n(\sin\theta) \tag{54}$$

The total autogravitational potential $\Omega = K + W$ coincides with the value given in the previous section and consequently with the classical theory.

5. Concluding Remarks

Classical methods to study the equilibrium figures of celestial bodies contain a convergence problem in a layer around dm. To solve this problem, two methods have been proposed one based on numerical integration formulae an other based on analytical develoments of the inverse of the distance. The solution to the problem following both methods coincides with the classical theory in the first order in amplitudes. On the other hand, both metods can be suitable to be extended to second and higher order to study the results concordance.

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RESEARCH ARTICLE

A note on the first order theories of equilibrium figures of celestial bodies

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One of main problems in celestial mechanics is the determination of the shape of the equilibrium configuration of celestial bodies. In this paper a model of a fluid mass rotating in space like a rigid body will be developed.
To this aim, the equipotential surfaces are developed by using the Neumann series with respect to the Clairaut coordinates, and from these developments, the equilibrium equations and the boundary conditions can be obtained. Classical methods involve convergence problems, and in this paper two methods are developed to solve this problem, one based on numerical quadrature methods and the other one based on an analytical development.

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1. Introduction

The study of the equilibrium configurations of celestial bodies is a classic problem in celestial mechanics, and they have been studied by classical authors. This paper focuses on the particular case of the study of the figures of equilibrium of rotating deformable bodies based on the use of the developments in Clairaut. Let us consider M as an isolated deformable mass with a uniform rotation $\vec{\omega}$. Let the rotating system of coordinates be defined by $OXYX$ axes where O is placed in the centre of masses of the body, OZ axe is parallel to $\vec{\omega}$ and OX, OY defines a direct trihedron with OZ. The potential in an internal point $\vec{r}$ of coordinate (x, y, z) is given by

$$\Psi = \Omega + V_c = G \int_M \frac{dm'}{\Delta} + \frac{\omega^2}{2}(x^2 + y^2) \tag{1}$$

where the first term is the so-called self-ravitational potential, and the second term is the centrifugal potential due to the rotation of the coordinate system. In this equation M denotes the mass of the body, dm' is the element of mass of an arbitrary internal point $\vec{r}\,'$ with coordinates (x', y', z'), and Δ is the distance between the point of vector radii $\vec{r}$ and $\vec{r}\,'$.

The condition of rigid rotation implies hydrostatic equilibrium $dP = \rho d\Psi$ and from this condition, and according to Kopal [7], [8] and Faulkner [3], this state

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implies the identification of the equipotential, isobaric, isothermal and isopycnic surfaces. To integrate this problem in a general case of mass distribution, a state equation is needed to connect the pressure and the density.

In Section 2 the coordinate system of Clairaut is defined and the classical potential development according to this coordinate system is given. From this development a set of integral equations for the amplitudes is obtained. Classical methods assumes that $U_n = K_n$ and $V_n = W_n$.

In section 3 we develop two main results; firstly, we show that the assumptions made in classical methods are not true to first order in the amplitudes and, secondly, we prove that, despite the above not be true, the development of the globlal potential for the first order in amplitudes is coincident with the classical theories.

In section 4 a new analytical method to arrange the potential is developed.

2. Development of the potential in Clairaut coordinates

To study the potential at a point in the primary component, classical methods use the Clairaut coordinate system (a, θ, λ) where a is a constant parameter on each equipotential surface. In this paper the parameter a was taken as the radius of the sphere with the same mass as the equipotential surface. The spherical coordinates (r, θ, λ) are connected to Clairaut ones by $r = r(a, \theta, \lambda)$. The equipotential surfaces are determined by a constant value of the parameter a. Since the Jacobian J of the transformation from spherical coordinates to Clairaut ones is of the form $J = \frac{\partial r'}{\partial a'}$, the element of mass dm' can be written, according to Clairaut coordinates, as

$$dm' = \rho(a')r(a', \theta', \lambda')^2 \left|\frac{\partial r}{\partial a'}\right| \cos\theta' d\theta' d\lambda' da'$$

To evaluate the self-gravitational potential Ω it is necessary to develop the inverse of the distance. Classical methods (Finlay [4], Kopal [7], Jardetzky [6], López [9]) are based on the development of the distance between two mass elements dm, dm' given by

$$\frac{1}{\Delta} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} = \begin{cases} \frac{1}{r}\sum\limits_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\gamma) & r > r' \\ \frac{1}{r'}\sum\limits_{n=0}^{\infty}\left(\frac{r}{r'}\right)^n P_n(\cos\gamma) & r < r' \end{cases} \tag{2}$$

where $P_n(\cos\gamma)$ are the Legendre polynomials.
Let (r, θ, λ) and (r', θ', λ') be the spherical coordinates of mass elements dm and dm'. The self-gravitational potential in a point of spherical coordinates (r, θ, λ) can be evaluated as

$$\Omega = U + V \tag{3}$$

where

$$U = G\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_r^{r_1} \frac{\rho(a')r'^2}{\Delta}\cos\theta' dr' d\theta' d\lambda'$$

$$V = G\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{r} \frac{\rho(a')r'^2}{\Delta}\cos\theta' da' d\theta' d\lambda' \tag{4}$$

and where r_1 is the minor radius of a sphere centred at 0 containing the mass distribution.
To evaluate these integrals it is convenient to replace $\frac{1}{\Delta}$ by its development in Legendre polynomial series.

$$U = \sum_{n=0}^{\infty} U_n r^n, \quad V = \sum_{n=0}^{\infty} V_n r^{-n-1} \tag{5}$$

where

$$\begin{aligned} U_n &= G \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_r^{r_1} r'^{1-n} P_n(\cos\gamma)\rho(r', \theta', \lambda') \cos\theta' dr' d\theta' d\lambda' \\ V_n &= G \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{r} r'^{2+n} P_n(\cos\gamma)\rho(r', \theta', \lambda') \cos\theta' dr' d\theta' d\lambda' \end{aligned} \tag{6}$$

Let us define K_n, W_n as

$$\begin{aligned} K_n &= G \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_a^{a_1} r'^{1-n} P_n(\cos\gamma)\rho(r', \theta', \lambda') \cos\theta' \frac{\partial r'}{\partial a'} da' d\theta' d\lambda' \\ W_n &= G \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{a} r'^{2+n} P_n(\cos\gamma)\rho(r', \theta', \lambda') \cos\theta' \frac{\partial r'}{\partial a'} da' d\theta' d\lambda' \end{aligned} \tag{7}$$

where a_1 the first root of the equation $\rho(a) = 0$.
Classical methods assumes that $U_n = K_n$ and $V_n = W_n$ and consequently,

$$U = \sum_{n=0}^{\infty} K_n r^n, \qquad V = \sum_{n=0}^{\infty} W_n r^{-n-1} \tag{8}$$

If so, however, then evidently

$$\begin{aligned} K_n &= \frac{G}{2-n} \int_a^{a_1} \rho(a') \frac{\partial}{\partial a'} \left[\int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r'^{2-n} P_n(\cos\gamma) \cos\theta' d\theta' d\lambda' \right] da', \quad n \neq 2 \\ K_2 &= G \int_a^{a_1} \rho(a') \frac{\partial}{\partial a'} \left[\int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log r' P_n(\cos\gamma) \cos\theta' d\theta' d\lambda' \right] da' \\ W_n &= \frac{G}{n+3} \int_0^{a} \rho(a') \frac{\partial}{\partial a'} \left[\int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r'^{n+3} P_n(\cos\gamma) \cos\theta' d\theta' d\lambda' \right] da' \end{aligned} \tag{9}$$

Let us assume that the radius vector r' of an equipotential surface can be developed as

$$r' = a' \left[1 + \sum_{n=0}^{\infty} \sum_{m=-n}^{m} f_{n,m}(a') Y_{n,m}(\theta', \lambda') \right] \tag{10}$$

where $f_{n,m}(a')$ are the so-called amplitudes, and $Y_{n,m}(\theta', \lambda')$ the spherical functions [1].

Spherical functions satisfy the orthogonality condition

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Y_{n,m}(\theta,\lambda)Y_{r,s}(\theta,\lambda)\cos\theta\, d\theta\, d\lambda = \delta_{n,r}\delta_{r,s} \tag{11}$$

where $\delta_{i,j}$ is the delta of Kronecher. On the other hand $P_n(\cos\gamma)$ satisfies [5].

$$P_n(\cos\gamma) = \frac{4\pi}{2n+1}\sum_{m=-n}^{n} Y_{n,m}(\theta,\lambda)Y_{n,m}(\theta',\lambda') \tag{12}$$

Due to reasons concerned with symmetry, in the particular case of only rotation, vector radius r' can be developed as $r' = a'\left(1+\sum_{k=0}^{\infty} f_{2k}(a')P_{2k}(\sin(\theta'))\right)$ where P_n are the Legendre polynomials, or in abbreviated form $r' = a'(1+\Sigma')$.
In order to evaluate the last integrals it is convenient to approach r'^p and $\log r'$ by

$$\begin{aligned} r'^p &= a'^p(1+p\Sigma' + \frac{1}{2}p(p-1)\Sigma'^2 + \frac{1}{6}p(p-1)(p-2)\Sigma'^3 + ..) \\ \log r' &= \log a + \Sigma' - \frac{1}{2}\Sigma'^2 + \frac{1}{3}\Sigma'^3 + ... \end{aligned} \tag{13}$$

The product of the Legendre polynomials for $m \le n$ is given by the Adams-Neumann formulae [2]

$$P_n(x)P_m(x) = \sum_{j=0}^{m}\frac{A_{m-j}A_jA_{n-j}}{A_{n+m-j}}\left\{\frac{2n+2m+1-4j}{2n+2m+1-2j}\right\}P_{n+m-2j}(x), \quad A_j = \frac{(2j-1)!!}{j!} \tag{14}$$

Replacing (2), (13) in (8), (9), and approximating r'^{2-n}, $\log r'$ and r'^{n+3} to an appropriate order in amplitudes, the self-gravitational potential can be written as

$$\Omega = 4\pi G\sum_{n=0}^{\infty}\frac{E_n(a)r^n + F_n(a)r^{-n-1}}{2n+1}P_n(\sin\theta) \tag{15}$$

Note that from the last equation $4\pi F_0(a) = M(a)$, where $M(a)$ is the mass contained in the equipotential surface of parameter a, and following Kopal [7] from this condition, to third order in amplitudes we have

$$f_0(a) = -\frac{1}{5}f_2^2(a) - \frac{2}{105}f_2^3(a) + ... \tag{16}$$

In the first order in aplitudes, functions $E_n(a)$, $F_n(a)$ can be written as.

$$\begin{aligned} E_0(a) &= \int_a^{a_1}\rho(a')a'\,da' \qquad & E_n(a) &= \int_a^{a_1}\rho(a')\frac{\partial}{\partial a'}\left[a'^{2-n}f_n(a')\right]da' \\ F_0(a) &= \int_0^{a}\rho(a')a'^2\,da' \qquad & F_n(a) &= \int_0^{a}\rho(a')\frac{\partial}{\partial a'}\left[a'^{n+3}f_n(a')\right]da' \end{aligned} \tag{17}$$

The centrifugal potential V_c is given by

$$V_c = \frac{1}{2}r^2\left[1 - P_2(\sin\theta)\right] \tag{18}$$

Replacing r^n and r^{-n-1} in (22), (18) by their developments, the total potential (1) can be writen as

$$\Psi(a) = \sum_{n=0}^{\infty} \Psi_n(a) P_n(\sin\theta) \tag{19}$$

and consequently

$$\Psi(a) = \Psi_0(a), \qquad \Psi_n(a) = 0 \qquad n \neq 0 \tag{20}$$

In the firts order in ω^2 we get [7] [8] [9]

$$\begin{aligned} &\frac{a^2 E_2(a)}{5} + \frac{a^{-3} F_2(a)}{5} - a^{-1} f_2(a) F_0(a) = \frac{\omega^2 a^2}{12\pi G} \\ &\frac{a^n E_n(a)}{2n+1} + \frac{a^{-n-1} F_n(a)}{2n+1} - a^{-1} f_n(a) F_0(a) = 0, \quad n = 2, 4, ... \end{aligned} \tag{21}$$

and from these integral equations, only $f_2(a)$ is not zero in the first order.
To get developments of $E_n(a)$ and $F_n(a)$ of order greather than one see [8].

3. First order theory: Numerical quadrature method

Unfortunately, the right-hand series do not converge in the layer defined by $r \in [r_{min}(a), r_{max}(a)]$ where $r_{min}(a) = \min\left\{r(a,\theta,\lambda) | \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], \lambda \in [0,\pi]\right\}$,$r_{max}(a) = \max\left\{r(a,\theta,\lambda) | \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], \lambda \in [0,\pi]\right\}$.
To solve this problem we can proceed as follows.
The potential Ω can be evaluated as

$$\Omega = \sum_{n=0}^{\infty} \left[U_n r^n + V_n r^{-n-1}\right] \tag{22}$$

To evaluate U_n and V_n it is more convenient to use of the Clairaut coordinates. Let us define $\Sigma = \sum_{n=0}^{\infty} f_n(a) P_n(\sin\theta)$ and $\Sigma' = \sum_{n=0}^{\infty} f_n(a') P_n(\sin\theta')$. The value of the vector radii r and r' of the equipotential surfaces that contain dm and dm', are given by $r = a(1+\Sigma)$, $r' = a'(1+\Sigma')$ and let (a, θ, λ) be the Clairaut coordinates of this surface in the (θ', λ') direction given by $(a(1+\Sigma)(1+\Sigma')^{-1}, \theta', \lambda')$.

$$U_n = \frac{G}{2-n}\left[\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{a(1+\Sigma)(1+\Sigma')^{-1}}^{a_1} \frac{\partial r'^{2-n}}{\partial a'} P_n(\cos\gamma)\rho(r',\theta',\lambda')\cos\theta' da' d\theta' d\lambda'\right] \tag{23}$$

if $n \neq 2$. For $n = 2$

$$U_2 = G\left[\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{a(1+\Sigma)(1+\Sigma')^{-1}}^{a_1}\frac{\partial \log r'}{\partial a'}P_n(\cos\gamma)\rho(r',\theta',\lambda')\cos\theta' dr' d\theta' d\lambda'\right] \tag{24}$$

$$V_n = \frac{G}{n+3}\left[\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a(1+\Sigma)(1+\Sigma')^{-1}}\frac{\partial r'^{n+3}}{\partial a'}P_n(\cos\gamma)\rho(r',\theta',\lambda')\cos\theta' da' d\theta' d\lambda'\right] \tag{25}$$

To evaluate V_n it is convenient to compute the integral

$$\int_0^{a(1+\Sigma)(1+\Sigma')^{-1}}\rho(a')\frac{\partial r'^{n+3}}{\partial a'}da' = \int_0^a \rho(a')\frac{\partial r'^{n+3}}{\partial a'}da' + \int_a^{a(1+\Sigma)(1+\Sigma')^{-1}}\rho(a')\frac{\partial r'^{n+3}}{\partial a'}da' \tag{26}$$

To evaluate the second integral, a numerical quadrature formula of an appropriate order can be used. To build up a first order theory in the amplitudes $f_n(a)$, the approach $a(1+\Sigma)(1+\Sigma')^{-1} = a(1+\Sigma-\Sigma')$ and the rectangle quadrature formula can be used

$$\int_a^{a(1+\Sigma)(1+\Sigma')^{-1}}\rho(a')\frac{\partial r'^{n+3}}{\partial a'}da' \approx \int_a^{a(1+\Sigma-\Sigma')}\rho(a')\frac{\partial r'^{n+3}}{\partial a'}da' \approx \rho(a')\frac{\partial r'^{n+3}}{\partial a'}(\Sigma-\Sigma') \tag{27}$$

In zero order in amplitudes $r'^{n+3} = a'^{n+3}$, and from them we have in first order

$$\int_a^{a(1+\Sigma)(1+\Sigma')^{-1}}\rho(a')\frac{\partial r'^{n+3}}{\partial a'}da' = (n+3)a^{n+2}\rho(a)\left[\sum_{n=0}^{\infty}f_n(a)(P_n(\sin\theta) - P_n(sin\theta'))\right] \tag{28}$$

From this result we have

$$\begin{aligned}
&\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a(1+\Sigma)(1+\Sigma')^{-1}}\frac{\partial r'^{n+3}}{\partial a'}P_n(\cos\gamma)\rho(r',\theta',\lambda')\cos\theta' dr' d\theta' d\lambda' = \\
&= \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a}\rho(a')\frac{\partial r'^{n+3}}{\partial a'}P_n(\cos\gamma)\cos\theta' da' d\theta' d\lambda' + \\
&+ \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(n+3)a^{n+2}\rho(a)\left[\sum_{n=0}^{\infty}f_n(a)(P_n(\sin\theta) - P_n(sin\theta'))\right]P_n(\cos\gamma)\cos\theta' d\theta' d\lambda'
\end{aligned} \tag{29}$$

If $n \neq 0$, the first integral can be approached in the first order in amplitudes by

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a}\rho(a')\frac{\partial r'^{n+3}}{\partial a'}P_n(\cos\gamma)\cos\theta' da' d\theta' d\lambda' =$$

$$= \int_0^{a}\rho(a')\frac{\partial}{\partial a'}\left[\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r'^{n+3}P_n(\cos\gamma)\cos\theta' d\theta' d\lambda'\right] da' =$$

$$= \int_0^{a}\rho(a')\frac{\partial}{\partial a'}\left[\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a'^{n+3}(1+(n+3)\sum_{k=0}^{\infty} f_k(a')P_k(\sin\theta))P_n(\cos\gamma)\cos\theta' d\theta' d\lambda'\right] da' =$$

$$= \frac{4\pi}{2n+1}(n+3)\int_0^{a}\rho(a')\frac{\partial}{\partial a'}\left[a'^{n+3}f_n(a')\right] da' P_n(\sin\theta) \qquad (30)$$

For $n = 0$ we have

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a}\rho(a')\frac{\partial r'^3}{\partial a'}\cos\theta' da' d\theta' d\lambda' = 12\pi\int_0^{a}\rho(a')a'^2 da' \qquad (31)$$

The value of the second integral is given by

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(n+3)a^{n+2}\rho(a)\left[\sum_{n=0}^{\infty} f_n(a)(P_n(\sin\theta)-P_n(\sin\theta'))\right]P_n(\cos\gamma)\cos\theta' d\theta' d\lambda' =$$

$$= -4\pi\frac{n+3}{2n+1}a^{n+2}f_n(a)\rho(a)P_n(\sin\theta), \quad n \neq 0 \qquad (32)$$

if $n = 0$ the integral is null.
Replacing (30), (32), in (29) we get

$$V_n = \frac{4\pi}{2n+1}G\int_0^{a}\rho(a')\frac{\partial}{\partial a'}\left[a'^{n+3}f_n(a')\right] da' - \frac{4\pi}{2n+1}Ga^{n+2}f_n(a)\rho(a)P_n(\sin\theta) \qquad (33)$$

if $n \neq 0$. If $n = 0$, its value is found by

$$V_0 = 4\pi G\int_0^{a}\rho(a')a'^2 da' \qquad (34)$$

The first integral of (33) coincides with the value of the classical theory.
The analyses of the corresponding U_n terms are similar. For $n \neq 0$ we have

$$U_n = \frac{4\pi}{2n+1}G\int_a^{a_1}\rho(a')\frac{\partial}{\partial a'}\left[a'^{2-n}f_n(a')\right] da' + \frac{4\pi}{2n+1}Ga^{1-n}f_n(a)\rho(a)P_n(\sin\theta) \qquad (35)$$

$$U_0 = 4\pi G\int_a^{a_1}\rho(a')a' da' \qquad (36)$$

Replacing r^p by $r^p = a^p(1 + p\sum_{k=0}^{\infty} f_n(a)P_n(\sin\theta))$ in (22) we have in the first order in the amplitudes

$$\Omega = 4\pi G\left[a^{-1}\int_0^a \rho(a')a'^2 da' - a^{-1}\sum_{n=0}^{\infty}\left[\int_0^a \rho(a')a'^2 da'\right] f_n(a)P_n(sin\theta)\right] +$$
$$+ 4\pi G\int_a^{a_1}\rho(a')a'da' + \sum_{n=1}^{\infty}\frac{4\pi G}{2n+1}\left\{a^{-n-1}\int_0^a \rho(a')\frac{\partial}{\partial a'}\left[a'^{2-n}f_n(a')\right]da' +\right.$$
$$\left. + a^n\int_a^{a_1}\rho(a')\frac{\partial}{\partial a'}\left[a'^{2-n}f_n(a')\right]da'\right\} P_n(\sin\theta) +$$
$$+ \sum_{n=1}^{\infty}\left[-a^{-n-1}\frac{2\pi}{2n+1}Ga^{n+2}f_n(a) + a^n\frac{2\pi}{2n+1}Ga^{1-n}f_n(a)\right]P_n(\sin\theta) \quad (37)$$

Note that the last sum in equation (37) is null. Replacing (17) in (37) we get

$$\Omega = 4\pi G\left[E_0(a) + a^{-1}F_0(a)\right] +$$
$$+ \sum_{n=1}^{\infty} 4\pi G\left\{\frac{1}{2n+1}\left[a^nE_n(a) + a^{-n-1}F_n(a)\right] - a^{-1}F_0(a)f_n(a)\right\}P_n(\sin\theta) \quad (38)$$

This result coincides with the classical expression of the potential [8].

4. First order theory: Analytical method

A second way, based on the analytical development of the inverse of the distance, can be formulated as follows:

$$\Omega = K + W \quad (39)$$

where

$$K = G\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_a^{a_1}\frac{\rho(a')r'^2}{\Delta}\frac{\partial r'}{\partial a'}\cos\theta' da'd\theta'd\lambda'$$
$$W = G\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a}\frac{\rho(a')r'^2}{\Delta}\frac{\partial r'}{\partial a'}\cos\theta' da'd\theta'd\lambda' \quad (40)$$

To evaluate K and W we cannot use $W = \sum_{n=0}^{\infty} W_n r^{-n-1}$ and $K = \sum_{n=0}^{\infty} K_n r^n$, where W_n and K_n are defined by (7) because the developments of $\frac{1}{\Delta}$ given by (2) do not converge in the layer defined by $r \in [r_{min}(a), r_{max}(a)]$ where $r_{min}(a) = \min\{r(a,\theta,\lambda)| \quad \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], \lambda \in [0,\pi]\}$, $r_{max}(a) = \max\{r(a,\theta,\lambda)|\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], \lambda \in [0,\pi]\}$.

To solve this problem, let us define

$$D(a,a') = \frac{1}{\sqrt{a^2 + a'^2 - 2aa'\cos\gamma}} \tag{41}$$

The inverse of the distance between dm and dm' can be developed to the second order in Σ, Σ'

$$\frac{1}{\Delta} = D(a,a') + D_a(a,a')a\Sigma + D_a(a,a')a'\Sigma' + \frac{1}{2}D_{aa}a^2\Sigma^2 + D_{aa'}aa'\Sigma\Sigma' + \frac{1}{2}D_{a'a'}a'^2\Sigma'^2 + .. \tag{42}$$

where subcript x denotes the partial derivative with respect to x.
On the other hand, we have

$$dm' = \rho(a')a'^2(1 + 3\Sigma' + a'\Sigma'_{a'} + 3\Sigma'^2 + 2a'\Sigma'\Sigma'_{a'} + ..)\cos\theta' da' d\theta' d\lambda' \tag{43}$$

To evaluate the potential integral inside the equipotential surface of dm, $D(a,a')$ can be evaluated by

$$D(a,a') = \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{a'}{a}\right)^n P_n(\cos\gamma) \tag{44}$$

while, for outside this surface, it can be evaluated by

$$D(a,a') = \frac{1}{a'}\sum_{n=0}^{\infty}\left(\frac{a}{a'}\right)^n P_n(\cos\gamma) \tag{45}$$

In order to evaluate W we have

$$\frac{1}{\Delta} = \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{a'}{a}\right)^n \{1 - (n+1)\Sigma + n\Sigma'\} P_n(\cos\gamma) \tag{46}$$

Replacing (43) and (44) in (40) we get

$$W = G\int_0^a \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{a'^{n+2}}{a^{n+1}} \{1 - (n+1)\Sigma + (n+3)\Sigma' + a'\Sigma'_{a'}\} P_n(\sin\theta')\rho'(a)\cos\theta' d\theta' d\lambda' da' \tag{47}$$

To evaluate K we can procced by a similar way

$$\frac{1}{\Delta} = \frac{1}{a'}\sum_{n=0}^{\infty}\left(\frac{a}{a'}\right)^n \{1 + n\Sigma - (n+1)\Sigma'\} P_n(\cos\gamma) \tag{48}$$

Repalcing(43) and (45) in (40) we get

$$K = G\int_a^{a_1} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{a^n}{a'^{n-1}} \{1 + n\Sigma + (2-n)\Sigma' + a'\Sigma'_{a'}\} P_n(\sin\theta')\rho'(a)\cos\theta' d\theta' d\lambda' da' \tag{49}$$

To evaluate (47) and (49), we get if $n \neq 0$

$$\int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Sigma P_n(\cos\gamma) d\theta' d\lambda' = 0, \qquad \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Sigma P_0(\cos\gamma) d\theta' d\lambda' = 4\pi\Sigma \tag{50}$$

and

$$\int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Sigma' P_n(\cos\gamma) d\theta' d\lambda' = \frac{4\pi}{2n+1} f_n(a') P_n(\sin\theta')$$

$$\int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a' \Sigma'_{a'} P_n(\cos\gamma) d\theta' d\lambda' = \frac{4\pi}{2n+1} a' f'_n(a') P_n(\sin\theta') \tag{51}$$

and consequently

$$W = 4\pi G \int_0^a \rho(a') a'^2 da' - 4\pi G \sum_{n=0}^{\infty} f_n(a) \int_0^a \rho(a') a'^2 da' P_n(\sin\theta) + \\ + \sum_{n=0}^{\infty} \frac{4\pi G}{2n+1} \int_0^a \rho(a') \frac{\partial}{\partial a'} \left[a'^{n+3} f_n(a') \right] da' \tag{52}$$

$$K = 4\pi G \int_a^{a_1} \rho(a') a' da' + \sum_{n=0}^{\infty} \frac{4\pi G}{2n+1} \int_a^{a_1} \rho(a') \frac{\partial}{\partial a'} \left[a'^{2-n} f_n(a') \right] da' \tag{53}$$

Replacing (52), (53), (17) in (39) we get

$$\Omega = K + W = 4\pi G \left[E_0(a) + a^{-1} F_0(a) \right] + \\ + \sum_{n=1}^{\infty} 4\pi G \left\{ \frac{1}{2n+1} \left[a^n E_n(a) + a^{-n-1} F_n(a) \right] - a^{-1} F_0(a) f_n(a) \right\} P_n(\sin\theta) \tag{54}$$

The total autogravitational potential $\Omega = K + W$ coincides with the value given in the previous section and consequently with the classical theory.

5. Concluding Remarks

Classical methods to study the equilibrium figures of celestial bodies contain a convergence problem in a layer around dm. To solve this problem, two methods have been proposed one based on numerical integration formulae an other based on analytical develoments of the inverse of the distance. The solution to the problem following both methods coincides with the classical theory in the first order in amplitudes.
On the other hand, both metods can be suitable to be extended to second and higher order to study the results concordance.

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