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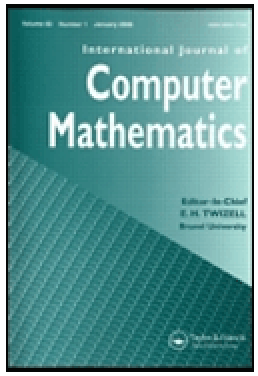
Istvan Szalkai, Della Vecchia Biancamaria



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Finding Better Weight Functions for Generalized Shepard's Operator on Infinite Intervals

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Abstract

In this paper, we investigate the difference of Shepard's generalized operators S_σ from the approximated set of data for various weight functions σ. Bounds are given for the sizes of the "bumps" shown in Fig. 1 and the best weight function σ for practical use is proposed in the last Section.

Keywords: Interpolation, Approximation, Generalized Shepard-method,
AMS classification: 41A05, 41A20, 41A36

1 Introduction

For any given set of datapoints $\{P_1, \ldots, P_M\} \subseteq \mathbb{R}^N$ in any dimension $N \geq 1$, real numbers $F_1, \ldots, F_M \in \mathbb{R}$ and fixed weight function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ we investigate the generalized Shepard operator $S_\sigma : \mathbb{R}^N \to \mathbb{R}^+$ defined for any $P \in \mathbb{R}^N$ as

$$S_\sigma^{(M)}(P) := \frac{\sum\limits_{i=1}^{M} F_i \sigma\left(d\left(P, P_i\right)\right)}{\sum\limits_{i=1}^{M} \sigma\left(d\left(P, P_i\right)\right)},$$

where $d : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is any distance function on $\mathbb{R}^N$. (For simplicity we omit the superscript M whenever it is clear from the context.)

The main advantage of the above simple formula is that it is applicable for any set of points $\{P_1, \ldots, P_M\} \subseteq \mathbb{R}^N$ (which is not our choice in general in practice). Let us highlight our main point of view: we consider S_σ for constructing a surface matching any given set of data[1)] $\{(P_1, F_1), \ldots, (P_M, F_M)\}$, i.e., we do not consider S_σ for approximating any pre-given function $f : \mathbb{R}^N \to \mathbb{R}$.

This method is widely applied, e.g., in geography for dimension $N = 2$ (see, e.g., [Katona (2002)]).

For exact approximation (that is $S_\sigma(P_i) = F_i$ for all $i \leq M$) σ must satisfy

$$\lim_{d \to 0+} \sigma(d) = +\infty. \tag{1}$$

Further we require

$$\lim_{d \to +\infty} \sigma(d) = 0 \tag{2}$$

since in our investigations $M \to \infty$ and so $d \to \infty$ (see [Szalkai (1999b)]).

In the present paper we restrict ourselves to dimension $N = 1$. However, the results we obtain can be used for any dimension, since any distortion of higher dimensional surfaces (defined by S_σ) can be detected in a suitable one-dimensional intersection.

The starting point of our investigation was the surprising diagram of $S_{1/d}$, shown in Fig. 1 (in dimension $N = 1$):

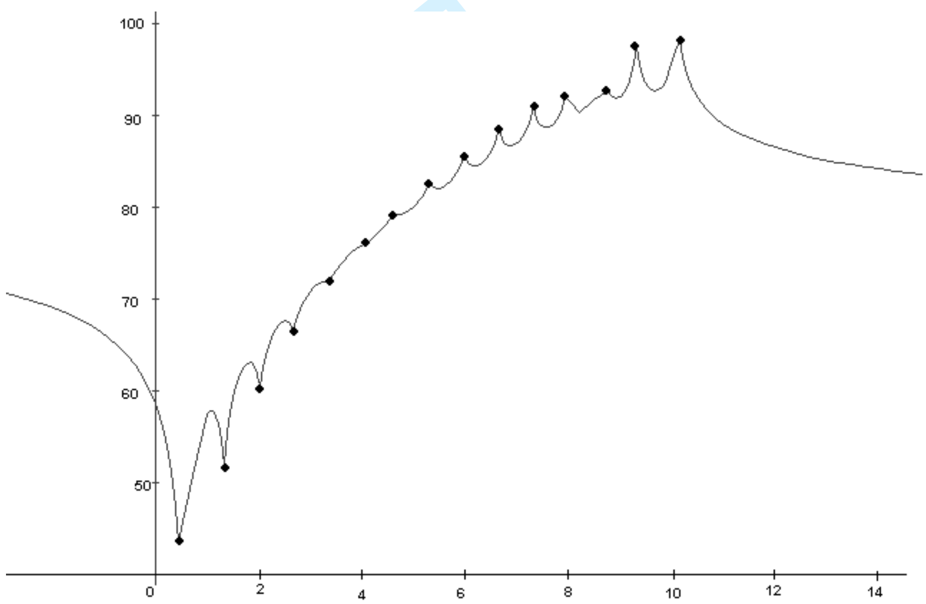

Figure 1: The graph of S_σ for $\sigma(d) = 1/d$ in dimension $N = 1$.

(A computer program for demonstrating and investigating different approximation methods is also in preparation in [Nagy (2010)].)

[1)] in practice, these data are obtained by measuring and not by using a formula

Black dots in the above figure show the pairs (P_i, F_i) for $i \le M$. What disturbs us is that the approximating formula $S_\sigma(P)$ has big differences ("waves") in many places despite the almost linear dataset. (In other words: S_σ tends to the average $\bar{F} := \frac{F_1+\ldots+F_M}{M}$ not only when $P \to \infty$ but even when P is inside the convex hull of the dataset $\{P_1, \ldots, P_M\}$.)

In this note we show that these "bumps" (big differences) are present almost in all cases. More precisely, we calculate the rate of these differences for several weight functions σ:

$\sigma_1(d) := 1/d^\alpha \quad (\alpha > 0)$ (Shepard's original formula),

$\sigma_2(d) := 1/d^\alpha \exp(-\lambda d^\beta) \quad (\alpha, \beta, \lambda > 0)$,

$\sigma_3(d) := \frac{1}{\ln^\beta(d+1)} \quad (\beta > 0)$,

$\sigma_4(d) := 1/d^\alpha \frac{1}{\ln^\beta(d+1)} \quad (\alpha, \beta > 0)$

(σ_1 is the original weight function of [Gordon and Wixom (1978)]. The others are our candidates for better approximation. We do not have so many choices since we have to ensure (1).)

For most of the investigated cases, the size of the differences goes to infinity when the number of the datapoints M tends to infinity. This latter assumption requires infinite domain for the approximation. This is why we investigate $\lim_{M\to\infty}$ in Questions 1 through 3.

Though everyday approximations are done on finite intervals, in most cases we cannot choose as many datapoints P_i as we like as, e.g., in the application [Katona (2002)]. This could result in the unexpected waves as in Fig. 1.

In the literature, numerous excellent properties of Shepard's original and generalized formulae are justified, see, e.g., in [Allasia (1995)], [Bojanic *et al.* (1999)], [DellaVecchia *et al.* (1996)], [DellaVecchia *et al.* (2004)], [Farwig (1986)], [Gál and Szabados (1999)], [Gordon and Wixom (1978)], [Hoschek and Lasser (1993)], [Mastroianni and Szabados (1997)], [Szabados (1991)], [Szalkai (1999a), (1999b)] or [Zhou (1998)]. These good approximation properties are proven either assuming a special set of datapoints $\{P_1, \ldots, P_M\}$, or by investigating the limit-approximation in the case when the number of the datapoints M tends to infinity on a fixed finite interval. Elimination of these restrictions is the main improvement of our analysis with respect to other investigations.

[Katona (2002)], [Láng-Lázi *et al.* (2006)] and [Szalkai (1996), (2000)] tried to apply Shepard's original formula in practice. We suggest using the weight functions that we will select in the Section "Conclusions".

1.1 Preliminary definitions

We are treating the $N = 1$-dimensional case[2)]. In the present investigation let us define the set of datapoints to be equidistant (they form an arithmetic progression), that is we fix $u, v > 0$ and we let

$$P_i := P_1 + (i-1)v \quad \text{and} \quad F_i := F_1 + (i-1)u, \text{ for } i = 1, \ldots, M \tag{3}$$

[2)] which can be embedded in some higher dimensional space (that is $P_i \in \mathbb{R}$ are in $N = 1$-dimension).

We investigate the difference of S_σ from the straight line $\ell(x)$

$$\ell(x) = F_1 + \tau u \quad \text{for } x = P_1 + \tau v \quad (\tau \in \mathbb{R})$$

(ℓ connects all the points (P_i, F_i)) at the point $x_\tau \in (P_1, P_2)$

$$x_\tau = P_1 + \tau v \qquad (\tau \in (0,1)),$$

that is we calculate

$$\Delta(x_\tau) = S_\sigma(x_\tau) - \ell(x_\tau)$$

for various weight functions σ. Our set of data $\{(P_i, F_i),\ i = 1, \ldots, M\}$ is equidistant, other sets of data are investigated in [Szalkai (1999b), Section 4.2].

The difference is

$$\begin{aligned}\Delta(x_\tau) &= S_\sigma(x_\tau) - \ell(x_\tau) \\ &= \frac{F_1 \sigma(\tau v) + \sum_{i=2}^{M} (F_1 + (i-1)u)\, \sigma\left((i-1-\tau)v\right)}{\sigma(\tau v) + \sum_{i=2}^{M} \sigma\left((i-1-\tau)v\right)} - (F_1 + \tau u) \\ &= u \left(\frac{\sum_{j=1}^{M-1} j \sigma\left((j-\tau)v\right)}{\sigma(\tau v) + \sum_{j=1}^{M-1} \sigma\left((j-\tau)v\right)} - \tau \right). \end{aligned} \tag{4}$$

We investigate the following questions for fixed $\tau \in (0,1)$ (i.e., $x_\tau \in (P_1, P_2)$ is fixed[3)]):

Question 1: Is $\lim_{M\to\infty} \Delta(x_\tau) = \infty$ or $\lim_{M\to\infty} \Delta(x_\tau) < \infty$?
In the latter case: what is the value of $\lim_{M\to\infty} \frac{\Delta(x_\tau)}{u}$ approximatively?

Question 2: For which weight functions σ do we have $\lim_{M\to\infty} S_\sigma(x_\tau) > F_2$ for some $x_\tau \in (P_1, P_2)$?
(This inequality is equivalent to $\lim_{M\to\infty} \frac{\Delta(x_\tau)}{u} > 1 - \tau$)

Question 3: For which point $x_\tau \in (P_1, P_2)$ is $\lim_{M\to\infty} \frac{\Delta(x_\tau)}{u}$ maximal?

[3)] The assumption $0 < \tau < 1$ is not a restriction in fact, since the limit $\lim_{M\to\infty} \Delta(x_*)$ we are discussing in this paper is the same for any fixed point $x_* \in (P_0, \infty)$. This is why we may restrict ourselves to the interval (P_0, P_1).

Similar questions might be investigated for the approximation in the finite interval $[0, a]$ (S_σ is invariant for vertical translation but not for vertical zooming).

The following well known results will be useful to our work:

Lemma 0: Let $a_0, a_1, \ldots \in \mathbb{R}_+$ with $a_j \to 0$. Then the fractions

$$\frac{\sum_{j=0}^{M} j a_j}{\sum_{j=0}^{M} a_j}$$

have a finite limit for $M \to \infty$ if and only if $\sum_{j=0}^{\infty} j a_j < \infty$. $\square$

2 Investigating the Weight Functions

Now we investigate the weight functions σ_1 through σ_4 in detail.

2.1 The weight function $\sigma_1(d) = 1/d^\alpha$

Since σ now is homogeneous, we have

$$\frac{\Delta(x_\tau)}{u} = \frac{\sum_{j=1}^{M-1} j\sigma\,(j-\tau)}{\sigma(\tau) + \sum_{j=1}^{M-1} \sigma\,(j-\tau)} - \tau = \frac{\sum_{j=1}^{M-1} \frac{j}{(j-\tau)^\alpha}}{\frac{1}{\tau^\alpha} + \sum_{j=1}^{M-1} \frac{1}{(j-\tau)^\alpha}} - \tau.$$

It is well known that the denominator is convergent iff $\alpha > 1$ while the numerator is convergent iff $\alpha > 2$.

This means that Shepard's original formula

$$S_\alpha(P) := \frac{\sum_{i=1}^{M} F_i \frac{1}{(d(P,P_i))^\alpha}}{\sum_{i=1}^{M} \frac{1}{(d(P,P_i))^\alpha}}$$

must have as large bumps as one likes for all $1 < \alpha \leq 2$, while the size of bumps is bounded for $2 < \alpha$:

Theorem 1. *For all $1 < \alpha \leq 2$ the limit $\lim_{M\to\infty} \frac{\Delta(x_\tau)}{u} = \infty$ diverges, while for $\alpha > 2$ we have*

$$\frac{\frac{1}{(1-\tau)^\alpha} + \zeta(\alpha-1) - 1}{\frac{1}{\tau^\alpha} + \frac{1}{(1-\tau)^\alpha} + \zeta(\alpha)} - \tau \leq \lim_{M\to\infty} \frac{\Delta(x_\tau)}{u} \leq \frac{\frac{1}{(1-\tau)^\alpha} + \zeta(\alpha-1) + \zeta(\alpha)}{\frac{1}{\tau^\alpha} + \frac{1}{(1-\tau)^\alpha} + \zeta(\alpha) - 1} - \tau, \quad (5)$$

where ζ is Riemann's zeta function.

Proof: In the case of $\alpha > 2$ in order to approximate the value of $\lim_{M\to\infty} \frac{\Delta(x_\tau)}{u}$ we write for the denominator

$$\frac{1}{\tau^\alpha} + \frac{1}{(1-\tau)^\alpha} + \sum_{j=2}^{M-1} \frac{1}{j^\alpha} < \frac{1}{\tau^\alpha} + \sum_{j=1}^{M-1} \frac{1}{(j-\tau)^\alpha} < \frac{1}{\tau^\alpha} + \frac{1}{(1-\tau)^\alpha} + \sum_{j=2}^{M-1} \frac{1}{(j-1)^\alpha},$$

i.e.,

$$\frac{1}{\tau^\alpha} + \frac{1}{(1-\tau)^\alpha} + \zeta(\alpha) - 1 \le \lim_{M\to\infty} (\text{den}) \le \frac{1}{\tau^\alpha} + \frac{1}{(1-\tau)^\alpha} + \zeta(\alpha)$$

and for the numerator

$$\frac{1}{(1-\tau)^\alpha} + \sum_{j=2}^{M-1} \frac{j}{j^\alpha} < \sum_{j=1}^{M-1} \frac{j}{(j-\tau)^\alpha} < \frac{1}{(1-\tau)^\alpha} + \sum_{j=2}^{M-1} \frac{j-1+1}{(j-1)^\alpha},$$

i.e.,

$$\frac{1}{(1-\tau)^\alpha} + \zeta(\alpha-1) - 1 \le \lim_{M\to\infty} (\text{num}) \le \frac{1}{(1-\tau)^\alpha} + \zeta(\alpha-1) + \zeta(\alpha),$$

which implies the estimation (5), answering Question 1. $\square$

Question 2 could be answered by the inequality

$$1 - \tau \le \frac{\frac{1}{(1-\tau)^\alpha} + \zeta(\alpha-1) - 1}{\frac{1}{\tau^\alpha} + \frac{1}{(1-\tau)^\alpha} + \zeta(\alpha)} - \tau,$$

i.e.,

$$\frac{1}{\tau^\alpha} + \frac{1}{(1-\tau)^\alpha} + \zeta(\alpha) \le \frac{1}{(1-\tau)^\alpha} + \zeta(\alpha-1) - 1,$$

or by the much simpler one

$$\frac{1}{\tau^\alpha} + 1 \le \zeta(\alpha-1) - \zeta(\alpha). \tag{6}$$

For each fixed α the left hand side has minimal value for $\tau = 1$, so (6) admits a solution for τ iff

$$2 \le \zeta(\alpha-1) - \zeta(\alpha). \tag{7}$$

From our computational experiments we learned that (7) holds for

$$2 < \alpha < 2.3617$$

and does not hold for $1 < \alpha < 2$ or $\alpha > 2.3617$.

For Question 3 we should find the maximal value(s) of

$$\frac{\Delta(x_\tau)}{u} := \frac{\frac{1}{(1-\tau)^\alpha} + \zeta(\alpha-1) - 1}{\frac{1}{\tau^\alpha} + \frac{1}{(1-\tau)^\alpha} + \zeta(\alpha)} - \tau$$

where $\tau \in (0,1)$ for each fixed $\alpha > 2$.

2.2 The weight function $\sigma_2(d) = 1/d^\alpha \exp(-\lambda d^\beta)$

In this case $\frac{\Delta(x_\tau)}{u}$ reads as

$$\frac{\Delta(x_\tau)}{u} = \frac{\sum_{j=1}^{M-1} j \frac{\exp(-\lambda((j-\tau)v)^\beta)}{((j-\tau)v)^\alpha}}{\frac{\exp(-\lambda(\tau v)^\beta)}{(\tau v)^\alpha} + \sum_{j=1}^{M-1} \frac{\exp(-\lambda((j-\tau)v)^\beta)}{((j-\tau)v)^\alpha}} - \tau = \frac{\sum_{j=1}^{M-1} j \frac{E^{(j-\tau)^\beta}}{(j-\tau)^\alpha}}{\frac{E^{\tau^\beta}}{\tau^\alpha} + \sum_{j=1}^{M-1} \frac{E^{(j-\tau)^\beta}}{(j-\tau)^\alpha}} - \tau \quad (8)$$

where

$$E := \exp(-\lambda v^\beta)$$

(v was defined in (3)).

Since $0 < E, \tau < 1$ we can easily prove

Theorem 2. $\lim_{M\to\infty} \frac{\Delta(x_\tau)}{u}$ *is convergent for all* $\alpha, \beta, \lambda > 0$, $\tau \in (0,1)$.

Proof. We use Lemma 0 for the sequence $a_j = \frac{E^{(j-\tau)^\beta}}{(j-\tau)^\alpha}$ $(1 \le j)$, $a_0 = \frac{E^{\tau^\beta}}{\tau^\alpha}$. The assumptions $a_j > 0$ and $a_j \to 0$ clearly hold since $|E| < 1$ and $1 \le j$.
The numerator can be estimated as

$$\sum_{j=1}^{\infty} j \frac{E^{(j-\tau)^\beta}}{(j-\tau)^\alpha} \le a_1 + \sum_{j=2}^{\infty} jE^{(j-1)^\beta} = a_1 + \sum_{i=1}^{\infty} (i+1)\, E^{i^\beta}. \quad (9)$$

Using the fact that

$$\lim_{i\to\infty} \frac{i^\beta}{\log_{\frac{1}{E}}(i)} = \infty,$$

we can find $i_0 \in \mathbb{N}$ such that $i^\beta > 3\log_{\frac{1}{E}}(i)$ for $i > i_0$. This proves (9) since

$$\begin{aligned} \sum_{i=1}^{\infty} (i+1)\, E^{i^\beta} &\le \sum_{i=1}^{i_0} (i+1)\, E^{i^\beta} + \sum_{i=i_0}^{\infty} (i+1)\, E^{3\log_{\frac{1}{E}}(i)} \\ &\le c + \sum_{i=i_0}^{\infty} \frac{i+1}{i^3}, \end{aligned}$$

which clearly converges. The denominator does not exceed the numerator so it converges as well. □

Now we present detailed calculations for the case $\alpha = \beta = 1$ (calculations for the general case of α and β are lengthy). In this case the numerator of (8) is

$$\begin{aligned} \sum_{j=1}^{M-1} j \frac{E^{(j-\tau)}}{(j-\tau)} &= \sum_{j=1}^{M-1} \left(1 + \frac{\tau}{j-\tau}\right) E^{(j-\tau)} = E^{-\tau} \sum_{j=1}^{M-1} E^j + \tau \sum_{j=1}^{M-1} \frac{E^{(j-\tau)}}{j-\tau} \\ &= \frac{E}{E^\tau} \frac{E^{M-1}-1}{E-1} + \tau \mathcal{I}_M(E,\tau), \end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_M(E,\tau) &:= \sum_{j=1}^{M-1}\frac{E^{(j-\tau)}}{(j-\tau)} = \frac{E^{1-\tau}}{1-\tau} + \sum_{j=2}^{M-1}\int_0 E^{(j-\tau-1)}dE \\ &= \frac{E^{1-\tau}}{1-\tau} + \int_0 E^{1-\tau}\sum_{J=0}^{M-3}E^J dE = \frac{E^{1-\tau}}{1-\tau} + \int_0 E^{1-\tau}\frac{E^{M-2}-1}{E-1}dE \\ &= \frac{E^{1-\tau}}{1-\tau} + \int_0^E x^{1-\tau}\frac{x^{M-2}-1}{x-1}dx,\end{aligned}$$

which has limit $(M \to \infty)$

$$\mathcal{I}_\infty(E,\tau) := \frac{E^{1-\tau}}{1-\tau} + \int_0^E x^{1-\tau}\frac{1}{1-x}dx.$$

So we finally get:

Theorem 3.

$$\mathcal{L}(E,\tau) := \lim_{M\to\infty}\frac{\Delta(x_\tau)}{u} = \frac{\frac{E^{1-\tau}}{1-E} + \tau\mathcal{I}_\infty(E,\tau)}{\frac{E^\tau}{\tau} + \mathcal{I}_\infty(E,\tau)} - \tau \quad \textit{for } \alpha=\beta=1. \tag{10}$$

This answers Question 1. □

It is easy to see that

$$\int x^{1-\tau}\frac{1}{1-x}dx = \frac{x^{1-\tau}\ {}_2F_1(1-\tau;1;2-\tau;x)-1}{1-\tau},$$

with ${}_2F_1(w,z,y,x)$ being the hypergeometric function, so that

$$\mathcal{I}_\infty(E,\tau) = \frac{E^{1-\tau}}{1-\tau}\ {}_2F_1(1-\tau;1;2-\tau;E). \tag{11}$$

Figures 2–4 below show 3D views and intersections of $\mathcal{L}$ vs. E and τ in different scaling. The hypergeometric function at the right-hand-side of (11) was computed by means of a routine included in the package of special functions by [Jin, Zhang (1996)]. Points 0 and 1 are excluded from the plots.

Since we are looking for the best approximating function S_σ including $E = \exp(-\lambda v^\beta)$, we can conclude in the case $\alpha=\beta=1$ that:

After estimating the largest or most common values of v we must choose λ such that

$$E = \exp(-\lambda v) < 0.6$$

which will make $\lim_{M\to\infty}\frac{\Delta(x_\tau)}{u}$ very small!

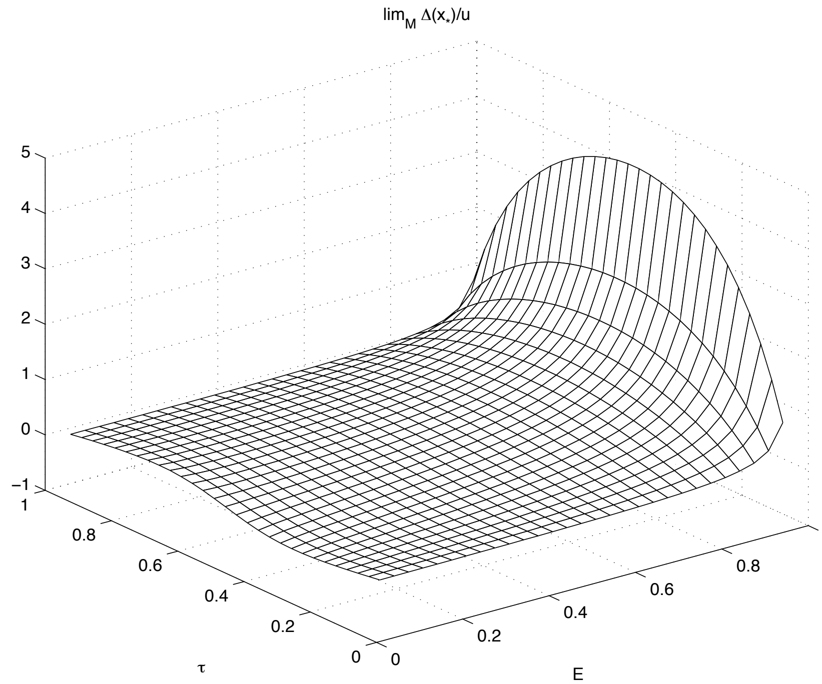

Figure 2: Plot $\lim_{\infty} \Delta(x_\tau)/u$ vs. τ and E, scale $[-1, 5]$.

Let us note that formula (10) for $\mathcal{L}(E,\tau)$ can be also written as

$$\begin{aligned}
\mathcal{L}(E,\tau) &= \frac{\tau\left[\frac{E^{\tau}}{\tau} + \mathcal{I}(E,\tau)\right] + \frac{E^{1-\tau}}{1-E} - E^{\tau}}{\frac{E^{\tau}}{\tau} + \mathcal{I}(E,\tau)} - \tau = \frac{\frac{E^{1-\tau}}{1-E} - E^{\tau}}{\frac{E^{\tau}}{\tau} + \frac{E^{1-\tau}}{1-\tau} + \int_0^E \frac{x^{1-\tau}}{1-x}dx} \\
&= \frac{\tau(1-\tau)\left[E^{1-\tau} - (1-E)\,E^{\tau}\right]}{(1-E)\left[(1-\tau)E^{\tau} + \tau E^{1-\tau} + \tau(1-\tau)\int_0^E \frac{x^{1-\tau}}{1-x}dx\right]}.
\end{aligned}$$

It is easy to see that, for fixed $E \in (0,1)$

$$\lim_{\tau\to 0} \mathcal{L}(E,\tau) = 0 \quad \text{and} \quad \lim_{\tau\to 1} \mathcal{L}(E,\tau) = 0,$$

which correspond to the fact that S_σ is exact (that is $S_\sigma(P_i) = F_i$).

For Question 2 we should solve the inequality

$$\frac{\frac{E^{1-\tau}}{1-E} + \tau\mathcal{I}(E,\tau)}{\frac{E^{\tau}}{\tau} + \mathcal{I}(E,\tau)} - \tau > 1 - \tau,$$

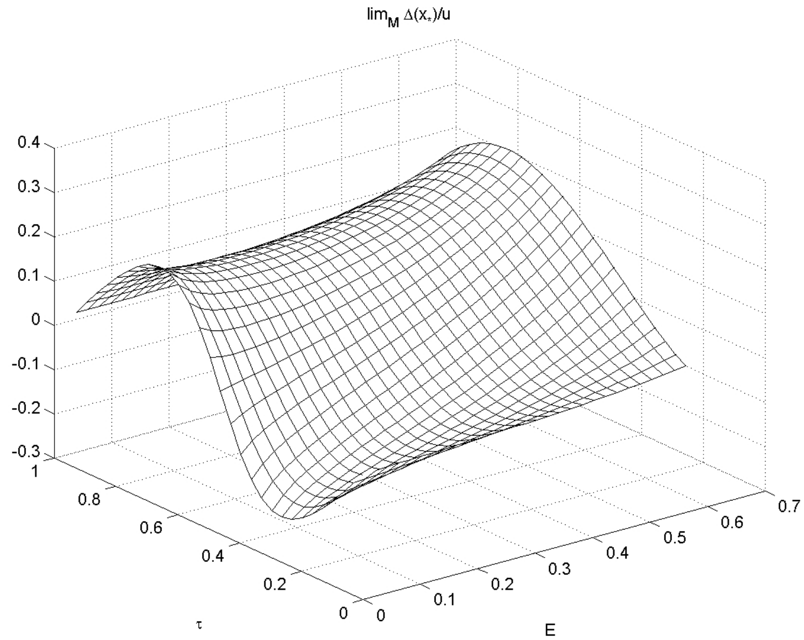

Figure 3: Plot $\lim_{\infty} \Delta(x_\tau)/u$ vs. τ and E, scale $[-0.3, 0.4]$.

that is

$$\frac{E^{1-\tau}}{1-E} + \tau \left(\frac{E^{1-\tau}}{1-\tau} + \int_0^E \frac{x^{1-\tau}}{1-x} dx \right) > \frac{E^\tau}{\tau} + \frac{E^{1-\tau}}{1-\tau} + \int_0^E \frac{x^{1-\tau}}{1-x} dx,$$

i.e.,

$$\frac{E^{1-\tau}}{1-E} E - \frac{E^\tau}{\tau} > (1-\tau) \int_0^E \frac{x^{1-\tau}}{1-x} dx,$$

or, using the hypergeometric function ${}_2F_1$,

$$\frac{E}{1-E} - \frac{E^{2\tau-1}}{\tau} > {}_2F_1(1-\tau; 1; 2-\tau; E) - 1.$$

Some more computer experiments are necessary for solving this inequality, we do not include them here.

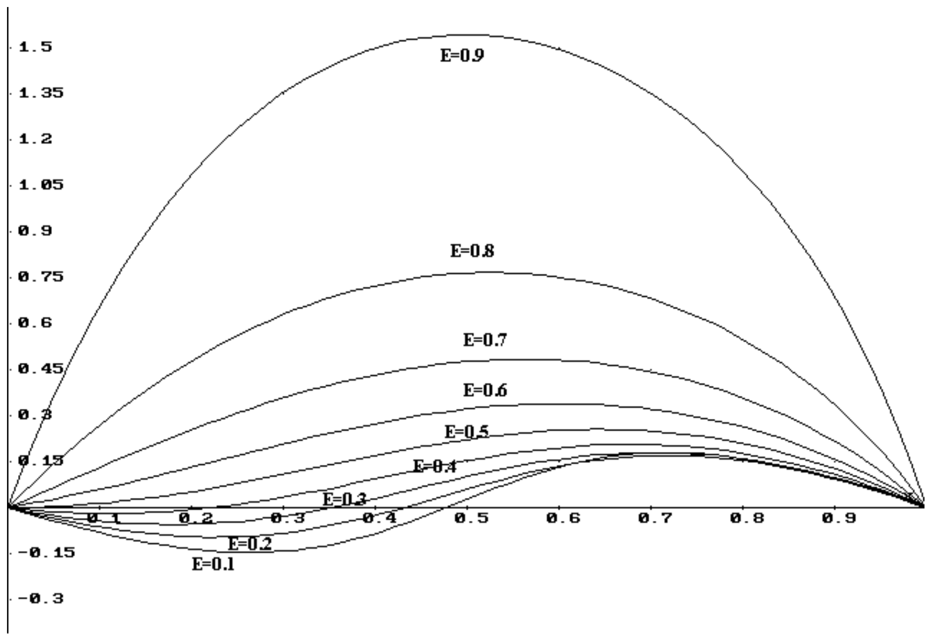

Figure 4: Plot $\lim_{\infty} \Delta(x_\tau)/u$ vs. τ for $E = 0.1, 0.2, \ldots, 0.9$.

2.3 The weight function $\sigma_3(d) = \frac{1}{\ln^\beta(\mathbf{d+1})}$

Now $\frac{\Delta(x_\tau)}{u}$ reads as

$$\frac{\Delta(x_\tau)}{u} = \frac{\sum\limits_{j=1}^{M-1} \frac{j}{\ln^\beta((j-\tau)v+1)}}{\frac{1}{\ln^\beta(\tau v+1)} + \sum\limits_{j=1}^{M-1} \frac{1}{\ln^\beta((j-\tau)v+1)}} - \tau.$$

Since

$$\sum_{j=1}^{\infty} \frac{1}{\ln^\beta((j-\tau)v+1)} \geq \sum_{j=2}^{\infty} \frac{1}{\ln^\beta(j^2)} - c = \infty$$

we see that

Theorem 4: *For the weight function* $\sigma(d) = \frac{1}{\ln^\beta(\mathrm{d+1})}$ *we have for all* $\beta > 0$

$$\lim_{M\to\infty} \frac{\Delta(x_\tau)}{u} = \infty,$$

answering Question 1. □

2.4 The weight function $\sigma_4(d) = 1/d^\alpha \frac{1}{\ln^\beta(d+1)}$

In this case we have

$$\frac{\Delta(x_\tau)}{u} = \frac{\sum_{j=1}^{M-1} \frac{j}{(j-\tau)^\alpha v^\alpha \ln^\beta((j-\tau)v+1)}}{\frac{1}{\tau^\alpha v^\alpha \ln^\beta(\tau v+1)} + \sum_{j=1}^{M-1} \frac{1}{(j-\tau)^\alpha v^\alpha \ln^\beta((j-\tau)v+1)}} - \tau$$
$$= \frac{\sum_{j=1}^{M-1} \frac{j}{(j-\tau)^\alpha \ln^\beta((j-\tau)v+1)}}{\frac{1}{\tau^\alpha \ln^\beta(\tau v+1)} + \sum_{j=1}^{M-1} \frac{1}{(j-\tau)^\alpha \ln^\beta((j-\tau)v+1)}} - \tau.$$

We will use the following fact from elementary calculus:

Lemma 5. *The sum*

$$\mathcal{L}(\alpha,\beta,v) := \sum_{\substack{j=1\\ jv\neq 1}}^{\infty} \frac{1}{j^\alpha \ln^\beta(jv)}, \ \ (v>0 \text{ fixed})$$

is convergent iff either $\alpha = 1$ *and* $\beta > 1$ or $\alpha > 1$ *and* $\beta > 0$. □

Now we can start answering Question 1:

$$\sum_{j=1}^{\infty} \frac{1}{(j-\tau)^\alpha \ln^\beta((j-\tau)v+1)} \geq \sum_{j=1}^{\infty} \frac{1}{j^\alpha \ln^\beta(j(v+1))} = \mathcal{L}(\alpha,\beta,v+1),$$

$$\sum_{j=1}^{\infty} \frac{1}{(j-\tau)^\alpha \ln^\beta((j-\tau)v+1)}$$
$$\leq \frac{1}{(1-\tau)^\alpha \ln^\beta((1-\tau)v+1)} + \sum_{\substack{j=2\\ (j-1)v\neq 1}}^{\infty} \frac{1}{(j-1)^\alpha \ln^\beta((j-1)v)}$$
$$= \frac{1}{(1-\tau)^\alpha \ln^\beta((1-\tau)v+1)} + \mathcal{L}(\alpha,\beta,v)$$

and

$$\sum_{j=1}^{M-1} \frac{j}{(j-\tau)^\alpha \ln^\beta((j-\tau)v+1)} \geq \sum_{j=1}^{\infty} \frac{1}{j^{\alpha-1} \ln^\beta(j(v+1))} = \mathcal{L}(\alpha-1,\beta,v+1),$$

$$\sum_{j=1}^{\infty} \frac{j}{(j-\tau)^{\alpha} \ln^{\beta}((j-\tau)v+1)}$$

$$\leq \frac{1}{(1-\tau)^{\alpha} \ln^{\beta}((1-\tau)v+1)} + \sum_{\substack{j=2 \\ (j-1)v\neq 1}}^{\infty} \frac{j-1}{(j-1)^{\alpha} \ln^{\beta}((j-1)v)}$$

$$+ \sum_{\substack{j=2 \\ (j-1)v\neq 1}}^{\infty} \frac{1}{(j-1)^{\alpha} \ln^{\beta}((j-1)v)}$$

$$= \frac{1}{(1-\tau)^{\alpha} \ln^{\beta}((1-\tau)v+1)} + \mathcal{L}(\alpha-1,\beta,v) + \mathcal{L}(\alpha,\beta,v),$$

which implies

Theorem 6: *For the weight function* $\sigma(d) = 1/d^{\alpha} \frac{1}{\ln^{\beta}(d+1)}$ *the limit* $\lim_{M\to\infty} \frac{\Delta(x_\tau)}{u}$ *is convergent if and only if* either $\alpha = 2$ *and* $\beta > 1$ or $\alpha > 2$ *and* $\beta > 0$.

In the above cases we have

$$\frac{\mathcal{L}(\alpha-1,\beta,v+1)}{\frac{1}{\tau^{\alpha}\ln^{\beta}(\tau v+1)} + \frac{1}{(1-\tau)^{\alpha}\ln^{\beta}((1-\tau)v+1)} + \mathcal{L}(\alpha,\beta,v)} - \tau \leq \lim_{M\to\infty} \frac{\Delta(x_\tau)}{u}$$

and

$$\lim_{M\to\infty} \frac{\Delta(x_\tau)}{u} \leq \frac{\frac{1}{(1-\tau)^{\alpha}\ln^{\beta}((1-\tau)v+1)} + \mathcal{L}(\alpha-1,\beta,v) + \mathcal{L}(\alpha,\beta,v)}{\frac{1}{\tau^{\alpha}\ln^{\beta}(\tau v+1)} + \mathcal{L}(\alpha-1,\beta,v+1)} - \tau. \qquad \square$$

3 Conclusions

In the previous sections we have seen that for most of the weight functions σ the relative size $\frac{\Delta(x_\tau)}{u}$ of the bumps may be convergent or divergent depending on its parameters. In general, the quicker $\sigma(d)$ tends to 0 as $d \to \infty$, the smaller $\frac{\Delta(x_\tau)}{u}$. In other words:

Conclusion: Among the investigated weight functions σ_1 through σ_4 we found

$$\sigma_2(d) := 1/d \exp(-\lambda d)$$

to be "smoothest", i.e., $\lim_{M\to\infty} \frac{\Delta(x_\tau)}{u}$ could be acceptably small for suitable λ.

For practical applications we recommend first to estimate the largest, or the most common values of v (the distances of the measuring datapoints, see (3)), then to choose λ as

$$\exp(-\lambda v) < 0.6.$$

(In the present paper we could make detailed computations only in the case $\alpha = \beta = 1$ for the function σ_2.)

Though we used the dataset (3) for our computations, we think that our conclusions above are valid also for any other dataset, since the "smoothness" of S_σ depends on the rate of (1) and (2) which is influenced by λ and v above rather than by the dataset.

In conclusion, we present some graphs of S_σ for some σ. In all examples in Figures 5 through 8, $M = 100$, $P_i = i$, $F_i = i$ $(1 \le i \le M)$.

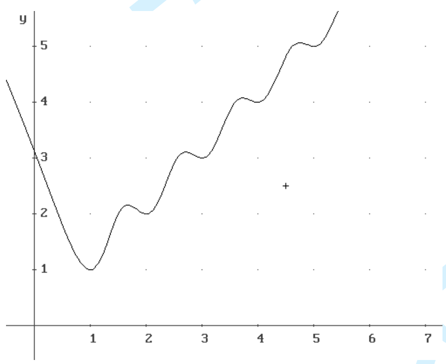

Figure 5: The graph of S_σ for $\sigma(d) = 1/d^{2.01}$.

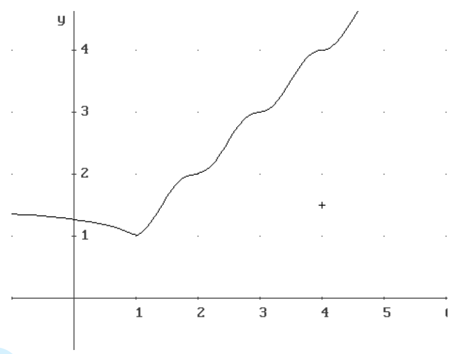

Figure 6: The graph of S_σ for $\sigma(d) = \exp(-d)/d$.

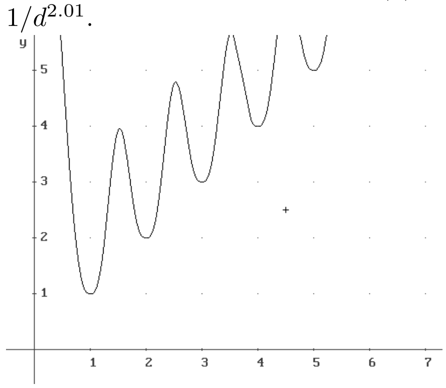

Figure 7: The graph of S_σ for $\sigma(d) = \frac{1}{\ln^3(d+1)}$.

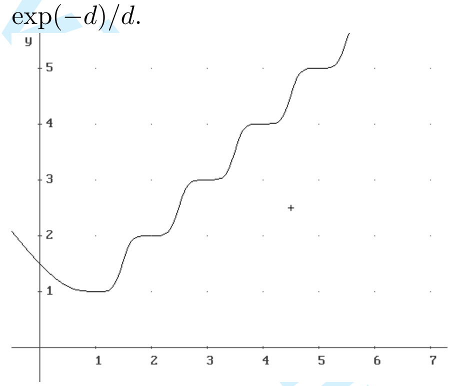

Figure 8: The graph of S_σ for $\sigma(d) = \frac{1}{d^2 \ln^{1.5}(d+1)}$.

Computational experiments were made by Derive 4.0 and Maple (Scientific Workplace 3.0).

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References

Allasia,G. (1995). A Class of Interpolating Positive Operators: Theory & Computational Aspects, *Approx. Th.* **(1995)**; 1-36.

Bojanic,R., Della Vecchia,B. and **Mastroianni,G.** (1999). On the Approximation of Bounded Functions with Discontinuities of the First Kind by Generalized Shepard Operators, *Acta Math. Hung.* **85;** 29-57.

Della Vecchia,B., Mastroianni,G. and **Szabados,J.** (1996). Balázs-Shepard Operators on Infinite Intervals I., *Annales Univ. Sci. Budapest, Sect. Comp.* **16;** 93-102.

Della Vecchia,B., Mastroianni,G. and **Szabados,J.** (2004). Weighted Approximation of Functions with Endpoints or Inner Singularities by Iterated Exponential Weights. *Submitted for publication.*

Farwig,R. (1986). Rate of Convergence of Shepard's Global Interpolation Formula, *Math. Comput.* **(1986)**; 577-590.

Gál,S.G. and **Szabados,J.** (1999). On the Preservation of Global Smoothness by Some Interpolation Operators, *Studia Sci.M. Hung.* **35**; 397-414.

Gordon,W.J. and **Wixom,J.A.** (1978). Shepard's Method of "Metric Interpolation" to Bivariate and Multivariate Interpolation, *Math.of Computation*, **32**; 253-264.

Hoschek,J. and **Lasser D.** (1993). Scattered Data Interpolation und Approximation, *Computer Aided Geometric Design, Chapter 9,* A.K.Peters Ltd.

Jin,J. and **Zhang,S.** (1996). *Computation of Special Functions,* Wiley-Interscience, New York

Katona E. (2002). Constructing Digital Elevation Models (in Hungarian), *Polygon* (2002); 35-55.

Láng-Lázi,M.,Heszberger,J.,Molnárné-Jobbágy,M. and **Viczián,G.** (2006). Spline Functions in Chemistry: Approximation of Surfaces Over Triangle-Domain. *Submitted for publication.*

Mastroianni,G. and **Szabados,J.** (1997). Balázs-Shepard Operators on Infinite Intervals II., *J.Approx. Th.* **90**; 1-8.

Nagy, Á. (2010). A computer-help for demonstrating and investigating approximating methods (in Hungarian), *Dissertation, Pannon Univ. (2010)*

Szabados, J. (1991). Direct and Converse Approximation Theorems for the Shepard Operator, *J.Approx. Th. & Appl.* **7**; 63-76.

Szalkai,I. (1996). SALT3DIM.exe - A Program for Handling 4 Component Mixtures, *Preprint of Univ.of Veszprém,* No.047.

Szalkai,I. (1999a). Handling Multicomponent Systems in R^n, I. Theoretical Results, *J.of Math. Chemistry*, **25**; 31-46.

Szalkai,I. (1999b). Scattered Data Interpolation via Improved Shepard's Method, *Preprint of Univ. of Veszprém,* No.076.

Szalkai,I. (2000). Handling Multicomponent Systems in R^n II: Computational Results, *Submitted for publication.*

Zhou,X.L. (1998). The Saturation Class of Shepard Operators, *Acta Math. Hung.* **80**; 293-310.