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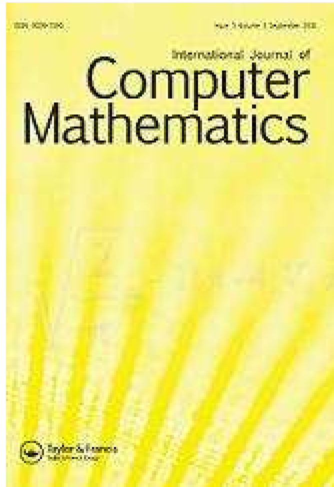

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RESEARCH ARTICLE

Artificial Satellites Preliminary Orbit Determination by modified high-order Gauss method

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In recent years, high-order methods have shown to be very useful in many practical applications, in which nonlinear systems arise. In this case, a classical method of positional astronomy have been modified in order to hold a nonlinear system in its establishments (that in the classical method is reduced to a single equation). At this point, high-order methods have been introduced in order to estimate the solutions of this system and, then, determine the orbit of the celestial body. We also have implemented a user friendly application, which will allow us to make a numerical and graphical comparison of the different methods with reference orbits, or user defined orbits.

Keywords: orbit determination, Gauss method, nonlinear systems, Newton's method, order of convergence, efficiency index

AMS Subject Classification: 65H10; 65L20

1. Introduction

Orbit determination is an old problem with new applications: at the early XIX century, Gauss designed a method to predict the future positions of asteroids, as Ceres, or other celestial bodies of our solar system with elliptical orbits. Nowadays, orbit determination methods are an essential tool in order to, by example, correct the position of artificial satellites in their orbits. The first step in kind of methods is to determine preliminary orbits, as the motion analyzed is under the premises of the two bodies problem, not taking into account any other force than mutual gravitational attraction between both bodies. Thereafter, perturbations and other variables must be taken into account in order to refine the preliminary orbit.

Recently, several authors have focused this problem from other points of view: in [3], Danchick improves the original work of Gauss introducing Newton-Raphson's method (instead of fixed-point method in the original scheme) and modifying the iterative process in order to widen the region of convergence. Moreover, Gronchi in [5] introduces a generalization of the geometric interpretation that provides Charlier's theory on Laplace's method of preliminary orbit determination from three observations. This generalization can to take into account topocentric observations and is useful to understand when there are multiple solutions and where they are

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located. This new theory works for both Laplace's and Gauss's methods. From another point of view, Gronchi et al. in [6] investigate a method to compute a finite set of preliminary orbits for solar system bodies using the first integral of the Kepler's problem, which is useful for the modern sets of astrometric observation.

The inertial system which the orbit is placed in is a geocentric system whose fundamental plane is the projection of the terrestrial equator to the celestial sphere, the perpendicular axis points to Celestial North and South Poles, and the fundamental direction point to the Vernal point γ, in Aries constellation.

If we focus on the orbital plane, it is possible to set a two-dimensional coordinate system, where the fundamental direction points to the perigee of the orbit, the closest point of the elliptical orbit to the focus and center of the system, the Earth. In order to place this orbit in the celestial sphere and determine completely the position of a body in the orbit, some elements (called orbital or keplerian elements) must be determined.

Then, the orbital elements are:

- Ω, (right ascension of the ascending node): defined as the equatorial angle between the Vernal point γ and the ascending node N; it orients the orbit in the equatorial plane.
- ω, (argument of the perigee): defined as the angle of the orbital plane, centered at the focus, between the ascending node N and the perigee of the orbit; it orients the orbit in its plane.
- i, (inclination): Dihedral angle between the equatorial and the orbital planes.
- a, (semi-major axis): Which sets the size of the orbit.
- e, (eccentricity): Which gives the shape of the ellipse.
- T_0, (perigee epoch): Time for the passing of the object over the perigee, to determine a reference origin in time. It can be denoted by a exact date, in Julian Days, or by the amount of time ago the object was over the perigee.

Different methods have been developed for this purpose (see [2, 4, 9]), constituting a fundamental element in navigation control, tracking and supervision of artificial satellites. By using these methods, from position and velocity coordinates for a given time, it is possible to determine those orbital elements for the preliminary orbit, which should be refined with later observations from ground stations, whose geographic coordinates are already known. In order to get this aim, some angles (or anomalies) must be determined on the planar orbit. Firstly, the object position in the ellipse can be determined by an angle, the true anomaly (ν), with center on the focus of the ellipse, which is the covered angle by a position vector, from its last perigee epoch ($\nu = 0$), to the observation instant. Another related angle with the previous one is the eccentric anomaly (E), whose center is on the center of the ellipse. This is the covered angle by a line from this center to the point where a circumference of radius a, the semi-major axis, is cut by a perpendicular line to X axis passing by the coordinates of the position vector, from its last Perigee epoch ($E = 0$) to the observation moment.

Using the Earth as the center of the coordinates system, it is useful to establish related units: the distance unit is the Earth radius (e.r.), approximately 6370 Km, and time unit is the minute, although some dates are described in Julian days (JD).

Some fundamental constants are the Earth gravitational constant, $k = 0.07436574 (e.r.)^{\frac{3}{2}}/min$ (see [4]), G, and the gravitational parameter $\mu = \frac{1}{m_{Earth}}(m_{Earth} + m_{Object}) \approx 1$. Then, modified time variable is introduced as $\tau = k(t_2 - t_1)$, where t_1 is an initial arbitrary time and t_2 is the observation time. So, τ is considered here as a measure of time difference, which will simplify calculations.

To estimate the velocity we can make use of the closed forms of the f and g series (see [4, 9]), $f = 1 - \frac{a}{|\vec{r_1}|}[1 - \cos(E_2 - E_1)]$ and $g = \tau - \frac{\sqrt{a^3}}{\mu}[(E_2 - E_1) - \sin(E_2 - E_1)]$, so we can express the rate respect two positions vectors and time as

$$\dot{\vec{r_1}} = \frac{\vec{r_2} - f \cdot \vec{r_1}}{g}. \tag{1}$$

So, it is clear that, known two position vectors and its corresponding observational instants, the main objective of the different methods that determine preliminary orbits is the calculation of the semi-major axis, a, and the eccentric anomalies difference, $E_2 - E_1$. When they have been calculated, it is possible to obtain by (1) the velocity vector corresponding to one of the known position vectors and, then, obtain the orbital elements.

Most of these methods have something in common: the need for finding the solution of a nonlinear equation or system, as in Gauss method. Usually, fixed point or secant methods are employed.

From the available input data, two position vectors and times for the observations, τ can be immediately deduced and other intermediate results as the difference in true anomalies, $(\nu_2 - \nu_1)$, deduced by $\cos(\nu_2 - \nu_1) = \frac{\vec{r_1} \cdot \vec{r_2}}{|\vec{r_1}| \cdot |\vec{r_2}|}$ and $\sin(\nu_2 - \nu_1) = \pm \frac{x_1 y_2 - x_2 y_1}{|x_1 y_2 - x_2 y_1|} \sqrt{1 - \cos^2(\nu_2 - \nu_1)}$, with positive sign for direct orbits, and negative for retrograde orbits.

Once the difference of true anomalies is obtained from the position vectors and times, the specific orbit determination method is used. In our particular case, we will introduce in the following section the classical Gauss method and, thereafter, we will modify it in order to estimate the value of the semi-major axis and eccentric anomalies by means of high-order iterative methods.

Let us also note that the inverse problem, it is the calculation of ephemeris (position an velocity in a given time) knowing the orbital elements, can be done easily, with direct operations that can be found in related bibliography (see [2, 4, 9]).

2. Gauss method of orbit determination

This method calculate a preliminary orbit of a celestial body by means of only two observations (position vectors). It is based on the relation between the areas

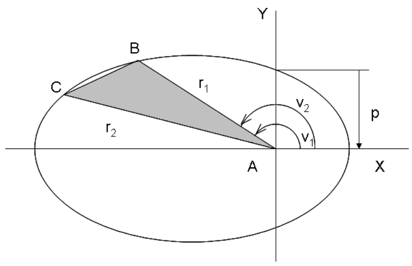

Figure 1. Ratio sector to triangle, in Gauss method.

of the sector ABC and the triangle ABC, as Figure 1 illustrates, delimited by both position vectors, $\vec{r_1}$ y $\vec{r_2}$. The ratio sector- triangle can be expressed as:

$$y = \frac{\sqrt{\mu p} \cdot \tau}{r_2 r_1 \sin(\nu_2 - \nu_1)} = \frac{\sqrt{\mu} \cdot \tau}{2\sqrt{a}\sqrt{r_2 r_1} \sin\left(\frac{E_2 - E_1}{2}\right) \cos\left(\frac{\nu_2 - \nu_1}{2}\right)}, \tag{2}$$

(with $(\nu_2 - \nu_1) \neq \pi$), and on the first

$$y^2 = \frac{m}{l+x} \tag{3}$$

and second

$$y^2(y-1) = mX. \tag{4}$$

Gauss equations, where the constants of the problem (based on the data and the previously made calculations are

$$l = \frac{r_2 + r_1}{4\sqrt{r_2 r_1}\cos\left(\frac{\nu_2-\nu_1}{2}\right)} - \frac{1}{2} \quad \text{and} \quad m = \frac{\mu\tau^2}{\left[2\sqrt{r_2 r_1}\cos\left(\frac{\nu_2-\nu_1}{2}\right)\right]^2}. \tag{5}$$

Moreover, also must be determined in the process the value of:

$$x = \frac{1}{2}\left[1 - \cos\left(\frac{E_2 - E_1}{2}\right)\right] = \sin^2\left(\frac{E_2 - E_1}{4}\right) \quad \text{and} \quad X = \frac{E_2 - E_1 - \sin(E_2 - E_1)}{\sin^3\left(\frac{E_2-E_1}{2}\right)}. \tag{6}$$

With these equations we present two different schemes to solve the problem: the classical method, which reduces first and second Gauss equations to a unique nonlinear equation, solved by fixed point method, and the modified Gauss scheme, which solve directly the nonlinear system formed by both Gauss equations.

2.1 *Classical Gauss method scheme*

In the classical method, an only nonlinear equation is obtained by, substituting second Gauss equation (3) into first equation (4):

$$y = 1 + X(l + x). \tag{7}$$

Then a fixed-point scheme is used to estimate the solution of (7), making a first estimation of the ratio , $y_0 = 1$, and by using the first Gauss equation to get $x_0 = \frac{m}{y_0^2} - l$.

From the definition of x in equation (6), it is possible to calculate cosine and sine of the half difference of eccentric anomalies, which is known to be between 0 an π radians, determining uniquely the difference of eccentric anomalies:

$$\cos\left(\frac{E_2 - E_1}{2}\right) = 1 - 2x_0 \quad \text{and} \quad \sin\left(\frac{E_2 - E_1}{2}\right) = +\sqrt{4x_0(1 - x_0)}. \tag{8}$$

Then, with equation (6), an estimation of X, X_0, can be calculated and used in the reduced nonlinear equation (7) in order to get a better estimation of the ratio $y_1 = 1 + X_0(l + x_0)$.

This iterative process gets new estimations of the ratio, until a given tolerance condition is satisfied. If there is convergence, the semi-major axis a, can be calculated by means of equation (2), from the last estimations of ratio and difference of eccentric anomalies, and the last phase of the process is then initiated, to determine velocity and orbital elements.

The Gauss method has some limitations, as the critical observation angles spread, in $\nu_2 - \nu_1 = \pi$, where the denominator of equation (2) vanish. Moreover, it is known that this method is only convergent to a coherent solution if the observation angles spread is less than 70^o, where this method has order of convergence 1. The ratio y grows with spread, leading to an invalid solution, if it converges. So this method is suitable for small spreads in observations, that is, observations which are close to each other.

2.2 *Modified Gauss schemes*

It is possible to make a different approach to the problem, solving the nonlinear system formed by both Gauss equations with different higher order iterative methods, instead of solving a unique nonlinear equation, which have the ratio y as unknown.

Firstly, it is necessary to establish the nonlinear system to be solved. In this case we can use the ratio $u = y$ and the difference of eccentric anomalies, $v = E_2 - E_1$, as our unknowns, and substitute (6) in first an second Gauss equations, (3) and (4), so the system $F(u, v) = 0$ becomes:

$$\begin{array}{r} u^2 + \frac{u^2}{2}\left(1 - \cos\frac{v}{2}\right) - m = 0, \\ u^3 + u^2 - m\frac{v - \sin v}{\sin^3\frac{v}{2}} = 0. \end{array} \quad (9)$$

Let us note that l and m are constants, with the input data, calculated with (5). Moreover, it is easy to check that the jacobian matrix $F'(u, v)$ associated to this system is ill-conditioned, so the iterative methods used to estimate its solutions must be robust enough. General information about iterative methods to solve nonlinear equations and systems can be found in [10].

Firstly, we will use Newton's method. Then, new estimations of the solution can be deduced with the following iterative scheme:

$$x^{(k+1)} = x^{(k)} - F'(x^{(k)})^{-1}F(x^{(k)}), \quad (10)$$

with convergence order up to 2. Also the well-known Jarrat's method (see [7]) will be employed, with forth order of convergence and iterative expression:

$$x^{(k+1)} = x^{(k)} - \frac{1}{2}(3F'(y^{(k)}) - F'(x^{(k)}))^{-1}(3F'(y^{(k)}) + F'(x_k))F'(x^{(k)}))^{-1}F(x^{(k)}) \quad (11)$$

where $y^{(k)} = x^{(k)} - \frac{2}{3}F'(x^{(k)})^{-1}F(x^{(k)})$.

Moreover, a new family of methods is introduced:

$$x^{(k+1)} = y^{(k)} - A^{-1}BF'(x^{(k)})^{-1}F(y^{(k)}), \quad (12)$$

where $y^{(k)} = x^{(k)} - cF'(x^{(k)})^{-1}F(x^{(k)})$, $A = a_1F'(x^{(k)}) + a_2F'(y^{(k)})$ and $B = b_1F'(x^{(k)}) + b_2F'(y^{(k)})$ which will be denoted by N_5 methods and whose convergence order will be proved to be five for some values of the parameters. In order to analyze the convergence of the new method we use Taylor expansions around the solution, whose notation was introduced in [1].

THEOREM 2.1 *Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be sufficiently differentiable at each point of an open neighborhood D of $\bar{x} \in \mathbb{R}^n$, that is a solution of the system $F(x) = 0$. Let us suppose that $F'(x)$ is continuous and nonsingular in $\bar{x}$. Then the sequence $\{x^{(k)}\}_{k\geq 0}$ obtained using the iterative expression (12) converges to $\bar{x}$ with order* 5 *if* $c=1$, $a_2 \neq 0$, $a_1 = -\frac{a_2}{5}$, $b_1 = \frac{3a_2}{5}$ *and* $b_2 = -a_1$ *and satisfies the error equation*

$$e_{k+1} = \left(-5C_2^4 - 4C_2^2C_3 + \frac{3}{2}C_2C_3C_2 - \frac{3}{2}C_3C_2 + 6C_2C_4\right) e_k^6 + O(e_k^7),$$

where $e_k = x(k) - \bar{x}$ *and* $C_k = (1/k!)[F(\bar{x})]^{-1}F(k)(\bar{x}), k = 2, 3, \ldots$.

Proof: Taylor's expansion around $x = \bar{x}$ gives

$$F(x^{(k)}) = F'(\bar{x})\left[e_k + C_2e_k^2 + C_3e_k^3 + C_4e_k^4 + C_5e_k^5 + C_6e_k^6\right] + O(e_k^7), \tag{13}$$

$$F'(x^{(k)}) = F'(\bar{x})\left[I + 2C_2e_k + 3C_3e_k^2 + 4C_4e_k^3 + 5C_5e_k^4 + 6C_6e_k^5\right] + O(e_k^6). \tag{14}$$

From (14), we have

$$\left[F'(x^{(k)})\right]^{-1} = \left[I + X_2e_k + X_3e_k^2 + X_4e_k^3 + X_5e_k^4\right][F'(\bar{x})]^{-1} + O(e_k^5),$$

with $X_2 = -2C_2$, $X_3 = 4C_2^2 - 3C_3$, $X_4 = 6C_3C_2 - 8C_2^3 + 6C_2C_3 - 4C_4$ and $X_5 = 16C_2^4 - 12C_3C_2^2 - 12C_2C_3c_2 + 8C_4C_2 + 9C_3^2 - 12C_2^2C_3 + 8C_2C_4 - 5C_5$. Then,

$$\begin{aligned} y^{(k)} - \bar{x} &= x^{(k)} - \bar{x} - c\left[F(x^{(k)})\right]^{-1}F(x^{(k)}) \\ &= (1-c)e_k + cC_2e_k^2 - cM_0e_k^3 - cM_1e_k^4 + O(e_k^5), \end{aligned} \tag{15}$$

where $M_0 = C_3 + X_2C_2 + X_3$ and $M_1 = C_4 + X_2C_3 + X_3C_2 + X_4$.

Again, by Taylor expansion:

$$F(y^{(k)}) = F'(\bar{x})\left[(1-c)e_k + Q_2e_k^2 + Q_3e_k^3 + Q_4e_k^4\right] + O(e_k^5), \tag{16}$$

where $Q_2 = (c + (1-c)^2)C_2$, $Q_3 = -cM_0 + 2c(1-c)C_2^2 + (1-c)^3C_3$ and $Q_4 = -cM_1 + c^2C_2^3 - 2c(1-c)C_2M_0 + 3c(1-c)^2C_3C_2 + (1-c)^4C_4$. Moreover,

$$F'(y^{(k)}) = F'(\bar{x})\left[I + 2(1-c)C_2e_k + T_2e_k^2 + T_3e_k^3 + T_4e_k^4\right] + O(e_k^5), \tag{17}$$

being $T_2 = 2cC_2^2 + 3(1-c)^2C_3$, $T_3 = -2cC_2M_0 + 6c(1-c)C_3C_2 + 4(1-c)^5C_4$ and $T_4 = -2cC_2M_1 + 3c^2C_3C_2^2 - 6c(1-c)C_3M_0 + 12c(1-c)^2C_4C_2 + 5(1-c)^4C_5$.

So, the expansion of matrix A is obtained combining (14) and (17):

$$\begin{aligned} A &= a_1F'(x^{(k)}) + a_2F'(y^{(k)}) \\ &= F'(\bar{x})\left[(a_1+a_2)I + L_1e_k + L_2e_k^2 + L_3e_k^3 + L_4e_k^4\right] + O(e_k^5), \end{aligned}$$

where $L_1 = 2(a_1 + a_2(1-c))C_2$, $L_2 = 3a_1C_3 + a_2T_2$, $L_3 = 4a_1C_4 + a_2T_3$ and $L_4 = 5a_1C_5 + a_2T_4$, and the inverse of A can be expressed as:

$$A^{-1} = \left[\frac{1}{a_1+a_2}I + Z_2e_k + Z_3e_k^2 + Z_4e_k^3 + Z_5e_k^4\right][F'(\bar{x})]^{-1} + O(e_k^5),$$

being $Z_2 = -\frac{L_1}{(a_1+a_2)^2}$, $Z_3 = -\frac{L_2}{(a_1+a_2)^2} - \frac{Z_2L_1}{a_1+a_2}$, $Z_4 = -\frac{L_3}{(a_1+a_2)^2} - \frac{Z_2L_2+Z_3L_1}{a_1+a_2}$ and $Z_5 = -\frac{L_4}{(a_1+a_2)^2} - \frac{Z_2L_3+Z_3L_2+Z_4L_1}{a_1+a_2}$.

Now, by (14) and (17), the Taylor expansion of matrix B is:

$$
\begin{aligned}
B &= b_1 F'(x^{(k)}) + b_2 F'(y^{(k)}) \\
&= F'(\bar{x}) \left[(b_1 + b_2) I + P_1 e_k + P_2 e_k^2 + P_3 e_k^3 + P_4 e_k^4 \right] + O(e_k^5),
\end{aligned}
$$

where $P_1 = 2(b_1 + b_2(1-c))C_2$, $P_2 = 3b_1 C_3 + b_2 T_2$, $P_3 = 4b_1 C_4 + b_2 T_3$ and $P_4 = 5b_1 C_5 + b_2 T_4$. Then,

$$A^{-1}B = HI + S_1 e_k + S_2 e_k^2 + S_3 e_k^3 + S_4 e_k^4 + O[e_k], \tag{18}$$

being $H = \frac{b_1+b_2}{a_1+a_2}$ and $S_1 = \frac{P_1}{a_1+a_2} + (b_1+b_2)Z_2$, $S_2 = \frac{P_2}{a_1+a_2} + Z_2 P_1 + (b_1+b_2)Z_3$, $S_3 = \frac{P_3}{a_1+a_2} + Z_2 P_2 + Z_3 P_1 + (b_1+b_2)Z_4$ and $S_4 = \frac{P_4}{a_1+a_2} + Z_2 P_3 + Z_3 P_2 + Z_4 P_1 + (b_1+b_2)Z_5$.

Now, the Taylor expansion of the remaining product of the iterative expression (12) is:

$$\left[F'(x^{(k)}) \right]^{-1} F(y^{(k)}) = (1-c)e_k + R_2 e_k^2 + R_3 e_k^3 + R_4 e_k^4 + O(e_k^5), \tag{19}$$

where $R_2 = Q_2 + (1-c)X_2$, $R_3 = Q_3 + X_2 Q_2 + (1-c)X_3$ and $R_4 = Q_4 + X_2 Q_3 + X_3 Q_2 + (1-c)X_4$.

And finally, combining (15), (18) and (19), we obtain:

$$
\begin{aligned}
e_{k+1} = {} & ((1-c) - H(1-c))e_k + (cC_2 - HR_2 - (1-c)S_1)e_k^2 \\
& + (-cM_0 - HR_3 - S_1 R_2 - (1-c)S_2)e_k^3 \\
& + (-cM_1 - HR_4 - S_1 R_3 - S_2 R_2 - (1-c)S_3)e_k^4 + O(e_k^5).
\end{aligned}
$$

So, the solution of the following linear system will provide us the conditions of the parameters to have convergence order five.

$$
\left.
\begin{aligned}
(1-c) - H(1-c) &= 0 \\
cC_2 - HR_2 - S_1(1-c) &= 0 \\
cM_0 + HR_3 + S_1 R_2 + S_2(1-c) &= 0 \\
cM_1 + HR_4 + S_1 R_3 + S_2 R_2 + S_3(1-c) &= 0
\end{aligned}
\right\}
$$

From the two first equations is directly obtained that $c = 1$ and $H = 1$. Then, from the third equation, the following condition must be satisfied in order to obtain convergence order four: $\frac{b_1}{a_1+a_2} - \frac{a_1(b_1+b_2)}{(a_1+a_2)^2} = 1$. Finally, the last equation gives us the condition to be satisfied for order five: $5a_1 + a_2 = 0$. So, the values of the parameters that ensure fifth-order convergence are: $a_1 = -\frac{a_2}{5}$, $b_1 = \frac{3a_2}{5}$ and $b_2 = -a_1$ and the resulting error equation is:

$$e_{k+1} = \left(-5C_2^4 - 4C_2^2 C_3 + \frac{3}{2} C_2 C_3 C_2 - \frac{3}{2} C_3 C_2 + 6C_2 C_4 \right) e_k^6 + O(e_k^7).$$

$\square$

In the last section, we will use a member of this family in order to compare the precision of the calculated orbit with the other methods. In particular, we will take

$a_2 = 5$ and its iterative expression is:

$$y^{(k)} = x^{(k)} - F'(x^{(k)})^{-1}F(x^{(k)}) \tag{20}$$
$$x^{(k+1)} = y^{(k)} - \left(-F'(x^{(k)}) + 5F'(y^{(k)})\right)^{-1}\left(3F'(x^{(k)}) + F'(y^{(k)})\right)F'(x^{(k)})^{-1}F(y^{(k)}).$$

Let us remark that this new uniparametric family of methods has better efficiency index than the well-known Jarrat's method and it is more efficient from the computational point of view, as it gets higher order of convergence with the same number of operations and only one more functional evaluation. In order to measure the efficiency of an iterative method, the efficiency index is defined as $I = p^{1/d}$ (see [8]), where p is the order of convergence and d is the total number of new functional evaluations (per iteration) required by the method. In the particular case of the modified Gauss method, the size of the nonlinear system involved is two; then, the respective efficiency indices are $I_{Newton} = 1.1225$, $I_{Jarrat} = 1.1487$ and $I_{N_5} = 1.1610$. In spite of this, we will see in the next section that the behavior of the new method N_5 is better, because of the ill-conditioned system to be solved.

All this Newton's variants applied to the nonlinear system appearing in Gauss method, (9), are expected to be at least so accurate as the classical scheme, but to drastically reduce the number of iterations needed to find a solution to the problem, as it will be seen later.

3. Comparing Gauss method schemes

Now, tests are needed to analyze the previously described schemes. For that purpose a graphical application has been developed with Matlab GUIDE (Graphical User Interface Development Environment) to make graphical and numerical comparison. The schemes presented will work with 200 digits of mantissa as they use variable precision arithmetics, so we can set more restrictive tolerances.

The reference or test orbits we will use, found on [4], are:

- Test Orbit I:

$$\vec{r_1} = [2.46080928705339, 2.04052290636432, 0.14381905768815] \text{ e.r.}$$
$$\vec{r_1} = [1.98804155574820, 2.50333354505224, 0.31455350605251] \text{ e.r.}$$
$$t_1 = 0\text{JD} \qquad t_2 = 0.01044412000000 \text{ JD}$$
$$\Omega = 30^o \quad \omega = 10^o \quad i = 15^o \quad a = 4\text{e.r.} \quad e = 0.2 \quad T_0 = 0m$$

- Test Orbit II:

$$\vec{r_1} = [-1.75981065999937, 1.68112802634201, 1.16913429510899] \text{ e.r.}$$
$$\vec{r_2} = [-2.23077219993536, 0.77453561301361, 1.34602197883025] \text{ e.r.}$$
$$t_1 = 0 \text{ JD} \qquad t_2 = 0.01527809000000 \text{ JD}$$
$$\Omega = 80^o \quad \omega = 60^o \quad i = 30^o \quad a = 3 \text{ e.r.} \quad e = 0.1 \quad T_0 = 0m$$

- Test Orbit III:

$$\vec{r_1} = [0.41136206679761, -1.66250000000000, 0.82272413359522] \text{ e.r.}$$
$$\vec{r_2} = [0.97756752977209, -1.64428006097667, -0.04236299091612] \text{ e.r.}$$
$$t_1 = 0 \text{ JD} \qquad t_2 = 0.01316924000000 \text{ JD}$$
$$\Omega = 120^o \quad \omega = 150^o \quad i = 60^o \quad a = 2 \text{ e.r.} \quad e = 0.05 \quad T_0 = 0m,$$

- Test Orbit IV:

$\vec{r_1} = [0.65241964490697, 3.80258035509303, 2.22750000000000]$ e.r.
$\vec{r_2} = [-1.35626966531604, 2.95849708305651, 3.05100082701246]$ e.r.
$t_1 = 0$ JD $\quad t_2 = 0.04622903000563$ JD
$\Omega = 45^o \quad \omega = 45^o \quad i = 45^o \quad a = 4.5$ e.r. $\quad e = 0.01 \quad T_0 = 0m$

By using the first test positions vectors and times, we can first compare the number of iterations and estimated accuracy of classical (C), Newton (N), Jarrat (J) and new fifth-order (N_5) schemes described in this paper with $a_2 = 5$, described in (20). As we can see in Table 1, with tolerance $= 10^{-100}$, higher order methods reduce significantly the number of iterations, getting even more accuracy than the classical scheme.

Scheme	**Iterations**	$\|\|\mathbf{x^{(k+1)}} - \mathbf{x^{(k)}}\|\|$
C	54	1.8364e-101
N	8	7.3258e-133
J	5	7.0e-200
N_5	4	1.0658e-108

Table 1. Comparison of different Gauss method schemes for reference orbit I

Due to limitations in number of digits and format in observations data, and to the last phase of calculations, some accuracy is lost, but it is hard to determine differences in errors in the presented schemes. As far as results can be represented, errors in the final results of the orbital elements, for classical Gauss method, are shown in Table 2, where the exact orbit elements are compared with the calculated ones by means of the classical method.

Errors	C	$N \ J \ N_5$
$\|\mathbf{a'} - \mathbf{a}\|$	3.3032e-070 e.r.	3.3032e-070 e.r.
$\|\mathbf{e'} - \mathbf{e}\|$	6.6064e-071	6.6064e-071
$\|\mathbf{T_0'} - \mathbf{T_0}\|$	2.3162e-048 min	2.0726e-069 min
$\|\mathbf{i'} - \mathbf{i}\|$	3.0000e-198^o	4.0000e-198^o
$\|\omega' - \omega\|$	1.6900e-069^o	1.6900e-069^o
$\|\mathbf{\Omega'} - \mathbf{\Omega}\|$	2.0000e-198^o	1.0000e-198^o

Table 2. Error in classical Gauss method for reference orbit I

Now we can compare the new schemes with the classical, seeing in Table 2 the differences between the calculated orbital elements by the classical method and each one of the modified methods. It can be observed that Jarrat's and new fifth-order methods obtain almost the same estimation of the solution than Newton's method. Nevertheless, the reduction in the number of iterations needed justifies the use of high-order methods.

If we vary tolerance from 10^{-100} up to 10^{-498}, we can compare in Table 3 how number of iterations grows, making it clear that solving the nonlinear system, instead of reducing it to a nonlinear equation, does not increase number of iterations so fast as the classical scheme.

Finally, in Table 4, we can compare the number of iterations needed for different test orbits with different spreads in observations $SP = \nu_2 - \nu_1$, to realize that the limitation of spread is still present, but overall process is made faster, not increasing iterations to find a solution in worse cases, that is, with bigger difference of true anomalies in observation.

Scheme	tol $= 10^{-100}$	tol $= 10^{-198}$	tol $= 10^{-498}$
C	54	106	172
N	8	9	10
J	5	5	6
N_5	4	5	5

Table 3. Iterations if varying tolerances, for reference orbit I

Scheme	Test Orbit I $SP = 12.23^o$	Test Orbit II $SP = 22.06^o$	Test Orbit III $SP = 31.46^o$	Test Orbit IV $SP = 30.29^o$
C	54	76	101	99
N	8	8	8	8
J	5	5	5	5
N_5	4	4	5	5

Table 4. Iterations needed for different spreads

4. Conclusion

A new approach to the problem of orbit determination is proposed, consisting in solving directly a nonlinear system formed by both Gauss equations, by means of well known iterative functions as Newton's and Jarrat's and a new method which have higher convergence order.

In the test of these variants of the Gauss methods, it is seen that they can reduce significantly the number of iterations, making the process faster, so it is possible to use more limiting tolerances to improve accuracy, without increasing much more the number of iterations. Some limitations of the classical scheme are still present in the alternatives introduced in this paper, such as spread limitation in observations, that is, the difference of true anomalies of observations. As the ratio y grows with spread, bigger spreads mean more iterations to find a solution, but in the proposed modified schemes this increment is very limited. If the difference is greater than 70^o, the process will probably lead to invalid solutions, which makes Gauss method suitable only for observations that are close enough.

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