A new intrinsic numerical method for PDE on surfaces [1]

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Abstract

In this note we shall introduce a simple, effective numerical method for solving partial differential equations for scalar and vector-valued data defined on surfaces. Even though we shall follow the traditional way to approximate the regular surfaces under consideration by triangular meshes, the key idea of our algorithm is to develop an intrinsic and unified way to compute directly the partial derivatives of functions defined on triangular meshes. We shall present examples in computer graphics and image processing applications.

Key words: gradient, laplacian, tangential lifting method, diffusion equations

1 Introduction

Numerical approaches to solve partial differential equations (PDE's) on surfaces have received growing interest over last decade. However, they are still not well-understood. Partial differential equations need to be solved intrinsically and numerically for data defined on 3D surfaces in many applications. For instance, such examples exist in texture synthesis (Turk[12], Witkin and Kass[13]), vector field visualization (Diewald, Preufer and Rumpf[**?**]), weathering (Dorsey and Hanrahan[6]) and cell-biology (Ayton, McWhirter, McMurty

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and Voth 2005). Usually, surfaces are presented by triangular or polygonal forms. Partial differential equations are then solved on these triangular or polygonal meshes with data defined on them. The use of triangular or polygonal meshes is very popular in all areas dealing with 3D models. However, it has not yet been a widely accepted method to compute differential characteristics such as principal directions, curvatures and Laplacians (Chen and Wu[2], Wu, Chen and Chi [3,4], Taubin[9,10]). This is because that there is no unified, simple and effective method to compute these first and second order differential characteristics of the triangular or polygonal surface and to solve PDE's for data defined on triangular or polygonal meshes. In Chen, Chi and Wu[5], the authors proposed a new intrinsic simple algorithm to handle this difficulty. In this note, we shall use this new technique to solve PDE's on surfaces.

In Bertalmio, Cheng, Osher and Sapiro[1] proposed a framework, the implicit surface algorithm, to solve variational problems and PDE's for scalar and vector-valued data defined on surfaces.Their key idea is to use, instead of a triangular or polygonal representation, an implicit representation. The surface under consideration is the zero-level set of a higher dimensional embedding function. Then they smoothly extend the original data on the surface to the 3D domain, adapt the PDE's accordingly, and implement all the numerical computations on the fixed Cartesian grid corresponding to the embedding function. The advantage of their method is the use of the Cartesian grid instead of a triangular mesh for the numerical implementation.

2 Our intrinsic algorithm for solving PDE's on surfaces

In ths section we first propose our discrete intrinsic algorithm for solving PDE's on regular surfaces. We divide our algorithm into two main steps: First. we approximate the given surface by a suitable triangular mesh according to the accuracy demand. Second, we use our new intrinsic differential method developed in Chen, Chi and Wu[5] to compute the numerical PDE on the fixed triangular mesh. The first step is now easy to implement since one can find some good and efficient algorithms in the public domain. The difficult part lies in the second step. Namely, how can one effectively compute differential quantities on functions on a triangular mesh?

Next, we shall compare our algorithm with the implicit algorithm proposed by Bertalmio, Cheng, Osher and Sapiro[1]. We list the key steps of these two algorithms about solving PDE's on surfaces as follows.

One can tell from this comparison that our method is much simpler and more intrinsic. In many applications, one usually starts with triangular meshes instead of regular surfaces. In this case, we do not need Step 1 in our intrinsic

Our intrinsic algorithm	The implicit surface algorithm
1. Obtain a triangular mesh approximated to the given surface.	1. Obtain an implicit representation of the given fixed surface.
	2. Extend smoothly the data on the surface to the 3D volume
	3. Adapt PDE's accordingly
2. Use our new intrinsic differential method to compute the numerical PDE's on the fixed triangular mesh.	4. Perform all the computations on the fixed Cartesian grid corresponding to the embedding function.

algorithm. However, in the implicit surface algorithm, one will need one extra processing step: Construct an accurate implicit surface from a triangular mesh. Note that the triangular mesh may compose of a lot of triangles. This will cost large computations to obtain the accurate implicit surface.

Next, we shall describe a new, simple and effective method to define the discrete gradient and the discrete LB operator on functions on a triangular mesh. In order to do so, we first recall the gradient and the LB operator on functions in a regular surface Σ in the 3D Euclidean space $\mathbb{R}^3$.

2.1 Gradient and LB operators on regular surfaces

We consider a parameterization $x : U \to \Sigma$ at a point p, where U is an open subset of the 2D Euclidean space $\mathbb{R}^2$. We can choose, at each point q of $x(U)$, a unit normal vector $N(q)$. The map $N : x(U) \to S^2$ is the local Gauss map from an open subset of the regular surface Σ to the unit sphere in the 3D Euclidean space $\mathbb{R}^3$. The Gauss map N is differentiable. Denote the tangent space of Σ at the point p by $T\Sigma_p = \{v \in \mathbb{R}^3 : v \perp N(p)\}$. The tangent space $T\Sigma_p$ is a linear space spanned by $\{x_u, x_v\}$ where u, v are coordinates for U.

The gradient ∇g of a smooth function g on Σ can be computed from

$$\nabla g = \frac{g_u G - g_v F}{EG - F^2} x_u + \frac{g_v E - g_u F}{EG - F^2} x_v \tag{1}$$

where E, F, and G are the coefficients of the first fundamental form and

$$\begin{cases} g_u = \frac{\partial g(x(u,v))}{\partial u} \\ g_v = \frac{\partial g(x(u,v))}{\partial v} \end{cases} \tag{2}$$

See do Carmo[7] for the details. Note that the gradient ∇g assigns to each point q in Σ a tangent vector $\nabla g(q)$ such that we have for all $v \in T\Sigma_q$,

$$\langle \nabla g(q), v \rangle_q = \frac{dg(\gamma(t))}{dt}\|_{t=0} \tag{3}$$

where the smooth curve $\gamma(t)$ is in Σ with $\gamma(0)$ and $\gamma'(0) = v$.

The LB operator $\triangle$ acting on the function g is defined by the integral duality

$$(\triangle g, \phi) = -(\nabla g, \nabla \phi) \tag{4}$$

for all smooth function ϕ on Σ. A direct computation yields the following local representation for the LB operator on a smooth function g:

$$\begin{array}{r} \nabla g = \frac{1}{\sqrt{EG-F^2}} \left[\frac{\partial}{\partial u}\left(\frac{G}{\sqrt{EG-F^2}}\frac{\partial g}{\partial u}\right) - \frac{\partial}{\partial u}\left(\frac{F}{\sqrt{EG-F^2}}\frac{\partial g}{\partial v}\right)\right] \\ + \frac{1}{\sqrt{EG-F^2}} \left[\frac{\partial}{\partial v}\left(\frac{E}{\sqrt{EG-F^2}}\frac{\partial g}{\partial v}\right) - \frac{\partial}{\partial v}\left(\frac{F}{\sqrt{EG-F^2}}\frac{\partial g}{\partial u}\right)\right] \end{array} \tag{5}$$

To move from regular surfaces to triangular meshes, one need to avoid the problem of local parametrization x around a vertex p. In other word, one does not have the fist fundamental form E, F, G and their derivatives for the computation of the gradient and the Laplacian operator of a function on a triangular mesh. To handle this problem, we give a novel method in Chen, Chi and Wu[5] to compute these differential quantities. The primary ideas were developed in Chen, Chi and Wu[4] where we try to estimate the discrete partial derivatives for 2D scattered data points.

2.2 A new discrete algorithm: local tangential lifting(LTL) method

In this section we shall describe a unified, simple and effective method to define the discrete gradient and the discrete Laplacian operator on functions on a triangular mesh. The primary ideas were developed in Chen, Chi and Wu[3,4] where we try to estimate the discrete partial derivatives of functions on 2D scattered data points. Indeed, the method that we shall use to develop our algorithm is divided into two main steps: first we lift the 1-neighborhood points to the tangent space and obtain a local tangential polygon. Second, we use some geometric idea to lift functions and vectors to the tangent space and then we can compute their derivatives in the 2D tangent space. This means that the lifting process allows us to reduce the 2D curved surface problem to the 2D Euclidean problem and hence the methods in [4] and[8] can be applied.

Consider a triangular mesh $S = (V, F)$, where $V = \{v_i | 1 \leq i \leq n_V\}$ is the list

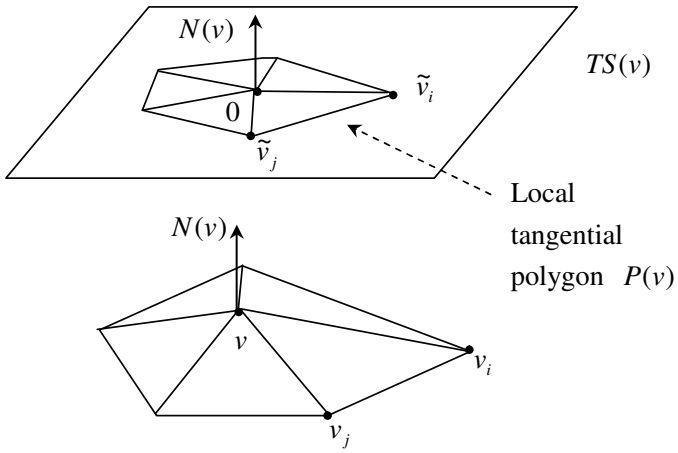

Fig. 1. The tangential polygon $P(v)$.

vertices and $F = \{f_k | 1 \leq k \leq n_F\}$ is the list of triangles. Next, we introduce the notion of the local tangential polygon $P(v)$ at the vertex v of S as follows:

(1) The normal vector $N(v)$ at the vertex v in S is given by

$$N(v) = \frac{\sum_{f \in T(v)} \omega_f N_f}{\left\| \sum_{f \in T(v)} \omega_f N_f \right\|} \tag{6}$$

where N_f is the unit normal to a triangle face f and the centroid weight is given in [2] by

$$\omega_f = \frac{\frac{1}{\|G_f - v\|^2}}{\sum_{\tilde{f} \in T(v)} \frac{1}{\|G_{\tilde{f}} - v\|^2}}. \tag{7}$$

Here, G_f is the centroid of the triangle face f determined by

$$G_f = \frac{v_i + v_j + v}{3}. \tag{8}$$

(2) The tangent plane $TS(v)$ of S at v is now determined by

$$TS(v) = \{w \in \mathbb{R}^3 | w \perp N(v)\}. \tag{9}$$

(3) The local tangential polygon $P(v)$ of v in $TS(v)$ is formed by the vertices $\tilde{v}_i$ which is the lifting vertex of v_i adjacent to v in S.

$$\tilde{v}_i = (v_i - v) - < v_i - v, N(v) > N(v). \tag{10}$$

as in figure 1.

Let h be a function on V. We will lift locally the function h to a function, denoted by $\tilde{h}_v$, on the vertices $\tilde{v}_i$ in $P(v)$ by simply setting

$$\tilde{h}_v(\tilde{v}_i) = h(v_i). \tag{11}$$

And $\tilde{h}_v(\vec{0}) = h(v)$ where $\vec{0}$ is the origin of $TS(v)$. One can then extend the function $\tilde{h}_v$ to a piecewise linear function, still denoted by $\tilde{h}_v$, on $P(v)$ as follows.

Consider a face f with vertices v, v_i and v_j in F. We obtain a lifting face $\tilde{f}$ with vertices $\vec{0}$, $\tilde{v}_i$ and $\tilde{v}_j$ in $P(v)$. Every point p in $\tilde{f}$ can be written as a linear combination of $\tilde{v}_i$ and $\tilde{v}_j$. That is, $p = a\tilde{v}_i + b\tilde{v}_j$ where $a, b \geq 0$ and $a + b \leq 1$. Then we define

$$\tilde{h}_v(p) = a\tilde{h}_v(\tilde{v}_i) + b\tilde{h}_v(\tilde{v}_j) + (1 - a - b)\tilde{h}_v(\vec{0}). \tag{12}$$

Hence, the extended function $\tilde{h}_v$ is affine on each triangle $\tilde{f}$ of $P(v)$ and is differentiable on $\tilde{f}$. The gradient $\nabla(\tilde{h}_v)_{\tilde{f}}$ of $\tilde{h}_v$ at the origin $\vec{0}$ can be obtained by

$$\nabla(\tilde{h}_v)_{\tilde{f}}(\vec{0}) = \alpha\tilde{v}_i + \beta\tilde{v}_j \tag{13}$$

where the coefficients α and β satisfy the relations:

$$\begin{cases} \tilde{h}(\tilde{v}_i) - \tilde{h}_v(\vec{0}) =< (\nabla\tilde{h}_v)_{\tilde{f}}(\vec{0}), v_i > \\ \tilde{h}(\tilde{v}_j) - \tilde{h}_v(\vec{0}) =< (\nabla\tilde{h}_v)_{\tilde{f}}(\vec{0}), v_j > \end{cases} \tag{14}$$

As easy computation gives

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} < \tilde{v}_i, \tilde{v}_i > & < \tilde{v}_i, \tilde{v}_j > \\ < \tilde{v}_i, \tilde{v}_j > & < \tilde{v}_j, \tilde{v}_j > \end{pmatrix}^{-1} \begin{pmatrix} \tilde{h}_v(\tilde{v}_i) - \tilde{h}_v(\vec{0}) \\ \tilde{h}_v(\tilde{v}_j) - \tilde{h}_v(\vec{0}). \end{pmatrix} \tag{15}$$

To obtain the gradient $\nabla h(v)$ of h on S at the vertex v, we use again the weighted combination method. Namely, we set

$$\nabla h(v) = (\nabla\tilde{h}_v)(\vec{0}) = \sum_{\tilde{f} \in P(v)} \omega_{\tilde{f}} (\nabla\tilde{h}_v)_{\tilde{f}}(\vec{0}) \tag{16}$$

with

$$\omega_{\tilde{f}} = \frac{\frac{1}{\|G_{\tilde{f}}\|^2}}{\sum_{f \in P(v)} \frac{1}{\|G_f\|^2}} \tag{17}$$

where $G_{\tilde{f}}$ is the centroid of the lifting triangle face $\tilde{f}$ and is determined by

$$G_{\tilde{f}} = \frac{\tilde{v}_i + \tilde{v}_j}{3}. \tag{18}$$

Next we explain how to obtain a good discrete Laplacian $\triangle h(v)$ of a function h on the triangular mesh S. From the discussions above, we obtain the gradient $\nabla h(v)$ of h on S at each vertex v. We can use the method of parallel transport

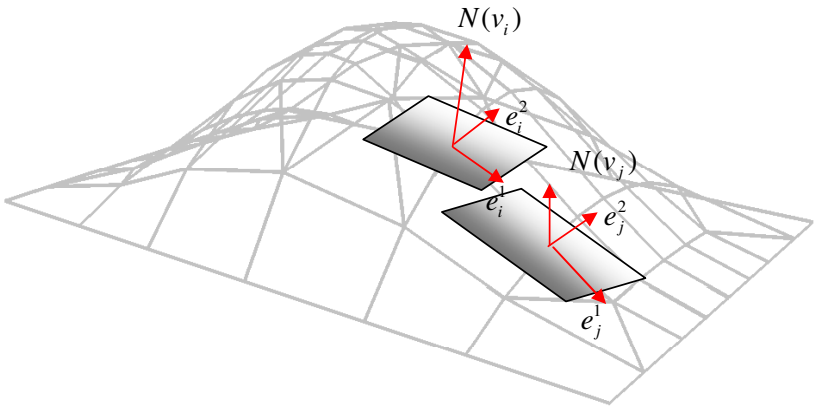

Fig. 2. parallel transport

to lift the vector $\nabla h(v_i)$ at v_i to a vector $\nabla\tilde{h}(\tilde{v}_i)$ in the tangential space $TS(v)$. The idea is to define a orthonormal linear map from $TS(v_i)$ to $TS(v)$. To do so, we choose an orthonormal basis for $TS(v)$ by

$$\begin{cases} N(v) \\ e_1 = \frac{(v_i - v) - <v_i - v, N(v)>N(v)}{\|(v_i - v) - <v_i - v, N(v)>N(v)\|} \\ e_2 = N(v) \times e_1 \end{cases} . \tag{19}$$

The corresponding orthonormal basis for $TS(v_i)$ is then given by

$$\begin{cases} N(v_i) \\ \tilde{e}_1 = \frac{(v_i - v) - <v_i - v, N(v_i)>N(v_i)}{\|(v_i - v) - <v_i - v, N(v_i)>N(v_i)\|} \\ \tilde{e}_2 = N(v_i) \times \tilde{e}_1 \end{cases} . \tag{20}$$

Then, the linear map L of the parallel transport is given by

$$L(w) = ae_1 + be_2 \in TS(v) \tag{21}$$

for $w = a\tilde{e}_1 + b\tilde{e}_2$ in $TS(v_i)$. See figure 2.

In this way, we can set the tangential gradient $\nabla\tilde{h}$ at $\tilde{v}_i$ by

$$\nabla\tilde{h}(\tilde{v}_i) = L(\nabla h(v_i)). \tag{22}$$

Hence we obtain a tangential gradient $\nabla\tilde{h}$ of h at each vertex $\tilde{v}_i$ in the tangential polygon $P(v)$ and we also set

$$\nabla\tilde{h}(\vec{0}) = \nabla h(v). \tag{23}$$

See figure 3.

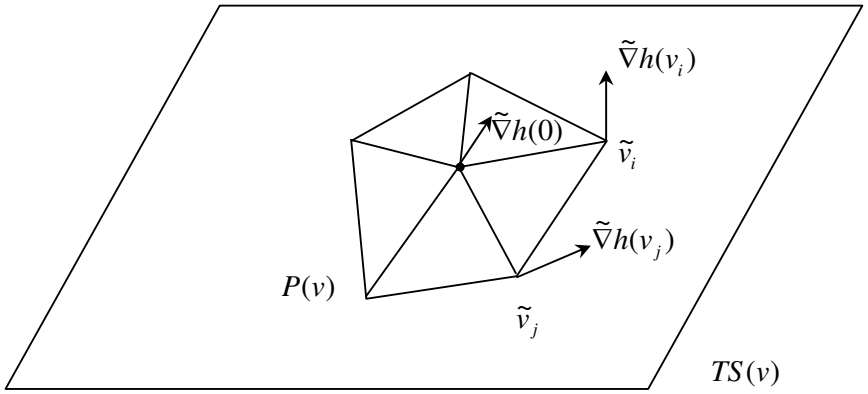

Fig. 3. $\nabla\tilde{h}$

Fix an orthonormal basis $\{e_1, e_2\}$ for $TS(v)$. The tangential gradient $\nabla\tilde{h}$ can be written as

$$\nabla\tilde{h}(\tilde{v}_i) = a(\tilde{v}_i)e_1 + b(\tilde{v}_i)e_2. \tag{24}$$

The coefficients $a(\tilde{v}_i)$ and $b(\tilde{v}_i)$ can now be viewed as functions on the vertices $\tilde{v}_i$ of the tangential polygon $P(v)$. As before, we can then obtain their gradients $(\nabla a)(\vec{0})$ and $(\nabla b)(\vec{0})$ at origin. Namely,

$$\begin{aligned} (\nabla a)(\vec{0}) &= \textstyle\sum_{\tilde{f}\in P(v)} \omega_{\tilde{f}} (\nabla a)_{\tilde{f}}(\vec{0}) \\ (\nabla b)(\vec{0}) &= \textstyle\sum_{\tilde{f}\in P(v)} \omega_{\tilde{f}} (\nabla b)_{\tilde{f}}(\vec{0}) \end{aligned} \tag{25}$$

with the centroid weights $\omega_{\tilde{f}}$ as in (17).

Put them in the matrix form to give

$$\begin{aligned} (\nabla a)(\vec{0}) &= a_{11}e_1 + a_{21}e_2 \\ (\nabla b)(\vec{0}) &= a_{12}e_1 + a_{22}e_2. \end{aligned} \tag{26}$$

Therefore we have the Laplacian $\triangle h(v)$ of h at the vertex v of S :

$$\triangle h(v) = a_{11} + a_{22}. \tag{27}$$

Theoretically the definition of the Laplacian $\triangle h(v)$ is independent of the choice of the orthonormal basis $\{e_1, e_2\}$.

3 Linear diffusion

As a simple example to illustrate our new algorithm, let us consider a linear diffusion equation on a regular surface Σ:

$$u_t - \triangle u = g \quad \text{on} \quad \Sigma \times I \tag{28}$$

for $u : \Sigma \times I \to \mathbb{R}$, $I \subset \mathbb{R}$, where $\triangle$ is the surface Laplacian on Σ and $g : \Sigma \times I \to \mathbb{R}$ is a smooth function on Σ. For the numerical implementation of our intrinsic algorithm, we take the regular surface Σ to be (1) the unit sphere S^2 or (2) a torus T^2. In the case of the sphere S^2, we consider the function

$$g(x) = x_1 \quad \text{for} \quad x = (x_1, x_2, x_3) \in S^2. \tag{29}$$

Figure 4 and figure 5 give the solution of (28) and (29) with initial functions $u(x, 0) = 0$. Different time steps are shown until the stationary solution is reached.

Consider the torus $T^2_{(a,r)} = ((a+r\cos x)\cos y, (a+r\cos x)\sin y, \sin x)$ for $x, y \in [0, 2\pi]$ with $a > r > 0$. we take $a = 2$, $r = 1$ and choose the function

$$g(x, y) = x \quad \text{for} \quad x, y \in [0, 2\pi] \tag{30}$$

Figure 6 and figure 7 give the solution of (28) and (30) with initial functions $u(x, 0) = 0$. As above, different timesteps are depicted until the stationary solution is reached.

4 Reaction-diffusion textures

The original idea about how reaction-diffusion equations can be used to create patterns was first introduced in (Turing[11]). The basic idea is to have a number of chemicals that diffuse at different rates and that react with each others. After the works of (Turk[12], Witkin and Kass[13]), the use of reaction-diffusion equations for texture synthesis attracted a lot of attentions in computer graphics. Turk , Witkin and Kass used these equations for planar textures and textures on surfaces. Then the patterns are analyzed by assigning a brightness value to the concentration of one of the "chemicals".

Consider two chemicals u_1 and u_2 on a surface Σ. In a simple isotropic model, we have

$$\begin{cases} \frac{\partial u_1}{\partial t} = f(u_1, u_2) + \alpha \triangle u_1 \\ \frac{\partial u_2}{\partial t} = g(u_1, u_2) + \beta \triangle u_2 \end{cases} \tag{31}$$

where α and β are two constants representing the diffusion rates and f and g are functions that describe the reaction. For simple isotropic patterns, Turing chose the functions f and g to be

$$\begin{cases} f(u_1, u_2) = s(16 - u_1 u_2) \\ g(u_1, u_2) = s(u_1 y_2 - y_2 - \gamma) \end{cases} \tag{32}$$

where s is a constant and γ is a random function giving the irregularities in the chemical concentration.

By using our intrinsic method described in the previous section, we can easily generate textures on surfaces without the elaborated schemes employed in (Turk 1991, Witkin and Kass 1991).

In the case of the sphere S^2, figure 8 and figure 9 give the solution of (31) and (32) with initial functions and Different timesteps are shown until the stationary solution is reached.

On the torus $T^2_{(a,r)} = ((a+r\cos x)\cos y, (a+r\cos x)\sin y, \sin x)$ for $x, y \in [0, 2\pi]$ figure 10 and figure 11 give the solution of (31) and (32) with initial functions $u_1(x, 0) = u_2(x, 0) = 1$ and $\alpha = 1$, $\beta = s = 2$, $\gamma = 0$. Different timesteps are shown until the stationary solution is reached.

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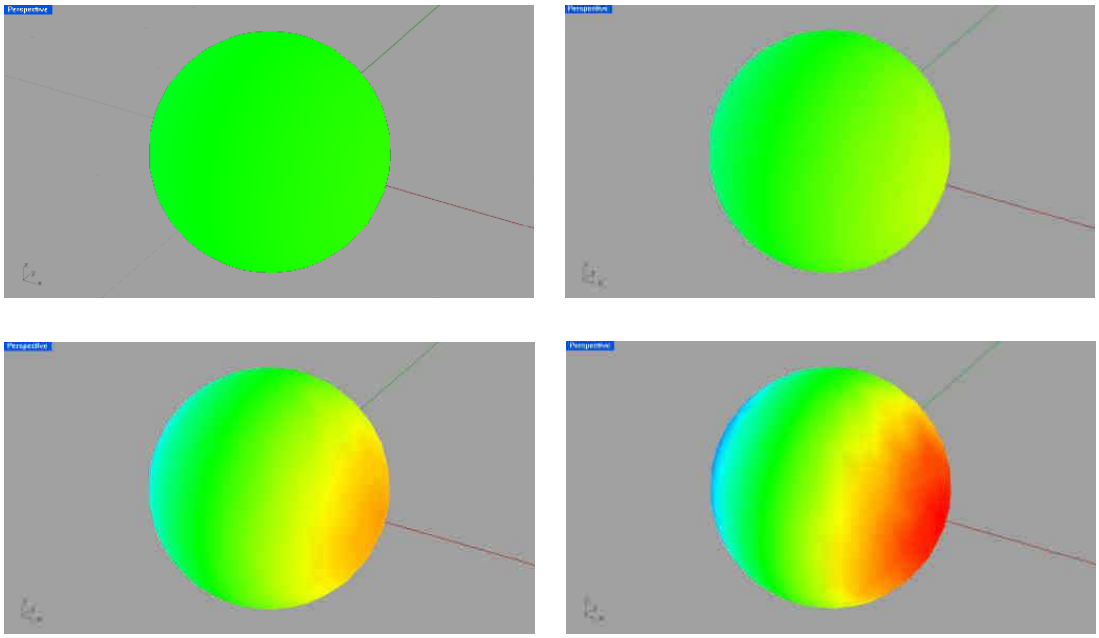

Fig. 4. sphere

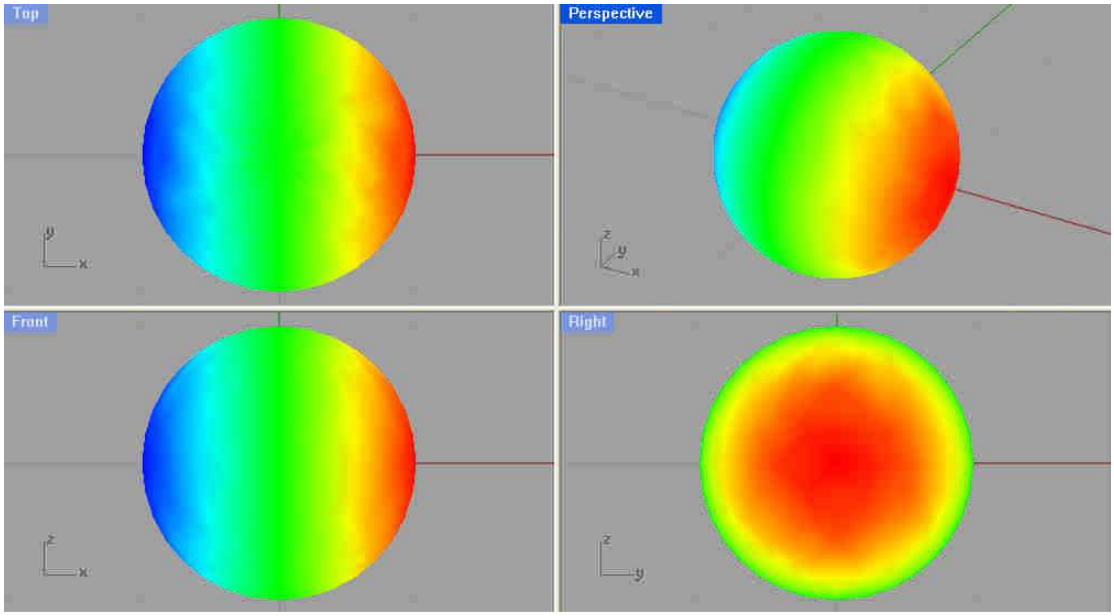

Fig. 5. sphere (stationary solution)

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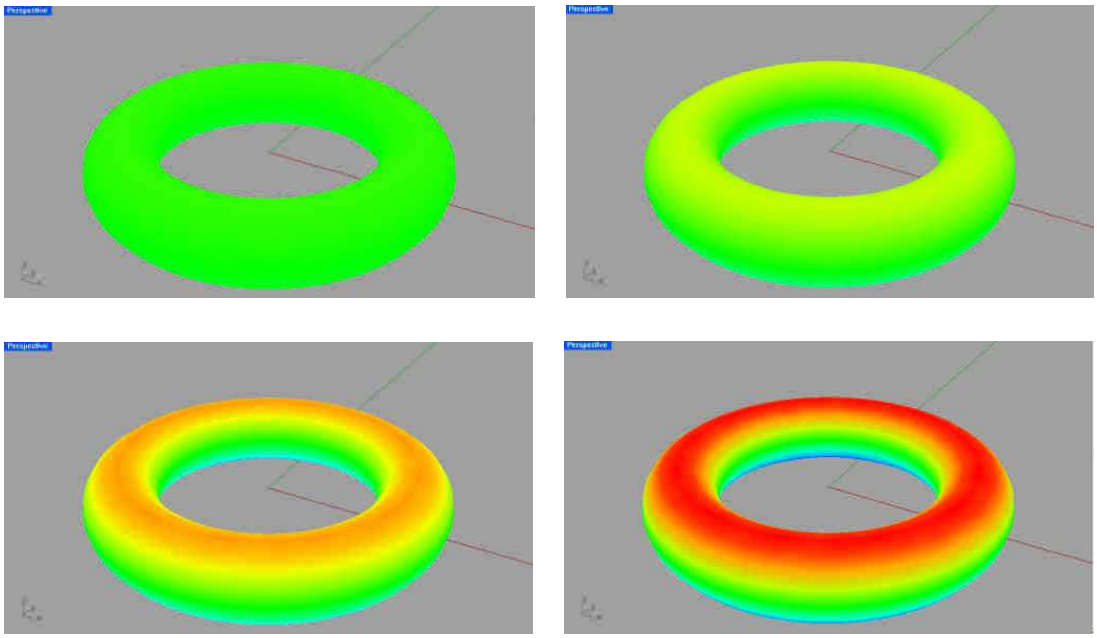

Fig. 6. torus

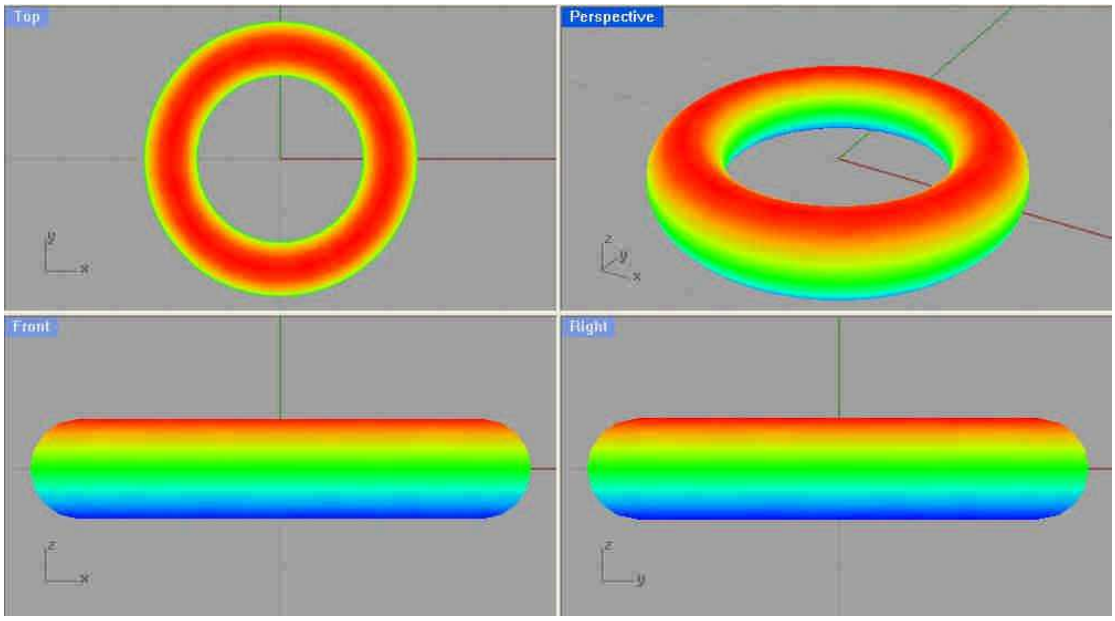

Fig. 7. torus (stationary solution)

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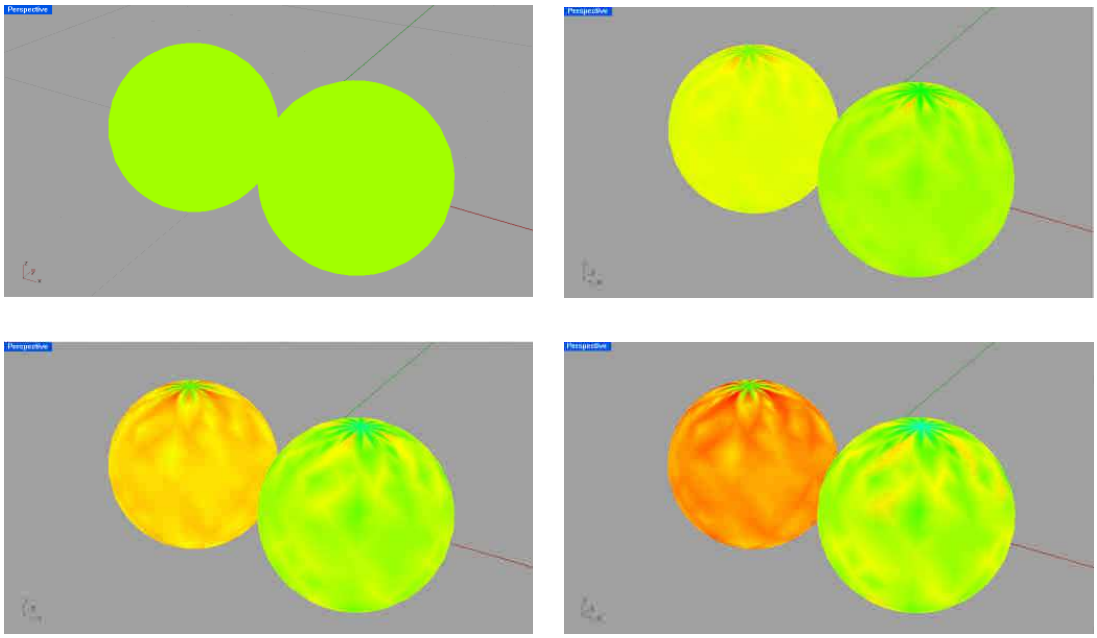

Fig. 8. sphere

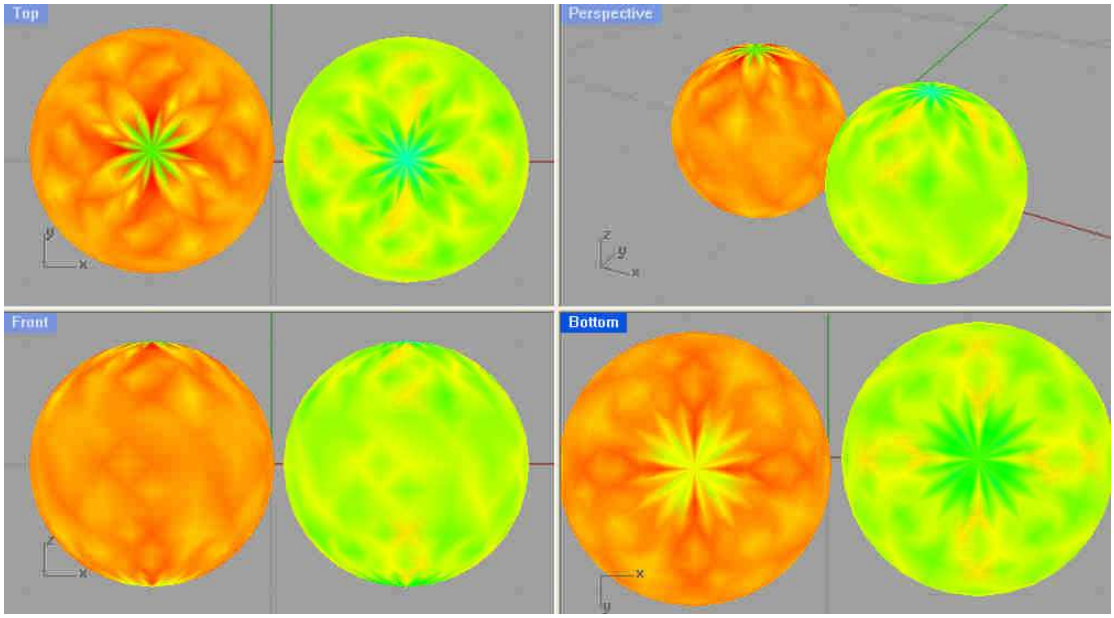

Fig. 9. sphere (stationary solution)

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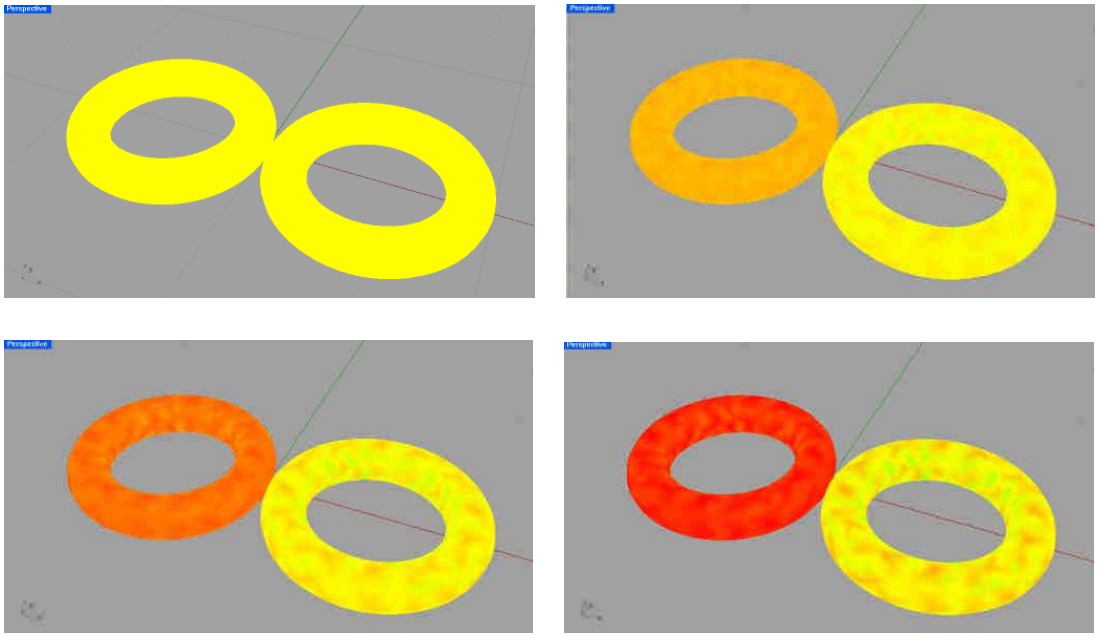

Fig. 10. torus

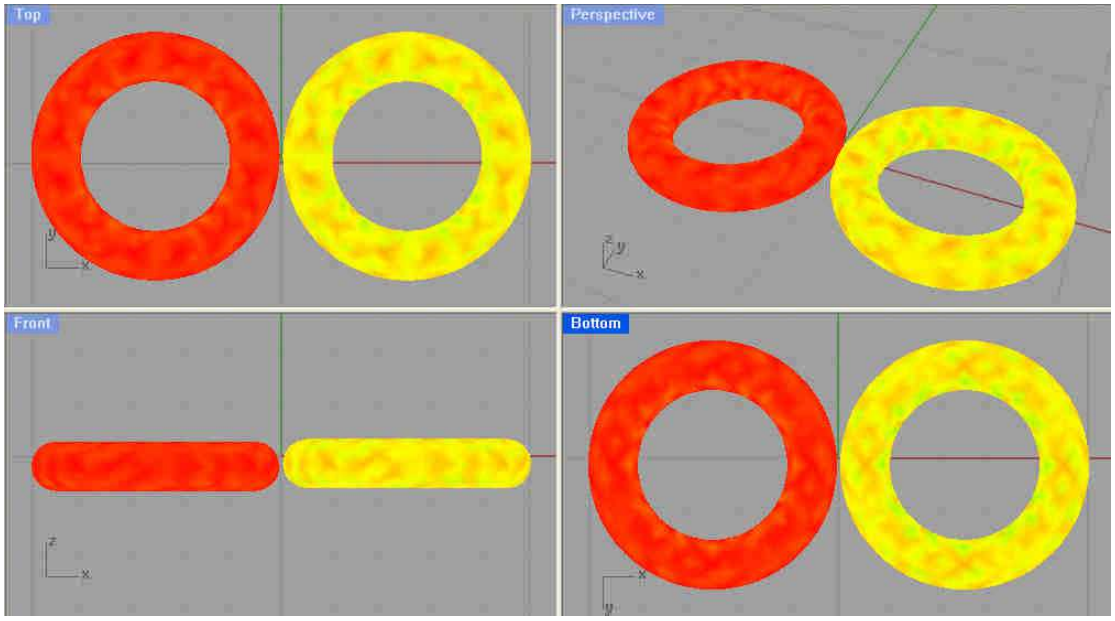

Fig. 11. torus (stationary solution)