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Numerical integration on multivariate scattered data by Lobachevsky splines

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Abstract

In this paper we investigate numerical integration on multivariate scattered data by a class of spline functions, called Lobachevsky splines. Starting from their interpolation properties, we focus on the construction of new quadrature and cubature formulas. The use of Lobachevsky splines takes advantages of their feature of being expressible in the multivariate setting as a product of univariate functions. Numerical results using Lobachevsky splines turn out to be interesting and promising for both accuracy and simplicity in computation. Finally, a comparison with radial basis functions (RBFs) confirms the validity of the proposed approach.

Key words: Lobachevsky splines, interpolation formulas, integration formulas, scattered data, radial basis functions.
2010 MSC: 65D05, 65D30, 65D32.

1. Introduction

In the last decades, the topic of numerical integration from scattered data has gained popularity in various areas of applied mathematics and scientific computing such as multivariate interpolation, approximation theory, and computer physics (see [7, 12, 15, 16, 18, 19]), although it has been much less developed with respect to the construction of integration formulas on nodes with a prefixed distribution (see, e.g., [11]).

Here, we consider the problem of constructing new integration formulas for multivariate scattered data by a class of spline functions, called *Lobachevsky splines* [4]. These consist in an infinite sequence of univariate spline functions depending on a shape parameter, which are compactly supported, strictly positive definite and enjoy noteworthy theoretical and computational properties, like the convergence of sequences of Lobachevsky splines to Gaussian functions and the convergence of sequences of their integrals and derivatives to integrals and derivatives of Gaussians.

Starting from the interpolation results in [4], where application of Lobachevsky splines shows good performance of stability and accuracy in univariate and multivariate scattered data interpolation, we here extend the use of Lobachevsky splines to numerical integration. The idea of using Lobachevsky splines to construct integration formulas on scattered data turns out to be quite natural, not only for the recent interest in *meshfree* or *meshless* numerical cubature by splines and radial basis functions (RBFs), but also for effectiveness and simplicity in computation of Lobachevsky spline integrals. In fact, Lobachevsky spline interpolation formulas take advantages of their feature of being expressible in the multivariate setting as a product of univariate functions. This makes simple the computation of multidimensional integrals: the integrand function is firstly approximated by a Lobachevsky spline interpolation formula, then the integrand is evaluated as a product of univariate integrals. This feature is interesting because other powerful and effective techniques of integration on scattered data such as RBFs, with the exception of the Gaussian, cannot be simply integrated and efficiently evaluated in high dimensions [18]. Moreover, we remark that Lobachevsky spline interpolation formulas are noteworthy because, besides being neither tensor product formulas (no grid/mesh is here considered) nor radial ones, they asymptotically behave like Gaussian interpolants (see [4]). This feature together with the uniform convergence of Lobachevsky splines to Gaussians allow us to give error estimates of cubature formulas, directly extending the results deduced in [18] for RBFs. Numerical tests point out that Lobachevsky splines

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are comparable in accuracy with Gaussians (and RBFs, in general), but they are usually much better conditioned than Gaussians.

The paper is organized as follows. In Section 2 we recall the analytic expressions of Lobachevsky splines and some of their properties. Section 3 refers to Lobachevsky spline interpolants. In Section 4 we construct new integration formulas based on Lobachevsky spline interpolants, which are expressed in the multivariate setting as a product of univariate functions. Section 5 contains an error analysis of Lobachevsky spline cubature formulas, discussing the role of two crucial parameters, i.e. the spectral norm of the inverses of interpolation matrices and the 1–norm of the weight vectors. In Section 6 we report several numerical results in order to show accuracy and stability of Lobachevsky spline interpolation formulas computing integration errors and condition numbers; all these experiments are also performed on RBFs for a comparison. Finally, Section 7 deals with conclusions and future work.

2. Lobachevsky splines

We consider a class of spline functions, called *Lobachevsky splines*, arising in probability theory [14, 17] and proposed for multivariate interpolation on scattered data in [1, 4] and for landmark-based image registration in [2, 5, 6].

Given an infinite sequence of random variables $X_1, X_2, \ldots$, which are independent and uniformly distributed on $[-a, a]$, $a \in \mathbb{R}^+$, Lobachevsky splines are the sequence of the density functions $\{f_n^*(x), n = 1, 2, \ldots\}$ of the reduced sum of the first n random variables

$$S_n^* = \frac{X_1 + X_2 + \cdots + X_n}{a\sqrt{\dfrac{n}{3}}}.$$

To get an explicit expression of f_n^* it is convenient referring to the density function f_n of the random variable $S_n = a\sqrt{\frac{n}{3}} S_n^*$. The functions f_n and f_n^* are related by

$$f_n^*(x) = a\sqrt{\frac{n}{3}}\, f_n\Big(a\sqrt{\frac{n}{3}}\, x\Big).$$

Both f_n^* and f_n have compact supports, which are on $[-\sqrt{3n}, \sqrt{3n}]$ and $[-na, na]$, $a \in \mathbb{R}^+$, respectively. Now, f_n is given by the convolution product

$$f_n(x) = \int_{-\infty}^{+\infty} f_1(u) f_{n-1}(x-u)du = \frac{1}{2a}\int_{-a}^{+a} f_{n-1}(x-u)du,$$

because by definition $f_1(x) = 1/(2a)$ for $-a \leq x \leq +a$, and $f_1(x) = 0$ elsewhere. Setting $x - u = t$, we get the recurrence formula

$$f_n(x) = \frac{1}{2a}\int_{x-a}^{x+a} f_{n-1}(t)dt, \qquad n = 2, 3, \ldots$$

Thus, reasoning by induction, we may express f_n, for $n = 1, 2, \ldots$, by the formula

$$f_n(x) = \frac{1}{(2a)^n (n-1)!}\sum_{k=0}^{n}(-1)^k \binom{n}{k}[x + (n-2k)a]_+^{n-1}, \tag{1}$$

where the truncated power function $(x)_+$ is defined as x for $x > 0$ and 0 for $x \leq 0$. Sometimes it may be convenient to consider a different form of (1), namely

$$f_n(x) = \frac{1}{(2a)^n (n-1)!}\sum_{k=0}^{\lfloor \frac{na+x}{2a} \rfloor}(-1)^k \binom{n}{k}[x + (n-2k)a]^{n-1}, \tag{2}$$

for $-na \leq x \leq na$, and $f_n(x) = 0$ otherwise, where $\lfloor \cdot \rfloor$ means the greatest integer less than or equal to the argument.

The piecewise function f_n is graphically represented by arcs of parabolas of degree $n-1$; the first $n-2$ derivatives of different arcs of parabolas are equal at the knots, i.e. $f_n \in C^{n-2}[-na, na]$. Moreover, f_n is an even function with support $[-na, na]$ and a strictly positive definite function for even $n \geq 2$.

From a computational viewpoint it is convenient to evaluate f_n starting from the pieces defined on $[-na, 0]$ and then obtain the pieces on $[0, na]$ by symmetry. Moreover, each piece on $[-na, 0]$ can be obtained by the preceding one by simply adding a term, as clearly appears from (2).

Lobachevsky splines satisfy a three-term recurrence relation for $n = 2, 3, \ldots$, namely,

$$f_n(x) = \frac{1}{2a(n-1)}\Big[(na+x)f_{n-1}(x+a) + (na-x)f_{n-1}(x-a)\Big]. \tag{3}$$

Note that, for computational reasons, f_1 in (3) must be taken in the interval $[-a, a[$.

From the central limit theorem in the local form (see, e.g., [14, 17]) we have that the sequence of Lobachevsky splines $\{f_n^*(x), n = 1, 2, \ldots\}$ converges for $n \to \infty$ to the standardized normal density function, i.e.

$$\lim_{n\to\infty} f_n^*(x) = \frac{1}{\sqrt{2\pi}} \exp\Big(\frac{-x^2}{2}\Big), \tag{4}$$

and moreover the convergence is uniform for all $x \in \mathbb{R}$.

Significant convergence properties are also satisfied by integrals and derivatives of Lobachevsky splines. From the central limit theorem for the convergence in distribution (see, e.g., [17]) we have that the sequence of distribution functions $\{\Phi_n^*(x), n = 1, 2, \ldots\}$ converges for $n \to \infty$ to the standardized normal distribution function, that is

$$\lim_{n\to\infty} \int_{-\infty}^{x} f_n^*(t)dt = \lim_{n\to\infty} \int_{-\infty}^{x} a\sqrt{\frac{n}{3}}\, f_n\Big(a\sqrt{\frac{n}{3}}\, t\Big)dt = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\Big(\frac{-t^2}{2}\Big)dt,$$

whereas the explicit form of $\Phi_n(x)$ is

$$\Phi_n(x) = \frac{1}{(2a)^n n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} [x + (n-2k)a]_+^n,$$

and

$$\Phi_n^*(x) = \Phi_n\Big(a\sqrt{\frac{n}{3}}\, x\Big).$$

The asymptotic behaviour of derivatives of Lobachevsky splines is described by the following result [8]

$$\lim_{n\to\infty} D^k f_n^*(x) = \lim_{n\to\infty} D^k\Big[a\sqrt{\frac{n}{3}}\, f_n\Big(a\sqrt{\frac{n}{3}}\, x\Big)\Big] = D^k\Big[\frac{1}{\sqrt{2\pi}} \exp\Big(\frac{-x^2}{2}\Big)\Big],$$

that is, k being a fixed integer, the sequence $D^k f_n^*(x)$, $n = k+2, k+3, \ldots$, of the k-th derivatives of $f_n^*(x)$ converges to the k-th derivative of the standardized normal density function.

Remark 2.1. *It is well-known in approximation theory that using Gaussians it is convenient to introduce a shape parameter α. This trick can be conveniently applied to Lobachevsky splines by rescaling x to αx and considering $f_n^*(\alpha x)$ instead of $f_n^*(x)$, the support being given by $[-\sqrt{3n}/\alpha, \sqrt{3n}/\alpha]$. Using the shape parameter α as a factor has the effect that a decrease of the shape parameter produces flat basis functions, while increasing α leads to more peaked (or localized) basis functions. The same result can be achieved considering $f_n(x)$ and acting on the parameter a. Numerical computations show that the approximation performance obtained by $f_n^*(\alpha x)$ or $f_n(x)$ are practically equivalent, if the values of the parameters α and a are conveniently chosen [10].*

Remark 2.2. *Since Lobachevsky splines are (univariate) strictly positive definite functions for even $n \geq 2$, we may construct multivariate strictly positive definite functions from univariate ones (see, e.g., [20]), expressing them as products of Lobachevsky splines [4].*

3. Interpolation by Lobachevsky splines

Let us consider a continuous function $g : \Omega \to \mathbb{R}$ on a compact domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, a set $\mathcal{X} = \{\mathbf{x}_i = (x_{1i}, x_{2i}, \ldots, x_{di}), i = 1, 2, \ldots, N\} \subset \Omega$ of *scattered data*, and the vector $\mathbf{g} = \{g(\mathbf{x}_i), i = 1, 2, \ldots, N\}$ of the corresponding function values. To shorten notations, we develop the argument for $d = 2$, since it is not restrictive.

For even $n \geq 2$, we construct the Lobachevsky spline interpolant of g at the nodes (x_{1i}, x_{2i})

$$F_n(x_1, x_2) = \sum_{j=1}^{N} c_j \phi_{n,j}(x_1, x_2), \qquad (x_1, x_2) \in \Omega, \tag{5}$$

requiring $F_n(x_{1i}, x_{2i}) = g(x_{1i}, x_{2i})$, $i = 1, 2, \ldots, N$. The interpolant F_n is a linear combination of products of univariate shifted and rescaled functions f_n^*

$$\phi_{n,j}(x_1, x_2) \equiv \phi_{n,j}(x_1, x_2; \alpha) = f_n^*(\alpha(x_1 - x_{1j}))f_n^*(\alpha(x_2 - x_{2j})), \tag{6}$$

with

$$f_n^*(\alpha(x_i - x_{ij})) = \sqrt{\frac{n}{3}} \frac{1}{2^n(n-1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left[\sqrt{\frac{n}{3}} \alpha(x_i - x_{ij}) + (n - 2k) \right]_+^{n-1}, \tag{7}$$

and $\alpha \in \mathbb{R}^+$ being a shape parameter. The coefficients $\mathbf{c} = \{c_j\}$ are computed by solving the linear system

$$A\mathbf{c} = \mathbf{g}, \tag{8}$$

where the interpolation matrix

$$A = \{a_{i,j}\} = \{\phi_{n,j}(x_{1i}, x_{2i})\}, \qquad i, j = 1, 2, \ldots, N, \tag{9}$$

is symmetric and depends on the choice of n and α in (6). Since Lobachevsky splines are strictly positive definite for any even $n \geq 2$, the interpolation matrix A in (9) is positive definite for any set of distinct nodes.

Finally, we remark that from (4) we have

$$\lim_{n \to \infty} f_n^*(\alpha x_1) f_n^*(\alpha x_2) = \frac{1}{2\pi} \exp\left(\frac{-\alpha^2(x_1^2 + x_2^2)}{2} \right).$$

Hence, Lobachevsky splines asymptotically behave like radial functions, though they are not radial in themselves.

4. Integration by Lobachevsky splines

Now we consider the problem of computing an approximate value of the integral

$$I(g) = \int_{\Omega} g(\mathbf{x})\, d\mathbf{x}, \qquad \mathbf{x} \in \Omega \subset \mathbb{R}^d.$$

A way to solve this problem consists in integrating a formula of type (5), which interpolates the given data, that is,

$$I(g) \approx I(F_n) = \int_{\Omega} \sum_{j=1}^{N} c_j \phi_{n,j}(\mathbf{x}) d\mathbf{x} = \sum_{j=1}^{N} c_j \int_{\Omega} \phi_{n,j}(\mathbf{x}) d\mathbf{x} = \sum_{j=1}^{N} c_j I(\phi_{n,j}). \tag{10}$$

For simplicity we limit us to consider numerical cubature on the unit square $\Omega = [0, 1]^2 \subset \mathbb{R}^2$, but it would be possible to work on hypercubes $\Omega \subset \mathbb{R}^d$, $d \geq 3$, as well. Hence, on $[0, 1]^2$ we have

$$I(\phi_{n,j}) = \int_0^1 \int_0^1 \phi_{n,j}(x_1, x_2)\, dx_1 dx_2 = \int_0^1 f_n^*(\alpha(x_1 - x_{1j}))\, dx_1 \int_0^1 f_n^*(\alpha(x_2 - x_{2j}))\, dx_2.$$

Thus, the problem is reduced to the evaluation of the two integrals

$$\int_0^1 f_n^*(\alpha(x_1 - x_{1j}))\, dx_1 \qquad \text{and} \qquad \int_0^1 f_n^*(\alpha(x_2 - x_{2j}))\, dx_2.$$

Considering the i-th integral, we obtain by (7)

$$\int_0^1 f_n^*(\alpha(x_i - x_{ij}))\, dx_i = \sqrt{\frac{n}{3}} \frac{1}{2^n(n-1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int_0^1 \left[\sqrt{\frac{n}{3}} \alpha(x_i - x_{ij}) + (n-2k) \right]_+^{n-1} dx_i. \tag{11}$$

Thus we have to compute integrals of the form

$$\int_0^1 \left[\sqrt{\frac{n}{3}} \alpha(x_i - x_{ij}) + (n-2k) \right]_+^{n-1} dx_i, \qquad j = 1, 2, \ldots, N. \tag{12}$$

Since

$$\left[\sqrt{\frac{n}{3}} \alpha(x_i - x_{ij}) + (n-2k) \right]_+^{n-1} = \begin{cases} \left[\sqrt{\frac{n}{3}} \alpha(x_i - x_{ij}) + (n-2k) \right]^{n-1}, & \text{if } x_i > x_{ij} - \sqrt{\frac{3}{n}}(n-2k)/\alpha, \\ 0, & \text{otherwise}, \end{cases} \tag{13}$$

we have to consider three cases:

$$\begin{aligned} &i) \quad x_{ij} - \sqrt{\frac{3}{n}}(n-2k)/\alpha \geq 1, \\ &ii) \quad x_{ij} - \sqrt{\frac{3}{n}}(n-2k)/\alpha \leq 0, \\ &iii) \quad 0 < x_{ij} - \sqrt{\frac{3}{n}}(n-2k)/\alpha < 1. \end{aligned} \tag{14}$$

Hence, according to which condition in (14) is satisfied, the integrals in (12) assume different values as follows:

- If the values of the parameters n, k, α are such that i) is satisfied, the function in (13) is identically equal to zero in $[0, 1]$. Hence, integral (12) is zero.
- If n, k, α are such that ii) is satisfied, the integration domain $[0, 1]$ is completely contained in the support of the function to be integrated and so integral (12) gives

$$\int_0^1 \left[\sqrt{\frac{n}{3}} \alpha(x_i - x_{ij}) + (n-2k) \right]^{n-1} dx_i = \frac{\left[\sqrt{\frac{n}{3}} \alpha(1 - x_{ij}) + (n-2k) \right]^n}{\sqrt{\frac{n}{3}} \alpha n} - \frac{\left[\sqrt{\frac{n}{3}} \alpha(-x_{ij}) + (n-2k) \right]^n}{\sqrt{\frac{n}{3}} \alpha n}.$$

- If n, k, α are such that iii) is satisfied, only a part of the integration domain $[0, 1]$ is contained in the support of the function to be integrated and therefore integral (12) is

$$\int_{x_{ij} - \sqrt{\frac{3}{n}}(n-2k)/\alpha}^1 \left[\sqrt{\frac{n}{3}} \alpha(x_i - x_{ij}) + (n-2k) \right]^{n-1} dx_i = \frac{\left[\sqrt{\frac{n}{3}} \alpha(1 - x_{ij}) + (n-2k) \right]^n}{\sqrt{\frac{n}{3}} \alpha n}.$$

Thus, we may approximate a double integral by evaluating two simple integrals, whose integrands are given by polynomial functions of degree $n-1$.

An alternative way to evaluate integral (11) of Lobachevsky splines is given by the relation

$$\int_0^1 f_n^*(\alpha(x_i - x_{ij}))\,dx_i = \frac{1}{\alpha}\left[\Phi_n^*(\alpha(1-x_{ij}) - \Phi_n^*(\alpha(-x_{ij}))\right]. \tag{15}$$

It can be easily shown that the formulation in (15) is numerically equivalent to (11) including the three cases in (14), but it is computationally more efficient.

5. Error analysis

In this section we develop an error analysis of the Lobachevsky spline integration formulas, which is firstly based on the uniform convergence of Lobachevsky splines to the Gaussian. Then, to get a more refined analysis, we discuss the role of some critical quantities.

By Hölder inequality, we have for $\Omega \subset \mathbb{R}^d$

$$|I(g) - I(F_n)| \le \|g - F_n\|_{L^1(\Omega)} \le \sqrt{\mathrm{meas}(\Omega)}\,\|g - F_n\|_{L^2(\Omega)} \le \mathrm{meas}(\Omega)\,\|g - F_n\|_\infty\,. \tag{16}$$

From the uniform convergence of the Lobachevsky splines to the Gaussian G (the standardized normal density function) we have for any $\epsilon \in \mathbb{R}^+$

$$|G - F_n| < \epsilon/2,$$

provided n is sufficiently large. Similarly, from the uniform convergence of Gaussian interpolant (see, e.g., [20]) it follows

$$|g - G| < \epsilon/2,$$

if the so-called *fill distance* of the interpolation nodes (the radius of the largest inner empty disc)

$$h = \sup_{\mathbf{x}\in\Omega} \min_{\mathbf{x}_i\in\mathcal{X}} \|\mathbf{x} - \mathbf{x}_i\|_2$$

is sufficiently small. Thus, by (16) $I(F_n)$ converges to $I(g)$ under the mentioned conditions on n and h. The convergence rate depends on the regularity degree of both the chosen Lobachevsky spline and the function g; in particular, for sufficiently regular g, $|I(g) - I(F_n)|$ may decrease exponentially as $h \to 0$ (see, e.g., [13]). Moreover, the convergence rate may depend on the form of the domain Ω, which is generally required to satisfy an *interior cone condition* (see, e.g., [9]).

Though the convergence result turns out to be interesting, in order to have more useful information for the implementation, we make a detailed analysis of the error of the integration formula (10), writing it by (8) in the form

$$I(g) \approx I(F_n) = \langle \mathbf{c}, \mathbf{I}\rangle = \left\langle A^{-1}\mathbf{g}, \mathbf{I}\right\rangle = \langle \mathbf{g}, \mathbf{w}\rangle = \sum_{j=1}^{N} w_j g_j, \tag{17}$$

with

$$A\mathbf{w} = \mathbf{I}, \tag{18}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^d$, and $\mathbf{I} = \{I(\phi_{n,j})\}_{1\le j\le N}$ is the vector of integrals of Lobachevsky splines.

In particular, we note that the quantity $\sum_j |w_j|$ plays a central role both in the study of the integration error and in the effect of perturbations in the data values [18]. Moreover, from the computational viewpoint the common trouble of Gaussian ill-conditioning is significantly reduced for Lobachevsky splines, and the errors obtained by approximating the integral $\mathbf{I} = \{I(\phi_{n,j})\}_{1\le j\le N}$ are not so much influenced as for Gaussians by the computed weights via the numerical solution of the linear system (8). In fact, for small values of n Lobachevsky splines are much better conditioned than Gaussians [4].

Now, referring to integration formulas in (17) and (18), we must take into account the presence of errors in the evaluation of the integrals of Lobachevsky splines, i.e.

$$\{\tilde{I}(\phi_{n,j})\} = \tilde{\mathbf{I}} \approx \mathbf{I},$$

which generates a vector $\tilde{\mathbf{w}}$ of approximate weights and, starting from (17), a perturbed integration formula

$$\tilde{I}(F_n) = \langle \mathbf{c}, \tilde{\mathbf{I}} \rangle = \langle \mathbf{g}, \tilde{\mathbf{w}} \rangle = \sum_{j=1}^{N} \tilde{w}_j g_j,$$

where $\tilde{\mathbf{w}} = A^{-1}\tilde{\mathbf{I}}$.

In error investigation it must be considered the parameter $\|A^{-1}\|_2$, which measures the sensitivity to perturbations of the interpolation processes based on the Gaussian or Lobachevsky splines. Now, in the literature lower bounds for the smallest eigenvalue of the interpolation matrix A of the Gaussian have been extensively studied (see, e.g., [20]), providing upper bounds for $\|A^{-1}\|_2$ in terms of the so-called *separation distance* of the nodes

$$q = \frac{1}{2} \min_{i \neq j} \left\| \mathbf{x}_i - \mathbf{x}_j \right\|_2 .$$

When q tends to zero, it happens that $\|A^{-1}\|_2$ diverges exponentially and A tends to become singular due to collapsing of two rows. Fortunately, the theoretical upper bounds for $\|A^{-1}\|_2$ are strongly conservative in the Gaussian interpolation and the use of the Gaussian for numerical cubature is not compromised, as it is clearly shown by numerical experiments. Considering the convergence property of Lobachevsky splines to the Gaussian, we can reasonably expect that the upper bounds for $\|A^{-1}\|_2$ relating to Lobachevsky splines are closed to Gaussian ones; in fact, numerical test show that in the case of Lobachevsky splines the values assumed by $\|A^{-1}\|_2$ are quite acceptable and, generally, considerably smaller than those of the Gaussian.

Hence, an error estimate of the Lobachevsky spline integration formula (17) may be expressed as follows

$$\begin{aligned} |I(g) - \langle \tilde{\mathbf{w}}, \mathbf{g} \rangle| &\le |I(g) - \langle \mathbf{w}, \mathbf{g} \rangle| + |\langle \mathbf{w} - \tilde{\mathbf{w}}, \mathbf{g} \rangle| = |I(g) - I(F_n)| + |A^{-1} \langle \mathbf{I} - \tilde{\mathbf{I}}, \mathbf{g} \rangle| \\ &\le \|g - F_n\|_{L^1(\Omega)} + \|A^{-1}(\mathbf{I} - \tilde{\mathbf{I}})\|_2 \|\mathbf{g}\|_2 \le \mathrm{meas}(\Omega) \, \|g - F_n\|_\infty + \|A^{-1}\|_2 \|\mathbf{I} - \tilde{\mathbf{I}}\|_2 \|\mathbf{g}\|_2 . \end{aligned}$$

Finally, we take into account the effect of perturbations on the data values, that is, we consider the perturbed data

$$\{\tilde{g}(x_j)\} = \tilde{\mathbf{g}} \approx \mathbf{g}.$$

Hence, we may estimate the error of the perturbed Lobachevsky spline integration formula $\langle \tilde{\mathbf{w}}, \tilde{\mathbf{g}} \rangle$ as follows

$$|I(g) - \langle \tilde{\mathbf{w}}, \tilde{\mathbf{g}} \rangle| \le |I(g) - \langle \mathbf{w}, \mathbf{g} \rangle| + |\langle \mathbf{w}, \mathbf{g} \rangle - \langle \tilde{\mathbf{w}}, \tilde{\mathbf{g}} \rangle| \le |I(g) - \langle \mathbf{w}, \mathbf{g} \rangle| + |\langle \mathbf{w}, \mathbf{g} - \tilde{\mathbf{g}} \rangle| + |\langle \mathbf{w} - \tilde{\mathbf{w}}, \tilde{\mathbf{g}} \rangle| , \tag{19}$$

where

a) since Lobachevsky integration formula is exact for every function $z(\mathbf{x})$ in the Lobachevsky spline space associated to $\mathcal{X} = \{\mathbf{x}_j, j = 1, 2, \ldots, N\}$, namely $I(z) = \langle \mathbf{w}, \mathbf{z} \rangle$, $z \in \mathrm{span}\{\phi_{n,j}\}_{1 \le j \le N}$, $\mathbf{z} = \{z(\mathbf{x}_j), j = 1, 2, \ldots, N\}$, we have

$$|I(g) - \langle \mathbf{w}, \mathbf{g} \rangle| \le |I(g) - \langle \mathbf{w}, \mathbf{z} \rangle| + |\langle \mathbf{w}, \mathbf{z} \rangle - \langle \mathbf{w}, \mathbf{g} \rangle| = |I(g) - I(z)| + |\langle \mathbf{w}, \mathbf{z} - \mathbf{g} \rangle| , \tag{20}$$

b)

$$|\langle \mathbf{w}, \mathbf{g} - \tilde{\mathbf{g}} \rangle| = \left| \sum_{j=1}^{N} w_j (g_j - \tilde{g}_j) \right| \le \left| \max_{1 \le j \le N} \{g_j - \tilde{g}_j\} \sum_{j=1}^{N} w_j \right| \le \max_{1 \le j \le N} |g_j - \tilde{g}_j| \sum_{j=1}^{N} \left| w_j \right| = \|\mathbf{g} - \tilde{\mathbf{g}}\|_\infty \|\mathbf{w}\|_1 , \tag{21}$$

being $\|\mathbf{w}\|_1 = \sum_{j=1}^{N} \left|w_j\right|$.

c)

$$|\langle \mathbf{w} - \tilde{\mathbf{w}}, \tilde{\mathbf{g}}\rangle| = |\langle A^{-1}\mathbf{I} - A^{-1}\tilde{\mathbf{I}}, \tilde{\mathbf{g}}\rangle| = |\langle A^{-1}(\mathbf{I} - \tilde{\mathbf{I}}), \tilde{\mathbf{g}}\rangle| \leq \|A^{-1}\|_2\|\mathbf{I} - \tilde{\mathbf{I}}\|_2\|\tilde{\mathbf{g}}\|_2. \tag{22}$$

Thus, using (20), (21), and (22), the error estimate (19) is

$$\begin{aligned}
|I(g) - \langle \tilde{\mathbf{w}}, \tilde{\mathbf{g}}\rangle| &\leq |I(g) - I(z)| + |\langle \mathbf{w}, \mathbf{z} - \mathbf{g}\rangle| + \|\mathbf{g} - \tilde{\mathbf{g}}\|_\infty\|\mathbf{w}\|_1 + \|A^{-1}\|_2\|\mathbf{I} - \tilde{\mathbf{I}}\|_2\|\tilde{\mathbf{g}}\|_2 \\
&\leq \mathrm{meas}(\Omega)\,\|g - z\|_\infty + \|\mathbf{z} - \mathbf{g}\|_\infty\|\mathbf{w}\|_1 + \|\mathbf{g} - \tilde{\mathbf{g}}\|_\infty\|\mathbf{w}\|_1 + \|A^{-1}\|_2\|\mathbf{I} - \tilde{\mathbf{I}}\|_2\|\tilde{\mathbf{g}}\|_2 \\
&\leq (\mathrm{meas}(\Omega) + \|\mathbf{w}\|_1)\,\|g - z\|_\infty + \|\mathbf{g} - \tilde{\mathbf{g}}\|_\infty\|\mathbf{w}\|_1 + \|A^{-1}\|_2\|\mathbf{I} - \tilde{\mathbf{I}}\|_2\|\tilde{\mathbf{g}}\|_2 \\
&\leq (\mathrm{meas}(\Omega) + \|\mathbf{w}\|_1)E_{\mathcal{X},\phi_n}(g) + \|\mathbf{g} - \tilde{\mathbf{g}}\|_\infty\|\mathbf{w}\|_1 + \|A^{-1}\|_2\|\mathbf{I} - \tilde{\mathbf{I}}\|_2\|\tilde{\mathbf{g}}\|_2,
\end{aligned}$$

where $E_{\mathcal{X},\phi_n}(g) = \inf_{z\in\mathrm{span}(\phi_{n,j})} \|g - z\|_\infty$.

It must be remarked once more that the above considered error estimates are theoretically interesting, but poorly significant for a particular integral evaluation being very conservative. Hence, a large number of numerical experiments offers a more precise picture of the performance of the proposed integration formulas.

6. Numerical results

In this section we report the results of an extensive and detailed investigation on testing and verifying accuracy and effectiveness of Lobachevsky spline integration formulas on various sets $\mathcal{X}$ of scattered data, which are given by Halton points [13] of dimension $N = (2^k + 1)^d$, $k = 3, 4$, contained in $\Omega = [0, 1]^d \subset \mathbb{R}^d$, for $d = 1, 2$. In particular, we consider integration formulas obtained by Lobachevsky spline interpolants for $n = 2, 4, 6, 8, 10$ (denoted by L2, L4, L6, L8, and L10, respectively). Moreover, to have an efficient evaluation of the integrals, we conveniently exploit some features of Lobachevsky spline functions such as (univariate) radial symmetry and separation of variables.

Here, as an example, we only report the results obtained by taking the well-known Franke's test function (see, e.g., [3])

$$\begin{aligned}
g(x_1, x_2) &= \frac{3}{4}\exp\left[-\frac{(9x_1 - 2)^2 + (9x_2 - 2)^2}{4}\right] + \frac{3}{4}\exp\left[-\frac{(9x_1 + 1)^2}{49} - \frac{9x_2 + 1}{10}\right] \\
&+ \frac{1}{2}\exp\left[-\frac{(9x_1 - 7)^2 + (9x_2 - 3)^2}{4}\right] - \frac{1}{5}\exp\left[-(9x_1 - 4)^2 - (9x_2 - 7)^2\right],
\end{aligned}$$

which, in the univariate case, is restricted setting $x_2 = 0.5$. However, we remark that several numerical experiments (not reported here for brevity) have been carried out using other test functions and the results show a uniform behavior.

First, in Tables 1 and 4 we point out integration errors obtained by varying the value of the shape parameter $\alpha \in [2, 10]$, whose variation influences the quality of approximation. Nevertheless, numerical tests show a good level of accuracy for any value of α also when we use Lobachevsky splines of low degree and smoothness, such as L2, L4 and L6. This highlights reliability and robustness of Lobachevsky spline integration formulas.

Then, using the `norm` function of MATLAB, we compute the values of two important parameters of Lobachevsky spline integration, i.e. the spectral norm $\|A^{-1}\|_2$, which measures the absolute conditioning of the interpolation and integration equations (see Tables 2 and 5), and the 1–norm $\|\tilde{\mathbf{w}}\|_1$, which leads to an approximation analysis and measures the conditioning of the integration formula (see Tables 3 and 6). In particular, for each of the Lobachevsky splines we observe that increasing the shape parameter produces peaked or localized basis functions with the effect of better conditioning, but losing accuracy (and vice versa). Note that, the values of quantities $\|A^{-1}\|_2$ and $\|\tilde{\mathbf{w}}\|_1$ strongly depend on the distribution of nodes, whereas they are independent of function values.

For a comparison we have also made several tests by considering some of the most known RBFs, that is,

$\exp(-\alpha^2 r^2/2)$,	Gaussian (G)
$(1 + \alpha^2 r^2/2)^{-1/2}$,	Inverse MultiQuadric (IMQ)
$(1 + \alpha^2 r^2/2)^{1/2}$,	MultiQuadric (MQ)
$(1 - \delta r)_+^4(4\delta r + 1)$,	Wendland C^2 compactly supported (W2)
$r^2 \log r$,	Thin Plate Spline (TPS)

where $\alpha, \delta \in \mathbb{R}^+$ are shape parameters, and $r = \|\mathbf{x} - \mathbf{x}_i\|_2$. We recall that G, IMQ and W2 are strictly positive definite functions, while MQ and TPS are conditionally strictly positive definite ones. Thus, in the latter cases the addition of a polynomial term of degree zero to MQ, and of degree one to TPS, is required to guarantee existence and uniqueness of the interpolant. Note that TPS is shape parameter free, since its use does not produce a significant change in results (see [13]). For details on the integration of RBFs we refer to the interesting paper [18], which explores potentialities and drawbacks of this approach. At our knowledge, this paper seems to be the first which gives a sistematic discussion on the topic.

Thus, in Table 7 we report errors obtained by using RBFs, while Tables 8 and 9 contain information on the conditioning of RBF interpolant and cubature formulas, respectively.

Analyzing Tables 4–9 we see that Lobachevsky splines are comparable in accuracy with Gaussians (and other RBFs in general), even if they are usually much better conditioned than Gaussians. Here, however, it is appropriate to remark that our numerical tests by varying the shape parameters α and δ are purely indicative, since only for Lobachevsky splines and Gaussians we may have a direct correspondence between their shape parameters. Moreover, as regards to compactly supported Wendland functions of smoothness 2 a comparison with Lobachevsky splines should be carried out taking the L4, which has the same degree of regularity. In such a case, without expecting to be too precise for the reasons described above, we may note comparable results for accuracy and stability, because both errors and conditioning have the same orders of magnitudine.

Finally, several experiments show that CPU times of Lobachevsky spline cubature formulas are in general much lower than those of RBF cubature formulas, with the exception of the Gaussian case. In fact, Lobachevsky spline (and Gaussian) integrals take advantages of being evaluated as products of univariate integrals.

N	α	L2	L4	L6	L8	L10
	2	3.2588E − 03	2.6771E − 03	2.3348E − 04	5.4080E − 03	6.8696E − 03
	4	2.4473E − 03	5.1717E − 04	3.0110E − 03	2.0553E − 03	1.9817E − 03
9	6	7.3948E − 03	1.1617E − 03	1.4853E − 03	6.3619E − 04	2.0463E − 04
	8	1.2175E − 02	2.9818E − 03	1.6647E − 03	1.4512E − 03	1.6162E − 03
	10	1.5159E − 02	5.7560E − 03	4.5567E − 03	4.8402E − 03	5.0095E − 03
	2	8.8379E − 04	9.6471E − 06	2.9493E − 04	1.8803E − 04	1.2822E − 04
	4	6.3940E − 04	8.0798E − 04	1.1069E − 04	7.4175E − 05	1.3221E − 04
17	6	2.2271E − 03	2.4944E − 04	1.4088E − 05	1.3420E − 05	1.3241E − 05
	8	3.2797E − 03	3.2388E − 04	1.3759E − 04	5.4048E − 05	2.7913E − 05
	10	4.5335E − 03	4.8002E − 04	2.5752E − 04	1.6262E − 04	1.7146E − 04

Table 1: Errors of Lobachevsky spline quadrature formulas computed on Halton points.

7. Conclusions

In this paper we propose the use of a class of piecewise polynomials, called Lobachevsky splines, in numerical integration of multivariate scattered data. Thus, after recalling analytic expressions of Lobachevsky splines and their noteworthy convergence properties, we construct new integration formulas based on Lobachevsky spline interpolants, which are expressed in the multivariate setting as a product of univariate functions. This feature makes effective and simple the computation of integrals, since each term of Lobachevsky splines are univariate polynomials and variables are independent and separable. Moreover, we have given an error analysis of Lobachevsky spline cubature formulas, discussing the role of two crucial parameters like the spectral norm of the inverses of interpolation matrices and the 1–norm of the computed weight vectors. Several numerical tests have confirmed the goodness of the proposed approach in univariate and bivariate setting, referring not only to accuracy (errors) and stability (conditioning) of Lobachevsky splines but also carring out a comparison with RBFs.

Finally, as regards to research and future work we are interested in efficiently extending the use of Lobachevsky spline integration formulas on scattered data to more general domains.

N	α	L2	L4	L6	L8	L10
9	2	7.7660E + 01	1.5678E + 04	1.0745E + 06	3.5978E + 07	8.9856E + 08
	4	4.1503E + 01	2.9330E + 03	2.7549E + 04	3.6286E + 05	4.2038E + 06
	6	2.9600E + 01	7.4263E + 02	9.9096E + 03	7.5550E + 04	8.0529E + 04
	8	2.4326E + 01	4.3273E + 02	2.5151E + 03	2.1736E + 03	1.7523E + 03
	10	2.9632E + 01	2.3321E + 02	2.5914E + 02	2.1365E + 02	1.8988E + 02
17	2	1.5533E + 02	1.2617E + 05	3.5986E + 07	5.9512E + 09	6.4206E + 11
	4	7.8054E + 01	1.5937E + 04	1.7169E + 06	4.1552E + 07	2.0475E + 09
	6	6.6357E + 01	7.3856E + 03	2.1477E + 05	4.7470E + 06	2.4506E + 07
	8	5.3073E + 01	3.5426E + 03	5.9873E + 04	1.5540E + 06	4.2012E + 07
	10	5.6034E + 01	1.5324E + 03	3.5706E + 04	1.1635E + 06	4.3087E + 06

Table 2: Values of the spectral norm $\|A^{-1}\|_2$ of Lobachevsky splines computed on Halton points.

N	α	L2	L4	L6	L8	L10
9	2	9.8807E − 01	9.9881E − 01	1.4723E + 00	2.5468E + 00	5.6239E + 00
	4	9.7761E − 01	1.0676E + 00	1.0711E + 00	1.7690E + 00	3.3156E + 00
	6	9.6054E − 01	9.8898E − 01	1.3932E + 00	3.9382E + 00	5.2754E + 00
	8	9.5308E − 01	9.7890E − 01	1.6376E + 00	1.7623E + 00	1.6551E + 00
	10	9.4306E − 01	9.6383E − 01	1.1750E + 00	1.1336E + 00	1.1115E + 00
17	2	9.9684E − 01	9.9975E − 01	1.0474E + 00	1.3452E + 00	2.4810E + 00
	4	9.9371E − 01	1.0100E + 00	1.2703E + 00	1.5682E + 00	3.1502E + 00
	6	9.9204E − 01	9.9842E − 01	1.1060E + 00	1.5953E + 00	3.5359E + 00
	8	9.8630E − 01	9.9716E − 01	1.2866E + 00	1.3107E + 00	6.6115E + 00
	10	9.8272E − 01	1.0010E + 00	1.1980E + 00	4.6108E + 00	3.5814E + 00

Table 3: Values of the 1–norm $\|\tilde{\mathbf{w}}\|_1$ of Lobachevsky splines computed on Halton points.

N	α	L2	L4	L6	L8	L10
81	2	7.8605E − 04	1.8290E − 03	1.9500E − 03	6.3251E − 03	1.1955E − 02
	4	2.0255E − 03	1.9810E − 03	1.0816E − 03	2.3345E − 03	5.2299E − 03
	6	4.2102E − 03	4.8743E − 04	1.7374E − 03	6.2329E − 04	1.2988E − 04
	8	7.9439E − 03	1.8623E − 03	1.2394E − 03	1.2890E − 03	1.3617E − 03
	10	1.6255E − 02	3.1382E − 03	2.8899E − 03	3.0567E − 03	3.1761E − 03
289	2	2.2706E − 04	7.7034E − 05	2.5455E − 04	2.1428E − 05	3.3971E − 03
	4	1.4565E − 04	5.5468E − 05	4.1333E − 04	1.8705E − 04	5.7711E − 04
	6	6.2325E − 04	2.7228E − 05	5.2692E − 05	1.5188E − 04	2.7640E − 05
	8	9.7914E − 04	1.2671E − 04	6.9888E − 05	1.0035E − 04	5.1256E − 05
	10	2.1137E − 03	3.2837E − 04	1.9814E − 04	1.3899E − 04	1.2249E − 04

Table 4: Errors of Lobachevsky spline cubature formulas computed on Halton points.

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N	α	L2	L4	L6	L8	L10
81	2	2.9725E + 02	4.5256E + 05	9.2843E + 07	5.2922E + 09	1.9512E + 11
	4	1.2441E + 02	2.5366E + 04	8.1775E + 05	1.7445E + 07	1.1485E + 08
	6	7.5543E + 01	6.2493E + 03	5.8514E + 04	6.2348E + 05	2.5916E + 06
	8	5.6686E + 01	2.0079E + 03	1.9298E + 04	2.0299E + 04	1.4742E + 04
	10	4.3990E + 01	9.1388E + 02	1.1248E + 03	8.4091E + 02	7.0674E + 02
289	2	8.6924E + 02	1.0258E + 07	2.2006E + 10	1.8086E + 13	3.1644E + 15
	4	3.3701E + 02	6.3173E + 05	2.2711E + 08	3.1951E + 10	3.0653E + 12
	6	2.0125E + 02	1.1183E + 05	1.7880E + 07	7.5518E + 08	2.1627E + 10
	8	1.4744E + 02	3.6117E + 04	1.9239E + 06	5.9256E + 07	1.0332E + 09
	10	1.1338E + 02	1.5675E + 04	5.4789E + 05	6.8735E + 06	5.6836E + 07

Table 5: Values of the spectral norm $\|A^{-1}\|_2$ of Lobachevsky splines computed on Halton points.

N	α	L2	L4	L6	L8	L10
81	2	9.9833E − 01	1.0001E + 00	1.3935E + 00	2.1302E + 00	2.9088E + 00
	4	9.9512E − 01	1.0097E + 00	1.0760E + 00	1.6840E + 00	2.2429E + 00
	6	9.8875E − 01	1.0355E + 00	1.1215E + 00	1.6633E + 00	2.2558E + 00
	8	9.7735E − 01	1.0150E + 00	1.1665E + 00	1.1924E + 00	1.1623E + 00
	10	9.5512E − 01	1.0159E + 00	1.0510E + 00	1.0447E + 00	1.0404E + 00
289	2	9.9975E − 01	1.0081E + 00	1.2204E + 00	2.5202E + 00	2.6853E + 01
	4	9.9928E − 01	1.0209E + 00	1.2406E + 00	2.0654E + 00	4.4176E + 00
	6	9.9805E − 01	1.0068E + 00	1.2512E + 00	1.9596E + 00	4.2948E + 00
	8	9.9669E − 01	1.0213E + 00	1.1889E + 00	1.7920E + 00	3.5613E + 00
	10	9.9342E − 01	1.0328E + 00	1.1858E + 00	1.8841E + 00	3.5574E + 00

Table 6: Values of the 1–norm $\|\tilde{\mathbf{w}}\|_1$ of Lobachevsky splines computed on Halton points.

N	α	G	IMQ	MQ	δ	W2
81	2	4.8924E − 03	1.6015E − 01	1.1403E − 01	0.2	4.5294E − 04
	4	8.9399E − 04	3.3075E − 04	1.2776E − 05	0.4	4.0482E − 04
	6	5.0700E − 04	1.4696E − 04	2.8271E − 04	0.6	3.8675E − 04
	8	1.6435E − 03	1.9427E − 04	1.6314E − 04	0.8	4.4893E − 04
	10	3.6736E − 03	3.7124E − 04	7.3634E − 05	1.0	6.3491E − 04
289	2	7.8848E − 02	6.3287E + 04	1.1464E + 05	0.2	8.1873E − 05
	4	2.5766E − 02	6.7067E − 03	3.8577E − 01	0.4	7.9784E − 05
	6	1.5309E − 04	8.1104E − 06	2.0180E − 04	0.6	8.3610E − 05
	8	3.2439E − 05	4.1534E − 05	8.2208E − 05	0.8	9.4982E − 05
	10	1.5733E − 04	6.1244E − 05	1.8593E − 05	1.0	1.1711E − 04

Table 7: Errors of RBF cubature formulas computed on Halton points (TPS = 1.5111E − 04, 1.6642E − 05 for N = 81 and 289, respectively).

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N	α	G	IMQ	MQ	δ	W2
	2	6.2590E + 15	3.1702E + 08	4.8519E + 09	0.2	4.4433E + 04
	4	1.3311E + 09	2.4902E + 04	2.5075E + 05	0.4	5.5839E + 03
81	6	4.3610E + 05	7.4148E + 02	5.0971E + 03	0.6	1.6659E + 03
	8	4.8384E + 03	1.5120E + 02	6.6564E + 02	0.8	7.0864E + 02
	10	4.0145E + 02	5.9656E + 01	2.0624E + 02	1.0	3.6633E + 02
	2	3.7212E + 17	3.2629E + 16	6.3110E + 16	0.2	5.1713E + 05
	4	3.0225E + 18	1.0935E + 09	2.1294E + 10	0.4	6.4648E + 04
289	6	5.1966E + 16	1.4766E + 06	1.9339E + 07	0.6	1.9158E + 04
	8	1.9877E + 11	5.0693E + 04	5.1188E + 05	0.8	8.0844E + 03
	10	1.2328E + 08	6.3926E + 03	5.2700E + 04	1.0	4.1406E + 03

Table 8: Values of the spectral norm of the inverses of RBF interpolation matrices computed on Halton points (TPS = 4.1077E + 02, 1.9983E + 03 for N = 81 and 289, respectively).

N	α	G	IMQ	MQ	δ	W2
	2	3.9032E + 01	2.4361E + 02	4.4889E + 02	0.2	9.9994E − 01
	4	6.4813E + 00	1.1400E + 00	3.8158E + 00	0.4	9.9975E − 01
81	6	1.6694E + 00	1.0290E + 00	1.0654E + 00	0.6	9.9945E − 01
	8	1.0972E + 00	1.0081E + 00	1.0179E + 00	0.8	9.9894E − 01
	10	1.0250E + 00	1.0016E + 00	1.0108E + 00	1.0	9.9829E − 01
	2	3.2302E + 02	9.6988E + 10	3.3028E + 10	0.2	1.0081E + 00
	4	3.0523E + 03	6.2388E + 04	3.2533E + 05	0.4	1.0081E + 00
289	6	8.2696E + 02	8.2031E + 01	5.1410E + 02	0.6	1.0081E + 00
	8	4.3699E + 01	3.3683E + 00	3.6460E + 01	0.8	1.0081E + 00
	10	6.5256E + 00	1.3148E + 00	1.9652E + 00	1.0	1.0081E + 00

Table 9: Values of the 1–norm of the RBF weight vectors computed on Halton points (TPS = 1.0006E + 00, 1.2362E + 00 for N = 81 and 289, respectively).

References

[1] G. Allasia, Scattered multivariate interpolation by a class of spline functions, in: J. Vigo-Aguiar et al. (Eds.), Proceedings of the International Conference on Computational and Mathematical Methods in Science and Engineering, vol. 1, 2009, pp. 73–79.

[2] G. Allasia, R. Cavoretto, A. De Rossi, B. Quatember, W. Recheis, M. Mayr, S. Demertzis, Radial basis functions and splines for landmark-based registration of medical images, in: T.E. Simos, G. Psihoyios, C. Tsitouras (Eds.), Proceedings of the International Conference on Numerical Analysis and Applied Mathematics, AIP Conference Proceedings, American Institute of Physics, vol. 1281, Melville, New York, 2010, pp. 716–719.

[3] G. Allasia, R. Besenghi, R. Cavoretto, A. De Rossi, Scattered and track data interpolation using an efficient strip searching procedure, Appl. Math. Comput. 217 (2011) 5949–5966.

[4] G. Allasia, R. Cavoretto, A. De Rossi, Lobachevsky spline functions and interpolation to scattered data, Department of Mathematics, University of Turin, Technical Report 1 (2011) 1–23, submitted for publication.

[5] G. Allasia, R. Cavoretto, A. De Rossi, A class of spline functions for landmark-based image registration, Math. Methods Appl. Sci. (2011), in press.

[6] G. Allasia, R. Cavoretto, A. De Rossi, A comparison of interpolation techniques using compactly supported functions for image registration, Department of Mathematics, University of Turin, Technical Report 1 (2012) 1–25, submitted for publication.

[7] A.Yu. Bezhaev, Cubature formulae on scattered meshes, Soviet J. Numer. Anal. Math. Modelling 6 (1991) 95–106.

[8] R. Brinks, On the convergence of derivatives of B-splines to derivatives of the Gaussian functions, Comput. Appl. Math. 27 (2008) 79–92.

[9] M.D. Buhmann, Radial Basis Functions: Theory and Implementation, Cambridge University Press, Cambridge, 2003.

[10] R. Cavoretto, Meshfree Approximation Methods, Algorithms and Applications, PhD Thesis, University of Turin, 2010.

[11] R. Cools, D. Laurie (Eds.), Numerical evaluation of integrals, J. Comput. Appl. Math. 112 (1999).

[12] G. Endrödi, Multidimensional spline integration of scattered data, Comput. Phys. Comm. 182 (2011) 1307–1314.

[13] G.E. Fasshauer, Meshfree Approximation Methods with MATLAB, World Scientific Publishing, Singapore, 2007.
[14] B.V. Gnedenko, The Theory of Probability, MIR, Moscow, 1976.
[15] D. Levin, Stable integration rules with scattered integration points, J. Comput. Appl. Math. 112 (1999) 181–187.
[16] A. Punzi, A. Sommariva, M. Vianello, Meshless cubature over the disk using thin-plate splines, J. Comput. Appl. Math. 221 (2008) 430–436.
[17] A. Rényi, Calcul des Probabilités, Dunod, Paris, 1966.
[18] A. Sommariva, M. Vianello, Numerical cubature on scattered data by radial basis functions, Computing 76 (2006) 295–310.
[19] A. Sommariva, M. Vianello, Meshless cubature by Green's formula, Appl. Math. Comput. 183 (2006) 1098–1107.
[20] H. Wendland, Scattered Data Approximation, Cambridge University Press, Cambridge, 2005.