The Bicube: An Interconnection of Two Hypercubes

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(*v3.0 released February 2013*)

We consider two definitions of the even-dimensional hypercube given in the literature. The labeled graphs obtained by two definitions are not same, but one is isomorphic to the other. By interconnecting two labeled graphs in such a way that each pair of vertices with the same label are joined by an edge, we construct a vertex-symmetric graph with the diameter about half that of a comparable hypercube. We extend the result to a general scheme for interconnecting two hypercubes to produce a network topology called the *bicube*. We show that the bicube preserves the vertex-symmetry, bipartiteness, hamiltonian and bipancyclic properties of the hypercube, and is highly edge-symmetric.

Keywords: Interconnection networks, hypercubes, diameter, vertex-symmetry, edge-symmetry

2010 AMS Subject Classification: 05C90; 68M10; 68R10

1. Introduction

Many interconnection network topologies have been proposed in the literature for connecting a large number of processors efficiently, for example, [3–7, 9, 12, 14]. The topology of an interconnection network can be represented as a graph in which vertices and edges correspond to nodes and communication links, respectively. In this paper, we will use standard terminology in graphs (see [2]). One popular topology is the hypercube graph which has 2^m vertices and each vertex has m edges incident to it. The hypercube is scalable, symmetric and has a logarithmic diameter, simple routing and broadcasting algorithms.

The diameter is an important measure for an interconnection network, because it is concerned with the maximum routing steps in transmitting a message. The hypercube does not have the smallest diameter that is possible with the degree of a vertex it has. To achieve the lower diameter than that of hypercube, a variety of hypercube variants appeared. There have been two major approaches taken so far in designing hypercube variants. In one approach, the degree of a vertex is increased by adding extra edges to create "shortcuts" between vertices, e.g. the folded hypercube [8], the enhanced hypercube [26] whose diameters are about half that of a comparable hypercube. In the other approach, the degree of a vertex is kept the same by just rearranging the hypercube edges, and the diameter can be reduced to about half, e.g. the twisted cube [12], crossed

cube [7], Möbius cube [4], Mcube [25], and multiply twisted cube [6]. Also, there were some theoretical works [5, 11, 29] that achieve even smaller diameters than the above networks, but the connection rules and routing algorithms are more complicated.

Another important measure for an interconnection network is symmetry of the network. Informally speaking, a vertex-symmetric (resp. edge-symmetric) graph looks the same from any vertex (resp. edge). The hypercube is both vertex-symmetric and edge-symmetric graph. In a symmetric network, it is easy to design parallel and communication algorithms, since it is irrelevant where the computation and/or communication starts or in which directions it will evolve. Unfortunately, the rearrangement or "twist" of the edges in the hypercube variants lose the high degree of symmetry of the hypercube, and may deteriorate the performance of the network as pointed in [1, 25]; asymmetry in a network affects message delays under heavy traffic, and may offset the advantage of lower diameter. It is easy to find that all the above twisted cube, crossed cube, Möbius cube, Mcube, and multiply twisted cube are not vertex-symmetric.

In this paper, we present hypercube variants using two definitions of the even-dimensional hypercube given in the literature. We show that two hypercubes obtained by different definitions can be combined to produce a graph with the diameter about half that of a comparable hypercube. The combined graph preserves the nice properties of the hypercube such as vertex-symmetry, bipartiteness, hamiltonian and bipancyclic properties. Also, the graph is highly edge-symmetric. We extend the result to a general scheme for interconnecting two m-dimensional hypercubes of every $m \geq 2$.

The organization of this paper is as follows. In the next section, we introduce some definitions and notation. In Section 3, we consider a network topology using two definitions of hypercube. In Section 4, we will give a construction scheme for interconnecting two hypercubes. Finally, in Section 5, concluding remarks of this paper will be given.

2. Preliminaries

We introduce definitions, notation, and some of hypercube variants concerned with this paper.

The usual definition of hypercube is as follows: An m-dimensional hypercube Q_m is an m-regular labeled graph on 2^m vertices. Each vertex u of Q_m is labeled by a distinct binary number $u_m u_{m-1} \cdots u_1$, and two vertices are connected by an edge if and only if their binary labels differ in exactly one bit position. An edge (u, v) of Q_m is called a *dimension-d edge* if the labels of u and v differ in dth bit position. The dimension d-edges in Q_m form a perfect matching for each d, $1 \leq d \leq m$. Moreover, removal of all the dimension d-edges in Q_m leaves two disjoint copies of Q_{m-1}. Conversely, Q_m can be constructed from two Q_{m-1}'s by adding a perfect matching.

For two vertices v and w in a graph, the *distance* between v and w is the length of the shortest path between v and w. The *diameter* of a graph G is the maximum distance between any two vertices of G.

Many interconnection networks can be constructed by connecting two lower dimensional networks. We represent the construction as follows. We are given two graphs G_0 and G_1 with n vertices each. We denote by V_i and E_i the vertex set and edge set of G_i, $i = 0, 1$, respectively. We let $V_0 = \{v_1, v_2, \ldots, v_n\}$ and $V_1 = \{w_1, w_2, \ldots, w_n\}$. With respect to a permutation $M = (i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$, we can "merge" the two graphs into a graph $G_0 \oplus_M G_1$ with $2n$ vertices in such a way that the vertex set $V = V_0 \cup V_1$ and the edge set $E = E_0 \cup E_1 \cup E_2$, where $E_2 = \{(v_j, w_{i_j}) : 1 \leq j \leq n\}$. We denote by $G_0 \oplus G_1$ a graph obtained by merging G_0 and G_1 w.r.t. an arbitrary permutation M. Obviously, E_2 is a perfect matching in $G_0 \oplus G_1$.

Vaidya *et al.* [27] introduced a class of hypercube-like interconnection networks, called *HL-graphs* which can be defined by applying the $\oplus$ operation repeatedly as follows: $HL_0 = \{K_1\}$; for $m \geq 1$, $HL_m = \{G_0 \oplus G_1 : G_0, G_1 \in HL_{m-1}\}$. Then, $HL_1 = \{K_2\}$; $HL_2 = \{C_4\}$; $HL_3 = \{Q_3, G(8,4)\}$. Here, C_4 is a cycle graph with 4 vertices, Q_3 is a 3-dimensional hypercube, and $G(8,4)$ is a recursive circulant which is isomorphic to twisted cube TQ_3 and Möbius ladder. An arbitrary graph which belongs to HL_m is called an *m-dimensional HL-graph*. Note that each graph in HL_m is m-regular and has 2^m vertices. Many well-known interconnection networks are members of HL-graphs including hypercube [17], twisted cube [12], twisted m-cube [9], crossed cube [7], Möbius cube [4], Mcube [25], multiply twisted cube [6], locally twisted cube [28], generalized twisted cube [3], and recursive circulant $G(2^m, 4)$ [20]. There have been many works on the fundamental properties of the above networks such as diameter, connectivity, hamiltonicity, and the embedding of other networks. Also, some literature [16, 19, 21–24] presented the generalized results on HL-graphs. Most of the interconnection networks in HL-graphs having smaller diameters than that of the hypercube are based on $G(8,4)$. That is, using $G(8,4)$ as a base case, those networks can be constructed inductively by the $\oplus$ operation specified in their definitions.

The m-dimensional folded hypercube, FH_m, is constructed by adding a complementary edge to Q_m between every pair of vertices with complementary labels. Due to the complementary edge, FH_m is an $(m+1)$-regular graph and its diameter is $\lceil m/2 \rceil$.

Two graphs G and H are *isomorphic* if there exists an isomorphism $f : V(G) \to V(H)$ such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H.

3. Interconnection of Two Even-Dimensional Hypercubes

In this section, we introduce an equivalent definition of hypercube in [13] that is different from the usual one given in Section 2. Then, we consider the symmetry of the graph constructed by combining two hypercubes obtained by mutually different definitions.

We can think of a graph with 2^m vertices whose vertices are labeled by m-bit binary numbers and two vertices are joined by an edge if their binary labels differ in exactly k bit positions. It becomes a regular graph of degree $\binom{m}{k}$. Of course, when $k = 1$, the graph is the very m-dimensional hypercube. For $k = m - 1$, the graph is m-regular, too. Such a graph was considered in [13] as follows:

DEFINITION 3.1 *An m-dimensional graph Q'_m is an m-regular labeled graph on 2^m vertices. Each vertex u of Q'_m is labeled by a distinct binary number $u_m u_{m-1} \cdots u_1$, and two vertices are connected by an edge if and only if their binary labels differ in exactly $m - 1$ bit positions.*

An edge (u, v) of Q'_m is called a *dimension-d edge* if the labels of u and v are the same in dth bit position. If m is an odd integer, every edge of Q'_m joins two vertices whose labels have the same parity. A binary label has *even parity* if it contains an even number of 1's and *odd parity* if it contains an odd number of 1's. Thus, the graph Q'_m for odd m is disconnected. It has two connected components of the same size. If m is an even integer, every edge of Q'_m joins two vertices whose labels are different in parity. The graph Q'_m is bipartite and its vertex set is partitioned into two partite sets: the set of vertices with even-parity labels and set of vertices with odd-parity labels.

Interestingly, it was shown in [13] that Q'_m is isomorphic to Q_m for every even m and Q'_m is isomorphic to two copies of FH_{m-1} for every odd m, $m \geq 2$. Although the isomorphic result on Q_m and Q'_m of even m is already known, we give a simple and

intuitive proof in the following for later use.

LEMMA 3.2 ([13]) Q'_m *is isomorphic to* Q_m *for any even* $m \geq 2$.

Proof. Let $f : V(Q'_m) \to V(Q_m)$ be a function such that

$$f(x) = \begin{cases} x' & \text{if the label of } x \text{ has even parity,} \\ x^c & \text{if the label of } x \text{ has odd parity,} \end{cases}$$

where x' is the vertex with the same label as x, and x^c is the vertex with a label complemented to the label of x. Obviously, f is a bijection. We claim that $(x, y) \in E(Q'_m)$ if and only if $(f(x), f(y)) \in E(Q_m)$. It suffices to show the 'only if' part since both graphs have the same number of edges. Let (x, y) be an edge of Q'_m and assume that the label of x has even parity and the label of y has odd parity. The fact that labels of x and y differ in $m - 1$ bit positions implies that labels of x' and y^c differ in one bit position. Thus, (x', y^c) is an edge of Q_m. This completes the proof. ∎

The graph with 2^m vertices obtained from Q_{m-1} and Q'_{m-1} by simply joining each pair of vertices of the same label via an edge, denoted by G_m, possesses interesting properties in symmetry and diameter.

DEFINITION 3.3   *For an odd integer* $m \geq 3$*, the graph* G_m *is defined as follows:* $V(G_m) = V(Q_{m-1}) \cup V(Q'_{m-1})$ *and* $E(G_m) = E(Q_{m-1}) \cup E(Q'_{m-1}) \cup E'$ *where* $E' = \{(u, v) : u \in V(Q_{m-1}), v \in V(Q'_{m-1})$, *and the labels of* u *and* v *are the same*$\}$.

We let dimension-d edges of G_m be the dimension-d edges of Q_{m-1} and Q'_{m-1}, $1 \leq d \leq m - 1$. And let dimension-m edges of G_m be the edges in E'. The label of each vertex in G_m is denoted by prefixing a bit 0 to each vertex label in Q_{m-1} and prefixing a bit 1 to each vertex label in Q'_{m-1} so that each vertex of G_m is labeled by a distinct m-bit binary number. The graph G_m is bipartite and consists of two partite sets, the set of vertices with even-parity labels and the set of vertices with odd-parity labels. Further, it is *equitable*, that is, the two partite sets are of the same size.

We say that a graph G is *vertex-symmetric* if any two vertices of G are *similar*, that is, if for any u, v in $V(G)$, there is an automorphism g of G such that $g(u) = v$. Informally speaking, every vertex of a vertex-symmetric graph looks exactly the same as every other vertex.

THEOREM 3.4 G_m *is vertex-symmetric for odd* $m \geq 3$.

Proof. Within this proof, we denote by $x_m x_{m-1} \cdots x_1$ the label of vertex x in G_m, and the label and the vertex itself are used interchangeably. Let u, v be any pair of vertices of G_m. We will show that there exists a bijection $g : V(G_m) \to V(G_m)$ such that $g(u) = v$ and for any edge (x, y) of G_m, $(g(x), g(y))$ is also an edge of G_m. Define an m-bit mask $M := M_m M_{m-1} \cdots M_1 = u$ XOR v, that is, $M = u_m u_{m-1} \cdots u_1$ XOR $v_m v_{m-1} \cdots v_1$. Here, XOR denotes the bitwise exclusive-OR operation. Notice that two labels x and y differ in k bit positions if and only if x XOR M and y XOR M differ in k bit positions. This is because $x_i = y_i$ if and only if x_i XOR $M_i = y_i$ XOR M_i. There are two cases depending on whether $u_m = v_m$ or not.

Case 1: $u_m = v_m$.
Remember $M_m = 0$ in this case. We define g as follows:

$$g(x) = x \text{ XOR } M.$$

It is straightforward to check that g is bijective and $g(u) = u$ XOR $M =$

u XOR (u XOR v) $= v$. Let (x, y) be an arbitrary edge of G_m. When (x, y) is an edge of subgraph Q_{m-1} (resp. Q'_{m-1}), we have $x_m = y_m = 0$ (resp. $x_m = y_m = 1$). Letting $g(x) = x'_m x'_{m-1} \cdots x'_1$ and $g(y) = y'_m y'_{m-1} \cdots y'_1$, we have $x'_m = y'_m = 0$ (resp. $x'_m = y'_m = 1$), and $x'_{m-1} \cdots x'_1$ and $y'_{m-1} \cdots y'_1$ differ in 1 (resp. $m-2$) bit position. Thus, $(g(x), g(y))$ is an edge of G_m. When (x, y) is a dimension-m edge, we have $x_m \neq y_m$ and $x_{m-1} x_{m-2} \cdots x_1 = y_{m-1} y_{m-2} \cdots y_1$. Then, $g(x)$ and $g(y)$ differ in mth bit position and are the same in all the other bit positions, thus $(g(x), g(y))$ is also a dimension-m edge of G_m.

Case 2: $u_m \neq v_m$.
In this case, $M_m = 1$. The function g is defined as follows:

$$g(x) = \begin{cases} x_m x_{m-1} x_{m-2} \cdots x_1 \text{ XOR } M \\ \quad \text{if } x_{m-1} x_{m-2} \cdots x_1 \text{ has the same parity as } u_{m-1} u_{m-2} \cdots u_1; \\ x_m \overline{x_{m-1} x_{m-2} \cdots x_1} \text{ XOR } M \quad \text{otherwise.} \end{cases}$$

Here, $\overline{x_{m-1} x_{m-2} \cdots x_1}$ denotes the complement of $x_{m-1} x_{m-2} \cdots x_1$. Obviously, g is bijective. It holds true that $g(u) = u$ XOR $M = u$ XOR (u XOR v) $= v$. Let (x, y) be an arbitrary edge of G_m. When $x_m = y_m$, we assume w.l.o.g. $x_{m-1} x_{m-2} \cdots x_1$ has the same parity as $u_{m-1} u_{m-2} \cdots u_1$. Of course, $y_{m-1} y_{m-2} \cdots y_1$ differs from $u_{m-1} u_{m-2} \cdots u_1$ in parity. If $x_m = y_m = 0$ (resp. $x_m = y_m = 1$), and $x_{m-1} x_{m-2} \cdots x_1$ and $y_{m-1} y_{m-2} \cdots y_1$ differ in 1 (resp. $m-2$) bit position, then $x_{m-1} x_{m-2} \cdots x_1$ and $\overline{y_{m-1} y_{m-2} \cdots y_1}$ differ in $m-2$ (resp. 1) bit positions. Then, we have $x'_m = y'_m = 1$ (resp. $x'_m = y'_m = 0$), and $x'_{m-1} \cdots x'_1$ and $y'_{m-1} \cdots y'_1$ differ in $m-2$ (resp. 1) bit positions, where $g(x) = x'_m x'_{m-1} \cdots x'_1$ and $g(y) = y'_m y'_{m-1} \cdots y'_1$. This implies $(g(x), g(y))$ is an edge of G_m. Finally, let $x_m \neq y_m$. Then, (x, y) is a dimension-m edge. We have $x_{m-1} x_{m-2} \cdots x_1 = y_{m-1} y_{m-2} \cdots y_1$. Whether $x_{m-1} x_{m-2} \cdots x_1$ has the same parity as $u_{m-1} u_{m-2} \cdots u_1$ or not, we can see that $x'_m \neq y'_m$ and $x'_{m-1} x'_{m-2} \cdots x'_1 = y'_{m-1} y'_{m-2} \cdots y'_1$. Thus, $(g(x), g(y))$ is an edge of G_m. This completes the proof. ∎

Remark 1   The automorphism g of G_m given in the proof of Theorem 3.4 preserves the partition $V(Q_{m-1})$ and $V(Q'_{m-1})$ of $V(G_m)$, that is, either $\{g(x) : x \in V(Q_{m-1})\} = V(Q_{m-1})$ and $\{g(y) : y \in V(Q'_{m-1})\} = V(Q'_{m-1})$ or $\{g(x) : x \in V(Q_{m-1})\} = V(Q'_{m-1})$ and $\{g(y) : y \in V(Q'_{m-1})\} = V(Q_{m-1})$.

Associated with any automorphism, there is an induced mapping on the edges obtained by considering the image of two end-vertices of any edge. Two edges (u, v) and (x, y) are *similar* if there exists an automorphism h of G such that maps (u, v) to (x, y), i.e., $h(u) = x$ & $h(v) = y$ or $h(u) = y$ & $h(v) = x$. Notice that the 'similarity' relation on the set of edges is an equivalence relation. The graph G_m keeps high degree of edge symmetry, precisely speaking, its edge set can be partitioned into two equivalence classes as follows.

THEOREM 3.5   *Any two dimension-m edges in G_m with odd $m \geq 3$ are similar, and any two edges which are not dimension-m edges are also similar.*

Proof. We first show that two dimension-m edges (u, v) and (a, b) are similar. Assume $u_m = 0$ and $a_m = 0$. Thus, $v_m = 1$ and $b_m = 1$. Letting an m-bit mask $M = u$ XOR a, we define automorphism h of G_m as follows:

$$h(x) = x \text{ XOR } M.$$

Then, $h(u) = u$ XOR $M = u$ XOR (u XOR a) $= a$. Furthermore, since $M = u$ XOR $a = v$ XOR b, we have $h(v) = v$ XOR $M = v$ XOR (v XOR b) $= b$. Here, $M_m = 0$. In exactly the

same way to Case 1 of the proof of Theorem 3.4, we can show that if (x, y) is an edge of G_m, then $(h(x), h(y))$ is also an edge of G_m.

Now, we are to show two edges (u, v) and (a, b) with $u_m = v_m = a_m = b_m = 0$ are similar. Here, (u, v) and (a, b) are edges of subgraph Q_{m-1}. We assume $u_i = 0$ and $v_i = 1$ for some $i < m$, and assume $a_j = 0$ and $b_j = 1$ for some $j < m$. Let $x^{i,j}$ be the m-bit binary number obtained from x by swapping ith bit and jth bit of x, possibly $i = j$ where $1 \leq i, j < m$. That is, $x^{i,j} = x_m \cdots x_{i+1} x_j x_{i-1} \cdots x_{j+1} x_i x_{j-1} \cdots x_1$. Define m-bit mask $M = u^{i,j}$ XOR a. Then, we have $M = u^{i,j}$ XOR a

$= u_m \cdots u_{i+1} u_j u_{i-1} \cdots u_{j+1} u_i u_{j-1} \cdots u_1$ XOR $a_m \cdots a_{i+1} a_i a_{i-1} \cdots a_{j+1} a_j a_{j-1} \cdots a_1$

$= u_m \cdots u_{i+1} u_j u_{i-1} \cdots u_{j+1} \overline{u_i} u_{j-1} \cdots u_1$ XOR $a_m \cdots a_{i+1} a_i a_{i-1} \cdots a_{j+1} \overline{a_j} a_{j-1} \cdots a_1$

$= v^{i,j}$ XOR b. An automorphism h of G_m is defined as follows:

$$h(x) = x^{i,j} \text{ XOR } M.$$

Then, we have $h(u) = u^{i,j}$ XOR $M = u^{i,j}$ XOR $(u^{i,j}$ XOR $a) = a$ and $h(v) = v^{i,j}$ XOR M $= v^{i,j}$ XOR $(v^{i,j}$ XOR $b) = b$. Observe that x and y differ in k bit positions if and only if $x^{i,j}$ and $y^{i,j}$ differ in k bit positions. Remembering the observation and $M_m = 0$, we can show in a very similar way to Case 1 of proof of Theorem 3.4 that for any edge (x, y) of G_m, $(h(x), h(y))$ is also an edge of G_m.

Recall that the 'similarity' relation on the set of edges is an equivalence relation. It suffices to show that for any edge (x, y) with $x_m = y_m = 1$, there is an edge of Q_{m-1} which is similar to (x, y). Let u be an arbitrary vertex with $u_m = 0$. By Theorem 3.4, there exists an automorphism g of G_m such that $g(x) = u$. Let $v = g(y)$. Since g is an automorphism, (u, v) is an edge of G_m. Furthermore, by Remark 1, v is also a vertex with $v_m = 0$. Thus, (x, y) is similar to the edge (u, v) of Q_{m-1}. This completes the proof. ∎

Remark 2   A graph is *edge-symmetric* if every pair of edges are similar. G_m is not edge-symmetric for all odd $m \geq 5$. There is a cycle of length 4 passing through an arbitrary dimension-p edge, $p < m$. We claim that there is no cycle of length 4 passing through a dimension-m edge (u, v). We assume $u \in V(Q_{m-1})$ and $v \in V(Q'_{m-1})$. The labels of u and any vertex $u' \in V(Q_{m-1})$ adjacent to u differ in 1 bit position, and the labels of v and any vertex $v' \in V(Q'_{m-1})$ adjacent v differ in $(m-1) - 1$ bit positions. Due to $m - 2 \neq 1$, u' is not adjacent to v' and thus there is no such cycle.

Among the diameters of m-dimensional HL-graphs, the diameter of Q_m, that is m, is the largest. But, to our surprise, the diameter of G_m is reduced to $(m + 1)/2$ which is about half that of the component graph Q_{m-1} (or Q'_{m-1}) for all odd $m \geq 7$. The diameter and other nice properties of G_m will be dealt with the generalized topology in the following section.

4.   Bicubes

The result in Section 3 suggests that two hypercubes can be combined to produce an interesting network topology. In this section, we present a scheme for interconnecting two (not necessarily even-dimensional) hypercubes and the properties of the resulting topology, called *bicube*.

We are to utilize the structure $Q_{m-1} \oplus Q'_{m-1}$ considered in Section 3. For convenience of description, we introduce some notation. Let $l(u)$ be the binary label of a hypercube vertex u and $c(u)$ be the complement of $l(u)$. We will let $p()$ be the parity function that maps binary labels of the hypercube vertices to 0 or 1. For a vertex u in Q_m with even m, $p(u)$ is defined to be 0 if the number of 1's in $l(u)$ is even, and 1 otherwise. We call

two vertices u and v are *lp-related* if either $l(u) = l(v)$ & $p(u) = p(v) = 0$ or $l(u) = c(v)$ & $p(u) = p(v) = 1$.

Interconnecting Scheme for Bicube BQ_m

/* two $(m-1)$-dimensional hypercubes are interconnected to produce BQ_m, $m \geq 3$ */

(1) Make two copies of Q_{m-1}, and let the copies be Q_{m-1}^0 and Q_{m-1}^1.

   Case 1: m odd.

   Connect each lp-related pair $u \in V(Q_{m-1}^0)$ and $v \in V(Q_{m-1}^1)$ by an edge.

   Case 2: m even.

   (a) Partition $V(Q_{m-1}^i)$ into $V(Q_{m-2}^{i0})$ and $V(Q_{m-2}^{i1})$; vertex $v \in V(Q_{m-1}^i)$ belongs to $V(Q_{m-2}^{ij})$ where j is the first bit of $l(v)$, $i, j \in \{0, 1\}$. Hereafter, exclude the first bit of every vertex label within this case.

   (b) Connect each lp-related pair $x \in V(Q_{m-2}^{00})$ and $y \in V(Q_{m-2}^{10})$ by an edge.

   (c) Connect each lp-related pair $w \in V(Q_{m-2}^{01})$ and $z \in V(Q_{m-2}^{11})$ by an edge.

(2) Prefix a bit 0 to each vertex label in Q_{m-1}^0, and prefix a bit 1 to each vertex label in Q_{m-1}^1 to produce BQ_m.

We let the dimension d-edges of the BQ_m be the dimension d-edges in two Q_{m-1}'s, $1 \leq d \leq m-1$, and let the dimension m-edge of the BQ_m be the edges connected between two Q_{m-1}'s. In Fig. 1, BQ_3, BQ_4, and BQ_5 are shown. As shown in Fig. 1, BQ_3 and BQ_4 are isomorphic to Q_3 and Q_4, respectively, but BQ_5 is not isomorphic to Q_5. In fact, BQ_m is not isomorphic to Q_m for every $m \geq 5$, which will be clear later in Theorem 4.8.

LEMMA 4.1   *BQ_m is isomorphic to G_m for odd $m \geq 3$.*

Proof. Similar to the proof of Lemma 3.2, if we complement all the vertex labels having odd parity (excluding the first bit) in the subgraph Q_{m-1}^1 of BQ_m, we can observe the labeled graph is same as G_m. ∎

We note that BQ_m with even m is isomorphic to $BQ_{m-1} \times K_2$. In the scheme, each of $Q_{m-2}^{00} \oplus Q_{m-2}^{10}$ and $Q_{m-2}^{01} \oplus Q_{m-2}^{11}$ is interconnected to be isomorphic to BQ_{m-1}. So, BQ_m with even m is of structure $BQ_{m-1} \oplus_I BQ_{m-1}$ where I is an identity permutation. Thus we have:

LEMMA 4.2   *BQ_m with even $m \geq 4$ is isomorphic to $BQ_{m-1} \times K_2$.*

We already know G_m of odd m is bipartite. By lemma 4.1 and Lemma 4.2 we have the following.

LEMMA 4.3   *BQ_m is bipartite for $m \geq 3$.*

In the next, we will discuss the symmetry and diameter of BQ_m.

THEOREM 4.4   *BQ_m is vertex-symmetric for $m \geq 3$.*

Proof. By Theorem 3.4 and Lemma 4.1, BQ_m with odd m is vertex-symmetric. BQ_m with even m is isomorphic to $BQ_{m-1} \times K_2$, which is vertex-symmetric obviously. ∎

THEOREM 4.5   *(a) In BQ_m with odd $m \geq 3$, any two dimension-m edges are similar, and any two edges which are not dimension-m edges are also similar.*
(b) In BQ_m with even $m \geq 4$, any two dimension-m edges are similar, any two dimension-$(m-1)$ edges are similar, and any two edges which are neither dimension-m edges nor dimension-$(m-1)$ edges are also similar.
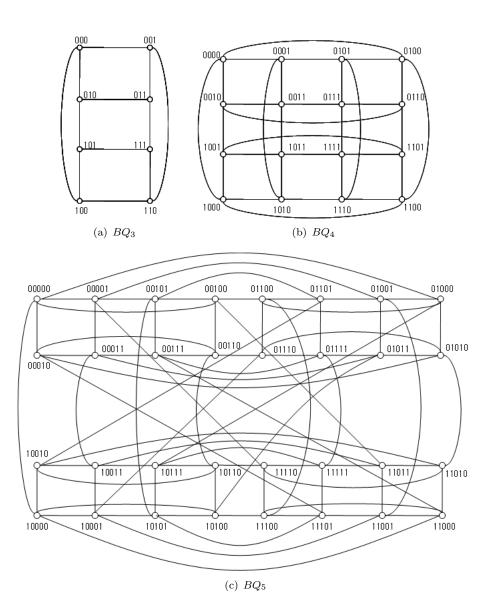
Figure 1. Illustration of the interconnecting scheme for BQ_m

Proof. Due to Theorem 3.5 and Lemma 4.1, the statement (a) holds true. To prove (b), we let m be even and denote by BQ^0_{m-1} (resp. BQ^1_{m-1}) the subgraph of BQ_m induced by vertices in Q^{00}_{m-2} or Q^{10}_{m-2} (resp. Q^{01}_{m-2} or Q^{11}_{m-2}). Since BQ_m is isomorphic to $BQ_{m-1} \times K_2$, for each edge in BQ^1_{m-1} there exists an edge in BQ^0_{m-1} similar to the edge, and vice versa. Recall that the 'similarity' relation on the set of edges is an equivalence relation.

We first show that any two dimension-m edges are similar. Two dimension-$(m-1)$ edges in BQ^0_{m-1} are similar within BQ^0_{m-1} by Theorem 3.5. It is not difficult to see that they are also similar in BQ_m. Moreover, for any dimension-$(m-1)$ edge in BQ^1_{m-1}, there is a dimension-$(m-1)$ edge in BQ^0_{m-1} which is similar to the edge. Thus, dimension-$(m-1)$ edges in BQ^0_{m-1} or BQ^1_{m-1}, which comprise dimension-m edges in BQ_m, are similar each other. In the same way, we can show that any two edges which are neither dimension-m edges nor dimension-$(m-1)$ edges are similar. The proof is omitted here.

It remains to show that any two dimension-$(m-1)$ edges (u, v) and (x, y) are similar. We assume that u and x are contained in BQ^0_{m-1}, and v and y are contained in BQ^1_{m-1}.

Since BQ_{m-1} is vertex-symmetric, there is an automorphism g_0 of BQ_{m-1}^0 mapping u to x. Furthermore, there exists an automorphism g_1 of BQ_{m-1}^1 such that $g_1(w) = z$ if and only if $g_0(w') = z'$, where w' and z' are the vertices in BQ_{m-1}^0 adjacent to w and z, respectively. Thus, an automorphism g of BQ_m can be defined such that $g(w) = g_0(w)$ for w in BQ_{m-1}^0 and $g(w) = g_1(w)$ for w in BQ_{m-1}^1. Then, we have $g(u) = x$ and $g(v) = y$. Therefore, (u, v) and (x, y) are similar in BQ_m. This completes the proof.   ∎

Remark 3   In BQ_m with even $m \geq 6$, a dimension-$(m-1)$ edge is not similar to any dimension-k edge with $k < m - 1$. The number of length 4 cycles passing through a dimension-$(m-1)$ edge is $m - 1$ while that number for a dimension-k edge is $m - 2$ for any $k < m - 1$.

LEMMA 4.6 BQ_m *of odd* $m \geq 3$ *is a spanning subgraph of* m-*dimensional folded hyper-cube*, FH_m.

Proof. In FH_m, there are two sets of edges E_1 and E_2 between two copies of Q_{m-1} where E_1 is a set of edges joining each pair of vertices of the same label and E_2 is a set of complementary edges. In BQ_m of odd m, any edge between two Q_{m-1}'s either joins two vertices of the same label, or joins two vertices with complementary labels. So, the lemma holds.   ∎

Now we consider the diameter of BQ_m. Let 0^k represent a series of k 0's. In the next, the label of a vertex and the vertex itself are used interchangeably. By virtue of the vertex-symmetry of BQ_m, it suffices to consider the distance between 0^m and other vertices. For two vertices v and w in BQ_m, the distance between v and w is denoted by $d(v, w)$, and the Hamming distance between v and w is denoted by $d_H(v, w)$.

LEMMA 4.7   *For* $v = 0^m$ *and* w *with* $d_H(v, w) = k$ *in* BQ_m *of odd* $m \geq 3$, *the following distance expressions hold:*
(a) For $k = m$, $d(v, w) = 3$.
(b) For $k = m - 1$ *and* $w \neq 01^{m-1}$, $d(v, w) = 2$.
(c) For $k = m - 1$ *and* $w = 01^{m-1}$, $d(v, w) = \min\{k, 4\}$.
(d) For $k \leq m - 2$, $d(v, w) = \min\{k, m - k + 1\}$.

Proof. Proof for (a). The label of w is 1^m. The path $(0^m, 0^{m-1}1, 1^{m-1}0, 1^m)$ of length 3 is the shortest path between v and w.
Proof for (b). When $k = m - 1$ and $w \neq 01^{m-1}$, then w is a vertex of Q_{m-1}^1. Therefore, the label of w is 1^p01^q for some p, q such that $p \geq 1$ and $p + q = m - 1$. The path $(0^m, 0^p10^q, 1^p01^q)$ of length 2 is the shortest path between v and w.
Proof for (c). Since $d_H(v, w) = k$, it is obvious that there is a path of length k between v and w. The distance between v and w is k when $k \leq 4$. There is a path $(0^m, 0^{m-1}1, 1^{m-1}0, 1^m, 01^{m-1})$ of length 4, which is the shortest path between v and w when $k \geq 5$.
Proof for (d). If $m = 3$, then $k \leq 1$; if $m = 5$, then $k \leq 3$. Thus, if $m = 3$ or $m = 5$, it is easy to see that $d(v, w) = k$ and $k \leq m-k+1$. We consider m with $m \geq 7$. For each i with $0 \leq i \leq m - 1$, let $S_i^0 = \{u \in V(Q_{m-1}^0) : d_H(0^m, u) = i\}$, and let $S_i^1 = \{u \in V(Q_{m-1}^1) : d_H(10^{m-1}, u) = i\}$. Let $V_i(v) = \{u : d(v, u) = i\}$. That is, $V_i(v)$ is the set of vertices which are at distance i from v. Note that for each $a \in S_i^0$ (resp. S_i^1), if i is even there is an edge (a, b) such that b is in S_i^1 (resp. S_i^0); otherwise there is an edge (a, b) such that b is in S_{m-i-1}^1 (resp. S_{m-i-1}^0). In Q_{m-1}^0 (resp. Q_{m-1}^1), the set of vertices adjacent to vertices in S_i^0 (resp. S_i^1) is $S_{i-1}^0 \cup S_{i+1}^0$ (resp. $S_{i-1}^1 \cup S_{i+1}^1$) for each i with $1 \leq i \leq m - 2$.
For $1 \leq i \leq 4$, $V_i(v)$ is obtained as follows:

$$V_1(v) = S_1^0 \cup S_0^1,$$

$V_2(v) = S_2^0 \cup S_1^1 \cup S_{m-2}^1$,
$V_3(v) = S_3^0 \cup S_{m-2}^0 \cup S_2^1 \cup S_{m-3}^1 \cup S_{m-1}^1$, and
$V_4(v) = S_4^0 \cup S_{m-3}^0 \cup S_{m-1}^0 \cup S_3^1 \cup S_{m-4}^1$.

We first consider the case of $m = 7$. In this case, we observe that $V_1(v), V_2(v), V_3(v)$, and $V_4(v)$ are mutually disjoint, and $\{v\} \cup V_1(v) \cup V_2(v) \cup V_3(v) \cup V_4(v)$ is equal to the set of vertices in BQ_m. From the condition of $k \leq m - 2$, we have $k \leq 5$. Since $d_H(v, w) = k$, w is in S_k^0 or S_{k-1}^1. If $k \leq 4$, w is in $V_k(v)$ from the fact that each of S_k^0 and S_{k-1}^1 is a subset of $V_k(v)$. Therefore we have $d(v, w) = k$. Note that $m - k + 1 = 8 - k \geq k$. If $k = 5$, then w is in $S_5^0 (= S_{m-2}^0)$ or $S_4^1 (= S_{m-3}^1)$. Since each of S_5^0 and S_4^1 is a subset of $V_3(v)$, we have $d(v, w) = 3$, which is equal to $m - k + 1$.

We next consider the case of $m \geq 9$. In this case, we claim that for each n with $5 \leq n \leq (m+1)/2$, $V_n(v) = S_n^0 \cup S_{m-n+1}^0 \cup S_{n-1}^1 \cup S_{m-n}^1$. We prove the claim by induction on n. If $n = 5$, it is easy to see that the claim holds. Assuming that for each n with $5 \leq n < p \leq (m+1)/2$ the claim holds, we will show that the claim holds for $n = p$. Let $T = \{u : d(v, u) \geq p\}$. Then from the assumption, $T = \bigcup_{i=p}^{m-p+1} S_i^0 \cup \bigcup_{i=p-1}^{m-p} S_i^1$. Since $V_p(v) = \{u \in T : u \text{ is adjacent to a vertex in } V_{p-1}(v)\}$ and $V_{p-1}(v) = S_{p-1}^0 \cup S_{m-p+2}^0 \cup S_{p-2}^1 \cup S_{m-p+1}^1$, we have $V_p(v) = S_p^0 \cup S_{m-p+1}^0 \cup S_{p-1}^1 \cup S_{m-p}^1$. Thus, the claim is proved.

By using the claim, we will show $d(v, w) = \min\{k, m - k + 1\}$. Note that $w \notin S_{m-1}^0 \cup S_{m-1}^1 \cup S_{m-2}^1$ since $k \leq m - 2$. If $k \leq 4$, as in the case of $m = 7$, it can be shown that $d(v, w) = k$, which is less than or equal to $m - k + 1$. From now on, it is assumed that $k \geq 5$. If w is a vertex of Q_{m-1}^0, then w is in S_k^0. From the claim, if $k \leq m - k + 1$ (that is, $k \leq (m+1)/2$), w is in $V_k(v)$ since S_k^0 is a subset of $V_k(v)$, which means $d(v, w) = k$; otherwise, S_k^0 is a subset of $V_p(v)$ such that $m - p + 1 = k$, which means $d(v, w) = p = m - k + 1$. If w is a vertex of Q_{m-1}^1, w is in S_{k-1}^1. From the claim, if $k - 1 \leq m - k$ (that is, $k \leq (m+1)/2$), $d(v, w) = k$ since S_{k-1}^1 is a subset of $V_k(v)$; otherwise, S_{k-1}^1 is a subset of $V_p(v)$ such that $m - p = k - 1$, which implies $d(v, w) = p = m - k + 1$. The proof is completed. ∎

THEOREM 4.8   *The diameter of BQ_m is $\lceil (m+1)/2 \rceil$ for $m \geq 7$. The diameters of BQ_3, BQ_4, BQ_5 and BQ_6 are 3, 4, 4 and 5, respectively.*

Proof. BQ_3 and BQ_4 are isomorphic to Q_3 and Q_4, respectively. And, it is easy to check the diameter of BQ_5 is 4. For BQ_m of $m \geq 7$ and odd, the maximum value in distance expression $\min\{k, m - k + 1\}$ in Lemma 4.7 is obtained when $k = (m+1)/2$. For BQ_m of $m \geq 6$ and even, the diameter of BQ_m is one more than the diameter of BQ_{m-1} from Lemma 4.2. Together with the above facts, the theorem holds. ∎

We note that the diameter of BQ_m is the same as that of FH_m for odd $m \geq 7$, even though BQ_m is a spanning subgraph of FH_m and the degree of BQ_m is one less than that of FH_m.

Next, we will consider hamiltonicity and pancyclicity of the BQ_m. Linear arrays and rings are two of the most important computational structures in interconnection networks. The embedding of linear arrays and rings into interconnection network has been studied in terms of paths and cycles in graphs. A graph is called *hamiltonian* if it has a cycle which contains all the vertices. An *s-t hamiltonian path* is a path which contains all the vertices joining a pair of vertices s and t. If a graph has an *s-t* hamiltonian path for all pairs of vertices s and t, the graph is called *hamiltonian-connected*. A bipartite graph is called *hamiltonian-laceable* if there is an *s-t* hamiltonian path for all pairs of vertices s and t, where s belongs to one set of the bipartition, and t belongs to the other. A graph G is called *pancyclic* if it contains a cycle of length l for every l such that $3 \leq l \leq |V(G)|$. And a graph G is called *bipancyclic* if it contains a cycle of length l for every even l,

$4 \le l \le |V(G)|$. Necessarily, a hamiltonian-connected graph and pancyclic graph is not bipartite. The hamiltonicity and pancyclicity of HL-graphs were considered in [19] as follows. For more discussion on hamiltonicity and its generalization, refer to [15, 18].

THEOREM 4.9 ([19])   *Every m-dimensional bipartite HL-graph is hamiltonian-laceable, and bipancyclic for $m \ge 2$.*

It is obvious that BQ_m is a bipartite HL-graph, so the lemma follows:

LEMMA 4.10   *BQ_m is hamiltonian-laceable and bipancyclic for $m \ge 3$.*

5.   Concluding remarks

In this paper, we considered two definitions of hypercube and their application to interconnecting hypercubes. It was shown that we can obtain a graph with diameter about half that of a comparable hypercube by just interconnecting two hypercubes. Also, the graph preserves desirable properties of the hypercube. Based on the above observation, we presented an interconnection network topology called the bicube. The diameter, vertex-symmetry, edge-symmetry, hamiltonicity, and bipancyclicity of the bicube are investigated. In further research of the bicube, we expect other nice properties can be found and many algorithms developed for hypercubes can be easily adapted to bicubes.

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