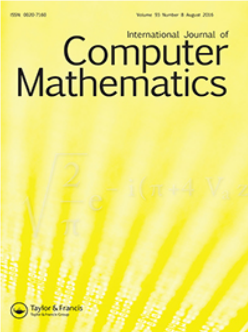

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Sufficient conditions for the existence of periodic solutions of the extended Duffing–Van der Pol oscillator

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In this paper, some aspects on the periodic solutions of the extended Duffing–Van der Pol oscillator are discussed. Doing different rescaling of the variables and parameters of the system associated with the extended Duffing–Van der Pol oscillator, we show that it can bifurcate one or three periodic solutions from a two-dimensional manifold filled by periodic solutions of the referred system. For each rescaling we exhibit concrete values for which these bounds are reached. Beyond that we characterize the stability of some periodic solutions. Our approach is analytical and the results are obtained using the averaging theory and some algebraic techniques.

Keywords: extended Duffing–Van der Pol oscillator; periodic solution; non-autonomous systems; averaging theory

2010 AMS Subject Classifications: Primary: 34C07; 34C15; 34C25; 34C29; 37C60

1. Introduction

1.1 *Setting the problem*

A large number of non-autonomous chaotic phenomena in physics, engineering, mechanics and biology, among others, are described by second-order differential systems of the form

$$\ddot{x} = g(x, \dot{x}, t) + \gamma(t), \tag{1}$$

where $g(x, \dot{x}, t)$ is a continuous function and $\gamma(t)$ is some external force. For instance, in biology, system (1) models the FHN neuron oscillator, and in engineering, system (1) is a model to the horizontal platform system.

The specific topic addressed in this paper concerns another particular case of Equation (1), namely, an extension of the forced Van der Pol equation with external excitation. Van der Pol's system also plays an important role in many applications in areas such as engineering, biology, physics and seismology (see [2] and references therein).

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The system associated with the forced Van der Pol equation with external excitation is characterized by system (1) with functions g and γ in the form

$$g(x, \dot{x}, t) = \rho_0(1 - x^2)\dot{x} - \frac{\mathrm{d}}{\mathrm{d}x}V(x), \quad \gamma(t) = \delta_0 \cos(\omega t), \tag{2}$$

where ρ_0 is the damping parameter, $V(x)$ is the potential function, and δ_0 and ω are the amplitude and angular frequency of the driving force $\gamma(t)$, respectively. We assume that ρ_0 is non-negative and δ_0 and ω are positive. The potential $V(x)$ can be approximated by a finite Taylor expansion in series

$$V_2(x) = \tfrac{1}{2}\omega_0^2 x^2, \quad V_4(x) = \tfrac{1}{2}\omega_0^2 x^2 + \tfrac{1}{4}\alpha_0 x^4, \quad V_6(x) = \tfrac{1}{2}\omega_0^2 x^2 + \tfrac{1}{4}\alpha_0 x^4 + \tfrac{1}{6}\lambda_0 x^6,$$

where ω_0 and λ_0 are non-zero and α_0 is a real number.

Almost all papers on forced excited Van der Pol systems deal with the potential V_2. However, some papers concerning the potential V_4 have shown a lot of interesting behaviours [7,12,25,27,28]. This case is usually referred to as Duffing–Van der Pol oscillator or ϕ^4-Van der Pol oscillator. Nevertheless, more recently, some papers were published taking into account the potential V_6 meanly addressing the problem of chaos control [13,17,22]. The dynamics considering the potential V_6 is more complex and rich than the corresponding cases considering the potentials V_2 and V_4 [24,26]. This is the case that we will regard. This case is quoted in the literature as the extended Duffing–Van der Pol oscillator or ϕ^6-Van der Pol oscillator.

In this paper, we will give an analytical treatment to system (1) in order to study its periodic solutions considering functions (2) and the potential V_6. Indeed, calling $y = \dot{x}$ we obtain

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\omega_0^2 x + \rho_0 y - \alpha_0 x^3 - \rho_0 x^2 y - \lambda_0 x^5 + \delta_0 \cos(\omega t). \end{aligned} \tag{3}$$

System (3) becomes simpler if we perform a rescaling $s = \omega t$ in the time t and another one $y = \omega_0 Y$ in the spatial variable y. In fact, calling again the new time s by t and the variable Y by y, after the rescaling, we have

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \rho y - \alpha x^3 - \rho x^2 y - \lambda x^5 + \delta \cos t, \end{aligned} \tag{4}$$

where the new parameters ρ, α, λ and δ are the respective old ones divided by ω_0^2. The study of the periodic solutions of the non-linear non-autonomous 2π-periodic differential system (4) will be the objective of this paper.

When α, λ and δ are zero, Equation (4) is referred to as unforced Van der Pol equation and has a unique stable periodic solution for ρ positive. Furthermore, if ρ is large, this periodic solution remains, and it describes a periodic oscillatory behaviour called relaxation oscillation. On the other hand, by considering δ non-zero, system (4) can present non-linear attractors from simple periodic solutions up to multi-periodicity and chaos. In [10], we can find a didactical discussion about these objects and some results using classical techniques are obtained for a special case of system (4) using particular values of ρ, δ, ω and initial conditions.

There exists an exhaustive list of papers in the literature studying the properties of system (4) when δ is zero. For δ positive, many open questions remain mainly due to the difficulty of integrating the system. In particular, in [19], Ma *et al.* study some aspects of robust practical synchronization for system (1) and apply the results to a particular case of system (4). In the same direction, in [14], Leung investigates synchronization processes between chaotic attractors considering $\alpha = \lambda = 0$ in system (4). In [2,11], many aspects of system (4) for the case where α and λ are zero and ρ is large are studied. Furthermore, in [8], Egami and Hirano provide sufficient conditions to the existence of one periodic solution under some analytical hypotheses for a general forced Van der Pol system considering ρ equal to zero.

Periodic solutions for system (4) were found in [17], where Liu and Yamaura investigate chaos control. Besides, in [5,6], results on the existence of periodic solutions for an autonomous special case of the extended Duffing–Van der Pol oscillator were obtained. In [16,30,31], we can also find some results on systems similar to system (4).

In this paper, we are concerned with periodic solutions of system (4). We will present sufficient conditions in order that this system possesses one or three periodic solutions, and we will provide conditions on the parameters ρ, α, λ and δ for which these bounds are realizable. In addition, we will prove that it is not possible to obtain different bounds of periodic solutions using the methodology presented in this paper.

The phase space of our non-autonomous differential system is (x, y, t) and its Poincaré map is defined in the (x, y)-space. When possible the stability of the periodic solutions will be studied. As usual a periodic solution is *stable* if the eigenvalues of the fixed point associated with its Poincaré map have a negative real part; otherwise, the periodic solution is *unstable*. Inside the unstable periodic solutions there are two types: the unstable saddle periodic solutions having an eigenvalue with a negative real part and the other with a positive real part, and the repeller with both eigenvalues having a negative real part.

We should note that the method used here for studying periodic solutions can be applied to any periodic non-autonomous differential system as done in [9,18]. In these papers, the authors applied the method used in this paper in order to guarantee the existence of periodic solutions in a periodic FitzHugh–Nagumo system and in the Vallis system, respectively.

The paper is organized as follows. In Section 1.2, the main results are stated and compared with other results. Also, some important points on the results are clarified. In Section 2.1, we prove the results. In Section 2.2, some aspects of the results are pointed out. Lastly, Section 3 is devoted to give a brief summary of the results that we use from the averaging theory.

1.2 *Statement of the main results*

In this section, we present our results. We will behave under an analytical approach and for this reason the results are valid for a large range of the parameters of system (4), different from the major part of the results dealing with numerical techniques. In addition, it is important to note that we are concerned with harmonic solutions in the sense that they do not bifurcate from periodic solutions with multiple periods.

We have the following results.

THEOREM 1 *Consider $\varepsilon > 0$ sufficiently small, $(\rho, \delta, \alpha, \lambda) = (\varepsilon r, \varepsilon d, \varepsilon^{n_2} a, \varepsilon^{n_3} \ell)$ with $n_2, n_3 > 1$ and $81d^2 > 48r^2 > 0$. Then system (4) has a 2π-periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ such that*

$$(x(0,\varepsilon), y(0,\varepsilon)) \to \left(0, \frac{2(36r)^{1/3}}{3\Gamma} + \frac{1}{3}\left(\frac{6}{r}\right)^{1/3}\Gamma\right),$$

when $\varepsilon \to 0$, where $\Gamma = (9d + \sqrt{81d^2 - 48r^2})^{1/3}$. Moreover, for r sufficiently small, this periodic solution is stable.

THEOREM 2 *Consider $\varepsilon > 0$ sufficiently small, $(\alpha, \delta, \rho, \lambda) = (\varepsilon a, \varepsilon d, \varepsilon^{n_1} r, \varepsilon^{n_3} \ell)$ with $n_1, n_3 > 1$ and $\alpha \neq 0$. Then system* (4) *has a 2π-periodic solution $(x(t, \varepsilon), y(t, \varepsilon))$ such that*

$$(x(0,\varepsilon), y(0,\varepsilon)) \to \left(\left(\frac{4d}{3a}\right)^{1/3}, 0\right),$$

when $\varepsilon \to 0$.

THEOREM 3 *Consider $\varepsilon > 0$ sufficiently small and $(\lambda, \delta, \rho, \alpha) = (\varepsilon \ell, \varepsilon d, \varepsilon^{n_1} r, \varepsilon^{n_2} a)$ with $n_1, n_2 > 1$. Then system* (4) *has a 2π-periodic solution $(x(t, \varepsilon), y(t, \varepsilon))$ such that*

$$(x(0,\varepsilon), y(0,\varepsilon)) \to \left(\left(\frac{8d}{5\ell}\right)^{1/5}, 0\right),$$

when $\varepsilon \to 0$.

We remark that the stability of the periodic solutions of Theorems 2 and 3 cannot be decided with the real part of the eigenvalues of their Poincaré map because these real parts are zero.

THEOREM 4 *Consider $\varepsilon > 0$ sufficiently small and $(x, y, \rho, \delta, \alpha, \lambda) = (\varepsilon X, \varepsilon Y, \varepsilon r, \varepsilon^2 d, \varepsilon^{n_2} a, \varepsilon^{n_3} \ell)$ with $\rho \neq 0$. Then system* (4) *has a 2π-periodic solution $(x(t, \varepsilon), y(t, \varepsilon))$ such that*

$$(x(0,\varepsilon), y(0,\varepsilon)) \to \left(O(\varepsilon), -\frac{\delta}{\varepsilon\rho} + O(\varepsilon)\right),$$

when $\varepsilon \to 0$. Moreover, this periodic solution is unstable.

Note that in Theorem 4 we have a periodic solution that comes from infinity. As far as we know, this kind of behaviour also has not been observed in the papers concerned with system (4). Nevertheless, this behaviour is common when we perform a rescaling in the parameters and variables as we made previously and can be observed also in [9,18].

THEOREM 5 *Consider $\varepsilon > 0$ sufficiently small, $(\alpha, \lambda, \delta, \rho) = (\varepsilon a, \varepsilon \ell, \varepsilon d, \varepsilon^{n_1} r)$ with $n_1 > 1$, $\alpha\lambda < 0$ and*

$$D = \frac{53747712 a^5 d^2}{78125 \ell^7} + \frac{20480 d^4}{\ell^4}.$$

Then system (4) *has a 2π-periodic solution if $D = 0$ and three 2π-periodic solutions if $D < 0$. Moreover, there are values of α, λ and δ such that for all ρ the system realizes the number of these periodic solutions.*

THEOREM 6 *Consider $\varepsilon > 0$ sufficiently small, $(\rho, \alpha, \delta, \lambda) = (\varepsilon r, \varepsilon a, \varepsilon d, \varepsilon^{n_3} \ell)$ with $n_3 > 1$ and $-324a^4r^2 - 9a^2d^2r^2 + 36a^2r^4 - d^2r^4 \neq 0$. Consider also the values*

$$\Delta_1 = 324a^4 + d^2r^2 + 9a^2(d^2 - 4r^2),$$

$$\Delta_2 = 2187a^4d^4 + 27d^4r^4 - 16d^2r^6 + 18a^2(27d^4r^2 - 72d^2r^4 + 32r^6).$$

The system

$$\begin{aligned} rx_0(-4 + x_0^2 + y_0^2) - 3ay_0(x_0^2 + y_0^2) &= 0, \\ 4d - ry_0(-4 + x_0^2 + y_0^2) - 3ax_0(x_0^2 + y_0^2) &= 0 \end{aligned}$$

has one solution (x_0^0, y_0^0) *if* $\Delta_1\Delta_2 = 0$ *or* $\Delta_2 > 0$, *and three solutions* (x_0^i, y_0^i) *if* $\Delta_2 < 0$ *for* $i = 1, 2, 3$; *we assume that all of them satisfy*

$$27\alpha^2((x_0^i)^2 + (y_0^i)^2)^2 + \rho^2(-4 + (x_0^i)^2 + (y_0^i)^2)(-4 + 3(x_0^i)^2 + 3(y_0^i)^2) \neq 0.$$

Then if $\Delta_1\Delta_2 = 0$ *or* $\Delta_2 > 0$, *system* (4) *has a* 2π*-periodic solution* $(x^0(t,\varepsilon),\ y^0(t,\varepsilon))$ *such that* $(x^0(0,\varepsilon), y^0(0,\varepsilon)) \to (x_0^0, y_0^0)$ *when* $\varepsilon \to 0$. *Additionally if* $\Delta_2 < 0$ *system* (4) *has three* 2π*-periodic solutions* $(x^i(t,\varepsilon), y^i(t,\varepsilon))$ *such that* $(x^i(0,\varepsilon), y^i(0,\varepsilon)) \to (x_0^i, y_0^i)$ *when* $\varepsilon \to 0$ *for* $i = 1, 2, 3$. *Moreover, there are values of* ρ, α *and* δ *such that for all* λ *the system realizes the number of these periodic solutions.*

THEOREM 7 *Consider* $\varepsilon > 0$ *sufficiently small,* $(\alpha, \rho, \lambda, \delta) = (\varepsilon a, \varepsilon r, \varepsilon\ell, \varepsilon d)$, $\alpha\rho \neq 0$, $(r^2 - 3a^2)(r^2 - 9a^2) \neq 0$ *and*

$$\begin{aligned} C &= 4(3a + 10\ell)(540a^3\ell - 9a^2(d^2 - 600\ell^2) - 120a\ell(d^2 - 150\ell^2) \\ &\quad + 50\ell^2(-7d^2 + 400\ell^2))r^6 + 6(a + 5\ell)(6a - d + 20\ell)(6a + d + 20\ell)r^8 \neq 0, \\ D &= 2066242608a^{14}d^6 - 3125d^8r^{12} - 531441a^{12}(3125d^8 + 96d^6r^2 + 1536d^4r^4 - 1024d^2r^6) \\ &\quad + 354294a^{10}(3125d^8r^2 + 616d^6r^4 - 1600d^4r^6) \\ &\quad + 18a^2r^{10}(9375d^8 + 5000d^6r^2 - 88000d^4r^4 + 102400d^2r^6 - 32768r^8) \\ &\quad + 2916a^6r^6(15625d^8 + 11700d^6r^2 - 45632d^4r^4 + 23552d^2r^6 - 2048r^8) \\ &\quad - 6561a^8r^4(46875d^8 + 24832d^6r^2 - 66816d^4r^4 + 12288d^2r^6 + 4096r^8) \\ &\quad + 81a^4r^8(-46875d^8 - 36000d^6r^2 + 275200d^4r^4 - 246784d^2r^6 + 61440r^8). \end{aligned}$$

The system

$$\begin{aligned} 2rx_0(-4 + x_0^2 + y_0^2) - y_0(x_0^2 + y_0^2)(6a + 5\ell(x_0^2 + y_0^2)) &= 0, \\ 8d + 8ry_0 - (x_0^2 + y_0^2)(6ax_0 + 2ry_0 + 5\ell x_0(x_0^2 + y_0^2)) &= 0 \end{aligned}$$

has one solution (x_0^0, y_0^0) *if* $D < 0$, *and three solutions* (x_0^i, y_0^i) *if* $D > 0$ *for* $i = 1, 2, 3$; *we assume that all of them satisfy*

$$\begin{aligned} &4\rho^2(-4 + (x_0^i)^2 + (y_0^i)^2)(-4 + 3(x_0^i)^2 + 3(y_0^i)^2) \\ &\quad + ((x_0^i)^2 + (y_0^i)^2)^2(6\alpha + \rho\lambda((x_0^i)^2 + (y_0^i)^2))(18\alpha + 25\lambda(x_0^i)^2 + 3(y_0^i)^2) \neq 0. \end{aligned}$$

Then, if $D < 0$ *system* (4) *has a* 2π*-periodic solution* $(x^0(t,\varepsilon), y^0(t,\varepsilon))$ *such that* $(x^0(0,\varepsilon), y^0(0,\varepsilon)) \to (x_0^0, y_0^0)$ *when* $\varepsilon \to 0$. *Furthermore, if* $D > 0$ *system* (4) *has three* 2π*-periodic solutions* $(x^i(t,\varepsilon), y^i(t,\varepsilon))$ *such that* $(x^i(0,\varepsilon),\ y^i(0,\varepsilon)) \to (x_0^i, y_0^i)$ *when* $\varepsilon \to 0$ *for* $i = 1, 2, 3$. *Moreover, there are values of* α, ρ, λ *and* δ *for which the system realizes the number of these periodic solutions.*

THEOREM 8 *Consider $\varepsilon > 0$ sufficiently small, $(\rho, \lambda, \delta, \alpha) = (\varepsilon r, \varepsilon \ell, \varepsilon d, \varepsilon^{n_2} a)$ with $n_2 > 1$. Consider the numbers*

$$\begin{aligned}
C &= 4000d^3\ell^3r^4 + 7200000d\ell^5r^4 + 60d^3\ell r^6 + 72000d\ell^3r^6 + (50d^4\ell^2r^4 - 220000d^2\ell^4r^4 \\
&\quad - 8000000\ell^6r^4 - 2800d^2\ell^2r^6 + 160000\ell^4r^6 - 6d^2r^8 + 2400\ell^2r^8), \\
N_2 &= \frac{-4500\ell^2r^4 + 3r^6}{3125d^2\ell^4}, \\
N_3 &= 5859375d^4\ell^6 + 18750000d^2\ell^6r^2 + 87500\ell^4(-7d^2 + 1200\ell^2)r^4 \\
&\quad + 25\ell^2(-7d^2 + 155600\ell^2)r^6 + 15600\ell^2r^8 + 9r^{10}, \\
N_4 &= 1171875d^8\ell^6(1250000\ell^4 + 8500\ell^2r^2 + r^4) - 2500d^6\ell^4(312500000000\ell^8 \\
&\quad + 5343750000\ell^6r^2 + 74625000\ell^4r^4 + 358375\ell^2r^6 + 69r^8) \\
&\quad - 3000d^4\ell^2r^2(218750000000\ell^{10} - 21625000000\ell^8r^2 - 686625000\ell^6r^4 \\
&\quad - 3715000\ell^4r^6 - 650\ell^2r^8 - 3r^{10}) - 1600\ell^2r^6(400000000000\ell^{10} \\
&\quad - 144900000000\ell^8r^2 - 2817000000\ell^6r^4 - 8880000\ell^4r^6 + 38700\ell^2r^8 \\
&\quad + 27r^{10}) + 12d^2r^4(-50000000000000\ell^{12} - 32250000000000\ell^{10}r^2 \\
&\quad - 662400000000\ell^8r^4 - 2803000000\ell^6r^6 + 5410000\ell^4r^8 + 19200\ell^2r^{10} + 9r^{12}) \\
N_5 &= 48828125d^8\ell^6 - 4687500d^6\ell^4r^4 + 6400\ell^2r^{10}(1600\ell^2 + r^2) - 16d^2r^8 \\
&\quad (2000000\ell^4 + 2900\ell^2r^2 + r^4) + d^4r^6(27500000\ell^4 + 60000\ell^2r^2 + 27r^4) \\
M_5 &= -25d^4\ell 2 + d^2(110000\ell 4 + 1400\ell 2r^2 + 3r^4) + 400(10000\ell 6 - 200\ell^4r^2 - 3\ell^2r^4).
\end{aligned}$$

We assume that $CM_5 \neq 0$. The system

$$\begin{aligned}
2rx_0(-4 + x_0^2 + y_0^2) - 5\ell y_0(x_0^2 + y_0^2)^2 &= 0, \\
8d - 2ry_0(-4 + x_0^2 + y_0^2) - 5\ell x_0(x_0^2 + y_0^2)^2 &= 0
\end{aligned}$$

has one solution (x_0^0, y_0^0) if $N_2 \leq 0$, $N_3 \geq 0$ or $N_4 \geq 0$ and $N_5 > 0$, and three solutions (x_0^i, y_0^i) if $N_5 < 0$ for $i = 1, 2, 3$; we assume that all of them satisfy

$$125\lambda^2((x_0^i)^2 + (y_0^i)^4) + 4\rho^2(-4 + (x_0^i)^2 + (y_0^i)^2)(-4 + 3(x_0^i)^2 + 3(y_0^i)^2) \neq 0.$$

Hence, if $N_2 \leq 0$, $N_3 \geq 0$ or $N_4 \geq 0$ and $N_5 > 0$, then system (4) *has a 2π-periodic solution $(x^0(t,\varepsilon), y^0(t,\varepsilon))$ such that $(x^0(0,\varepsilon), y^0(0,\varepsilon)) \to (x_0^0, y_0^0)$ when $\varepsilon \to 0$. If $N_5 < 0$ then system* (4) *has three 2π-periodic solutions $(x^i(t,\varepsilon), y^i(t,\varepsilon))$ such that $(x^i(0,\varepsilon), y^i(0,\varepsilon)) \to (x_0^i, y_0^i)$ when $\varepsilon \to 0$ for $i = 1, 2, 3$. Moreover, there are values of ρ, λ and δ such that for all α system* (4) *has one or three periodic solutions.*

We note that in Theorems 5, 6, 7 and 8 we do not say anything about the kind of stability of the periodic solutions because we do not have the explicit expressions of the real part of the eigenvalues of their Poincaré maps.

We remark that the periodic solutions provided in Theorems 1 until 8 exist when we take small values for the parameters of system (4) obeying some relations among them. For instance, Theorem 1 states the existence of one periodic solution for system (4) if each parameter of this system is small and α and λ are much smaller than ρ and δ. This assertion becomes more clear if we observe the replacement done in each theorem and take into account that ε is sufficiently small. As far as we know, periodic solutions of system (4) whose parameters have this characteristic have not been observed in the literature. Besides, it seems that the simultaneous bifurcation of three harmonic periodic solutions in the extended Duffing–Van der Pol system is also new.

Furthermore, we note that the periodic solutions presented in the present paper are different from those stated in [8]. Indeed, in the referred paper, the authors ask for $V_6'(0) < 0$, and in our case, we have $V_6'(0) = 0$.

2. Proof and discussion of the results

2.1 *Proof of the results*

In order to apply the averaging theory described in Section 3 in systems (4), we start performing a rescaling of the variables x and y and of the parameters ρ, α, λ and δ as follows:

$$
\begin{aligned}
&x = \varepsilon^{m_1} X, \quad y = \varepsilon^{m_2} Y, \\
&\rho = \varepsilon^{n_1} r, \quad \alpha = \varepsilon^{n_2} a, \quad \lambda = \varepsilon^{n_3} \ell, \quad \delta = \varepsilon^{n_4} d,
\end{aligned} \tag{5}
$$

where ε is positive and sufficiently small and m_i and n_j are non-negative integers, for $i = 1, 2$ and $j = 1, 2, 3, 4$. We recall that since $\delta > 0$, $\rho \geq 0$ and $\lambda \neq 0$ we have $d > 0$, $r \geq 0$ and $\ell \neq 0$.

In the new variables (X, Y), system (4) is written as

$$
\begin{aligned}
\frac{\mathrm{d}X}{\mathrm{d}t} &= \varepsilon^{-m_1+m_2} Y, \\
\frac{\mathrm{d}Y}{\mathrm{d}t} &= -\varepsilon^{m_1-m_2} X + \varepsilon^{n_1} rY - \varepsilon^{2m_1+n_1} rX^2 Y - \varepsilon^{3m_1-m_2+n_2} aX^3 - \varepsilon^{5m_1-m_2+n_3} \ell X^5 + \varepsilon^{-m_2+n_4} d \cos t.
\end{aligned} \tag{6}
$$

In order to have non-negative powers of ε, we must impose the conditions

$$
m_1 = m_2 = m \quad \text{and} \quad n_4 \geq m, \tag{7}
$$

where m is a non-negative integer.

Hence, with conditions (7), system (6) becomes

$$
\begin{aligned}
\frac{\mathrm{d}X}{\mathrm{d}t} &= Y, \\
\frac{\mathrm{d}Y}{\mathrm{d}t} &= -X + \varepsilon^{n_1} rY - \varepsilon^{2m+n_1} rX^2 Y - \varepsilon^{2m+n_2} aX^3 - \varepsilon^{4m+n_3} \ell X^5 + \varepsilon^{-m+n_4} d \cos t.
\end{aligned} \tag{8}
$$

In this paper, we will find periodic solutions of system (8) depending on the parameters r, a, ℓ and d and the powers m, n_i of ε, for $i = 1, 2, 3, 4$. Then, we will go back through rescaling (5) to ensure the existence of periodic solutions in system (4).

From now on we assume that the values n_1, $m^2 + n_2^2$, $m^2 + n_3^2$ and $n_4 - m$ are positive and observe that considering these conditions each power of ε in system (8) becomes positive. The reason for a such assumption will be explained later on in Section 2.2. Now we shall apply the averaging theory described in Section 3. Thus following the notation of the mentioned section and denoting again the variables (X, Y) by (x, y), we have $\mathbf{x} = (x, y)^{\mathrm{T}}$, and system (13) corresponding to system (8) can be written as

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}) = (y, -x)^{\mathrm{T}}. \tag{9}$$

So the solution $\mathbf{x}(t, \mathbf{z}) = (x(t, \mathbf{z}), y(t, \mathbf{z}))$ of system (9) such that $\mathbf{x}(0, \mathbf{z}) = \mathbf{z} = (x_0, y_0)$ is

$$x(t, \mathbf{z}) = x_0 \cos t + y_0 \sin t,$$
$$y(t, \mathbf{z}) = y_0 \cos t - x_0 \sin t.$$

It is clear that the origin of coordinates of $\mathbb{R}^2$ is a global isochronous centre for system (9) whose circular periodic solutions starting on $\mathbf{z} = (x_0, y_0) \in \mathbb{R}^2$ are parametrized by the above functions $x(t, \mathbf{z})$ and $y(t, \mathbf{z})$. Therefore, through every initial condition (x_0, y_0) in $\mathbb{R}^2$ passes a 2π-periodic solution of system (9).

Physically speaking, system (9) models a simple harmonic oscillator and its solutions $(x(t, \mathbf{z}), y(t, \mathbf{z}))$ describe the wave behaviour of this oscillator. In this direction, the problem of perturbation of the global centre (9) is equivalent to the problem of perturbation of a simple harmonic oscillator introducing a damping parameter as external force and considering a potential V by taking small values for ρ, α, λ and δ as stated in Theorems 1–8.

We note that using the notation of Section 3, the fundamental matrix $Y(t, \mathbf{z})$ of system (9) satisfying that $Y(0, \mathbf{z})$ is the identity of $\mathbb{R}^2$ is written as

$$Y(t, \mathbf{z}) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

From the averaging theory we are interested in the simple zeros of the function

$$f(\mathbf{z}) = (f_1(\mathbf{z}), f_2(\mathbf{z})) = \int_0^{2\pi} Y^{-1}(t, \mathbf{z}) F_1(t, \mathbf{x}(t, \mathbf{z}, 0))\, \mathrm{d}t, \tag{10}$$

where $F_1(t, \mathbf{x})$ are the terms of order 1 on ε of the vector field associated with system (8)

$$(y, -x + \varepsilon^{n_1} ry - \varepsilon^{2m+n_1} rx^2 y - \varepsilon^{2m+n_2} ax^3 - \varepsilon^{4m+n_3} \ell x^5 + \varepsilon^{-m+n_4} d \cos t)^{\mathrm{T}}.$$

Now we prove our results.

Proof of Theorem 1 First we take $n_1 = n_4 = 1, m = 0$ and consider $n_2, n_3 > 1$. So rescaling (5) becomes $(x, y, \rho, \delta, \alpha, \lambda) = (X, Y, \varepsilon r, \varepsilon d, \varepsilon^{n_2} a, \varepsilon^{n_3} d)$ and calling again (X, Y) by (x, y) we verify the hypotheses of Theorem 1. Now the vector field of system (4) becomes

$$(y, -x + \varepsilon ry - \varepsilon rx^2 y - \varepsilon^{n_2} ax^3 - \varepsilon^{n_3} \ell x^5 + \varepsilon d \cos t)^{\mathrm{T}}.$$

Hence, as we stated before, the terms of order 1 in this vector field are given by $F_1(t, \mathbf{x}) = (0, ry - rx^2 y + d \cos t)$. Then since $\mathbf{z} = (x_0, y_0)$, the function $f(x_0, y_0) = (f_1(x_0, y_0), f_2(x_0, y_0))$ given in Equation (10) turns into the form

$$f_1(x_0, y_0) = \int_0^{2\pi} -(\sin t)(d \cos t + r(y_0 \cos t - x_0 \sin t)$$
$$- r(y_0 \cos t - x_0 \sin t)(x_0 \cos t + y_0 \sin t)^2)\, \mathrm{d}t$$

$$
\begin{aligned}
&= -\frac{1}{4}\pi r x_0(-4 + x_0^2 + y_0^2),\\
f_2(x_0, y_0) &= \int_0^{2\pi} (\cos t)(d\cos t + r(y_0\cos t - x_0\sin t)\\
&\quad - r(y_0\cos t - x_0\sin t)(x_0\cos t + y_0\sin t)^2)\,\mathrm{d}t\\
&= \frac{1}{4}(4d - r y_0(-4 + x_0^2 + y_0^2)).
\end{aligned}
$$

The zeros of functions f_1 and f_2 are a pair of conjugate complex vectors and a real pair (x_0^0, y_0^0) satisfying $x_0^0 = 0$ and

$$
y_0^0 = \frac{2}{3}\frac{(36r)^{1/3}}{(9d + \sqrt{81d^2 - 48r^2})^{1/3}} + \frac{1}{3}\left(\frac{6}{r}\right)^{1/3}(9d + \sqrt{81d^2 - 48r^2})^{1/3}.
$$

Note that y_0^0 is real because $81d^2 > 48r^2 > 0$ by assumption. We observe that the expression of $y_0^0 = y_0^0(r, d)$ does not change when we go back through rescaling (5) taking $r = \varepsilon^{-1}\rho$ and $d = \varepsilon^{-1}\delta$ according to the hypotheses of Theorem 1.

Moreover, we denote $\Gamma = (9d + \sqrt{81d^2 - 48r^2})^{1/3}$ and observe that the matrix $M = (m_{ij}) = \partial(f_1, f_2)/\partial(x_0, y_0)$ at (x_0^0, y_0^0) is diagonal and its elements are $m_{11} = -\pi r\Gamma_+/36$ and $m_{22} = \pi r\Gamma_-$, where

$$
\Gamma_\pm = -12 \pm \frac{6^{1/3}24r^{2/3}}{\Gamma^2} \pm \frac{6^{2/3}\Gamma^2}{r^{2/3}}.
$$

The determinant Π_0 of M written in a power series of r around $r = 0$ is

$$
\Pi_0 = \frac{3}{8}(2\pi^2 d(4d)^{1/3})r^{2/3} + \frac{(\pi r)^2}{3} + O(r^{7/3}),
$$

which is positive for $|r|$ sufficiently small because $d > 0$. So the averaging theory described in Section 3 guarantees the existence of a 2π-periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ such that $(x(0,\varepsilon), y(0,\varepsilon))$ tends to (x_0^0, y_0^0) when $\varepsilon \to 0$.

On the other hand, the trace Σ_0 of M at (x_0^0, y_0^0) is

$$
\Sigma_0 = -2^{2/3}\pi(2d)^{2/3}r^{1/3} - \frac{2\pi r}{3} + O(r^{4/3}).
$$

Thus for $|r|$ sufficiently small, the trace Σ_0 of M is negative. So the real part of the eigenvalues of the matrix M are both negative and then the periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ is stable for each ε sufficiently small. So Theorem 1 is proved. ■

Proof of Theorem 2 Now we take $n_2 = n_4 = 1$ and consider $n_1, n_3 > 1$. So we are under the hypotheses of Theorem 2. Now, taking $m = 0$ and doing rescaling (5), the vector field of system (8) is written as

$$
(y, -x + \varepsilon^{n_1} r y - \varepsilon^{n_1} r x^2 y - \varepsilon a x^3 - \varepsilon^{n_3}\ell x^5 + \varepsilon d\cos t)^{\mathrm{T}}.
$$

Thus, we have $F_1(t, \mathbf{x}) = (0, -ax^3 + d\cos t)$, and $f(x_0, y_0) = (f_1(x_0, y_0), f_2(x_0, y_0))$ becomes

$$\begin{aligned} f_1(x_0, y_0) &= \int_0^{2\pi} -(\sin t)(d\cos t - a(x_0\cos t + y_0\sin t)^3)\,\mathrm{d}t \\ &= \frac{3}{4}a\pi y_0(x_0^2 + y_0^2), \\ f_2(x_0, y_0) &= \int_0^{2\pi} (\cos t)(d\cos t - a(x_0\cos t + y_0\sin t)^3)\,\mathrm{d}t \\ &= \frac{1}{4}\pi(4d - 3ax_0(x_0^2 + y_0^2)). \end{aligned}$$

The real zero (x_0^0, y_0^0) of f_1 and f_2 is

$$(x_0^0, y_0^0) = \left(\left(\frac{4d}{3a}\right)^{1/3}, 0\right).$$

Again the values (x_0^0, y_0^0) does not depend on ε when we go back to the original parameters α and δ through rescaling (5). Besides, the matrix $M = \partial(f_1, f_2)/\partial(x_0, y_0)$ at (x_0^0, y_0^0) now is written as

$$M = \frac{1}{\varepsilon}\begin{pmatrix} 0 & \left(\left(\dfrac{3}{4}\right)\alpha\delta^2\right)^{1/3}\pi \\ -3\left(\left(\dfrac{3}{4}\right)\alpha\delta^2\right)^{1/3}\pi & 0 \end{pmatrix},$$

and then the determinant Π_0 of M is

$$\Pi_0 = \frac{3\pi^2}{2\varepsilon^2}\left(\frac{9\alpha^2\delta^4}{2}\right)^{1/3} > 0.$$

Therefore, the averaging theory implies the existence of a 2π-periodic solution $(x(t, \varepsilon), y(t, \varepsilon))$ such that $(x(0, \varepsilon), y(0, \varepsilon))$ tends to (x_0^0, y_0^0) when $\varepsilon \to 0$. ■

We observe that since the trace Σ_0 of M is zero and the determinant Π_0 is positive, then the real part of both eigenvalues of M is zero. So we cannot decide about the stability of the periodic solution of Theorem 2. In what follows, we see that the same occurs in Theorem 3.

Proof of Theorem 3 We start considering $n_3 = n_4 = 1, n_1, n_2 > 1$ and $m = 0$. Therefore, we are under the assumptions of Theorem 3. Doing rescaling (5) the vector field of system (8) becomes

$$(y, -x + \varepsilon^{n_1}ry - \varepsilon^{n_1}rx^2y - \varepsilon^{n_2}ax^3 - \varepsilon\ell x^5 + \varepsilon d\cos t)^{\mathrm{T}}.$$

So we conclude that $F_1(t, \mathbf{x}) = (0, -\ell x^5 + d\cos t)$ and then we get

$$\begin{aligned} f_1(x_0, y_0) &= \int_0^{2\pi} -(\sin t)(d\cos t - \ell(x_0\cos t + y_0\sin t)^5)\,\mathrm{d}t \\ &= \frac{5}{8}\ell\pi y_0(x_0^2 + y_0^2)^2, \end{aligned}$$

$$f_2(x_0, y_0) = \int_0^{2\pi} (\cos t)(d \cos t - \ell(x_0 \cos t + y_0 \sin t)^5)\, \mathrm{d}t$$
$$= \frac{1}{8}\pi(8d - 5\ell x_0(x_0^2 + y_0^2)^2).$$

Now the real zero (x_0^0, y_0^0) of f_1 and f_2 is

$$(x_0^0, y_0^0) = \left(\left(\frac{8d}{5\ell}\right)^{1/5}, 0\right).$$

We note that as in Theorems 1 and 2 the root (x_0^0, y_0^0) does not depend on ε after we perform rescaling (5). However, using Equation (5), the matrix $M = \partial(f_1, f_2)/\partial(x_0, y_0)$ at (x_0^0, y_0^0) is

$$M = \frac{1}{\varepsilon}\begin{pmatrix} 0 & \left(\left(\frac{5}{8}\right)\lambda\delta^4\right)^{1/5}\pi \\ -5\left(\left(\frac{5}{8}\right)\lambda\delta^4\right)^{1/5}\pi & 0 \end{pmatrix},$$

whose determinant is

$$\Pi_0 = \frac{5\pi^2}{2\varepsilon^2}\left(\frac{25\lambda^2\delta^8}{2}\right)^{1/5} > 0.$$

Then the averaging method states the existence of a 2π-periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ such that $(x(0,\varepsilon), y(0,\varepsilon))$ tends to (x_0^0, y_0^0) when $\varepsilon \to 0$. The stability of this periodic solution is unknown because the trace Σ_0 of M is zero and the determinant $\Pi_0 > 0$ is positive just as in Theorem 2. ∎

Proof of Theorem 4 In what follows, we assume that $n_1 = m = 1$ and $n_4 = 2$. Now the expression of the vector field of system (8) is

$$(y, -x + \varepsilon r y - \varepsilon^3 r x^2 y - \varepsilon^{n_2+2} a x^3 - \varepsilon^{n_3+4}\ell x^5 + \varepsilon d \cos t)^{\mathrm{T}},$$

and then $F_1(t, \mathbf{x}) = (0, ry + d\cos t)$. In this case, we obtain

$$f_1(x_0, y_0) = \int_0^{2\pi} -(\sin t)(d\cos t + r(y_0 \cos t - x_0 \sin t))\, \mathrm{d}t$$
$$= \pi r x_0,$$
$$f_2(x_0, y_0) = \int_0^{2\pi} (\cos t)(d\cos t + r(y_0 \cos t - x_0 \sin t))\, \mathrm{d}t$$
$$= \pi(d + r y_0).$$

It is immediate that the only zero (x_0^0, y_0^0) of f_1 and f_2 in this case is $(x_0^0, y_0^0) = (0, d/r)$. So after going back through rescaling (5) taking $r = \varepsilon^{-1}\rho$ and $d = \varepsilon^{-2}\delta$, we have

$$(x_0^0, y_0^0) = \left(0, -\frac{\delta}{\varepsilon\rho}\right).$$

In addition, the matrix $M = \partial(f_1, f_2)/\partial(x_0, y_0)$ at (x_0^0, y_0^0) is diagonal and its elements are both given by $\rho\pi/\varepsilon$. Consequently, M has determinant $\Pi_0 = (\rho\pi/\varepsilon)^2 > 0$ and trace $\Sigma_0 = 2\rho\pi/\varepsilon >$

0. Then the averaging theory described in Section 3 assures the existence of a 2π-periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ such that $(x(0,\varepsilon), y(0,\varepsilon))$ tends to (x_0^0, y_0^0) when $\varepsilon \to 0$. Moreover, since ρ is positive this periodic solution is unstable. Then, Theorem 4 is proved. ■

Proof of Theorem 5 Now we take $m = 0$, $n_2 = n_3 = n_4 = 1$ and $n_1 > 1$. The expression of the vector field of system (8) in this case is

$$(y, -x + \varepsilon^{n_1} ry - \varepsilon^{n_1} rx^2 y - \varepsilon ax^3 - \varepsilon \ell x^5 + \varepsilon d \cos t)^{\mathrm{T}},$$

and we get $F_1(t, \mathbf{x}) = (0, -ax^3 - \ell x^5 + d \cos t)$. Hence, function $f = (f_1, f_2)$ is written as

$$\begin{aligned} f_1(x_0, y_0) &= \int_0^{2\pi} -(\sin t)(d \cos t - a(x_0 \cos t + y_0 \sin t)^3 - \ell(x_0 \cos t + y_0 \sin t)^5)\, \mathrm{d}t \\ &= \frac{1}{8}\pi y_0(x_0^2 + y_0^2)(6a + 5\ell(x_0^2 + y_0^2)), \\ f_2(x_0, y_0) &= \int_0^{2\pi} (\cos t)(d \cos t - a(x_0 \cos t + y_0 \sin t)^3 - \ell(x_0 \cos t + y_0 \sin t)^5)\, \mathrm{d}t \\ &= \frac{1}{8}\pi(8d - x_0(x_0^2 + y_0^2)(6a + 5\ell(x_0^2 + y_0^2))). \end{aligned}$$

In order to find the zeros of $f(x_0, y_0)$, we compute a Gröbner basis $\{b_k(x_0, y_0), k = 1, 2, 3\}$ in the variables x_0 and y_0 for the set of polynomials $\{\bar{f}_1(x_0, y_0), \bar{f}_2(x_0, y_0)\}$, where $\bar{f}_{1,2} = (8/\pi) f_{1,2}$. Then, we look for the zeros of each b_k, $k = 1, 2, 3$. It is a known fact that the zeros of a Gröbner basis of $\{\bar{f}_1(x_0, y_0)), \bar{f}_2(x_0, y_0)\}$ are the zeros of $\bar{f}_1$ and $\bar{f}_2$, and consequently, zeros of f_1 and f_2 too. For more information about a Gröbner basis, see [1,15].

The Gröbner basis is

$$\begin{aligned} b_1(x_0, y_0) &= dy_0, \\ b_2(x_0, y_0) &= (6ax_0^2 + 5\ell x_0^4)y_0 + (6a + 10\ell x_0^2)y_0^3 + 5\ell y_0^5, \\ b_3(x_0, y_0) &= -8d + 6ax_0^3 + 5\ell x_0^5 + (6ax_0 + 10\ell x_0^3)y_0^2 + 5\ell x_0 y_0^4. \end{aligned}$$

The only zero of $b_1(x_0, y_0)$ is $y_0^0 = 0$ since $d = \varepsilon^{-1}\delta > 0$. So replacing y_0^0 into $b_3(x_0, y_0)$ and simplifying the new expressions, we reduce our problem to find a zero of the polynomial

$$p(x_0) = \frac{1}{5\ell} b_3(x_0, 0) = -\frac{8d}{r\ell} + \frac{6a}{5\ell} x_0^3 + x_0^5. \tag{11}$$

As we know from algebra, there is no general formula to provide the roots of a quintic polynomial as polynomial (11). Nevertheless, some techniques may make this task easier. An elegant one can be found in [29]. In this paper, a way is provided to study the number of roots (complex and real, including multiplicity) of a polynomial of any order by only performing some calculations on the coefficients of the considered polynomial. We will present a brief summary of the algorithm of this method in the appendix.

By the Fundamental Theorem of Algebra, polynomial (11) has five roots taking into account their multiplicities and once we want to apply the averaging theory, we are only interested in the real simple roots of polynomial (11). Indeed, applying the method due to [29] and summarized in the appendix, we obtain

$$D_4 = \frac{384ad^2}{\ell^3},$$
$$D_5 = \frac{53747712}{78125}\frac{a^5d^2}{\ell^7} + 20480\frac{d^4}{\ell^4}.$$

Therefore as described in the appendix, since $\alpha\lambda = \varepsilon^2 a\ell < 0$, by hypothesis we have that D_4 is negative. So polynomial (11) has a unique simple real root if $D_5 = D = 0$ according to statement (5) of the appendix. On the other hand, statement (3) says that polynomial (11) has three simple real roots if $D_5 = D < 0$. Moreover, D and D_4 do not change when we replace (a, ℓ, d) by $(\varepsilon^{-1}\alpha, \varepsilon^{-1}\lambda, \varepsilon^{-1}\delta)$ using rescaling (5).

Now using the property on zeros of the Gröbner basis cited previously, the function $f(x_0, y_0)$ has a zero $(x_0^0, 0)$ if $D = 0$ and three zeros $(x_0^i, 0)$ if $D < 0$, with $i = 1, 2, 3$. Besides the matrix $M = \partial(f_1, f_2)/\partial(x_0, y_0)$ evaluated at $y_0^0 = 0$ after rescaling (5) is

$$M = \frac{1}{\varepsilon}\begin{pmatrix} 0 & \frac{\pi}{8}(6\alpha x_0^2 + 5\lambda x_0^4) \\ -\frac{\pi}{8}(18\alpha x_0^2 + 25\lambda x_0^4) & 0 \end{pmatrix},$$

whose determinant is

$$\Pi(x_0) = \frac{1}{\varepsilon^2}\left(\frac{27}{16}(\alpha\pi x_0^2)^2 + \frac{15}{4}\alpha\lambda(\pi x_0^3)^2 + \frac{125}{64}(\lambda\pi x_0^4)^2\right).$$

Thus by computing a Gröbner basis for polynomials $\{\Pi(x_0), f_1(x_0, 0), f_2(x_0, 0)\}$ in the variables α, δ, λ and x_0, we get the polynomials $\bar{b}_1(\alpha, \delta, \lambda, x_0) = \delta$ and $\bar{b}_2(\alpha, \delta, \lambda, x_0) = 6\alpha + 5\lambda x_0^2$. So $\bar{b}_1(\alpha, \delta, \lambda, x_0) > 0$ because $\delta > 0$ and it means that we cannot have f_1, f_2 and Π equal to zero simultaneously. Consequently, $\Pi(x_0^i) \neq 0$ for all $i = 0, 1, 2, 3$. Then using the averaging theory we conclude the existence of a 2π-periodic solution if $D = 0$ and three 2π-periodic solutions if $D < 0$.

Now we exhibit the values of α, λ and δ for which system (4) has one or three 2π-periodic solutions. On the one hand, if consider $a = \frac{25}{18}$, $\ell = -1$ and $d = \frac{5}{12}$, we obtain $a\ell = -\frac{25}{18} < 0$ and $D = 0$. On the other hand, taking $d = 18a/31$ and $\ell = -42a/155$, we get $a\ell = -42a^2/155 < 0$ and $D = -\frac{1425339825408}{823543} < 0$. So Theorem 5 is proved. ■

We note that by assumption in Theorem 5 we have $a\ell < 0$. So considering the notation of the appendix, the coefficients $D_2 = -6a/5\ell$ and $D_3 = 384ad^2/\ell^3$ corresponding to polynomial (11) are positive and D_4 is negative. Thus the only possible configurations of roots of polynomial (11) are those one listed in statements (3) and (5) of the appendix. This fact means that we cannot have two or five periodic solutions when we consider $m = 0$, $n_2 = n_3 = n_4 = 1$ and $n_1 > 1$.

Proof of Theorem 6 We start fixing the values $m = 0$, $n_1 = n_2 = n_4 = 1$ and $n_3 > 1$. Now the vector field of system (8) is written as

$$(y, -x + \varepsilon ry - \varepsilon rx^2y - \varepsilon ax^3 - \varepsilon^{n_3}\ell x^5 + \varepsilon d\cos t)^{\mathrm{T}},$$

and then $F_1(t,\mathbf{x}) = (0, ry - rx^2y - ax^3 + d\cos t)$. With this expression of F_1, function (10) is as follows:

$$\begin{aligned}
f_1(x_0,y_0) &= \int_0^{2\pi} -(\sin t)(d\cos t + r(y_0\cos t - x_0\sin t) \\
&\quad - r(y_0\cos t - x_0\sin t)(x_0\cos t + y_0\sin t)^2 - a(x_0\cos t + y_0\sin t)^3)\,\mathrm{d}t \\
&= -\frac{1}{4}\pi(rx_0(-4 + x_0^2 + y_0^2) - 3ay_0(x_0^2 + y_0^2)), \\
f_2(x_0,y_0) &= \int_0^{2\pi} (\cos t)(d\cos t + r(y_0\cos t - x_0\sin t) - r(y_0\cos t - x_0\sin t)(x_0\cos t + y_0\sin t)^2 \\
&\quad - a(x_0\cos t + y_0\sin t)^3)\,\mathrm{d}t \\
&= \frac{1}{4}\pi(4d - ry_0(-4 + x_0^2 + y_0^2) - 3ax_0(x_0^2 + y_0^2)).
\end{aligned}$$

In order to find zeros (x_0^*, y_0^*) of $f(x_0,y_0)$ as before, we compute a Gröbner basis $\{b_k(x_0,y_0), k = 1,\ldots,14\}$ in the variables x_0 and y_0 now for the set of polynomials $\{\bar{f}_1(x_0,y_0), \bar{f}_2(x_0,y_0)\}$ with $\bar{f}_{1,2} = \mp(4/\pi)f_{1,2}$. We will look for zeros of two elements of the Gröbner basis for the polynomials $\{\bar{f}_1(x_0,y_0), \bar{f}_2(x_0,y_0)\}$ in the variables x_0 and y_0. These polynomials are

$$\begin{aligned}
b_1(x_0,y_0) &= 144a^2dr^3 - 4d^3r^3 + (108a^2d^2r^2 + 144a^2r^4 - 4d^2r^4)y_0 \\
&\quad + 144a^2dr^3y_0^2 + (81a^4d^2 + 18a^2d^2r^2 + d^2r^4)y_0^3, \\
b_2(x_0,y_0) &= 216a^3dr^2 + (-324a^4r^2 - 9a^2d^2r^2 + 36a^2r^4 - d^2r^4)x_0 \\
&\quad + (27a^3d^2r + 216a^3r^3 + 3ad^2r^3)y_0 + (243a^5d + 54a^3dr^2 + 3adr^4)y_0^2.
\end{aligned}$$

We must observe that $b_1(x_0,y_0)$ depends only on y_0. Then for each zero y_0^* of $b_1(y_0)$, the second polynomial $b_2(x_0,y_0)$ provides a zero x_0^* associated with y_0^* because the coefficient of x_0 in $b_2(x_0,y_0)$ is not zero by the hypothesis. Now we will look for zeros of $b_1(y_0)$. Indeed, the discriminant Δ of the cubic polynomial $b_1(y_0)$ is written as

$$\begin{aligned}
\Delta &= -16d^2r^6(324a^4 + d^2r^2 + 9a^2(d^2 - 4r^2))^2 \\
&\quad \times (2187a^4d^4 + 27d^4r^4 - 16d^2r^6 + 18a^2(27d^4r^2 - 72d^2r^4 + 32r^6)) \\
&= -16d^2r^6\Delta_1^2\Delta_2.
\end{aligned}$$

So if Δ_1 or Δ_2 is zero, then Δ is zero and consequently $b_1(y_0)$ has a simple real root y_0^0 because this polynomial has no root with multiplicity 3. Actually $b_1'''(y_0) = 6(81a^4d^2 + 18a^2d^2r^2 + d^2r^4) > 0$. The same occurs if $\Delta_2 > 0$, since this condition implies $\Delta < 0$. On the other hand, if $\Delta_2 < 0$, then $\Delta > 0$ and consequently polynomial $b_1(y_0)$ has three simple real roots y_0^i, $i = 1,2,3$. Additionally, replacing each value y_0^i into $b_2(x_0,y_0)$ we obtain the respective values x_0^i, for $= 0,1,2,3$. We note that coming back through rescaling (5), the signs of Δ, Δ_1 and Δ_2 does not change because each monomial composing Δ has the same degree.

Now we will verify the condition $M = \det((\partial f/\partial \mathbf{z})(x_0^i, y_0^i)) \neq 0$ for $i = 0,1,2,3$. In fact M is

$$\frac{-\pi}{4\varepsilon}\begin{pmatrix} -6\alpha x_0y_0 + \rho(-4 + 3x_0^2 + y_0^2) & 2\rho x_0y_0 - 3\alpha(x_0^2 + 3y_0^2) \\ 2\rho x_0y_0 + 3\alpha(3x_0^2 + y_0^2) & 6\alpha x_0y_0 + \rho(-4 + x_0^2 + 3y_0^2) \end{pmatrix},$$

whose determinant Π now is

$$\Pi(x_0,y_0) = \frac{\pi^2}{16\varepsilon^2}(27\alpha^2(x_0^2 + y_0^2)^2 + \rho^2(-4 + x_0^2 + y_0^2)(-4 + 3x_0^2 + 3y_0^2)).$$

However for each $i = 0, 1, 2, 3$ the determinant $\Pi(x_0^i, y_0^i) \neq 0$ by hypothesis. Therefore, using the averaging theory described in Section 3, system (4) has a 2π-periodic solution if $\Delta_1\Delta_2 = 0$ or $\Delta_2 > 0$ and three 2π-periodic solutions if $\Delta_2 < 0$.

Now we present concrete values of α, ρ and δ for which we have one or three periodic solutions. Indeed taking $a = r = 1$ and $d = 6$, we have $\Delta_2 = 3452544 > 0$. Meantime if we consider $d = 6a$ and

$$r = -\frac{9a}{\sqrt{-9 + 4\sqrt{6}}},$$

we obtain $\Delta_2 = -34012224a^8/(9 - 4\sqrt{6})^2 < 0$. We also observe that the values $(r, a, d) = ((\frac{555}{4})\sqrt{\frac{154073}{9622}}, 185, 22)$ and $(r, a, d) = (585\sqrt{3}, 195, 585\sqrt{2})$ make Δ_1 and Δ_2 equal to zero, respectively. This ends the proof of Theorem 6. ■

Proof of Theorem 7 For this case, we assume $m = 0$ and $n_i = 1$, $i = 1, 2, 3, 4$. Then, the vector field of system (8) is written as

$$(y, -x + \varepsilon ry - \varepsilon rx^2y - \varepsilon ax^3 - \varepsilon\ell x^5 + \varepsilon d\cos t)^{\mathrm{T}},$$

and we have $F_1(t, \mathbf{x}) = (0, ry - rx^2y - ax^3 - \ell x^5 + d\cos t)$. So it follows that

$$\begin{aligned}
f_1(x_0, y_0) = \int_0^{2\pi} & -(\sin t)(d\cos t + r(y_0\cos t - x_0\sin t) - r(y_0\cos t - x_0\sin t)\\
&(x_0\cos t + y_0\sin t)^2 - a(x_0\cos t + y_0\sin t)^3 - \ell(x_0\cos t + y_0\sin t)^5)\,\mathrm{d}t\\
= & -\frac{1}{8}\pi(2rx_0(-4 + x_0^2 + y_0^2) - y_0(x_0^2 + y_0^2)(6a + 5\ell(x_0^2 + y_0^2)),\\
f_2(x_0, y_0) = \int_0^{2\pi} & (\cos t)(d\cos t + r(y_0\cos t - x_0\sin t) - r(y_0\cos t - x_0\sin t)\\
&(x_0\cos t + y_0\sin t)^2 - a(x_0\cos t + y_0\sin t)^3 - \ell(x_0\cos t + y_0\sin t)^5)\,\mathrm{d}t\\
= & \frac{1}{8}\pi(8d + 8ry_0 - (x_0^2 + y_0^2)(6ax_0 + 2ry_0 + 5\ell x_0(x_0^2 + y_0^2))).
\end{aligned}$$

We will find zeros of $f(x_0, y_0)$ through the zeros of a Gröbner basis $\{b_k(x_0, y_0)\}$ of it. As before, instead of $f(x_0, y_0)$, the Gröbner basis will be related with the functions $\bar{f}_{1,2} = \mp(8/\pi)f_{1,2}$.

One of the elements of the Gröbner basis in the variables x_0 and y_0 is

$$\begin{aligned}
b_1(x_0, y_0) = {} & 144a^2dr^5 - 4d^3r^5 + 960ad\ell r^5 + 1600d\ell^2r^5 + (108a^2d^2r^4\\
&+ 960ad^2\ell r^4 + 2000d^2\ell^2r^4 + 144a^2r^6 - 4d^2r^6 + 960a\ell r^6\\
&+ 1600\ell^2r^6)y_0 + (120ad^3\ell r^3 + 500d^3\ell^2r^3 + 144a^2dr^5\\
&+ 1200ad\ell r^5 + 2400d\ell^2r^5)y_0^2 + (81a^4d^2r^2 + 540a^3d^2\ell r^2 + 900a^2d^2\ell^2r^2\\
&+ 18a^2d^2r^4 + 300ad^2\ell r^4 + 900d^2\ell^2r^4 + d^2r^6)y_0^3\\
&+ (-450a^2d^3\ell^2r - 1500ad^3\ell^3r + 50d^3\ell^2r^3)y_0^4 + 625d^4\ell^4y_0^5.
\end{aligned}$$

We note that again $b_1(x_0, y_0)$ depends only on y_0. Moreover, the Gröbner basis has another element $b_2(x_0, y_0)$ which is linear on x_0 and the coefficients depend on y_0, where the coefficient C of x_0 is not zero by hypotheses. Its expression is too large and we will omit it here. It can be easily obtained by an algebraic manipulator. So for each zero y_0^* of $b_1(y_0)$, the second polynomial $b_2(x_0, y_0)$ provides a zero x_0^* associated with y_0^*.

Now we will look for zeros of $b_1(y_0)$. Indeed using again the appendix, the conditions on a, r, d and ℓ that provide three zeros, (x_0^i, y_0^i), where $i = 1, 2, 3$ for $b_1(y_0)$, are $D_5 = -D_5^1 D_5^2 D < 0$, where D is given in Theorem 7 and D_5^1 and D_5^2 are the values

$$D_5^1 = \frac{110075314176a^8 r^{20}}{d^{20}(-9a^2 + r^2)^{28}},$$
$$D_5^2 = (59049a^{10}d^4 + 16r^{14} + 36a^2r^{10}(11d^2 - 4r^2) - 81a^4r^6(d^4 - 12d^2r^2 + 16r^4) - 6561a^8(3d^4r^2 - 4d^2r^4) + 729a^6(3d^4r^4 - 28d^2r^6 + 16r^8))^2.$$

By hypothesis $ar = \varepsilon^{-2}\alpha\rho \neq 0$ and $D_5^2 \neq 0$, then D_5^1 and D_5^2 are positive. Therefore, in order to have D_5 negative, we must have D positive. On the other hand, if D_5 is positive then according to statement (2) of the appendix, we have a unique zero (x_0^0, y_0^0) for $b_1(y_0)$ because the value

$$D_2 = -\frac{1296a^2r^4(-3a^2 + r^2)^2}{d^2(-9a^2 + r^2)^4}$$

is negative if $(r^2 - 3a^2)(r^2 - 9a^2) \neq 0$. In addition by considering $a = \varepsilon^{-1}\alpha$, $r = \varepsilon^{-1}\rho$ and $d = \varepsilon^{-1}\delta$, we get

$$M = \frac{-\pi}{8\varepsilon}\begin{pmatrix} 2\rho(-4 + 3x_0^2 + y_0^2) - \Gamma_1 & \Gamma_2 - \Gamma_4 - 5\lambda\Gamma_3(x_0^2 + 5y_0^2) \\ \Gamma_2 + \Gamma_4 + 5\lambda\Gamma_3(5x_0^2 + y_0^2) & 2\rho(-4 + x_0^2 + 3y_0^2) - \Gamma_1 \end{pmatrix},$$

where $\Gamma_1 = 2x_0y_0(3\alpha + 5\ell(x_0^2 + y_0^2))$, $\Gamma_2 = -4\rho x_0 y_0$, $\Gamma_3 = x_0^2 + y_0^2$ and $\Gamma_4 = 6\alpha(x_0^2 + 3y_0^2)$. Moreover, determinant Π of M is written as

$$\Pi(x_0, y_0) = \frac{1}{\varepsilon^2}(4\rho^2(-4 + x_0^2 + y_0^2)(-4 + 3x_0^2 + 3y_0^2) + (x_0^2 + y_0^2)^2(6\alpha + \rho\lambda(x_0^2 + y_0^2)(18\alpha + 25\lambda x_0^2 + 3y_0^2))).$$

However, by hypothesis $\Pi(x_0^i, y_0^i) \neq 0$ for $i = 0, 1, 2, 3$ and hence we conclude the first part of Theorem 7.

Now we provide values for α, ρ, λ and δ for which we have one or three 2π-periodic solutions. First, we observe that taking $(a, r, d) = (1, 2, \frac{8}{3})$, we obtain

$$D = -\frac{2770035802112}{6561} < 0.$$

However, if we consider

$$(a, r, d, \ell) = \left(\frac{1}{\sqrt{3}}, 1, 1, -\frac{1}{(5\sqrt{3})}\right),$$

then $D = 2752$. So Theorem 7 is proved. ■

Now we will prove that under the conditions $m = 0$ and $n_i = 1$, $i = 1, 2, 3, 4$, and the assumptions of Theorem 7, we cannot have two or five periodic solutions by using the averaging theory and rescaling (5). Indeed, we observe that the only possibility to have five periodic solutions is that statement (1) of the appendix holds. However, this condition does not hold since D_2 is negative for any a, r and d satisfying the hypotheses of Theorem 7.

In addition, we note that the only condition that provides two periodic solutions according to the appendix is statement (7), and it needs $D_5 = D_4 = E_2 = 0$. Nevertheless, we start considering this condition and will find a Gröbner basis for the set of polynomials $\{D_4, D_5, E_2\}$

in the variables a, d and r. First, we will consider the factor D_5^2 of D_5 because $D_5^1 \neq 0$ by hypothesis. In this case, we obtain 55 polynomials in the Gröbner basis, where the first one is $g(d,r) = r^{62}\bar{g}(d,r)$ and

$$\bar{g}(d,r) = 91125d^{12} - 438800d^{10}r^2 + 919360d^8r^4 - 1038464d^6r^6 + 660992d^4r^8 - 225280d^2r^{10} + 32768r^{12}.$$

Since $r \neq 0$ we must find a zero of $\bar{g}(d,r)$. So solving $\bar{g}(d,r) = 0$ in the variable r we obtain six pairs of values

$$r_{\pm}^i = \pm d\sqrt{q_0^i},$$

where $i = 1, \ldots, 6$ and for each i the value q_0^i is the ith root of the polynomial

$$q(x) = 91125 - 438800x + 919360x^2 - 1038464x^3 + 660992x^4 - 225280x^5 + 32768x^6.$$

This polynomial has only complex roots. Indeed, if we study the function $q'(x)$ which is given by

$$q'(x) = -438800 + 1838720x - 3115392x^2 + 2643968x^3 - 1126400x^4 + 196608x^5,$$

we can apply the method described in the appendix to show that the respective values D_2 and D_5 related to $q'(x)$ are negative and positive, respectively. Therefore, $q'(x)$ has only one real root q_0' whose approximate value is $q_0' = \frac{106703}{100000}$. Additionally, the approximate value of $q(x)$ evaluated in q_0' is $q(q_0') = \frac{16558687}{10000} > 0$. So if we observe that the coefficient of x^6 is positive, it follows that the minimum value of $q(x)$ is positive. Then each $r_{\pm}^i$, $i = 1, \ldots, 6$, is a complex value and the correspondent Gröbner basis has no zeros. Then we cannot have a common real zero of D_4, D_5^2 and E_2 and so statement (7) in the appendix does not hold. Consequently, it is not possible to obtain two periodic solutions. The proof considering the factor D of D_5 instead of D_5^2 leads to the same polynomial $g(d,r)$ and then we have the same conclusion.

Proof of Theorem 8 In what follows we take $m = 0$, $n_1 = n_3 = n_4 = 1$ and $n_2 > 1$. So the vector field of system (8) becomes

$$(y, -x + \varepsilon ry - \varepsilon rx^2y - \varepsilon^{n_2}ax^3 - \varepsilon\ell x^5 + \varepsilon d\cos t)^{\mathrm{T}},$$

and we obtain $F_1(t,\mathbf{x}) = (0, ry - rx^2y - \ell x^5 + d\cos t)$. So $f(x_0,y_0)$ is written as

$$\begin{aligned} f_1(x_0,y_0) &= \int_0^{2\pi} -(\sin t)(d\cos t + r(y_0\cos t - x_0\sin t) - r(y_0\cos t - x_0\sin t) \\ &\quad (x_0\cos t + y_0\sin t)^2 - \ell(x_0\cos t + y_0\sin t)^5)\,\mathrm{d}t \\ &= -\frac{1}{8}\pi(2rx_0(-4 + x_0^2 + y_0^2) - 5\ell y_0(x_0^2 + y_0^2)^2), \\ f_2(x_0,y_0) &= \int_0^{2\pi} (\cos t)(d\cos t + r(y_0\cos t - x_0\sin t) - r(y_0\cos t - x_0\sin t) \\ &\quad (x_0\cos t + y_0\sin t)^2 - \ell(x_0\cos t + y_0\sin t)^5)\,\mathrm{d}t \\ &= \frac{1}{8}\pi(8d - 2ry_0(-4 + x_0^2 + y_0^2) - 5\ell x_0(x_0^2 + y_0^2)^2). \end{aligned}$$

Again we will find zeros of $f(x_0,y_0)$ through the roots of a Gröbner basis $\{b_k(x_0,y_0)\}$ of $f(x_0,y_0)$. We will find a Gröbner basis of the functions $\bar{f}_{1,2} = \mp(8/\pi)f_{1,2}$.

An element of the Gröbner basis in the variables x_0 and y_0 is

$$b_1(x_0,y_0) = -4d^3r^5 + 1600dL^2r^5 + (2000d^2\ell^2r^4 - 4d^2r^6 + 1600L^2R^6)y_0 + (500d^3\ell^2r^3 + 2400d\ell^2r^5)y_0^2 + (900d^2\ell^2r^4 + d^2r^6)y_0^3 + 50d^3\ell^2r^3y_0^4 + 625d^4\ell^4y_0^5.$$

As before, $b_1(x_0,y_0)$ depends only on y_0 and another element $b_2(x_0,y_0)$ of the Gröbner basis is linear on x_0 with coefficients depending on y_0, where the coefficient C of x_0 in $b_2(x_0,y_0)$ is not zero by hypotheses. We will not present the expression of $b_2(x_0,y_0)$ in order to avoid large expressions. Then for each zero y_0^* of $b_1(y_0)$ we have a second zero x_0^* through $b_2(x_0,y_0)$ related to y_0^*.

In order to apply the method described in [29] and the appendix for $b_1(y_0)$, we will perform the translation $y_0 = \varphi - (2r^3/d(125\ell)^2)$. With this translation we obtain a new polynomial b_1^* in the form

$$b_1^*(\varphi) = \varphi^5 + \frac{4500\ell^2r^4 - 3r^6}{3125d^2\ell^4}\varphi^3 + \frac{2k_1}{\bar{k}_1d^3\ell^6}\varphi^2 + \frac{4r^4k_2}{\bar{k}_2d^4\ell^8}\varphi - \frac{4r^5k_3}{\bar{k}_3d^5\ell^{10}},$$

where

$$\begin{aligned}
k_1 &= 156250d^2\ell^4r^3 + 750000\ell^4r^5 - 13500\ell^2r^7 + r^9,\\
\bar{k}_1 &= 390625,\\
k_2 &= 39062500d^2\ell^6 - 390625\ell^4(d^2 - 80\ell^2)r^2 - 1500000\ell^4r^4 + 13500\ell^2r^6 + 3r^8,\\
\bar{k}_2 &= 48828125,\\
k_3 &= 48828125d^4\ell^6 - 781250d^2(25000\ell^8 - 500\ell^6r^2 + 3\ell^4r^4) + 2r^4\\
&\quad (156250000\ell^6 - 3750000\ell^4r^2 + 22500\ell^2r^4 + 9r^6),\\
\bar{k}_3 &= 30517578125.
\end{aligned}$$

We observe that the translation performed does not change the number or kind of the zeros of the original polynomial b_1. Now we will apply the method of the appendix for b_1^*. Indeed, we have

$$\begin{aligned}
D_2 &= N_2,\\
D_3 &= -\frac{48r^6}{9765625d^6\ell^{10}}N_3,\\
D_4 &= -\frac{16r^{12}}{3814697265625d^{12}\ell^{18}}N_4,\\
D_5 &= \frac{256r^{20}}{1490116119384765625d^{20}\ell^{26}}N_5M_5.
\end{aligned}$$

Therefore, if D_5 is negative then b_1^* and consequently b_1 have exactly three zeros, and hence function $f = (f_1,f_2)$ has exactly three zeros (x_0^i,y_0^i), $i = 1,2,3$. On the other hand, if D_5 is positive and one of the values D_2, D_3 or D_4 is non-positive, then f has exactly one zero (x_0^0,y_0^0). Moreover, using rescaling (5) the matrix M is the same as the one of the proof of Theorem 7 taking $a = 0$. In addition, determinant Π of M is written as

$$\Pi(x_0,y_0) = \frac{1}{\varepsilon^2}(125\lambda^2(x_0^2 + y_0^4) + 4\rho^2(-4 + x_0^2 + y_0^2)(-4 + 3x_0^2 + 3y_0^2)).$$

By hypothesis $\Pi(x_0^i,y_0^i) \neq 0$ for $i = 0,1,2,3$, then we have the first part of Theorem 8 proved.

Now we exhibit values of ρ, λ and δ for which we have either one or three periodic solutions. First, we consider

$$(r,\ell,d) = \left(\sqrt{31}, 1, \frac{1}{125}\sqrt{1547875 - 226851\sqrt{35}}\right).$$

With these values we obtain

$$N_3 = \frac{1}{625}(206837701817900 - 10419995302263\sqrt{35}),$$
$$N_5 = \frac{2(-6494914780517287154582 44 + 10963983255479402755852 5\sqrt{35})}{48828125},$$

which are positive. On the other hand, considering $(r,\ell,d) = (\sqrt{31}/4, 1, 1)$ we get $N_5 = -\frac{258261575}{1048576} < 0$. This ends the proof of Theorem 8. ■

Now we will show that if $m = 0$, $n_1 = n_3 = n_4 = 1$ and $n_2 > 1$, then we cannot have two periodic solutions by using the averaging theory described in Section 3 and rescaling (5). Indeed in order that function $b_1^*(\varphi)$, given in the proof of Theorem 8, has two periodic solutions, we need statement (7) of the appendix. We start considering the conditions $N_5 = N_4 = E_2 = 0$ and will see that these equalities imply that N_3 cannot be negative. We claim that N_5 and M_5 are factors of D_5 as we saw in the last proof. Also we will not present the expression of E_2 in order to avoid its large expression. It can be obtained by an algebraic manipulator such as Mathematica. Additionally, we note that $N_4 = N_5 = 0$ implies $D_4 = D_5 = 0$, which are necessary conditions for having exactly two real roots.

We start replacing the condition $N_4 = N_5 = E_2 = 0$ for other ones easier when $r > 0$. It is easy to see that if $r = 0$ we have $N_3 = 0$ and consequently we cannot get two roots for b_1^*. Now we obtain a Gröbner basis for the set of polynomials $\{N_4, N_5, E_2\}$ in the variables d, r and ℓ. This basis has 12 elements, where 2 of them are

$$g_1(r,\ell) = 125\cdot 10^8\ell^8 + 2\cdot 10^8\ell^6 r^2 - 100000\ell^4 r^4 - 10400\ell^2 r^6 - 3r^8,$$
$$g_2(d,r,\ell) = 25d^4\ell^2 - 110000d^2\ell^4 - 4000000\ell^6 - 1400d^2\ell^2 r^2 + 80000\ell^4 r^2 - 3d^2 r^4 + 1200\ell^2 r^4.$$

In addition, the resultant of these two polynomials with respect to the variable r has the factor

$$g_3(d,\ell) = d^8 - 1240000d^6\ell^2 - 269440000d^4\ell^4 + 6144\cdot 10^6 d^2\ell^6 + 28672\cdot 10^8\ell^8.$$

Consequently, we will replace the problem of finding a zero of N_4, N_5 and E_2 by the problem of finding a zero of g_1, g_2 and g_3. Moreover, we suppose that we have a common zero between each g_i and N_3, $i = 1,2,3$. With this supposition and computing the resultant between g_1 and N_3, we obtain a new polynomial g_4 in the variables r and d given by

$$\begin{aligned} g_4(r,d) = {} & 1423828125d^{16} + 25974000000d^{14}r^2 - 47933388000000d^{12}r^4 \\ & - 478810245120000d^{10}r^6 + 1711832243200000d^8 r^8 \\ & - 12559106086010880 0d^6 r^{10} - 1334616713986048000d^4 r^{12} \\ & + 5191865574606503936d^2 r^{14} - 4405603330689073152r^{16}. \end{aligned}$$

However, the resultant between g_3 and g_4 is written as Kr^{128}, where K is a positive constant. This polynomial cannot be zero since $r \geq 0$. It means that when we consider each g_i zero, N_3 must be positive or negative, exclusively. But taking the values $(r, \ell, d) = (r_0, \ell_0, 1)$ we have $g_i \equiv 0$, for each $i = 1, 2, 3$ and N_3 is approximately $\frac{8738483}{2500} > 0$, where r_0 and ℓ_0 are roots of the polynomials

$$\bar{r}(x) = -125 - 44400x^2 + 184640x^4 - 237568x^6 + 86016x^8,$$

and

$$\bar{l}(x) = 1 - 1240000x^2 - 269440000x^4 + 6144000000x^6 + 2867200000000x^8,$$

respectively. So N_3 is positive when $g_i \equiv 0$, for each $i = 1, 2, 3$, and then D_3 is negative. Hence using the factor N_5 of D_5 we cannot obtain two periodic solutions. The proof that cannot exists two periodic solutions for the case that we consider the factor M_5 of D_5 instead of N_5 is similar and we will omit it here.

We remark that we cannot prove analytically the non-existence of five periodic solutions for system (8) considering $m = 0$, $n_1 = n_3 = n_4 = 1$ and $n_2 > 1$. Actually, using the algebraic manipulator Mathematica, we obtained evidences that this number of periodic solutions cannot happen, but an analytical treatment is not trivial.

2.2 *Discussion of the results*

Now we discuss some aspects of the averaging method presented in Section 3 in order to find periodic solutions in system (4). In fact by using the averaging theory and rescaling (5), we shall see that we cannot prove that system (4) has periodic solutions except in the cases presented in Theorems 1–8.

We start studying the possible values to the powers of ε in system (8). This is important because these powers play an important role in the averaging theory described in Section 3 because they determine the terms of order 1 that we are interested in. Indeed, we note that each one of the five different powers of ε in system (8) must be non-negative. Therefore, considering these powers as zero or positive, we have 32 possible combinations of these 5 different powers of ε. Actually some of them are not algebraically possible. Table 1 exhibits only the possible case.

Observing Table 1 we see that in fact we have only 18 cases. Moreover, in order to apply the averaging theory for one of the 18 cases, we must integrate the equations of the non-perturbed part of the vector field of system (8). This is not a simple task because the major part of these equations are non-linear and non-autonomous. Indeed in each case from 1 to 8 and from 11 to 16 we could not integrate the equations even using the algebraic manipulator Mathematica or Maple. The non-integrable cases from Table 1 are listed as follows:

Case 1: $F_0(t, x) = (y, -x + ry - ax^3 - rx^2y - \ell x^5 + d\cos t)$,
Case 2: $F_0(t, x) = (y, -x + ry - ax^3 - rx^2y - \ell x^5)$,
Case 3: $F_0(t, x) = (y, -x + ry - ax^3 - rx^2y + d\cos t)$,
Case 4: $F_0(t, x) = (y, -x + ry - ax^3 - rx^2y)$,
Case 5: $F_0(t, x) = (y, -x + ry - rx^2y - \ell x^5 + d\cos t)$,
Case 6: $F_0(t, x) = (y, -x + ry - rx^2y - \ell x^5)$,
Case 7: $F_0(t, x) = (y, -x + ry - rx^2y + d\cos t)$,
Case 8: $F_0(t, x) = (y, -x + ry - rx^2y)$,
Case 11: $F_0(t, x) = (y, -x - ax^3 - \ell x^5 + d\cos t)$,
Case 12: $F_0(t, x) = (y, -x - ax^3 - \ell x^5)$,
Case 13: $F_0(t, x) = (y, -x - ax^3 + d\cos t)$,
Case 14: $F_0(t, x) = (y, -x - ax^3)$,
Case 15: $F_0(t, x) = (y, -x - \ell x^5 + d\cos t)$,
Case 16: $F_0(t, x) = (y, -x - \ell x^5)$.

Table 1. Here, when appear $\varnothing$, this means that the corresponding cases cannot occur.

Possible combination of powers of ε for the system (9)					
$n_1 = 0$	$2m + n_1 = 0$	$2m + n_2 = 0$	$4m + n_3 = 0$	$-m + n_4 = 0$	C1
				$-m + n_4 > 0$	C2
			$4m + n_3 > 0$	$-m + n_4 = 0$	C3
				$-m + n_4 > 0$	C4
		$2m + n_2 > 0$	$4m + n_3 = 0$	$-m + n_4 = 0$	C5
				$-m + n_4 > 0$	C6
			$4m + n_3 > 0$	$-m + n_4 = 0$	C7
				$-m + n_4 > 0$	C8
	$2m + n_1 > 0$	$2m + n_2 = 0$		$\varnothing$	
		$2m + n_2 > 0$	$4m + n_3 = 0$	$\varnothing$	
			$4m + n_3 > 0$	$-m + n_4 = 0$	C9
				$-m + n_4 > 0$	C10
$n_1 > 0$	$2m + n_1 = 0$		$\varnothing$		
	$2m + n_1 > 0$	$2m + n_2 = 0$	$4m + n_3 = 0$	$-m + n_4 = 0$	C11
				$-m + n_4 > 0$	C12
			$4m + n_3 > 0$	$-m + n_4 = 0$	C13
				$-m + n_4 > 0$	C14
		$2m + n_2 > 0$	$4m + n_3 = 0$	$-m + n_4 = 0$	C15
				$-m + n_4 > 0$	C16
			$4m + n_3 > 0$	$-m + n_4 = 0$	C17
				$-m + n_4 > 0$	C18

Note: The notation Ci, $i = 1, \ldots, 18$ enumerate each possible case.

In particular, we observe that cases 12, 14 and 16 turn the non-perturbed part of system (8) into a Hamiltonian system. In [3], Buica and Llibre provide a method to apply the averaging theory in planar systems when the system is Hamiltonian, but the expressions become also too much complicated and again we could not integrate the equations in cases 12, 14 and 16.

In cases 9, 10 and 17, we can integrate the expressions but in these cases the hypotheses of the averaging method do not apply. In these cases, we have the following expressions for $F_0(t, \mathbf{x})$.

Case 9: $F_0(t, \mathbf{x}) = (y, -x + ry + d\cos t)$,
Case 10: $F_0(t, \mathbf{x}) = (y, -x + ry)$,
Case 17: $F_0(t, \mathbf{x}) = (y, -x + d\cos t)$.

Solution $x(t)$ in case 9 is

$$C^- \,\mathrm{e}^{(1/2)t(r-\sqrt{-4+r^2})} + C^+ \,\mathrm{e}^{(1/2)t(r+\sqrt{-4+r^2})} - \left(\frac{d}{r}\right)\sin t,$$

where

$$C^{\mp} = \frac{\mp 2d + r((\pm r + \sqrt{-4 + r^2})x_0 \mp 2y_0)}{2r\sqrt{-4 + r^2}}.$$

In order that $x(t)$ be periodic, we must have C^- and C^+ equal to zero. But these conditions are verified only if we take $x_0 = 0$ and $y_0 = -d/r$. So x_0 and y_0 are fixed and then the unperturbed system $\dot{x} = F_0(t, x)$ has no sub-manifold of periodic solutions. Consequently, we cannot apply the averaging theory in this case.

On the other hand, in case 10 solution $x(t)$ is written as

$$C^- \,\mathrm{e}^{(1/2)t(r-\sqrt{-4+r^2})} + C^+ \,\mathrm{e}^{(1/2)t(r+\sqrt{-4+r^2})} - \left(\frac{d}{r}\right)\sin t,$$

where

$$C^{\mp} = \frac{(\pm r + \sqrt{-4 + r^2})x_0 \mp 2y_0}{2r\sqrt{-4 + r^2}}.$$

Again we need to choose C^- and C^+ equal to zero, but this happens only if $(x_0, y_0) = (0, 0)$. So we cannot apply the averaging theory because $(0, 0)$ is the equilibrium point of the non-perturbed part of system (8) in case 10.

Finally, in case 17 solutions $x(t)$ and $y(t)$ have the form

$$x(t) = x_0 \cos t + y_0 \sin t + \frac{d}{2} t \cos t,$$
$$y(t) = y_0 \cos t - x_0 \sin t + \frac{d}{2}(t \cos t + \sin t).$$

It is immediate to note that these solutions are non-periodic because d is positive.

Case 18 provides positive results and Theorems 1–8 are based on this case. In fact, we observe that in case 18 the vector field of system (8) is

$$(y, -x + \varepsilon^{n_1} ry - \varepsilon^{2m+n_1} rx^2 y - \varepsilon^{2m+n_2} ax^3 - \varepsilon^{4m+n_3} \ell x^5 + \varepsilon^{-m+n_4} d \cos t)^{\mathrm{T}}.$$

As we commented before, the periodic solutions of our problem correspond to the simple zeros of function (10). So in order to calculate the zeros of the function $f(\mathbf{z})$ given in Equation (10), we must determine $F_1(t, \mathbf{x})$, where $F_1(t, \mathbf{x})$ is determined by the terms of order 1 on ε of the vector field of system (8). However, we see that the expression of $F_1(t, \mathbf{x})$ depends on the values m and n_i, $i = 1, 2, 3, 4$. In fact, if m is positive and observing condition (7), the only terms of the vector field of order 1 on ε can be generated by powers n_1 and $-m + n_4$, where these two values can be 1 or greater than 1. This implies four possibilities. On the other hand, if m is zero the powers of ε in the vector field of system (8) depend on n_1, n_2, n_3 and n_4. Again, each one of these powers can be 1 greater than 1. So, if m is zero we have 16 possibilities for F_1. Then case 18 has 20 subcases corresponding to the different possibilities of $F_1(t, \mathbf{x})$. These subcases are presented in Table 2.

Table 2. Possible expressions for $F_1(t, \mathbf{x})$ when $m^2 + n_2^2$, $m^2 + n_3^2$, $n_4 - m$ and n_1 are positives.

Sub.	Conditions	Second coordinate of $F_1(t, \mathbf{x})$
1	$m = 0, n_i = 1, i = 1, 2, 3, 4$	$ry - ax^3 - rx^2y - \ell x^5 + d \cos t$
2	$m = 0, n_i = 1, i = 1, 2, 3, n_4 > 1$	$ry - ax^3 - rx^2y - \ell x^5$
3	$m = 0, n_i = 1, i = 1, 2, 4, n_3 > 1$	$ry - ax^3 - rx^2y + d \cos t$
4	$m = 0, n_i = 1, i = 1, 3, 4, n_2 > 1$	$ry - rx^2y - \ell x^5 + d \cos t$
5	$m = 0, n_i = 1, i = 2, 3, 4, n_1 > 1$	$-ax^3 - \ell x^5 + d \cos t$
6	$m = 0, n_1 = n_2 = 1, n_3, n_4 > 1$	$ry - ax^3 - rx^2y$
7	$m = 0, n_1 = n_3 = 1, n_2, n_4 > 1$	$ry - rx^2y - \ell x^5$
8	$m = 0, n_1 = n_4 = 1, n_2, n_3 > 1$	$ry - rx^2y + d \cos t$
9	$m = 0, n_2 = n_3 = 1, n_1, n_4 > 1$	$-ax^3 - \ell x^5$
10	$m = 0, n_2 = n_4 = 1, n_1, n_3 > 1$	$-ax^3 + d \cos t$
11	$m = 0, n_3 = n_4 = 1, n_1, n_2 > 1$	$-\ell x^5 + d \cos t$
12	$m = 0, n_1 = 1, n_i > 1, i = 2, 3, 4$	$ry - rx^2y$
13	$m = 0, n_2 = 1, n_i > 1, i = 1, 3, 4$	$-ax^3$
14	$m = 0, n_3 = 1, n_i > 1, i = 1, 2, 4$	$-\ell x^5$
15	$m > 0, n_4 = 1, n_i > 1, i = 1, 2, 3$	$d \cos t$
16	$m > 0, n_i > 1, i = 1, 2, 3, 4$	0
17	$m > 0, n_1 = 1, -m + n_4 = 1$	$ry + d \cos t$
18	$m > 0, n_1 = 1, -m + n_4 > 1$	ry
19	$m > 0, n_1 > 1, -m + n_4 = 1$	$d \cos t$
20	$m > 0, n_1 > 1, -m + n_4 > 1$	0

Notes: We exhibit only the second coordinate of $F_1(t, \mathbf{x})$ because the first one has no terms depending on ε.

Theorems 1–8 correspond to subcases 8, 10, 11, 17, 5, 3, 1 and 4, respectively. These are the only subcases of case 18 where the averaging method provides positive results. Indeed, in cases 15, 16, 19 and 20, function (10) does not have zeros. On the other hand, in cases 2, 6, 7, 13, 14, 18, the only zero of function (10) is $(x_0, y_0) = (0, 0)$ that corresponds to the equilibrium point of the system, and consequently, in these subcases the system does not have periodic solutions. In cases 9 and 12, function (10) has real zeros different from $(0, 0)$, but they are non-isolated and then the Jacobian of function (10) at the zero is zero, and consequently, the averaging theory cannot be applied.

3. The averaging theory for periodic solutions

Now we present the basic results on the averaging theory of first order that we need to prove our results.

Consider the problem of bifurcation of T-periodic solutions from differential systems of the form

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 R(t, \mathbf{x}, \varepsilon), \tag{12}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. Here the functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $R : \mathbb{R} \times \Omega \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are $\mathcal{C}^2$, T-periodics in the first variable and Ω is an open subset of $\mathbb{R}^n$. One of the main assumptions is that the unperturbed system

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}) \tag{13}$$

has a manifold of periodic solutions. A solution to this problem is given using the averaging theory.

Indeed, assume that there is an open set V with $\bar{V} \subset D \subset \Omega$ and such that for each $\mathbf{z} \in \bar{V}$, $\mathbf{x}(\cdot, \mathbf{z}, 0)$ is T-periodic, where $\mathbf{x}(\cdot, \mathbf{z}, 0)$ is the solution of the unperturbed system (13) with $\mathbf{x}(0) = \mathbf{z}$. Answer to the problem of bifurcation of T-periodic solutions from $\mathbf{x}(\cdot, \mathbf{z}, 0)$ is given in the following theorem.

THEOREM 9 *We assume that there exists an open set V with $\bar{V} \subset D$ and such that for each $z \in \bar{V}$, $\boldsymbol{x}(\cdot, z, 0)$ is T-periodic and consider the function $f : \bar{V} \to \mathbb{R}^n$ given by*

$$f(z) = \int_0^{\mathrm{T}} Y^{-1}(t, z) F_1(t, \boldsymbol{x}(t, z, 0))\, \mathrm{d}t.$$

Then the following statements hold.

(a) If there exists $a \in V$ with $f(a) = 0$ and $\det((\partial f/\partial z)(a)) \neq 0$, then there exists a T-periodic solution $\varphi(\cdot, \varepsilon)$ of system (12) such that $\varphi(0, \varepsilon) \to a$ as $\varepsilon \to 0$.

(b) The type of stability of the periodic solution $\varphi(\cdot, \varepsilon)$ is given by the eigenvalues of the Jacobian matrix $M = ((\partial f/\partial z)(a))$.

For a proof of Theorem 9(a), see [4, Corollary 1].

In fact, the result of Theorem 9 is a classical result due to Malkin [20] and Roseau [21]. For a shorter proof of Theorem 9(a), see [4].

For additional information on the averaging theory, see [23].

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Appendix. Root classification for a quintic polynomial

In this section, we present a brief summary of the results about the number and multiplicities of the real/complex roots for a quintic polynomial with arbitrary coefficients presented in [29]. Indeed, consider the polynomial $P(x) = x^5 + px^3 + qx^2 + ux + v$. So, the following table gives the number of real and complex roots and multiplicities of roots of $P(x)$ in all cases:

(1)	$D_5 > 0 \wedge D_4 > 0 \wedge D_3 > 0 \wedge D_2 > 0$	$\{1, 1, 1, 1, 1\}$
(2)	$D_5 > 0 \wedge (D_4 \leq 0 \vee D_3 \leq 0 \vee D_2 \leq 0)$	$\{1\}$
(3)	$D_5 < 0$	$\{1, 1, 1\}$
(4)	$D_5 = 0 \wedge D_4 > 0$	$\{2, 1, 1, 1\}$
(5)	$D_5 = 0 \wedge D_4 < 0$	$\{2, 1\}$
(6)	$D_5 = 0 \wedge D_4 = 0 \wedge D_3 > 0 \wedge E_2 \neq 0$	$\{2, 2, 1\}$
(7)	$D_5 = 0 \wedge D_4 = 0 \wedge D_3 > 0 \wedge E_2 = 0$	$\{3, 1, 1\}$
(8)	$D_5 = 0 \wedge D_4 = 0 \wedge D_3 < 0 \wedge E_2 \neq 0$	$\{1\}$
(9)	$D_5 = 0 \wedge D_4 = 0 \wedge D_3 < 0 \wedge E_2 = 0$	$\{3\}$
(10)	$D_5 = 0 \wedge D_4 = 0 \wedge D_3 = 0 \wedge D_2 \neq 0 \wedge F_2 \neq 0$	$\{3, 2\}$
(11)	$D_5 = 0 \wedge D_4 = 0 \wedge D_3 = 0 \wedge D_2 \neq 0 \wedge F_2 = 0$	$\{4, 1\}$
(12)	$D_5 = 0 \wedge D_4 = 0 \wedge D_3 = 0 \wedge D_2 = 0$	$\{5\}$

where

$$\begin{aligned}
D_2 &= -p,\\
D_3 &= -12p^3 - 45q^2 + 40pu,\\
D_4 &= -4p^3q^2 - 27q^4 + 12p^4u + 117pq^2u - 88p^2u^2 + 160u^3 - 40p^2qv - 300quv + 125pv^2,\\
D_5 &= -4p^3q^2u^2 - 27q^4u^2 + 16p^4u^3 + 144pq^2u^3 - 128p^2u^4 + 256u^5\\
&\quad + 16p^3q^3v + 108q^5v - 72p^4quv - 630pq^3uv + 560p^2qu^2v - 1600qu^3v\\
&\quad + 108p^5v^2 + 825p^2q^2v^2 - 900p^3uv^2 + 2250q^2uv^2 + 2000pu^2v^2 - 3750pqv^3 + 3125v^4,\\
E_2 &= 16p^4q^2 - 48p^5u + 60p^2q^2u + 160p^3u^2 + 900q^2u^2 - 1100p^3qv\\
&\quad - 3375q^3v + 1500pquv + 625p^2v^2,\\
F_2 &= 3q^2 - 8pu.
\end{aligned}$$

The polynomials D_i, $i = 2, 3, 4, 5$, E_2 and F_2 form a discriminant system which is sufficient for the classification of roots of the polynomial $P(x)$, which is described by the right column of the table. For instance, $\{1, 1, 1\}$ means three real simple roots and a pair of complex roots, and $\{3, 1, 1\}$ means a real triple root plus two real simple roots.