# Strong metric dimension of rooted product graphs 

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#### Abstract

Let $G$ be a connected graph. A vertex $w$ strongly resolves a pair $u, v$ of vertices of $G$ if there exists some shortest $u-w$ path containing $v$ or some shortest $v-w$ path containing $u$. A set $W$ of vertices is a strong resolving set for $G$ if every pair of vertices of $G$ is strongly resolved by some vertex of $W$. The smallest cardinality of a strong resolving set for $G$ is called the strong metric dimension of $G$. It is known that the problem of computing this invariant is NP-hard. This suggests finding the strong metric dimension for special classes of graphs or obtaining good bounds on this invariant. In this paper we study the problem of finding exact values or sharp bounds for the strong metric dimension of rooted product of graphs and express these in terms of invariants of the factor graphs.


Keywords: Strong metric dimension; rooted product graphs; strong metric basis; strong resolving set.

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## 1 Introduction

A generator of a metric space is a set $S$ of points in the space with the property that every point of the space is uniquely determined by its distances from the elements of $S$. Given a simple and connected graph $G=(V, E)$, we consider the metric $d_{G}: V \times V \rightarrow \mathbb{R}^{+}$, where $d_{G}(x, y)$ is the length of a shortest path between $x$ and $y .\left(V, d_{G}\right)$ is clearly a metric space. A vertex $v \in V$ is said to distinguish two vertices $x$ and $y$ if $d_{G}(v, x) \neq d_{G}(v, y)$. A set $S \subset V$ is said to be a metric generator for $G$ if any pair of vertices of $G$ is distinguished by some element of $S$. A minimum generator is called a metric basis, and its cardinality the metric dimension of $G$, denoted by $\operatorname{dim}(G)$. Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension of a graph was introduced by Slater in [23, 24], where the metric generators were called locating sets. The concept of metric dimension of a graph was introduced independently by Harary and Melter in [9], where metric generators were called resolving sets. Applications of this invariant to the navigation of robots in networks are discussed in [13] and applications to chemistry
in $[11,12]$. This invariant was studied further in a number of other papers including for example, $[2,3,4,6,7,10,16,18,20,25,26,27,28]$. Several variations of metric generators including resolving dominating sets [1], independent resolving sets [5], local metric sets [18], and strong resolving sets $[14,17,22]$, etc. have been introduced and studied.

In this article we are interested in the study of strong resolving sets [17, 22]. A vertex $w \in V(G)$ strongly resolves two vertices $u, v \in V(G)$ if $d_{G}(w, u)=d_{G}(w, v)+d_{G}(v, u)$ or $d_{G}(w, v)=d_{G}(w, u)+d_{G}(u, v)$, i.e., there exists some shortest $w-u$ path containing $v$ or some shortest $w-v$ path containing $u$. A set $S$ of vertices in a connected graph $G$ is a strong metric generator for $G$ if every two vertices of $G$ are strongly resolved by some vertex of $S$. The smallest cardinality of a strong resolving set of $G$ is called strong metric dimension and is denoted by $\operatorname{dim}_{s}(G)$. So, for example, $\operatorname{dim}_{s}(G)=n-1$ if and only if $G$ is the complete graph of order $n$. For the cycle $C_{n}$ of order $n$ the strong metric dimension is $\operatorname{dim}_{s}\left(C_{n}\right)=\lceil n / 2\rceil$ and if $T$ is a tree with $l(T)$ leaves, its strong metric dimension equals $l(T)-1$ (see [22]). A strong metric basis of $G$ is a strong metric generator for $G$ of cardinality $\operatorname{dim}_{s}(G)$.

Given a simple graph $G=(V, E)$, we denote two adjacent vertices $u, v$ by $u \sim v$. The neighborhood of a vertex $v$ of $G$ is $N_{G}(v)=\{u \in V(G): u \sim v\}$ and the degree of $v$ is $\delta_{G}(v)=\left|N_{G}(v)\right|$. The open neighborhood of a set $S$ of vertices of $G$ is $N_{G}(S)=\bigcup_{v \in S} N_{G}(v)$ and the closed neighborhood of $S$ is $N_{G}[S]=N_{G}(S) \cup S$. The subgraph induced by a set $X$ will be denoted by $\langle X\rangle$. A vertex $u$ of $G$ is maximally distant from $v$ if for every vertex $w$ in the open neighborhood of $u, d_{G}(v, w) \leq d_{G}(u, v)$. If $u$ is maximally distant from $v$ and $v$ is maximally distant from $u$, then we say that $u$ and $v$ are mutually maximally distant. The boundary of $G=(V, E)$ is defined as $\partial(G)=\{u \in V$ : there exists $v \in V$ such that $u, v$ are mutually maximally distant $\}$. For some basic graph classes, such as complete graphs $K_{n}$, complete bipartite graphs $K_{r, s}$, cycles $C_{n}$ and hypercube graphs $Q_{k}$, the boundary is simply the whole vertex set. It is not difficult to see that this property holds for all 2 -antipodal ${ }^{1}$ graphs and also for all distance-regular graphs. Notice that the boundary of a tree consists exactly of the set of its leaves. A vertex of a graph is a simplicial vertex if the subgraph induced by its neighbors is a complete graph. Given a graph $G$, we denote by $\sigma(G)$ the set of simplicial vertices of $G$. Notice that $\sigma(G) \subseteq \partial(G)$.

We use the notion of strong resolving graph introduced in [17]. The strong resolving graph ${ }^{2}$ of $G$ is a graph $G_{S R}$ with vertex set $V\left(G_{S R}\right)=\partial(G)$ where two vertices $u, v$ are adjacent in $G_{S R}$ if and only if $u$ and $v$ are mutually maximally distant in $G$.

There are some families of graphs for which its strong resolving graph can be obtained relatively easy. For instance, we emphasize the following cases.

- If $\partial(G)=\sigma(G)$, then $G_{S R} \cong K_{|\partial(G)|}$. In particular, $\left(K_{n}\right)_{S R} \cong K_{n}$ and for any tree $T$ with $l(T)$ leaves, $(T)_{S R} \cong K_{l(T)}$.
- For any 2-antipodal graph $G$ of order $n, G_{S R} \cong \bigcup_{i=1}^{\frac{n}{2}} K_{2}$. In particular, $\left(C_{2 k}\right)_{S R} \cong$ $\bigcup_{i=1}^{k} K_{2}$.
- $\left(C_{2 k+1}\right)_{S R} \cong C_{2 k+1}$.

[^0]A set $S$ of vertices of $G$ is a vertex cover of $G$ if every edge of $G$ is incident with at least one vertex of $S$. The vertex cover number of $G$, denoted by $\alpha(G)$, is the smallest cardinality of a vertex cover of $G$. We refer to an $\alpha(G)$-set in a graph $G$ as a vertex cover of cardinality $\alpha(G)$. Oellermann and Peters-Fransen [17] showed that the problem of finding the strong metric dimension of a connected graph $G$ can be transformed to the problem of finding the vertex cover number of $G_{S R}$. The following result will be an important tool of this article.

Theorem 1. [17] Let $G$ be connected graph. A set $W \subset V(G)$ is a strong metric generator for $G$ if and only if $W$ is a vertex cover for $G_{S R}$.

It was shown in [17] that the problem of computing $\operatorname{dim}_{s}(G)$ is NP-hard. This suggests finding the strong metric dimension for special classes of graphs or obtaining good bounds on this invariant. An efficient procedure for finding the strong metric dimension of distance hereditary graphs was described in [15]. In this paper we study the problem of finding exact values or sharp bounds for the strong metric dimension of rooted product of graphs and express these in terms of invariants of the factor graphs. Notice that the metric dimension of rooted product graphs has been recently studied in [28].


Figure 1: The rooted product graphs $P_{4} \circ C_{3}$ and $C_{3} \circ_{v} P_{4}$, where $v$ has degree two.
A rooted graph is a graph in which one vertex is labeled in a special way so as to distinguish it from other vertices. The special vertex is called the root of the graph. Let $G$ be a labeled graph on $n$ vertices. Let $\mathcal{H}$ be a sequence of $n$ rooted graphs $H_{1}, H_{2}, \ldots, H_{n}$. The rooted product graph $G(\mathcal{H})$ is the graph obtained by identifying the root of $H_{i}$ with the $i^{\text {th }}$ vertex of $G$ [8]. In this paper we consider the particular case of rooted product graph where $\mathcal{H}$ consists of $n$ isomorphic rooted graphs [21]. More formally, assuming that $V(G)=\left\{u_{1}, \ldots, u_{n}\right\}$ and that the root vertex of $H$ is $v$, we define the rooted product graph $G \circ_{v} H=(V, E)$, where $V=V(G) \times V(H)$ and

$$
E=\bigcup_{i=1}^{n}\left\{\left(u_{i}, b\right)\left(u_{i}, y\right): b y \in E(H)\right\} \cup\left\{\left(u_{i}, v\right)\left(u_{j}, v\right): u_{i} u_{j} \in E(G)\right\}
$$

Note that for any $x \in V(G)$ the subgraph $H_{x}=\langle\{x\} \times V(H)\rangle$ of $G \circ_{v} H$ is isomorphic to $H$. Given $x \in V(G), v \in V(H)$ and $B \subset V(G) \times V(H)$ we will denote by $B_{x}$ the set of element of $B$ whose first component is $x$, i.e., $B_{x}=B \cap(\{x\} \times V(H))$.

If $H$ is a vertex transitive graph, then $G \circ_{v} H$ does not depend on the choice of $v$, up to isomorphism. In such a case we will denote the rooted product by $G \circ H$. Figure 1 shows the case of the rooted product graphs $P_{4} \circ C_{3}$ and $C_{3} \circ_{v} P_{4}$, where $v$ has degree two. We also recall that the corona product $G \odot H$ is defined as the graph obtained from $G$ and $H$ by taking one copy of $G$ and $n$ copies of $H$ and joining by an edge each vertex from the
$i^{\text {th }}$-copy of $H$ with the $i^{\text {th }}$-vertex of $G$. If $G$ and $H$ are connected graphs of order $n \geq 2, H$ is a connected graph of order $t \geq 2$, then we note that the corona product graph $G \odot H$ is a particular case of a rooted product graph, i.e., $G \odot H \cong G \circ_{v}\left(K_{1}+H\right)$, where $v$ denotes the vertex of $K_{1}$. Metric dimension and strong metric dimension of corona product graphs were studied in [27] and [14], respectively.

We emphasize that given $a, b \in V(G)$ and $x, y, v \in V(H)$ it follows, $d_{G \circ_{v} H}((a, x),(a, y))=$ $d_{H}(x, y)$ and if $a \neq b$, then $d_{G \circ_{v} H}((a, x),(b, y))=d_{H}(x, v)+d_{G}(a, b)+d_{H}(v, y)$.

This article is composed by two main sections. In Section 2 we obtain closed formulae for the strong metric dimension of some classes of rooted product graphs while Section 3 is devoted to obtain tight bounds for the strong metric dimension of rooted product graphs.

## 2 Closed formulae

We start by stating the following easily verified lemmas.
Lemma 2. Let $G$ and $H$ be two connected graphs. Let $a, b \in V(G), a \neq b, x, y, v \in V(H)$ and let $M(v)$ be the set of vertices of $H$ which are maximally distant from $v$. Then $(a, x)$ and $(b, y)$ are mutually maximally distant vertices in $G \circ_{v} H$ if and only if $x, y \in M(v)$.

Proof. (Sufficiency) Suppose that $(a, x)$ and $(b, y)$ are not mutually maximally distant vertices in $G \circ_{v} H$. So, there exists a vertex $\left(a, x^{\prime}\right) \in N_{G \circ_{v} H}(a, x)$ such that $d_{G \circ_{v} H}\left(\left(a, x^{\prime}\right),(b, y)\right)>$ $d_{G \circ_{v} H}((a, x),(b, y))$, or there exists $\left(b, y^{\prime}\right) \in N_{G \circ_{v} H}(b, y)$ such that $d_{G \circ_{v} H}\left((a, x),\left(b, y^{\prime}\right)\right)>$ $d_{G \circ_{v} H}((a, x),(b, y))$. We consider, without loss of generality, that $\left(a, x^{\prime}\right) \in N_{G \circ_{v} H}(a, x)$ and $d_{G \circ_{v} H}\left(\left(a, x^{\prime}\right),(b, y)\right)>d_{G \circ_{v} H}((a, x),(b, y))$. So we have,

$$
\begin{aligned}
d_{H}\left(x^{\prime}, v\right) & =d_{G \circ_{v} H}\left(\left(a, x^{\prime}\right),(b, y)\right)-d_{G}(a, b)-d_{H}(v, y) \\
& >d_{G \circ_{v} H}((a, x),(b, y))-d_{G}(a, b)-d_{H}(v, y) \\
& =d_{H}(x, v) .
\end{aligned}
$$

Thus, $d_{H}\left(x^{\prime}, v\right)>d_{H}(x, v)$. Since $x^{\prime} \in N_{H}(x)$ and $x \in M(v)$, we have a contradiction.
(Necessity) Let us suppose that $x \notin M(v)$. So, there exists $x^{\prime \prime} \in N_{H}(x)$ such that $d_{H}\left(x^{\prime \prime}, v\right)>d_{H}(x, v)$. Thus, $d_{G \circ_{v} H}((a, x),(b, y))=d_{H}(x, v)+d_{G}(a, b)+d_{H}(v, y)<d_{H}\left(x^{\prime \prime}, v\right)+$ $d_{G}(a, b)+d_{H}(v, y)=d_{G \circ_{v} H}\left(\left(a, x^{\prime \prime}\right),(b, y)\right)$. Hence, there exists a vertex $\left(a, x^{\prime \prime}\right) \in N_{G \circ_{v} H}((a, x))$ such that $d_{G \circ_{v} H}((a, x),(b, y))<d_{G \circ_{v} H}\left(\left(a, x^{\prime \prime}\right),(b, y)\right)$, which is a contradiction since $(a, x)$ and $(b, y)$ are mutually maximally distant.

Lemma 3. Let $G$ and $H$ be two connected nontrivial graphs. Let $v, x, y$ be vertices of $H$ such that $x, y \neq v$. For every vertex a of $G$ we have that $(a, x)$ and $(a, y)$ are mutually maximally distant vertices in $G \circ_{v} H$ if and only if the vertices $x$ and $y$ are mutually maximally distant in $H$.

Proof. The result follows directly from the fact that for every vertex $c$ of $G$ and every vertex $z \neq v$ of $H$ we have that $w \in N_{H}(z)$ if and only if $(c, w) \in N_{G \circ_{v} H}(c, z)$ and also that $d_{G \circ_{v} H}((a, x),(a, y))=d_{H}(x, y)$ for every $x, y$ of $H$.

Lemma 4. Let $H$ be a connected graph, let $v \in V(H)$ and let $M(v)$ be the set of vertices of $H$ which are maximally distant from $v$. Then $M(v) \subseteq \partial(H)$.

Proof. Let $u \in M(v)$. If $v$ is not maximally distant from $u$, then there exists a vertex $y_{1} \in N(v)$ such that $d\left(y_{1}, u\right)>d(v, u)$. So $u$ is maximally distant from $y_{1}$. By repeating this argument, since $H$ is finite, we will find a vertex $y_{i}$ such that $y_{i}$ and $u$ are mutually maximally distant. Therefore, $u \in \partial(H)$.

Proposition 5. Let $G$ be a connected graph of order $n \geq 2$ and let $H$ be a connected graph.
(i) If $v \in \partial(H)$, then $\partial\left(G \circ_{v} H\right)=V(G) \times(\partial(H)-\{v\})$.
(ii) If $v \notin \partial(H)$, then $\partial\left(G \circ_{v} H\right)=V(G) \times \partial(H)$.

Proof. Let $(x, y)$ and $\left(x^{\prime}, y^{\prime}\right)$ be two mutually maximally distant vertices in $G \circ_{v} H$. Since $(V(G) \times\{v\}) \cap \partial\left(G \circ_{v} H\right)=\emptyset$, it follows $y, y^{\prime} \neq v$. We differentiate two cases.

Case 1: $x=x^{\prime}$. By Lemma 3 we conclude that $(x, y)$ and ( $x^{\prime}, y^{\prime}$ ) are mutually maximally distant in $G \circ_{v} H$ if and only if $y$ and $y^{\prime}$ are mutually maximally distant in $H$.

Case 2: $x \neq x^{\prime}$. By Lemma 2 the vertices $(x, y)$ and $\left(x^{\prime}, y^{\prime}\right)$ are mutually maximally distant in $G \circ_{v} H$ if and only if $y, y^{\prime} \in M(v)$. Note that, by Lemma 4, $y, y^{\prime} \in \partial(H)$.

According to the above cases we conclude that if $(x, y) \in \partial\left(G \circ_{v} H\right)$, then $y \in \partial(H)-\{v\}$. Moreover, if $y \in \partial(H)-\{v\}$, then for every $x \in V(G)$ we have $(x, y) \in \partial\left(G \circ \circ_{v} H\right)$.

Therefore, if $v \in \partial(H)$, then $\partial\left(G \circ_{v} H\right)=V(G) \times(\partial(H)-\{v\})$ and if $v \notin \partial(H)$, then $\partial\left(G \circ_{v} H\right)=V(G) \times \partial(H)$.

Proposition 6. Let $G$ be a connected graph of order $n \geq 2$ and let $H$ be a connected graph.
(i) If $v \in \sigma(H)$, then $\sigma\left(G \circ_{v} H\right)=V(G) \times(\sigma(H)-\{v\})$.
(ii) If $v \notin \sigma(H)$, then $\sigma\left(G \circ_{v} H\right)=V(G) \times \sigma(H)$.

Proof. Note that $(x, v)$ is not simplicial in $G \circ_{v} H$. Since the following assertions are equivalent, the result immediately follows.

- The vertex $(x, y) \in V(G) \times(V(H)-\{v\})$ is simplicial in $G \circ_{v} H$.
- For $x \in V(G)$ and $y \neq v$ the vertex $(x, y)$ is simplicial in $H_{x}$.
- The vertex $y \in V(H)-\{v\}$ is simplicial in $H$.

Theorem 7. Let $G$ be a connected graph of order $n \geq 2$ and let $H$ be a connected graph such that $\partial(H)=\sigma(H)$.
(i) If $v \in \partial(H)$, then $\operatorname{dim}_{s}\left(G \circ_{v} H\right)=n(|\partial(H)|-1)-1$.
(ii) If $v \notin \partial(H)$, then $\operatorname{dim}_{s}\left(G \circ_{v} H\right)=n|\partial(H)|-1$.

Proof. Since $\partial(H)=\sigma(H)$, as a direct consequence of Proposition 5 and Proposition 6 we obtain that if $v \notin \partial(H)$, then $\partial\left(G \circ_{v} H\right)=V(G) \times \partial(H)=\sigma\left(G \circ_{v} H\right)$ and if $v \in \partial(H)$, then $\partial\left(G \circ_{v} H\right)=V(G) \times(\partial(H)-\{v\})=\sigma\left(G \circ_{v} H\right)$. Hence, if $v \notin \partial(H)$, then $\left(G \circ_{v} H\right)_{S R} \cong$ $K_{n|\partial(H)|}$ and, if $v \in \partial(H)$, then $\left(G \circ_{v} H\right)_{S R} \cong K_{n(|\partial(H)|-1)}$. Therefore, the result follows by Theorem 1.

We emphasize the following particular cases of Theorem 7.

Corollary 8. Let $G$ be a connected graph of order $n \geq 2$.
(i) For any complete graph of order $n^{\prime}, \operatorname{dim}_{s}\left(G \circ K_{n^{\prime}}\right)=n\left(n^{\prime}-1\right)-1$.
(ii) For any tree $T$ with $l(T)$ leaves,

$$
\operatorname{dim}_{s}\left(G \circ_{v} T\right)= \begin{cases}n(l(T)-1)-1, & \text { if } v \text { is a leaf of } T \\ n \cdot l(T)-1, & \text { if } v \text { is an inner vertex of } T\end{cases}
$$

(iii) Let $G^{\prime}$ be a connected graph of order $n^{\prime}$ and let $H=G^{\prime} \odot\left(\bigcup_{i=1}^{r} K_{t_{i}}\right)$, where $r \geq 2$, $t_{i} \geq 1$. Then

$$
\operatorname{dim}_{s}\left(G \circ_{v} H\right)= \begin{cases}n \sum_{i=1}^{r} t_{i}-n-1, & \text { if } v \in \bigcup_{i=1}^{r} V\left(K_{t_{i}}\right) \\ n \sum_{i=1}^{r} t_{i}-1, & \text { if } v \in V\left(G^{\prime}\right)\end{cases}
$$

Theorem 9. Let $G$ be a connected graph of order $n \geq 2$ and let $H$ be a connected graph such that $H_{S R} \cong \bigcup_{i=1}^{\frac{|\partial(H)|}{2}} K_{2}$. Let $v \in V(H)$ and let $M(v)$ be the set of vertices of $H$ which are maximally distant from $v$. Let $i(v)$ be the set of isolated vertices of the subgraph of $H_{S R}$ induced by $M(v)$.
(i) If $v \notin \partial(H)$, then $\operatorname{dim}_{s}\left(G \circ_{v} H\right)=\frac{n(|\partial(H)|+|M(v)|-|i(v)|)-|M(v)|+|i(v)|}{2}$.
(ii) If $v \in \partial(H)$, then $\operatorname{dim}_{s}\left(G \circ_{v} H\right)=\frac{n(|\partial(H)|+|M(v)|-|i(v)|)-|M(v)|+|i(v)|-2}{2}$.

Proof. Let $V(G)=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ be the vertex set of $G$ and let $B$ be a vertex cover for $\left(G \circ_{v} H\right)_{S R}$. First we note that by premiss for every $a \in \partial(H)$ there exists exactly one vertex $a^{\prime} \in \partial(H)$ such that $a$ and $a^{\prime}$ are adjacent in $H_{S R}$. We consider the set $i^{\prime}(v) \subset \partial(H)$ defined in the following way: $a^{\prime} \in i^{\prime}(v)$ if and only if there exists $a \in i(v)$ such that $a$ and $a^{\prime}$ are mutually maximally distant in $H$. Note that $\left|i^{\prime}(v)\right|=|i(v)|$ and, if $v \in \partial(H)$ and $v, v^{\prime}$ are mutually maximally distant, then $v \in i^{\prime}(v)$ and $v^{\prime} \in i(v)$. Also, since there are no edges in $H_{S R}$ connecting vertices belonging to $M(v) \cup i^{\prime}(v)$ to vertices belonging to $\partial(H)-M(v) \cup i^{\prime}(v)$, by Lemmas 2 and 3 we conclude that there are no edges in $\left(G \circ_{v} H\right)_{S R}$ connecting vertices belonging to $V(G) \times\left(\partial(H)-\left(M(v) \cup i^{\prime}(v)\right)\right.$ to vertices belonging to $V(G) \times\left(M(v) \cup i^{\prime}(v)\right)$. With this idea in mind, we proceed to prove the results.

In order to prove (i) we consider that $v \notin \partial(H)$. Note that in this case by Proposition 5 (ii), $\partial\left(G \circ_{v} H\right)=V(G) \times \partial(H)$. By Lemma 3 we have that for every mutually maximally distant vertices $a, a^{\prime} \in \partial(H)-\left(M(v) \cup i^{\prime}(v)\right)$ and every $j \in\{1, \ldots, n\}$ the vertices $\left(x_{j}, a\right)$ and $\left(x_{j}, a^{\prime}\right)$ are mutually maximally distant in $G \circ_{v} H$ and, as a consequence, $\left(x_{j}, a\right) \notin B$ if and only if $\left(x_{j}, a^{\prime}\right) \in B$. Thus, the subgraph of $\left(G \circ_{v} H\right)_{S R}$ induced by $V(G) \times\left(\partial(H)-M(v) \cup i^{\prime}(v)\right)$ is composed by $\frac{n}{2}\left(|\partial(H)|-|M(v)|-\left|i^{\prime}(v)\right|\right)$ components isomorphic to $K_{2}$.

On the other hand, by Lemma 2 we have that $\left(x_{j}, a\right),\left(x_{k}, a\right)$ are mutually maximally distant in $G \circ_{v} H$, for every $a \in M(v)$ and $j \neq k$. Thus, if $\left(x_{j}, a\right) \notin B$ for some $j$, then $\left(x_{k}, a\right) \in B$ for every $k \neq j$. Moreover, as above, Lemma 3 allows us to conclude that given two mutually maximally distant vertices $a, a^{\prime} \in M(v) \cup i^{\prime}(v)$ it follows that $\left(x_{j}, a\right) \notin B$ if and
only if $\left(x_{j}, a^{\prime}\right) \in B$. Thus, $B$ contains exactly $(n-1) \left\lvert\, M(v)+\frac{\left|M(v) \cup i^{\prime}(v)\right|}{2}\right.$ vertices belonging to $V(G) \times\left(M(v) \cup i^{\prime}(v)\right)$. Therefore,

$$
\begin{aligned}
|B| & =\frac{n(|\partial(H)|-|M(v)|-|i(v)|)}{2}+(n-1)|M(v)|+\frac{|M(v)|+|i(v)|}{2} \\
& =\frac{n(|\partial(H)|+|M(v)|-|i(v)|)-|M(v)|+|i(v)|}{2}
\end{aligned}
$$

The proof of (i) is complete.
From now on we suppose $v \in \partial(H)$. Note that in this case by Proposition 5 (i) we have $\partial\left(G \circ_{v} H\right)=V(G) \times(\partial(H)-\{v\})$. To prove (ii) we proceed by analogy to the proof of (i). In this case we obtain that the subgraph of $\left(G \circ_{v} H\right)_{S R}$ induced by $V(G) \times\left(\partial(H)-\left(M(v) \cup i^{\prime}(v)\right)\right.$ is composed by $\frac{n}{2}\left|\partial(H)-M(v) \cup i^{\prime}(v)\right|=\frac{n}{2}(|\partial(H)|-|M(v)|-|i(v)|)$ components isomorphic to $K_{2}$ and $B$ contains exactly $(n-1)|M(v)|+\frac{\left|\left(M(v)-\left\{v^{\prime}\right\}\right) \cup\left(i^{\prime}(v)-\{v\}\right)\right|}{2}=(n-1)|M(v)|+\frac{|M(v)|+|i(v)|-2}{2}$ vertices of $G \circ_{v} H$ belonging to $V(G) \times\left(M(v) \cup\left(i^{\prime}(v)-\{v\}\right)\right)$. Thus,

$$
\begin{aligned}
|B| & =\frac{n(|\partial(H)|-|M(v)|-|i(v)|)}{2}+(n-1)|M(v)|+\frac{|M(v)|+|i(v)|-2}{2} \\
& =\frac{n(|\partial(H)|+|M(v)|-|i(v)|)-|M(v)|+|i(v)|-2}{2}
\end{aligned}
$$

The proof of (ii) is complete.
We conjecture that if $v \notin \partial(H)$, then $i(v)=i^{\prime}(v)=\emptyset$. In order to show a particular case of Theorem 9 where $i(v) \neq \emptyset$ we consider the graph $H$ shown in the left hand side of Figure 2 where $\partial(H)=\left\{a, a^{\prime}, b, b^{\prime}, v, v^{\prime}\right\}, M(v)=i(v)=\left\{a, v^{\prime}\right\}$ and $i^{\prime}(v)=\left\{a^{\prime}, v\right\}$. In the case of the graph $H$ shown in the right hand side of Figure 2 we have $\partial(H)=\left\{a, a^{\prime}, b, b^{\prime}, v, v^{\prime}\right\}$, $M(v)=\left\{a, a^{\prime}, v^{\prime}\right\}, i(v)=\left\{v^{\prime}\right\}$ and $i^{\prime}(v)=\{v\}$. In both cases

$$
B=\left(V(G)-\left\{u_{n}\right\}\right) \times(M(v) \cup\{b\}) \cup\left\{\left(u_{n}, a\right),\left(u_{n}, b\right)\right\}
$$

is a strong metric basis of $G \circ_{v} H$ for any graph $G$ with vertex set $V=\left\{u_{1}, u_{2}, \ldots, u_{n}\right\}$.


Figure 2: In left hand side graph $i(v)=\left\{a, v^{\prime}\right\}$ and $i^{\prime}(v)=\left\{a^{\prime}, v\right\}$. In right hand side graph $i(v)=\left\{v^{\prime}\right\}$ and $i^{\prime}(v)=\{v\}$.

Corollary 10. Let $G$ be a connected graph of order $n \geq 2$ and let $H$ be a connected 2antipodal graph of order $n^{\prime}$. Then $\operatorname{dim}_{s}(G \circ H)=\frac{n n^{\prime}}{2}-1$.

Theorem 11. Let $C_{t}$ be a cycle of order $t \geq 3$. For any connected graph $G$ of order $r \geq 2$,

$$
\operatorname{dim}_{s}\left(G \circ C_{t}\right)=r\left\lceil\frac{t}{2}\right\rceil-1
$$

Proof. Let $V(G)=\left\{x_{1}, x_{2}, \ldots, x_{r}\right\}$ and $V\left(C_{t}\right)=\left\{y_{0}, y_{1}, \ldots, y_{t-1}\right\}$ be the vertex sets of $G$ and $C_{t}$, respectively. We assume $y_{0} \sim y_{1} \sim \ldots \sim y_{t-1} \sim y_{0}$ in $C_{t}$ and from now on all the operations with the subscripts of $y_{i}$ are done modulo $t$. Since $C_{t}$ is a vertex transitive graph, we can take without loss of generality $v=y_{0}$ as the root of $C_{t}$.

If $t$ be an even number, then $C_{t}$ is 2 -antipodal. So the result follows by Corollary 10. Now let $t$ be an odd number. Note that exactly two vertices $y_{\left\lceil\frac{t}{2}\right\rceil}$ and $y_{\left\lfloor\frac{t}{2}\right\rfloor}$ are maximally distant from $v$ in $C_{t}$. So, from Lemma 2 we have that every vertex $\left(x_{i}, y_{l}\right)$ is mutually maximally distant from $\left(x_{j}, y_{k}\right)$ in $G \circ C_{t}$, with $j \neq i$ and $l, k \in\left\{\left\lceil\frac{t}{2}\right\rceil,\left\lfloor\frac{t}{2}\right\rfloor\right\}$. Moreover, from Lemma 3 we have that for every $i \in\{1,2, \ldots, r\},\left(x_{i}, y_{k}\right)$ is mutually maximally distant from $\left(x_{i}, y_{k+\left\lfloor\frac{t}{2}\right\rfloor}\right)$ and $\left(x_{i}, y_{k+\left\lceil\frac{t}{2}\right\rceil}\right)$ in $G \circ C_{t}$ with $k \in\left\{1,2, \ldots,\left\lfloor\frac{t}{2}\right\rfloor-1,\left\lceil\frac{t}{2}\right\rceil+1, \ldots, t-1\right\}$. Also, the vertex $\left(x_{i}, y_{\left\lfloor\frac{t}{2}\right\rfloor}\right)$ is mutually maximally distant from $\left(x_{i}, y_{t-1}\right)$ and the vertex $\left(x_{i}, y_{\left\lceil\frac{t}{2}\right\rceil}\right)$ is mutually maximally distant from $\left(x_{i}, y_{1}\right)$. Thus, we obtain that the graph $\left(G \circ C_{t}\right)_{S R}$ is isomorphic to a graph with set of vertices $U \cup\left(\bigcup_{i=1}^{r} V_{i}\right)$ where $\langle U\rangle$ is isomorphic to a complete $r$-partite graph $K_{2,2, \ldots, 2}$ and for every $i \in\{1, \ldots, r\},\left\langle V_{i}\right\rangle$ is isomorphic to a path graph $P_{t-1}$. Notice that the leaves of $P_{t-1}$ belong to $U$, so for every $i \in\{1, \ldots, r\},\left|V_{i} \cap U\right|=2$. Thus, we have the following:

$$
\begin{aligned}
\operatorname{dim}_{s}\left(G \circ C_{t}\right) & =\alpha\left(\left(G \circ C_{t}\right)_{S R}\right) \\
& =\alpha(\langle U\rangle)+(r-1) \alpha\left(P_{t-3}\right)+\alpha\left(P_{t-1}\right) \\
& =2(r-1)+(r-1) \frac{t-3}{2}+\frac{t-1}{2} \\
& =r\left\lceil\frac{t}{2}\right\rceil-1 .
\end{aligned}
$$

The proof is complete.
We recall that the clique number of a graph $H$, denoted by $\omega(H)$, is the number of vertices in a maximum clique in $H$. Two distinct vertices $x, y$ are called true twins if $N_{H}[x]=N_{H}[y]$. We say that $X \subset V(H)$ is a twin-free clique in $H$ if the subgraph induced by $X$ is a clique and for every $u, v \in X$ it follows $N_{H}[u] \neq N_{H}[v]$, i.e., the subgraph induced by $X$ is a clique and it contains no true twins. We say that the twin-free clique number of $H$, denoted by $\varpi(H)$, is the maximum cardinality among all twin-free cliques in $H$. So, $\omega(H) \geq \varpi(H)$.

Theorem 12. [14] Let $G$ be a connected graph of order r. Let $H$ be a graph of order $t$ and maximum degree $\Delta$. If $\Delta \leq t-2$ or $r \geq 2$, then $\operatorname{dim}_{s}(G \odot H)=r t-\varpi(H)$.

Given a vertex $v$ of a graph $H$, we denote by $H-v$ the graph obtained by removing $v$ from $H$. Now, if $v$ is a vertex of $H$ of degree $n-1$, then the rooted product graph $G \circ_{v} H$ is isomorphic to the corona product graph $G \odot(H-v)$. So, as a direct consequence of Theorem 12 we obtain the following result.

Corollary 13. Let $G$ be a connected graph of order $r \geq 2$. Let $H$ be a connected graph of order $t \geq 2$ and let $v$ be a vertex of $H$ of degree $t-1$. Then $\operatorname{dim}_{s}\left(G \circ_{v} H\right)=r(t-1)-\varpi(H-v)$.

The next result gives the exact value for the strong metric dimension of $G \odot H$ when $H$ is a triangle free graph.

Theorem 14. [14] Let $G$ be a connected graph of order $r$ and let $H$ be a triangle free graph of order $t \geq 3$ and maximum degree $\Delta$. If $r \geq 2$ or $\Delta \leq t-2$, then

$$
\operatorname{dim}_{s}(G \odot H)=r t-2
$$

As a direct consequence of Theorem 14 we have the following.
Corollary 15. Let $G$ be a connected graph of order $r \geq 2$. Let $H$ be a connected graph of order $t \geq 2$ and let $v$ be a vertex of $H$ of degree $t-1$. If $H-v$ is a triangle free graph. Then

$$
\operatorname{dim}_{s}\left(G \circ_{v} H\right)=r(t-1)-2
$$

As the next theorem shows, the strong metric dimension of $G \odot H$ depends on the diameter of $H$.

Theorem 16. [14] Let $G$ be a connected graph of order $r$. Let $H$ be a graph of order $t$ and maximum degree $\Delta$.
(i) If $H$ has diameter two and either $\Delta \leq t-2$ or $r \geq 2$, then

$$
\operatorname{dim}_{s}(G \odot H)=(r-1) t+\operatorname{dim}_{s}(H)
$$

(ii) If $H$ is not connected or its diameter is greater than two, then

$$
\operatorname{dim}_{s}(G \odot H)=(r-1) t+\operatorname{dim}_{s}\left(K_{1}+H\right)
$$

Therefore, as a consequence of Theorem 16 we obtain the following result for $G \circ_{v} H$.
Corollary 17. Let $G$ be a connected graph of order $r \geq 2$. Let $H$ be a graph of order $t \geq 2$ and let $v$ be a vertex of $H$ of degree $t-1$.
(i) If $H-v$ has diameter two, then

$$
\operatorname{dim}_{s}\left(G \circ_{v} H\right)=(r-1)(t-1)+\operatorname{dim}_{s}(H-v)
$$

(ii) If $H-v$ has diameter greater than two, then

$$
\operatorname{dim}_{s}\left(G \circ_{v} H\right)=(r-1)(t-1)+\operatorname{dim}_{s}(H) .
$$

The strong metric dimension of $G \odot H$ depends on the existence or not of true twins in $H$. In this sense, the following result was presented in [14].

Theorem 18. [14] Let $G$ be a connected graph of order $r$ and let $H$ be a graph of order $t$. Let $c(H)$ be the number of vertices of $H$ having degree $t-1$.
(i) If $H$ has no true twins and $r \geq 2$, then

$$
\operatorname{dim}_{s}(G \odot H)=r t-\omega(H) .
$$

(ii) If the only true twins of $H$ are vertices of degree $t-1$ and $r \geq 2$, then

$$
\operatorname{dim}_{s}(G \odot H)=r t+c(H)-1-\omega(H)
$$

Our next result is an interesting consequence of Theorem 18.
Corollary 19. Let $G$ be a connected graph of order $r \geq 2$. Let $H$ be a connected graph of order $t \geq 2$ and let $v$ be a vertex of $H$ of degree $t-1$. Let $c(H-v)$ be the number of vertices of $H-v$ having degree $t-2$.
(i) If $H-v$ has no true twins, then

$$
\operatorname{dim}_{s}\left(G \circ_{v} H\right)=r(t-1)-\omega(H-v) .
$$

(ii) If the only true twins of $H-v$ are vertices of degree $t-2$, then

$$
\operatorname{dim}_{s}\left(G \circ_{v} H\right)=r(t-1)+c(H-v)-1-\omega(H-v)
$$

## 3 Tight bounds

Lemma 20. Let $G$ and $H$ be two connected graphs. Given $x \in V(G), v \in V(H)$ and $a$ strong metric basis $B$ of $G \circ_{v} H$ let $B_{x}=B \cap(\{x\} \times V(H))$ and let $M(v)$ be the set of vertices of $H$ which are maximally distant from $v$. Then the following assertions hold.
(i) $\left|B_{x}\right| \geq \operatorname{dim}_{s}(H)-1$.
(ii) If $B_{x} \supset\{x\} \times M(v)$, then $\left|B_{x}\right| \geq \operatorname{dim}_{s}(H)$.
(iii) If $v$ does not belong to any strong metric basis of $H$, then $\left|B_{x}\right| \geq \operatorname{dim}_{s}(H)$.

Proof. First we consider a pair $(x, y),\left(x, y^{\prime}\right)$ of adjacent vertices in $\left(H_{x}\right)_{S R}$, where $y, y^{\prime} \neq v$. Since $B$ is a vertex cover of $\left(G \circ_{v} H\right)_{S R}$, either $(x, y) \in B_{x}$ or $\left(x, y^{\prime}\right) \in B_{x}$. Thus, $B_{x} \cup\{(x, v)\}$ is a vertex cover of $\left(H_{x}\right)_{S R}$. Note that $(x, v) \notin \partial\left(G \circ_{v} H\right)$ and, as a consequence, $(x, v) \notin B_{x}$. Hence, $\left|B_{x}\right|+1=\left|B_{x} \cup\{(x, v)\}\right| \geq \operatorname{dim}_{s}\left(H_{x}\right)=\operatorname{dim}_{s}(H)$. Therefore, (i) follows.

Now we suppose $B_{x} \supset\{x\} \times M(v)$. If $(x, y)$ and $(x, v)$ are adjacent in $\left(H_{x}\right)_{S R}$, then $y \in M(v)$. So the edge $\{(x, y),(x, v)\}$ of $\left(H_{x}\right)_{S R}$ is covered by $(x, y) \in B_{x}$. Thus, $B_{x}$ is a vertex cover of $\left(H_{x}\right)_{S R}$ and, as a result, $\left|B_{x}\right| \geq \operatorname{dim}_{s}(H)$. Therefore, (ii) follows.

Finally, suppose that $v$ does not belong to any strong metric basis of $H$. Since the function $f:\{x\} \times V(H) \rightarrow V(H)$, where $f(x, y)=y$, is a graph isomorphism and $B_{x} \cup\{(x, v)\}$ is a strong metric generator for $H_{x}$, the set

$$
A=f\left(B_{x} \cup\{(x, v)\}\right)=\{v\} \cup\left\{u:(x, u) \in B_{x}\right\}
$$

is a strong metric generator for $H$. Thus, since $v$ does not belong to any strong metric basis of $H,|A|>\operatorname{dim}_{s}(H)$. Taking into account that $(x, v) \notin B_{x}$ we obtain $\left|B_{x}\right|=\mid B_{x} \cup$ $\{(x, v)\}\left|-1=|A|-1 \geq \operatorname{dim}_{s}(H)\right.$. The proof is complete.

Theorem 21. Let $G$ be a connected graph of order $n \geq 2$ and let $H$ be a connected graph.
(i) If $v \in V(H)$ belongs to a strong metric basis of $H$, then

$$
n \cdot \operatorname{dim}_{s}(H)-1 \leq \operatorname{dim}_{s}\left(G \circ_{v} H\right) \leq(|\partial(H)|-1)(n-1)+\operatorname{dim}_{s}(H)-1
$$

(ii) If $v \in V(H)$ does not belong to any strong metric basis of $H$, then

$$
n \cdot \operatorname{dim}_{s}(H) \leq \operatorname{dim}_{s}\left(G \circ_{v} H\right) \leq \begin{cases}|\partial(H)|(n-1)+\operatorname{dim}_{s}(H), & \text { if } v \notin \partial(H), \\ (|\partial(H)|-1)(n-1)+\operatorname{dim}_{s}(H), & \text { if } v \in \partial(H) .\end{cases}
$$

Proof. Let $W$ be a strong metric basis of $H$ such that $v \in W$ and let $B$ be a strong metric basis of $G \circ_{v} H$. Since $v$ belongs to a metric basis of $H$, we have $v \in \partial(H)$. Suppose there exists $x \in V(G)$ such that $(x, u) \notin B_{x}$ for some $u \in M(v)$. By Lemma 20 (i) we obtain $\left|B_{x}\right| \geq \operatorname{dim}_{s}(H)-1$. Moreover, by Lemma 2 we have that for $x^{\prime} \in V(G)-\{x\}$ and $u^{\prime} \in M(v)$ the vertices $(x, u)$ and $\left(x^{\prime}, u^{\prime}\right)$ are mutually maximally distant in $G \circ_{v} H$. Hence, since $(x, u) \notin B_{x}$ and $B$ is a vertex cover of $\left(G \circ_{v} H\right)_{S R}$, for every $x^{\prime} \in V(G)-\{x\}$ we have $B_{x^{\prime}} \supset\left\{x^{\prime}\right\} \times M(v)$. So, according to Lemma 20 (ii) we have $\left|B_{x^{\prime}}\right| \geq \operatorname{dim}_{s}(H)$. Therefore,

$$
\operatorname{dim}_{s}\left(G \circ_{v} H\right)=|B|=\left|B_{x}\right|+\sum_{x^{\prime} \in V(G)-\{x\}}\left|B_{x^{\prime}}\right| \geq n \cdot \operatorname{dim}_{s}(H)-1
$$

On the other hand, since $v \in \partial(H)$, Proposition 5 (ii) leads to $\partial\left(G \circ \circ_{v} H\right)=V(G) \times$ $(\partial(H)-\{v\})$. We will show that $S=\partial\left(G \circ_{v} H\right)-P$ is a vertex cover for $\left(G \circ_{v} H\right)_{S R}$, where $P=\{a\} \times(\partial(H)-W \cup\{v\})$ and $a \in V(G)$. Let $(x, y)$ and $\left(x^{\prime}, y^{\prime}\right)$ be two adjacent vertices in $\left(G \circ_{v} H\right)_{S R}$. If $x \neq a$ or $x^{\prime} \neq a$, then $(x, y) \in S$ or $\left(x^{\prime}, y^{\prime}\right) \in S$. Now let, $x=x^{\prime}=a$. Since $H_{a} \cong H$ and $W$ is a vertex cover for $H,\{a\} \times W$ is a vertex cover for $H_{a}$ and, as a consequence, $(x, y) \in\{a\} \times W \subset S$ or $\left(x^{\prime}, y^{\prime}\right) \in\{a\} \times W \subset S$. Hence, $S$ is a vertex cover for $\left(G \circ_{v} H\right)_{S R}$. Therefore,

$$
\operatorname{dim}_{s}\left(G \circ_{v} H\right) \leq|S|=(|\partial(H)|-1)(n-1)+\operatorname{dim}_{s}(H)-1
$$

The proof of (i) is complete.
From now on we assume that $v$ does not belong to any strong metric basis of $H$. The lower bound of (ii) is a direct consequence of Lemma 20 (iii). Suppose $v \notin \partial(H)$. In this case, by Proposition 5 (i) we conclude $\partial\left(G \circ_{v} H\right)=V(G) \times \partial(H)$. By analogy with the proof of the upper bound of (i) we show that $S^{\prime}=\partial\left(G \circ_{v} H\right)-P^{\prime}$ is a vertex cover for $\left(G \circ_{v} H\right)_{S R}$, where $P^{\prime}=\{a\} \times\left(\partial(H)-W^{\prime}\right), a \in V(G)$ and $W^{\prime}$ is a strong metric basis of $H$. Hence,

$$
\operatorname{dim}_{s}\left(G \circ_{v} H\right) \leq\left|S^{\prime}\right|=|\partial(H)|(n-1)+\operatorname{dim}_{s}(H)
$$

Finally, for the case $v \in \partial(H)$ we have $\partial\left(G \circ_{v} H\right)=V(G) \times(\partial(H)-\{v\})$ and proceeding by analogy with the proof of the upper bound of (i) we show that $S^{\prime \prime}=\partial\left(G \circ_{v} H\right)-P^{\prime \prime}$ is a vertex cover for $\left(G \circ_{v} H\right)_{S R}$, where $P^{\prime \prime}=\{a\} \times\left(\partial(H)-W^{\prime \prime}\right), a \in V(G)$ and $W^{\prime \prime}$ is a strong metric basis of $H$. Thus, in this case

$$
\operatorname{dim}_{s}\left(G \circ_{v} H\right) \leq\left|S^{\prime \prime}\right|=(|\partial(H)|-1)(n-1)+\operatorname{dim}_{s}(H)
$$

The proof of (ii) is complete.
As Corollary 8 shows, the bounds of Theorem 21 (i) are tight and the upper bound $\operatorname{dim}_{s}\left(G \circ_{v} H\right) \leq|\partial(H)|(n-1)+\operatorname{dim}_{s}(H)$ of Theorem 21 (ii) is tight. To show the tightness of the upper bound $\operatorname{dim}_{s}\left(G \circ_{v} H\right) \leq(|\partial(H)|-1)(n-1)+\operatorname{dim}_{s}(H)$ we consider the graph $J$ shown in Figure 3. Notice that any strong metric basis of $J$ is formed by the vertices $y_{2}$, $y_{4}$ and three vertices of the set $\left\{y_{1}, y_{3}, y_{5}, x_{6}\right\}$.


Figure 3: The graph $J$ and its strong resolving graph $J_{S R}$.

Remark 22. Let $G$ be a connected graph of order $n$. Let $v$ be the vertex of the graph $J$ denoted by $w$. Then $\operatorname{dim}_{s}\left(G \circ_{v} J\right)=(\partial(J)-1)(n-1)+\operatorname{dim}_{s}(J)$.

Proof. Let $V=\left\{u_{1}, u_{2}, \ldots, u_{n}\right\}$ be the set of vertices of $G$. From Figure 3 we have that there exits six vertices $y_{1}, y_{2}, y_{3}, y_{4}, y_{5}$ and $x_{6}$ which are maximally distant from $v$. So, by using Lemma 2, we have that every two vertices $\left(u_{i}, y\right),\left(u_{j}, y^{\prime}\right) \in V \times\left\{y_{1}, y_{2}, y_{3}, y_{4}, y_{5}, x_{6}\right\}$, where $i \neq j$, are mutually maximally distant. Moreover, by Lemma 3 for every two mutually maximally distant vertices $z, z^{\prime}$ in $J$ we have that $\left(u_{i}, z\right),\left(u_{i}, z^{\prime}\right)$ are mutually maximally distant in $G \circ_{v} J$ for every vertex $u_{i}$ of $G$. Thus, $\left(G \circ_{v} J\right)_{S R}$ is isomorphic to $K_{6 n}$. Therefore, $\operatorname{dim}_{s}\left(G \circ_{v} J\right)=6 n-1=(\partial(J)-1)(n-1)+\operatorname{dim}_{s}(J)$.

To see the tightness of the lower bound of Theorem 21 (ii) we define the family $\mathcal{F}$ of graphs $H$ containing a vertex of degree one not belonging to any strong metric basis of $H$. We begin with the cycle $C_{t}$, where $t$ is an odd number such that $t \geq 5$, with set of vertices $X=\left\{x_{1}, x_{2}, \ldots, x_{t}\right\}$. To obtain a graph $H_{t, p, r} \in \mathcal{F}$ we add the sets of vertices $Y=\{y\}, W=\left\{w_{1}, w_{2}, \ldots, w_{p}\right\}$ and $Z=\left\{z_{1}, z_{2}, \ldots, z_{r}\right\}$, where $p, r \geq 1$, and edges $y x_{t}$, $x_{1} x_{t-1}, x_{\left\lfloor\frac{t}{2}\right\rfloor} w_{i}$, for every $i \in\{1,2, \ldots, p\}$, and $x_{\left\lceil\frac{t}{2}\right\rceil} z_{j}$, for every $j \in\{1,2, \ldots, r\}$. Notice that vertices of $Y \cup W \cup Z$ have degree one in $H_{t, p, r}$ and they are mutually maximally distant between them. Also, for any vertex $a \in N_{H_{t, p, r}}\left(x_{1}\right), d_{H_{t, p, r}}\left(a, z_{j}\right) \leq d_{H_{t, p, r}}\left(x_{1}, z_{j}\right)$, where $j \in$ $\{1,2, \ldots, r\}$. Similarly, for any vertex $b \in N_{H_{t, p, r}}\left(x_{t-1}\right), d_{H_{t, p, r}}\left(b, w_{i}\right) \leq d_{H_{t, p, r}}\left(x_{t-1}, w_{i}\right)$, where $i \in\{1,2, \ldots, p\}$. Moreover, we can observe that $x_{k}$ and $x_{k+\left\lfloor\frac{t}{2}\right\rfloor}$ are mutually maximally distant for every $k \in 2,3, \ldots\left\lfloor\left\lfloor\frac{t}{2}\right\rfloor-1\right.$. So, $\left(H_{t, p, r}\right)_{S R}$ is formed by $\left\lfloor\frac{t}{2}\right\rfloor-1$ connected components, that is, $\left\lfloor\frac{t}{2}\right\rfloor-2$ connected components isomorphic to $K_{2}$ and also, a connected component isomorphic to a graph with set of vertices $Y \cup W \cup Z \cup\left\{x_{1}, x_{t-1}\right\}$ where $\langle Y \cup W \cup Z\rangle$ is isomorphic to $K_{|Y \cup W \cup Z|}, x_{1}$ is adjacent to every vertex $z_{j}, j \in\{1,2, \ldots, r\}$, and $x_{t-1}$ is adjacent to every vertex $w_{i}, i \in\{1,2, \ldots, p\}$. Notice that every $\alpha\left(\left(H_{t, p, r}\right)_{S R}\right)$-set is formed only by the vertices of $W \cup Z$ and one vertex from each subgraph isomorphic to $K_{2}$. Therefore,

$$
\operatorname{dim}_{s}\left(H_{t, p, r}\right)=\frac{t-5}{2}+p+r
$$

and $y$ is a vertex of degree one not belonging to any strong metric basis of $H_{t, p, r}$. The graphs $H_{9,3,4}$ and $\left(H_{9,3,4}\right)_{S R}$ are shown in Figure 4.


Figure 4: The graphs $H_{9,3,4}$ and $\left(H_{9,3,4}\right)_{S R}$. The set $S=\left\{w_{1}, w_{2}, w_{3}, w_{4}, z_{1}, z_{2}, z_{3}, x_{2}, x_{3}\right\}$ is a strong metric basis of $H_{9,3,4}$.

Remark 23. Let $G$ be a connected graph of order $n$. Let $v$ be the vertex of degree one not belonging to any strong metric basis of the graph $H_{t, p, r} \in \mathcal{F}$. Then

$$
\operatorname{dim}_{s}\left(G \circ_{v} H_{t, p, r}\right)=n\left(\frac{t-5}{2}+p+r\right)=n \cdot \operatorname{dim}_{s}\left(H_{t, p, r}\right)
$$

Proof. Let $V$ be the vertex set of $G$ and let $H_{t, p, r} \in \mathcal{F}$ with set of vertices $W \cup X \cup Y \cup Z$, where $W=\left\{w_{1}, w_{2}, \ldots, w_{p}\right\}, X=\left\{x_{1}, x_{2}, \ldots, x_{t}\right\}, Y=\{y\}$ and $Z=\left\{z_{1}, z_{2}, \ldots, z_{r}\right\}$. Since every vertex $u \in W \cup Z$ is maximally distant from $v$, by Lemma 2 , we have that every two different vertices $(x, y),\left(x^{\prime}, y^{\prime}\right) \in V \times(W \cup Z), x \neq x^{\prime}$, are mutually maximally distant. Moreover, by Lemma 3 for every two mutually maximally distant vertices $v_{i}, v_{j}$ in $H_{t, p, r}$ we have that $\left(u, v_{i}\right),\left(u, v_{j}\right)$ are mutually maximally distant in $G \circ_{v} H_{t, p, r}$ for every vertex $u$ of $G$. Thus, $\left(G \circ_{v} H_{t, p, r}\right)_{S R}$ is formed by $n \frac{t-5}{2}+1$ connected components, i. e., $n \frac{t-5}{2}$ connected components isomorphic to $K_{2}$ and one connected component isomorphic to a graph $G_{1}$ with set of vertices $V \times\left(W \cup Z \cup\left\{x_{1}, x_{t-1}\right)\right\}$ where $\langle V \times(W \cup Z)\rangle$ is isomorphic to $K_{n|W \cup Z|}$ and for every $u \in V,\left(u, x_{1}\right)$ is adjacent to every vertex $\left(u, z_{j}\right), j \in\{1,2, \ldots, r\}$, and $\left(u, x_{t-1}\right)$ is adjacent to every vertex $\left(u, w_{i}\right), i \in\{1,2, \ldots, p\}$. Since in $G_{1}$ every vertex of $\langle V \times(W \cup Z)\rangle$ has a neighbor not belonging to $V \times(W \cup Z)$ we have that $\alpha\left(G_{1}\right)=n|W \cup Z|$. Therefore, we obtain that

$$
\operatorname{dim}_{s}\left(G \circ_{v} H_{t, p, r}\right)=\alpha\left(\left(G \circ_{v} H_{t, p, r}\right)_{S R}\right)=n|W \cup Z|+n \frac{t-5}{2}=n\left(\frac{t-5}{2}+p+r\right) .
$$

According to the Remark 23 we have that for every graph $H \in \mathcal{F}$ and any connected graph $G$ of order $n, \operatorname{dim}_{s}\left(G \circ_{v} H\right)=n \cdot \operatorname{dim}_{s}(H)$ where $v$ is the vertex of degree one not belonging to any strong metric basis of the graph $H$.

The next result from [19] will be useful to prove Proposition 25.
Lemma 24. [19] For every connected graph $G$, $\operatorname{dim}_{s}(G) \geq|\sigma(G)|-1$.
Proposition 25. Let $G$ be a connected graph of order $n \geq 2$ and let $v$ be a vertex of a graph $H$. If $v$ does not belong to the boundary of $H$ and there exists a vertex different from $v$, of degree one in $H$, not belonging to any strong metric basis of $H$, then

$$
\operatorname{dim}_{s}\left(G \circ_{v} H\right) \geq n\left(\operatorname{dim}_{s}(H)+1\right)-1 .
$$

Proof. Let $w$ be a vertex of degree one in $H$ not belonging to any strong metric basis of $H$. Notice that the vertices of the set $A=\left\{\left(u_{i}, w\right): i \in\{1,2, \ldots, n\}\right\}$ are also vertices of degree one in $G \circ_{v} H$. Thus, they are simplicial vertices and from Lemma 24 we have that at least all but one vertices of $A$ belongs to every strong metric basis of $G \circ_{v} H$. Thus,

$$
\operatorname{dim}_{s}\left(G \circ_{v} H\right)=\alpha\left(\left(G \circ_{v} H\right)_{S R}\right) \geq n \alpha(\langle\partial(H)\rangle)+|A|-1=n \alpha\left(H_{S R}\right)+n-1=n\left(\operatorname{dim}_{s}(H)+1\right)-1 .
$$

As the following remark shows, the above bound is tight.
Remark 26. Let $G$ be a connected graph of order $n$. Let $v$ be the vertex of the graph $H_{t, p, r} \in \mathcal{F}$ adjacent to the vertex of degree one not belonging to any strong metric basis of $H_{t, p, r}$. Then

$$
\operatorname{dim}_{s}\left(G \circ_{v} H_{t, p, r}\right)=n\left(\frac{t-5}{2}+p+r+1\right)-1=n\left(\operatorname{dim}_{s}\left(H_{t, p, r}\right)+1\right)-1
$$

Proof. Let $V$ be the vertex set of $G$. Now, according to the construction of the family $\mathcal{F}$, let the graph $H_{t, p, r}$ with set of vertices $W \cup X \cup Y \cup Z$, where $W=\left\{w_{1}, w_{2}, \ldots, w_{p}\right\}$, $X=\left\{x_{1}, x_{2}, \ldots, x_{t}\right\}, Y=\{y\}$ and $Z=\left\{z_{1}, z_{2}, \ldots, z_{r}\right\}$. Since every vertex $y \in W \cup Y \cup$ $Z$ is maximally distant from $v$, by Lemma 2, we have that every two different vertices $(x, y),\left(x^{\prime}, y^{\prime}\right) \in V \times(W \cup Y \cup Z), x \neq x^{\prime}$, are mutually maximally distant. Moreover, by Lemma 3 for every two mutually maximally distant vertices $v_{i}, v_{j}$ in $H_{t, p, r}$ we have that $\left(u, v_{i}\right),\left(u, v_{j}\right)$ are mutually maximally distant in $G \circ_{v} H_{t, p, r}$ for every vertex $u$ of $G$. Thus, $\left(G \circ_{v}\right.$ $\left.H_{t, p, r}\right)_{S R}$ is formed by $n \frac{t-5}{2}+1$ connected components, that is, $n \frac{t-5}{2}$ connected components isomorphic to $K_{2}$ and one connected component isomorphic to a graph $G_{1}$ with set of vertices $V \times\left(W \cup Y \cup Z \cup\left\{x_{1}, x_{t-1}\right)\right\}$ where $\langle V \times(W \cup Y \cup Z)\rangle$ is isomorphic to $K_{n|W \cup Y \cup Z|}$ and for every $u \in V,\left(u, x_{1}\right)$ is adjacent to every vertex $\left(u, z_{j}\right), j \in\{1,2, \ldots, r\}$, and $\left(u, x_{t-1}\right)$ is adjacent to every vertex $\left(u, w_{i}\right), i \in\{1,2, \ldots, p\}$. Notice that $\alpha\left(G_{1}\right)=n|W \cup Y \cup Z|-1$. Therefore, we obtain that
$\operatorname{dim}_{s}\left(G \circ_{v} H_{t, p, r}\right)=\alpha\left(\left(G \circ_{v} H_{t, p, r}\right)_{S R}\right)=n|W \cup Y \cup Z|-1+n \frac{t-5}{2}=n\left(\frac{t-5}{2}+p+r+1\right)-1$.

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[^0]:    ${ }^{1}$ The diameter of $G=(V, E)$ is defined as $D(G)=\max _{u, v \in V}\{d(u, v)\}$. We recall that $G=(V, E)$ is 2-antipodal if for each vertex $x \in V$ there exists exactly one vertex $y \in V$ such that $d_{G}(x, y)=D(G)$.
    ${ }^{2}$ In fact, according to [17] the strong resolving graph $G_{S R}^{\prime}$ of a graph $G$ has vertex set $V\left(G_{S R}^{\prime}\right)=V(G)$ and two vertices $u, v$ are adjacent in $G_{S R}^{\prime}$ if and only if $u$ and $v$ are mutually maximally distant in $G$. So, the strong resolving graph defined here is a subgraph of the strong resolving graph defined in [17] and it can be obtained from the latter graph by deleting its isolated vertices.

