# Optimal Newton-Secant like methods without memory for solving nonlinear equations with its dynamics

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Received: date / Accepted: date

**Abstract** We construct two optimal Newton-Secant like iterative methods for solving non-linear equations. The proposed classes have convergence order four and eight and cost only three and four function evaluations per iteration, respectively. These methods support the Kung and Traub conjecture and possess a high computational efficiency. The new methods are illustrated by numerical experiments and a comparison with some existing optimal methods. We conclude with an investigation of the basins of attraction of the solutions in the complex plane.

**Keywords** Multi-point iterative methods; Newton-Secant method; Kung and Traub's conjecture.

## 1 Introduction

A main tool for solving nonlinear problems is the approximation of simple roots  $x^*$  of a nonlinear equation  $f(x^*) = 0$  with a scalar function  $f : D \subset \mathbb{R} \to \mathbb{R}$  which is defined on an open interval D (see e.g. [28,30,31,39] and the references therein). The secant method is a simple root-finding algorithm which can be traced back to a historic precursor called "rule of double false position" [29]. A modern way to view the secant method would be to replace the derivative in the Newton-Raphson method  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$  by a finitedifference approximation. The Newton-Raphson method is one of the most widely used algorithms for finding roots. It is of second order and requires two evaluations for each iteration step, one evaluation of f and one of f'.

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Newton-Raphson iteration is an example of a one-point iteration, i.e. in each iteration step the evaluations are taken at one point. Multiple-point methods evaluate at several points in each iteration step and in principle allow for a higher convergence order with a lower number of function evaluations. Kung and Traub [20] conjectured that no multi-point method without memory with k evaluations could have a convergence order larger than  $2^{k-1}$ . A multi-point method with convergence order  $2^{k-1}$  is called optimal.

In this paper we construct two new optimal multi-point methods. We present a two-point iteration with convergence order four which requires two evaluations of f and one evaluation of f' and a three-point iteration with convergence order eight which requires three evaluations of f and one evaluation of f'. Both methods combine the Newton and Secant methods and utilize the idea of weight functions to obtain optimality in the sense of Kung and Traub. For an alternative construction of an optimal three-point method with convergence order eight which also uses carefully chosen weight functions, see [23].

For well known two-point methods without memory one can consult e.g. Jarrat [18], King [19] and Ostrowski [28]. Bi et al. [8] developed an optimal three-point iterative method with convergence order eight. Wang and Liu used weight functions to construct optimal three-point methods [21] and [41] and optimal convergence order eight was achieved by Geum and Kim [15] and [16] utilizing parametric weight functions. Based on rational interpolation and weight functions, Sharma et al. introduced two three-point methods [33,34], see also Cordero et al. [12]-[14] and Soleymani et al. [35], Babajee et al. [7], Thukral and Petkovic [38] and for recent studies the interested reader is referred to Chun and Lee [10] and Petkovic et al. [30] and Neta [24] has demonstrated methods of eight and sixteen order of convergence. Alberto et al. [1] have analyzed a different anomalies in a Jarrat family of iterative root-finding methods. In [9] Chun et al. introduced weight functions with a parameter into an iteration process to increase the order of the convergence and enhance the behavior of the iteration process. In [22] Lotfi and Salimi pointed to serious errors that presented in the paper entitled "A family of optimal iterative methods with fifth and tenth order convergence for solving nonlinear equations" as well.

The paper is organized as follows: Section 2 is devoted to the construction and convergence analysis of a new two-point method with optimal convergence order four and a new three-point method with optimal convergence order eight. Computational aspects, comparisons and dynamic behavior with other methods are illustrated in Section 3.

# 2 Development of multi-point methods

#### 2.1 Optimal two-point method

In this section we construct a new optimal two-point class of iterative methods for solving nonlinear equations. The Newton-Secant method is given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
  

$$x_{n+1} = x_n - \frac{f^2(x_n)}{(f(x_n) - f(y_n))f'(x_n)}, \quad (n = 0, 1, \ldots),$$
(1)

where  $x_0$  is an initial approximation of  $x^*$ . The convergence order of (1) is three and with three evaluations it is not optimal. We intend to increase the order of convergence and extend (1) by an additional step

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
  

$$z_n = x_n - \frac{f^2(x_n)}{(f(x_n) - f(y_n))f'(x_n)},$$
  

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}.$$
(2)

Method (2) uses four function evaluations with order of convergence four. Therefore, this method is not optimal. In order to decrease the number of function evaluations, we approximate  $f(z_n)$  by an expression based on  $f(x_n)$ ,  $f(y_n)$  and  $f'(x_n)$ . Taylor expansion of f at  $x_n$  yields

$$f(z_n) = f(x_n) + f'(x_n)(z_n - x_n) + \frac{1}{2}f''(x_n)(z_n - x_n)^2 + O((z_n - x_n)^3), \quad (3)$$

and similarly we have

$$f(y_n) = f(x_n) + f'(x_n)(y_n - x_n) + \frac{1}{2}f''(x_n)(y_n - x_n)^2 + O((y_n - x_n)^3).$$
(4)

Using Newton's method and (4), we obtain

$$\frac{1}{2}f^{''}(x_n) \approx \frac{f(y_n)(f^{'}(x_n))^2}{f^2(x_n)}.$$
(5)

According to (2), we have

$$z_n - x_n = -\frac{f^2(x_n)}{(f(x_n) - f(y_n))f'(x_n)}.$$
(6)

Substituting (5) and (6) into (3), we obtain

$$f(z_n) \approx f(x_n) - \frac{f^2(x_n)}{f(x_n) - f(y_n)} + \frac{f(y_n)f^2(x_n)}{(f(x_n) - f(y_n))^2}.$$
(7)

Substituting (7) into (2), yields

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n - \frac{f^2(x_n)}{(f(x_n) - f(y_n))f'(x_n)},$$

$$x_{n+1} = z_n - \left[1 - \frac{f(x_n)}{f(x_n) - f(y_n)} \left(1 + \frac{f(y_n)}{f(x_n) - f(y_n)}\right)\right] \frac{f(x_n)}{f'(x_n)}.$$
(8)

Although we reduced the number of function evaluations compared to (2), the convergence order of (8) is not yet four. In order to increase it, we consider an appropriate weight function, namely  $\phi(t_n)$ , as follows:

$$y_{n} = x_{n} - \frac{f(x_{n})}{f'(x_{n})},$$

$$z_{n} = x_{n} - \frac{f^{2}(x_{n})}{(f(x_{n}) - f(y_{n}))f'(x_{n})},$$

$$x_{n+1} = z_{n} - \left[1 - \frac{f(x_{n})}{f(x_{n}) - f(y_{n})}\left(1 + \frac{f(y_{n})}{f(x_{n}) - f(y_{n})}\right)\right] \frac{f(x_{n})}{f'(x_{n})}\phi(t_{n}),$$
(9)

where  $t_n = \frac{f(y_n)}{f(x_n)}$ . In the following theorem, we provide sufficient conditions on the weight function  $\phi(t_n)$  which imply that method (9) has convergence order four.

**Theorem 1** Let  $D \subseteq \mathbb{R}$  be an open interval,  $f : D \to \mathbb{R}$  four times continuously differentiable and let  $x^* \in D$  be a simple zero of f. If the initial point  $x_0$  is sufficiently close to  $x^*$ , then the method defined by (9) converges to  $x^*$  with order at least four if the weight function  $\phi : \mathbb{R} \to \mathbb{R}$  is two times continuously differentiable and satisfies the conditions

$$\phi(0)=0 \quad, \quad \phi^{'}(0)=-rac{1}{2} \quad ext{and} \quad |\phi^{''}(0)|<\infty.$$

Proof Let  $e_n := x_n - x^*$ ,  $e_{n,y} := y_n - x^*$ ,  $e_{n,z} := z_n - x^*$  and  $c_n := \frac{f^{(n)}(x^*)}{n!f'(x^*)}$ for  $n \in \mathbb{N}$ . Using the fact that  $f(x^*) = 0$ , Taylor expansion of f at  $x^*$  yields

$$f(x_n) = f'(x^*) \left( e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 \right) + O(e_n^5)$$
(10)

and

$$f'(x_n) = f'(x^*) \left( 1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 \right) + O(e_n^4).$$
(11)

Therefore

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + \left(2c_2^2 - 2c_3\right)e_n^3 + O(e_n^4),$$

and hence

$$e_{n,y} = y_n - x^* = c_2 e_n^2 + O(e_n^3).$$

For  $f(y_n)$  we also have

$$f(y_n) = f'(x^*) \left( c_2 e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 \right) + O(e_n^5),$$
(12)

therefore, by substituting (10), (11) and (12) into (2), we get

$$e_{n,z} = z_n - x^* = c_2^2 e_n^3 + O(e_n^4).$$

From (10) and (12), we obtain

$$t_n = \frac{f(y_n)}{f(x_n)} = c_2 e_n + (-3c_2^2 + 2c_3)e_n^2 + (8c_2^3 - 10c_2c_3 + 3c_4)e_n^3 + O(e_n^4).$$
(13)

Expanding  $\phi$  at 0, yields

$$\phi(t_n) = \phi(0) + \phi'(0)t_n + \frac{1}{2}\phi''(0)t_n^2 + O(t_n^3).$$
(14)

Substituting (10)-(14) into (9), we obtain

$$e_{n+1} = x_{n+1} - x^* = R_2 e_n^2 + R_3 e_n^3 + R_4 e_n^4 + O(e_n^5),$$

where

$$R_{2} = 2c_{2}\phi(0),$$

$$R_{3} = c_{2}^{2}(1 + 2\phi'(0)),$$

$$R_{4} = -c_{2}c_{3} + c_{2}^{3}(\frac{5}{2} + \phi''(0)).$$
(15)

By setting  $R_2 = R_3 = 0$ , the convergence order becomes four. Obviously

$$\begin{aligned}
\phi(0) &= 0 \quad \Rightarrow \quad R_2 = 0, \\
\phi'(0) &= -\frac{1}{2} \quad \Rightarrow \quad R_3 = 0, \\
|\phi''(0)| &< \infty \quad \Rightarrow \quad R_4 \neq 0.
\end{aligned} \tag{16}$$

Consequently, the error equation becomes

$$e_{n+1} = R_4 e_n^4 + O(e_n^5),$$

which finishes the proof of the theorem.

# 2.2 Optimal three-point method

In this section we construct a new optimal three-point method based on the two-point method (9). We extend method (9) by a Newton step and get

$$y_{n} = x_{n} - \frac{f(x_{n})}{f'(x_{n})},$$

$$z_{n} = x_{n} - \frac{f^{2}(x_{n})}{(f(x_{n}) - f(y_{n}))f'(x_{n})},$$

$$v_{n} = z_{n} - \left[1 - \frac{f(x_{n})}{f(x_{n}) - f(y_{n})}\left(1 + \frac{f(y_{n})}{f(x_{n}) - f(y_{n})}\right)\right] \frac{f(x_{n})}{f'(x_{n})}\phi(t_{n}),$$

$$x_{n+1} = v_{n} - \frac{f(v_{n})}{f'(v_{n})},$$
(17)

where  $\phi(t_n)$  is a weight function as in Theorem 1.

Method (17) evaluates functions for five times with order of convergence eight, so the method is not optimal. In order to reduce the number of function

evaluation, we approximate  $f'(v_n)$  by an expression which is based on  $f(x_n)$ ,  $f(y_n)$ ,  $f(v_n)$ , and  $f'(x_n)$ , namely its linear approximation

$$f'(v_n) \approx f'(x_n) + \frac{f'(z_n) - f'(x_n)}{z_n - x_n} (v_n - x_n).$$
(18)

We approximate  $f'(z_n)$  by expressions which were calculated above. The Taylor expansion of f at  $y_n$  yields

$$f(z_n) = f(y_n) + f'(y_n)(z_n - y_n) + \frac{1}{2}f''(y_n)(z_n - y_n)^2 + O((z_n - y_n)^3), \quad (19)$$

and

$$f'(z_n) = f'(y_n) + f''(y_n)(z_n - y_n) + O((z_n - y_n)^2).$$
(20)

According to (19), we have

$$f'(y_n) \approx \frac{f(z_n) - f(y_n)}{z_n - y_n} - \frac{1}{2}f''(y_n)(z_n - y_n).$$
(21)

On the other hand, we have

$$f''(y_n) \approx 2f[z_n, x_n, x_n] = \frac{2\left(f[z_n, x_n] - f'(x_n)\right)}{z_n - x_n},$$
(22)

where  $f[z_n, x_n] = \frac{f(z_n) - f(x_n)}{z_n - x_n}$ . Substituting (21) and (22) into (20), we obtain

$$f'(z_n) \approx f[z_n, y_n] + \left(f[z_n, x_n] - f'(x_n)\right) \frac{z_n - y_n}{z_n - x_n},\tag{23}$$

where  $f[z_n, y_n] = \frac{f(z_n) - f(y_n)}{z_n - y_n}$ . In a next step we replace  $f(z_n)$  by an approximation to reduce the number of function evaluations. Taylor expansion of f at  $x_n$  yields

$$f(z_n) = f(x_n) + f'(x_n)(z_n - x_n) + \frac{1}{2}f''(x_n)(z_n - x_n)^2 + \frac{1}{6}f'''(x_n)(z_n - x_n)^3 + O((z_n - x_n)^4),$$
(24)

and similarly we have

$$f(v_n) = f(x_n) + f'(x_n)(v_n - x_n) + \frac{1}{2}f''(x_n)(v_n - x_n)^2 + \frac{1}{6}f'''(x_n)(v_n - x_n)^3 + O((v_n - x_n)^4).$$
(25)

From (25), we calculate

$$\frac{1}{6}f^{'''}(x_n) \approx \left[\frac{f(v_n) - f(x_n)}{v_n - x_n} - f^{'}(x_n) - \frac{f(y_n)\left(f^{'}(x_n)\right)^2}{f^2(x_n)}(v_n - x_n)\right] \frac{1}{(v_n - x_n)^2}$$
(26)

Plugging (5) and (26) into (24), we obtain

$$f(z_n) \approx f(x_n) + f'(x_n)(z_n - x_n) + \frac{f(y_n)(f'(x_n))^2}{f^2(x_n)}(z_n - x_n)^2 + \left[f[v_n, x_n] - f'(x_n) - \frac{f(y_n)(f'(x_n))^2}{f^2(x_n)}(v_n - x_n)\right] \frac{(z_n - x_n)^3}{(v_n - x_n)^2}.$$
(27)

Then, by replacing (23) into (18), we get

$$f'(v_n) \approx f'(x_n) + \frac{f[z_n, y_n] + \left(f[z_n, x_n] - f'(x_n)\right)\frac{z_n - y_n}{z_n - x_n} - f'(x_n)}{z_n - x_n}(v_n - x_n),$$
(28)

where we can plug (27) instead of  $f(z_n)$  in (28) as well. The following scheme evaluates functions for four times

$$y_{n} = x_{n} - \frac{f(x_{n})}{f'(x_{n})},$$

$$z_{n} = x_{n} - \frac{f^{2}(x_{n})}{(f(x_{n}) - f(y_{n}))f'(x_{n})},$$

$$v_{n} = z_{n} - \left[1 - \frac{f(x_{n})}{f(x_{n}) - f(y_{n})}\left(1 + \frac{f(y_{n})}{f(x_{n}) - f(y_{n})}\right)\right]\frac{f(x_{n})}{f'(x_{n})}\phi(t_{n}),$$

$$x_{n+1} = v_{n} - f(v_{n})\left(f'(x_{n}) + \frac{f[z_{n}, y_{n}] + \left(f[z_{n}, x_{n}] - f'(x_{n})\right)\frac{z_{n} - y_{n}}{z_{n} - x_{n}} - f'(x_{n})}{(x_{n} - x_{n})}\right)^{-1},$$
(29)

where  $f(z_n)$  is evaluated from (27) and  $t_n = \frac{f(y_n)}{f(x_n)}$ .

Method (29) is not still optimal. Therefore we introduce a second weight function as follows:

$$y_{n} = x_{n} - \frac{f(x_{n})}{f'(x_{n})},$$

$$z_{n} = x_{n} - \frac{f^{2}(x_{n})}{(f(x_{n}) - f(y_{n}))f'(x_{n})},$$

$$v_{n} = z_{n} - \left[1 - \frac{f(x_{n})}{f(x_{n}) - f(y_{n})}\left(1 + \frac{f(y_{n})}{f(x_{n}) - f(y_{n})}\right)\right] \frac{f(x_{n})}{f'(x_{n})}\phi(t_{n}),$$

$$x_{n+1} = v_{n} - f(v_{n})$$

$$\left(f'(x_{n}) + \frac{f[z_{n}, y_{n}] + \left(f[z_{n}, x_{n}] - f'(x_{n})\right)\frac{z_{n} - y_{n}}{z_{n} - x_{n}} - f'(x_{n})}(v_{n} - x_{n})\right)^{-1}\psi(s_{n}),$$
(30)
where  $f(z_{n})$  is evaluated from (27) and  $t_{n} = \frac{f(y_{n})}{z_{n}}$  and  $s_{n} = \frac{f(v_{n})}{z_{n}}$ 

where  $f(z_n)$  is evaluated from (27) and  $t_n = \frac{f(y_n)}{f(x_n)}$  and  $s_n = \frac{f(v_n)}{f(x_n)}$ .

In the following theorem we prove that method (30) is of convergence order eight if the weight functions  $\phi(t_n)$  and  $\psi(s_n)$  satisfy the stated conditions in the following theorem.

**Theorem 2** Let  $D \subseteq \mathbb{R}$  be an open interval,  $f: D \to \mathbb{R}$  eight times continuously differentiable and let  $x^* \in D$  be a simple zero of f. If the initial point  $x_0$  is sufficiently close to  $x^*$ , then the method defined by (30) converges to  $x^*$  with order at least eight if the weight function  $\phi: \mathbb{R} \to \mathbb{R}$  is two times continuously

differentiable,  $\psi : \mathbb{R} \to \mathbb{R}$  is continuously differentiable and they satisfy the conditions of Theorem 1 and moreover

$$\phi^{''}(0) = -\frac{5}{2}, \quad \psi(0) = 1 \quad \text{and} \quad \psi^{'}(0) = 1.$$

Proof Let  $e_n := x_n - x^*$ ,  $e_{n,y} := y_n - x^*$ ,  $e_{n,z} := z_n - x^*$ ,  $c_n := \frac{f^{(n)}(x^*)}{n!f'(x^*)}$  for  $n \in \mathbb{N}$ . Using the fact that  $f(x^*) = 0$ , Taylor expansion of f at  $x^*$  yields

$$f(x_n) = f'(x^*)(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + c_8e_n^8) + O(e_n^9), \quad (31)$$

and

$$f'(x_n) = f'(x^*)(1 + 2c_2e_n + 3c_3e_n^2 + \dots + 9c_9e_n^8) + O(e_n^9).$$
(32)

According to Theorem 1, we get

$$e_{n,y} = y_n - x^* = c_2 e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + O(e_n^5),$$

and

$$e_{n,v} = v_n - x^* = \left( \left(\frac{5}{2} + \phi''(0)\right)c_2^3 - c_2c_3 \right)e_n^4 + O(e_n^5)$$

By using Taylor's theorem for  $f(y_n)$  and  $f(v_n)$  at  $x^*$ , we have

$$f(y_n) = f'(x^*) \left[ c_2 e_n^2 - 2(c_2^2 - c_3) e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4) e_n^4 -2(6c_2^4 - 12c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5) e_n^5 + \left( 28c_2^5 - 73c_2^3c_3 + 37c_2c_3^2 + 34c_2^2c_4 - 17c_3c_4 - 13c_2c_5 + 5c_6 \right) e_n^6 \right] + O(e_n^7),$$
(33)

and

$$f(v_n) = f'(x^*) \left[ c_2 c_3 e_n^4 + \frac{1}{4} \left( 29 c_2^4 + 8 c_2^2 c_3 - 8 c_3^2 \right) e_n^5 + \left( -60.75 c_2^5 + 54 c_2^3 c_3 + 6 c_2 c_3^2 + 3 c_2^2 c_4 - 7 c_3 c_4 - 3 c_2 c_5 \right) e_n^6 + \left( 1243 c_2^6 - 2166 c_2^4 c_3 + 332 c_2^3 c_4 + 8 c_2^2 (77 c_3^2 + 2 c_5) + 8(2 c_3^3 - 3 c_4^2 - 5 c_3 c_5) + 16 c_2 (4 c_3^2 c_4 - c_6)) e_n^7 \right] + O(e_n^8).$$

$$(34)$$

Also

$$f(z_n) = f'(x^*) \left[ c_2^2 e_n^3 + 3c_2(-c_2^2 + c_3)e_n^4 + (6c_2^4 - 13c_2^2c_3 + 2c_3^2 + 4c_2c_4)e_n^5 + (-8c_2^5 + 33c_2^3c_3 - 18c_2c_3^2 - 18c_2^2c_4 + 5c_3c_4 + 5c_2c_5)e_n^6 + (3c_2^6 - \frac{175}{4}c_2^4c_3 + 48c_2^3c_4 + c_2^2(64c_3^2 - 23c_5) + (-8c_3^3 + 3c_4^2 + 6c_3c_5) + c_2(-50c_3c_4))e_n^7 \right] + O(e_n^8).$$
(35)

Moreover, for  $f'(v_n)$ , we also have

$$f'(v_n) = f'(x^*) \left[ 1 - c_2 c_3 e_n^3 + \left( c_2^2 c_3 - 2c_3^2 - c_2 c_4 \right) e_n^4 \right. \\ \left. + \frac{1}{4} \left( 48c_2^5 - 197c_2^3 c_3 + 104c_2 c_3^2 + 60c_2^2 c_4 - 20c_3 c_4 - 4c_2 c_5 \right) e_n^5 \right. \\ \left. - \frac{1}{2} \left( 243c_2^6 - 412c_2^4 c_3 - 16c_3^3 + 39c_2^3 c_4 + 6c_4^2 + 12c_3 c_5 \right. \\ \left. + c_2^2 \left( 165c_3^2 + 4c_5 \right) + c_2 \left( -20c_3 c_4 + 2c_6 \right) \right) e_n^6 \right] + O(e_n^7).$$

$$(36)$$

From (31) and (34), we calculate

$$s_n = \frac{f(v_n)}{f(x_n)} = (-c_2c_3) e_n^3 + (7.25c_2^4 + 3c_2^2c_3 - 2c_3^2 - 2c_2c_4) e_n^4 + (-68c_2^5 + 51c_2^3c_3 + 5c_2^2c_4 - 7c_3c_4 + 9c_2c_3^2 - 3c_2c_5) e_n^5 + O(e_n^6).$$
(37)

Expanding  $\psi$  at 0, yields

$$\psi(s_n) = \psi(0) + \psi'(0)s_n + \frac{1}{2}\psi''(0)s_n^2 + O(s_n^3).$$
(38)

By substituting (31)-(38) into (30), we obtain

$$e_{n+1} = x_{n+1} - x^* = R_4 e_n^4 + R_5 e_n^5 + R_6 e_n^6 + R_7 e_n^7 + R_8 e_n^8 + O(e_n^9),$$

where

$$R_{4} = -\frac{1}{2}c_{2}(-1+\psi(0))((5+2\phi''(0))c_{2}^{2}-2c_{3}),$$

$$R_{5} = 0,$$

$$R_{6} = 0,$$

$$R_{7} = -\frac{1}{4}c_{2}^{2}(-2c_{3}+c_{2}^{2}(5+2\phi''(0))))$$

$$(-2c_{3}(-1+\psi'(0))+c_{2}^{2}(5+2\phi''(0))\psi'(0)),$$

$$R_{8} = \frac{1}{4}c_{2}^{2}c_{3}(29c_{2}^{3}+4c_{2}c_{3}-4c_{4}).$$
(39)

To ensure convergence order eight for the three-point method (30), it is necessary to have  $R_i = 0$ , (i = 4, 5, 6, 7). Obviously

$$\psi(0) = 1 \implies R_4 = 0, \psi'(0) = 1, \phi''(0) = -\frac{5}{2} \implies R_7 = 0.$$
(40)

It is clear that  $R_8 \neq 0$ , thus the error equation becomes

$$e_{n+1} = R_8 e_n^8 + O(e_n^9),$$

and method (30) has convergence order eight, which proves the theorem.

In what follows, we give some concrete explicit representations of (30) by choosing different weight functions satisfying the provided condition for the weight functions  $\phi(t_n)$  and  $\psi(t_n)$  in Theorems 1 and 2.

**Method 1.** Choose the weight functions  $\phi(t_n)$  and  $\psi(s_n)$  as follows:

$$\phi(t_n) = -\frac{1}{2}t_n - \frac{5}{4}t_n^2 \quad \text{and} \quad \psi(s_n) = \frac{1+2s_n}{1+s_n},\tag{41}$$

where  $t_n = \frac{f(y_n)}{f(x_n)}$  and  $s_n = \frac{f(v_n)}{f(x_n)}$ . The functions  $\phi(t_n)$  and  $\psi(s_n)$  in (41) satisfy the assumptions of Theorem 2 denoted by SLSS, so

$$y_{n} = x_{n} - \frac{f(x_{n})}{f'(x_{n})},$$

$$z_{n} = x_{n} - \frac{f^{2}(x_{n})}{(f(x_{n}) - f(y_{n}))f'(x_{n})},$$

$$v_{n} = z_{n} - \left[1 - \frac{f(x_{n})}{f(x_{n}) - f(y_{n})} \left(1 + \frac{f(y_{n})}{f(x_{n}) - f(y_{n})}\right)\right] \left(\frac{-f(y_{n})}{2f(x_{n})} - \frac{5}{4} \left(\frac{f(y_{n})}{f(x_{n})}\right)^{2}\right) \frac{f(x_{n})}{f'(x_{n})},$$

$$x_{n+1} = v_{n} - f(v_{n}) \left(\frac{f(x_{n}) + 2f(v_{n})}{f(x_{n}) + f(v_{n})}\right) \times \left(f'(x_{n}) + \frac{f[z_{n}, y_{n}] + \left(f[z_{n}, x_{n}] - f'(x_{n})\right)\frac{z_{n} - y_{n}}{z_{n} - x_{n}} - f'(x_{n})}(v_{n} - x_{n})\right)^{-1},$$

$$(42)$$

where  $f(z_n)$  is evaluated by (27).

**Method 2.** Choose the weight functions  $\phi(t_n)$  and  $\psi(s_n)$  as follows:

$$\phi(t_n) = t_n + \frac{9t_n}{5t_n - 6}$$
 and  $\psi(s_n) = \frac{1}{1 - s_n}$ , (43)

where  $t_n = \frac{f(y_n)}{f(x_n)}$  and  $s_n = \frac{f(v_n)}{f(x_n)}$ . The functions  $\phi(t_n)$  and  $\psi(s_n)$  in (43) satisfy the assumptions of Theorem 2 and we get

$$y_{n} = x_{n} - \frac{f(x_{n})}{f'(x_{n})},$$

$$z_{n} = x_{n} - \frac{f^{2}(x_{n})}{(f(x_{n}) - f(y_{n}))f'(x_{n})},$$

$$v_{n} = z_{n} - \left[1 - \frac{f(x_{n})}{f(x_{n}) - f(y_{n})}\left(1 + \frac{f(y_{n})}{f(x_{n}) - f(y_{n})}\right)\right] \left(\frac{f(y_{n})}{f(x_{n})} + \frac{9f(y_{n})}{5f(y_{n}) - 6f(x_{n})}\right) \frac{f(x_{n})}{f'(x_{n})},$$

$$x_{n+1} = v_{n} - f(v_{n}) \left(\frac{f(x_{n})}{f(x_{n}) - f(v_{n})}\right) \times \left(f'(x_{n}) + \frac{f(z_{n}, y_{n}] + \left(f(z_{n}, x_{n}] - f'(x_{n})\right)\frac{z_{n} - y_{n}}{z_{n} - x_{n}} - f'(x_{n})}(v_{n} - x_{n})\right)^{-1},$$

$$(44)$$

where  $f(z_n)$  is evaluated by (27).

**Method 3.** Choose the weight functions  $\phi(t_n)$  and  $\psi(s_n)$  as follows:

$$\phi(t_n) = \frac{t_n}{5t_n - 2}$$
 and  $\psi(s_n) = 1 + \frac{2s_n}{2 + 5s_n}$ , (45)

where  $t_n = \frac{f(y_n)}{f(x_n)}$  and  $s_n = \frac{f(v_n)}{f(x_n)}$ . The functions  $\phi(t_n)$  and  $\psi(s_n)$  in (45) satisfy the assumptions of Theorem 2 and we get

$$y_{n} = x_{n} - \frac{f(x_{n})}{f'(x_{n})},$$

$$z_{n} = x_{n} - \frac{f^{2}(x_{n})}{(f(x_{n}) - f(y_{n}))f'(x_{n})},$$

$$v_{n} = z_{n} - \left[1 - \frac{f(x_{n})}{f(x_{n}) - f(y_{n})}\left(1 + \frac{f(y_{n})}{f(x_{n}) - f(y_{n})}\right)\right]\left(\frac{f(y_{n})}{5f(y_{n}) - 2f(x_{n})}\right)\frac{f(x_{n})}{f'(x_{n})},$$

$$x_{n+1} = v_{n} - f(v_{n})\left(1 + \frac{2f(v_{n})}{2f(x_{n}) + 5f(v_{n})}\right) \times \left(f'(x_{n}) + \frac{f[z_{n}, y_{n}] + \left(f[z_{n}, x_{n}] - f'(x_{n})\right)\frac{z_{n} - y_{n}}{z_{n} - x_{n}} - f'(x_{n})}(v_{n} - x_{n})\right)^{-1},$$
(46)

where  $f(z_n)$  is evaluated by (27).

**Method 4.** Choose the weight functions  $\phi(t_n)$  and  $\psi(s_n)$  as follows:

$$\phi(t_n) = -\frac{6t_n + t_n^2}{4} + \frac{t_n}{1 + t_n} \quad \text{and} \quad \psi(s_n) = (1 + s_n)^{\frac{s_n + 1}{2s_n + 1}}, \tag{47}$$

where  $t_n = \frac{f(y_n)}{f(x_n)}$  and  $s_n = \frac{f(v_n)}{f(x_n)}$ . The functions  $\phi(t_n)$  and  $\psi(s_n)$  in (47) satisfy the assumptions of Theorem 2 and we get

$$y_{n} = x_{n} - \frac{f(x_{n})}{f'(x_{n})},$$

$$z_{n} = x_{n} - \frac{f^{2}(x_{n})}{(f(x_{n}) - f(y_{n}))f'(x_{n})},$$

$$v_{n} = z_{n} - \left[1 - \frac{f(x_{n})}{f(x_{n}) - f(y_{n})}\left(1 + \frac{f(y_{n})}{f(x_{n}) - f(y_{n})}\right)\right] \times \left(\frac{-6f(y_{n})}{4f(x_{n})} - \frac{f^{2}(y_{n})}{4f^{2}(x_{n})} + \frac{f(y_{n})}{f(x_{n}) + f(y_{n})}\right)\frac{f(x_{n})}{f'(x_{n})},$$

$$(48)$$

$$x_{n+1} = v_{n} - f(v_{n})\left(\left(1 + \frac{f(v_{n})}{f(x_{n})}\right)\frac{f'(v_{n}) + f(x_{n})}{2f'(v_{n}) + f'(x_{n})}\right) \times \left(f'(x_{n}) + \frac{f[z_{n}, y_{n}] + \left(f[z_{n}, x_{n}] - f'(x_{n})\right)\frac{z_{n} - y_{n}}{z_{n} - x_{n}} - f'(x_{n})}(v_{n} - x_{n})\right)^{-1},$$

where  $f(z_n)$  is evaluated by (27).

In the next section we apply the new methods (42), (44), (46) and (48) to several benchmark examples and compare them with existing three-point methods which have the same order of convergence and the same computational efficiency index equal to  $\sqrt[\theta]{r} = 1.682$  for the convergence order r = 8 which is optimal for  $\theta = 4$  function evaluations per iteration [28,39].

## 3 Numerical performance and dynamic behavior

## 3.1 Numerical results

In this section we test and compare our proposed methods with some existing methods. We compare our Methods 1-4 with the following related three-point methods.

**W. Bi, H. Ren and Q. Wu method.** The method by Bi et al. [8] denoted by BRW is  $f(x_n)$ 

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(y_n)}{f'(x_n)} \cdot \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)} H(t_n),$$
(49)

with weight function

$$H(t_n) = \frac{1}{(1 - \alpha t_n)^2}, \quad \alpha = 1,$$
(50)

and  $t_n = \frac{f(z_n)}{f(x_n)}$  and  $\beta = -\frac{1}{2}$ .

Wang and Liu method. The method by Wang and Liu [41] denoted by WL is  $f(x_n)$ 

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
  

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)} G(t_n),$$
  

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} (H(t_n) + V(t_n)W(s_n)),$$
  
(51)

with weight functions

$$G(t_n) = \frac{1 - t_n}{1 - 2t_n}, \quad H(t_n) = \frac{5 - 2t_n + t_n^2}{5 - 12t_n}, \quad V(t_n) = 1 + 4t_n, \quad W(s_n) = s_n,$$
(52)
and  $t_n = \frac{f(y_n)}{f(x_n)}$  and  $s_n = \frac{f(z_n)}{f(y_n)}.$ 

Sharma and Sharma method. The Sharma and Sharma method [33] denoted by SS is

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
  

$$z_n = y_n - \frac{f(y_n)}{f'(x_n)} \cdot \frac{f(x_n)}{f(x_n) - 2f(y_n)},$$
  

$$x_{n+1} = z_n - \frac{f[x_n, y_n]f(z_n)}{f[x_n, z_n]f[y_n, z_n]} W(t_n),$$
(53)

where weight functions are

$$W(t_n) = 1 + \frac{t_n}{1 + \alpha t_n}, \quad \alpha = 1,$$
(54)

and  $t_n = \frac{f(z_n)}{f(x_n)}$ .

**Babajee et al. method.** The method by Babajee et al., see [7], denoted by BCST, is

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \cdot \left(1 + \left(\frac{f(x_n)}{f'(x_n)}\right)^5\right),$$

$$z_n = y_n - \frac{f(y_n)}{f'(x_n)} \cdot \left(1 - \frac{f(y_n)}{f(x_n)}\right)^{-2},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} \cdot \frac{1 + \left(\frac{f(y_n)}{f(x_n)}\right)^2 + 5\left(\frac{f(y_n)}{f(x_n)}\right)^4 + \frac{f(z_n)}{f(x_n)}}{\left(1 - \frac{f(y_n)}{f(x_n)} - \frac{f(z_n)}{f(x_n)}\right)^2}.$$
(55)

**Cordero et al. method.** The method by Cordero et al., see [14], denoted by CFGT, is

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f^3(x_n) - 2f^2(x_n)f(y_n) - f(x_n)f^2(y_n) - \frac{1}{2}f^3(y_n)}{f'(x_n) + 3f(z_n)} \cdot \frac{f(y_n)}{f(z_n, y_n) + f[z_n, x_n, x_n](z_n - y_n)},$$
(56)

with the divided differences  $f[z_n, y_n] = \frac{f(z_n) - f(y_n)}{z_n - y_n}$ ,  $f[z_n, x_n, x_n] = \frac{f[z_n, x_n] - f'(x_n)}{z_n - x_n}$ .

**Cordero et al. method.** The method by Cordero et al., see [13], denoted by CTV, is

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(y_n)}{f'(x_n)} \cdot \frac{f(x_n)}{f(x_n) - 2f(y_n)},$$

$$x_{n+1} = v_n - \frac{f(z_n)}{f'(x_n)} \cdot \frac{\gamma(v_n - z_n)}{\beta_1(v_n - z_n) + \beta_2(y_n - x_n) + \beta_3(z_n - x_n)},$$
(57)

where

$$v_n = z_n - \frac{f(z_n)}{f'(x_n)} \cdot \left(\frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)} + \frac{1}{2} \frac{f(z_n)}{f(y_n) - 2f(z_n)}\right)^2,$$

and  $\gamma, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$  such that  $\gamma = 3(\beta_2 + \beta_3) \neq 0$ .

Thukral and Petkovic method. The method by Thukral and Petkovic., see [38], denoted by TP, is

$$y_{n} = x_{n} - \frac{f(x_{n})}{f'(x_{n})},$$

$$z_{n} = y_{n} - \frac{f(y_{n})}{f'(x_{n})} \cdot \frac{f(x_{n}) + \beta f(y_{n})}{f(x_{n}) + (\beta - 2)f(y_{n})}, \quad (\alpha, \beta \in \mathbb{R})$$

$$x_{n+1} = z_{n} - \frac{f(z_{n})}{f'(x_{n})} \cdot \left(H(t_{n}) + \frac{f(z_{n})}{f(y_{n}) - \alpha f(z_{n})} + \frac{4f(z_{n})}{f(x_{n})}\right),$$
(58)

with weight functions

$$H(t_n) = \frac{5 - 2\beta - (2 - 8\beta + 2\beta^2)t_n + (1 + 4\beta)t_n^2}{5 - 2\beta - (12 - 12\beta + 2\beta^2)t_n},$$
(59)

where  $t_n = \frac{f(y_n)}{f(x_n)}$ .

**Chun and Lee method.** The method by Chun and Lee., see [10], denoted by CL, is

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
  

$$z_n = y_n - \frac{f(y_n)}{f'(x_n)} \cdot \frac{1}{\left(1 - \frac{f(y_n)}{f(x_n)}\right)^2},$$
  

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} \cdot \frac{1}{(1 - H(t_n) - J(s_n) - P(u_n))^2},$$
(60)

with weight functions

$$H(t_n) = -\beta - \gamma + t_n + \frac{t_n^2}{2} - \frac{t_n^3}{2}, \quad J(s_n) = \beta + \frac{s_n}{2}, \quad P(u_n) = \gamma + \frac{u_n}{2}, \quad (61)$$

where 
$$t_n = \frac{f(y_n)}{f(x_n)}$$
,  $s_n = \frac{f(z_n)}{f(x_n)}$ ,  $u_n = \frac{f(z_n)}{f(y_n)}$  and  $\beta, \gamma \in \mathbb{R}$ .

The three-point method (30), more precisely, the explicitly proposed methods (42), (44), (46) and (48), are now tested on a number of nonlinear equations. To obtain a high accuracy and avoid the loss of significant digits, we employed multi-precision arithmetic with 1800 significant decimal digits in the programming package of Mathematica 8. In order to compare them with the methods (49), (51), (53), (55), (56), (57), (58) and (60) we choose the initial value  $x_0$ using the Mathematica command FindRoot [17, pp. 158-160] and compute the error, the computational order of convergence, (COC) by the approximate formula [42]

$$\operatorname{COC} \approx \frac{\ln |(x_{n+1} - x^*)/(x_n - x^*)|}{\ln |(x_n - x^*)/(x_{n-1} - x^*)|}.$$

and the approximated computational order of convergence, (ACOC) by the formula [11]

ACOC 
$$\approx \frac{\ln |(x_{n+1} - x_n)/(x_n - x_{n-1})|}{\ln |(x_n - x_{n-1})/(x_{n-1} - x_{n-2})|}$$

It is worth noting although the former formula, COC, has been used in the recent years, nevertheless, the later, ACOC, is more practical. Here we have collect and use both of them for checking the accuracy of the considered methods. Moreover, we should note that the results for these formula are generally different from the exact convergence order of the method. The reason is that in the error equations of the methods, we have some coefficients that depend on  $c_k$ , and these  $c_k$ s may vanish or vary for different kinds of examples. See the out puts in the Tables 1 and 2. We should be careful about these events. Indeed, it does not contradicts our discussed theory since all of the formulas are provided approximately and behave asymptotically.

Table 1:  $f(x) = \sin(x) - \frac{x}{100}, x^* = 0, x_0 = 0.1$  $|x_2 - x^*|$ COC Methods  $|x_1 - x^*|$  $|x_3 - x^*|$ ACOC (42)0.949e - 140.486e - 1570.314e - 173311.0000 11.0000 (44)0.929e - 140.387e - 1570.252e - 173411.000011.0000 0.877e - 140.204e - 1570.223e - 1737(46)11.0000 11.0000 0.971e - 140.629e - 1570.531e - 1732(48)11.0000 11.0000

Table 2:							
$f(x) = \tan^{-1}(x), x^* = 0, x_0 = 0.1$							
Methods	$ x_1 - x^* $	$ x_2 - x^* $	$ x_3 - x^* $	COC	ACOC		
(42)	0.769e - 12	0.424e - 134	0.610e - 1479	11.0000	11.0000		
(44)	0.758e - 12	0.361e - 134	0.103e - 1479	11.0000	11.0000		
(46)	0.728e - 12	0.232e - 134	0.819e - 1482	11.0000	10.9999		
(48)	0.782e - 12	0.509e - 134	0.455e - 1478	11.0000	11.0000		

Table 2

In Table 1 and 2 our new three-point methods (42), (44), (46) and (48) with weight functions (41), (43), (45) and (47) are tested on two nonlinear equations.

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$\begin{array}{c c c c c c c c c c c c c c c c c c c $			Table 3:				
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	$f(x) = e^{\sin(x)} - 1 - \frac{x}{5}, x^* = 0, x_0 = 0.1$						
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	Methods	$ x_1 - x^* $	$ x_2 - x^* $	$ x_3 - x^* $	COC	ACOC	
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	(42)	0.551e - 9	0.735e - 83	0.982e - 748	9.0000	9.0000	
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	(44)	0.346e - 9	0.628e - 85	0.132e - 766	9.0000	9.0000	
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	(46)	0.543e - 10	0.362e - 93	0.943e - 842	9.0000	9.0000	
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	(48)	0.768e - 9	0.226e - 81	0.373e - 734	9.0000	9.0000	
$      \begin{array}{c} (53) \\ (53) \\ (55) \\ (55) \\ (55) \\ (56) \\ (56) \\ (57) \\ (57) \\ (58) \\ (58) \\ (58) \\ (58) \\ (58) \\ (58) \\ (56) \\ (57) \\ (57) \\ (58) \\ (58) \\ (58) \\ (57) \\ (58$	(49)	0.123e - 9	0.162e - 89	0.181e - 808	9.0000	9.0000	
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	(51)	0.266e - 8	0.108e - 68	0.831e - 552	8.0000	7.9999	
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	(53)	0.589e - 9	0.128e - 74	0.673e - 600	8.0000	7.9999	
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	(55)	0.672e - 9	0.199e - 74	0.119e - 598	8.0000	7.9999	
(58)   0.109e - 7   0.524e - 63   0.140e - 505   8.0000   7.9999	(56)	0.125e - 9	0.175e - 89	0.380e - 808	9.0000	9.0000	
	(57)	0.815e - 9	0.263e - 73	0.315e - 598	8.0000	7.9999	
(60) $0.542e - 9$ $0.133e - 74$ $0.178e - 599$ $8.0000$ $8.0000$	(58)	0.109e - 7	0.524e - 63	0.140e - 505	8.0000	7.9999	
	(60)	0.542e - 9	0.133e - 74	0.178e - 599	8.0000	8.0000	

Table 4:

$f(x) = \ln(1 - x + x)$	$(x^2) + 4\sin(1-x), x$	$x^* = 1, x_0 = 1.1$
-------------------------	-------------------------	----------------------

Methods	$ x_1 - x^* $	$ x_2 - x^* $	$ x_3 - x^* $	COC	ACOC
(42)	0.225e - 12	0.274e - 117	0.162e - 1061	9.0000	8.9999
(44)	0.300e - 12	0.473e - 116	0.284e - 1051	9.0000	8.9999
(46)	0.469e - 12	0.395e - 114	0.845e - 1033	9.0000	8.9999
(48)	0.159e - 12	0.966e - 119	0.577e - 1075	9.0000	8.9999
(49)	0.423e - 12	0.134e - 114	0.445e - 1037	9.0000	8.9999
(51)	0.295e - 11	0.629e - 96	0.265e - 773	8.0000	7.9999
(53)	0.172e - 11	0.581e - 98	0.984e - 760	8.0000	7.9999
(55)	0.357e - 11	0.343e - 95	0.251e - 797	8.0000	7.9999
(56)	0.423e - 12	0.135e - 114	0.474e - 1037	9.0000	8.9999
(57)	0.179e - 11	0.839e - 98	0.195e - 788	8.0000	7.9999
(58)	0.829e - 11	0.977e - 92	0.362e - 739	8.0000	7.9999
(60)	0.211e - 11	0.494e - 97	0.250e - 782	8.0000	7.9999

In Tables 3 and 4 we compare our new method with the methods (49), (51), (53), (55), (56), (57), (58) and (60).

#### 3.2 Dynamic behavior

We already observed that all methods converge if the initial guess is chosen suitably. We now investigate the stability region. In other words, we numerically approximate the domain of attraction of the zeros as a qualitative measure of stability. To answer the important question on the dynamical behavior of the algorithms, we investigate the dynamics of the new methods and compare them with common and well-performing methods from the literature. It turns out that only one method, namely CFGT, has better stability than ours. In the following we recall some basic concepts such as basin of attraction. For more details one can consult [2]-[6], [25]-[27], [32,36,37,40].

Let  $G : \mathbb{C} \to \mathbb{C}$  be a rational map on the complex plane. For  $z \in \mathbb{C}$ , we define its orbit as the set  $orb(z) = \{z, G(z), G^2(z), \ldots\}$ . A point  $z_0 \in \mathbb{C}$  is called periodic point with minimal period m if  $G^m(z_0) = z_0$ , where m is the smallest integer with this property. A periodic point with minimal period 1 is called fixed point. Moreover, a point  $z_0$  is called attracting if  $|G'(z_0)| < 1$ , repelling if  $|G'(z_0)| > 1$ , and neutral otherwise. The Julia set of a nonlinear map G(z), denoted by J(G), is the closure of the set of its repelling periodic points. The complement of J(G) is the Fatou set F(G), where the basin of attraction of the different roots lie [7], [14].

For the dynamical point of view, in fact, we take a  $256 \times 256$  grid of the square  $[-3,3] \times [-3,3] \in \mathbb{C}$  and assign a color to each point  $z_0 \in D$  according to the simple root to which the corresponding orbit of the iterative method starting from  $z_0$  converges, and we mark the point as black if the orbit does not converge to a root, in the sense that after at most 100 iterations it has a distance to any of the roots, which is larger than  $10^{-3}$ . In this way, we distinguish the attraction basins by their color for different methods.

We have tested several different examples, and the results on the performance of the tested methods were similar. Therefore we merely report the general observation here for  $f(z) = z^3 - 1/z$ . A visual inspection of the simulations indicates that for some examples the SLSS method (see Fig. 1) seems to produce a larger basin of attraction than the BCST, SS, CTV, TP, CL, BRW, WL methods (see Figs. 2-5 and Figs. 7-9), but it seems to be smaller than that of the CFGT method (see Fig. 6). We stop here for a moment. Although we were able to ignore the method CFGT, however, we should note that it is a very good example to discuss some aspects of our algorithms. It is wellknown that any good algorithm should study these three concepts: accuracy, efficiency, and stability. All the work in this study have the same efficiency, four functional evaluations per iterate. On the other hand, comparing CFGT and method (44) reveal another fact: while a method may have a slightly better accuracy, see Table 4 and compare numerical results for methods (44) and (56), the other method may have produce a little better stability. Therefore, we cannot conclude which one is better in action. One has better accuracy, and the other has better stability. On the whole, finding such examples could make deeper understanding of devising new algorithms and it can be left for future works. Note that some points belong to no basin of attraction; these are starting points for which the methods do not converge, denoted by black points. These exceptional points constitute the Julia set of methods, so named in honor of G. Julia, a French mathematician who published an important memoir on this subject in 1918. Here, we would like to tell a little more about these black points. We have said that these point do not converge to the roots. This statement is true only for the given number of iterations, say 100 here. If we increase the number of iteration, they might converge to a root, and the basins or Fatou set might be larger.

Test problem  $f(z) = z^3 - \frac{1}{z}$ 



Fig. 1 SLSS



Fig. 2 BRW



Fig. 3 WL



Fig. 4 SS

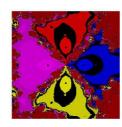


Fig. 5 BCST

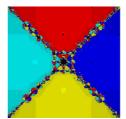


Fig. 6 CFGT



Fig. 7 CTV



Fig. 8 TP



**Fig. 9** CL

# 4 Conclusion

Two new optimal classes of two-point and three-point methods without memory have been developed which use only three and four function evaluations per iteration, respectively. Both methods are based on the Newton and Secant methods. A numerical comparison with other well-known optimal multi-point methods shows that our new classes are a valuable alternative to existing optimal multi-point methods. In addition, a numerical investigation of the basins of attraction of the solutions illustrate that the stability region of our method it typically larger than that of other methods. Indeed, among the eight compared methods, only one shows a larger stability region than our proposed methods.

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