# Richardson extrapolation for the iterated Galerkin solution of Urysohn integral equations with Green's kernels 

Gobinda RAKSHIT* Akshay S. RANE ${ }^{\dagger}$ and Kshitij PATIL ${ }^{\ddagger}$


#### Abstract

We consider a Urysohn integral operator $\mathcal{K}$ with kernel of the type of Green's function. For $r \geq 1$, a space of piecewise polynomials of degree $\leq r-1$ with respect to a uniform partition is chosen to be the approximating space and the projection is chosen to be the orthogonal projection. Iterated Galerkin method is applied to the integral equation $x-\mathcal{K}(x)=f$. It is known that the order of convergence of the iterated Galerkin solution is $r+2$ and, at the above partition points it is $2 r$. We obtain an asymptotic expansion of the iterated Galerkin solution at the partition points of the above Urysohn integral equation. Richardson extrapolation is used to improve the order of convergence. A numerical example is considered to illustrate our theoretical results.


Key Words : Urysohn integral operator, Green's kernel, Galerkin method, Richardson extrapolation.
AMS subject classification : 45G10, 65B05, 65J15, 65R20

[^0]
## 1 Introduction

Let $\mathcal{X}=L^{\infty}[0,1]$ and consider the following Urysohn integral operator

$$
\begin{equation*}
\mathcal{K}(x)(s)=\int_{0}^{1} \kappa(s, t, x(t)) d t, \quad s \in[0,1], x \in \mathcal{X} \tag{1.1}
\end{equation*}
$$

where $\kappa(s, t, u)$ is a real valued continuous function defined on $\Omega=[0,1] \times[0,1] \times \mathbb{R}$. Then $\mathcal{K}$ is a compact operator from $L^{\infty}[0,1]$ to $C[0,1]$. Consider the Urysohn integral equation

$$
\begin{equation*}
x(s)-\int_{0}^{1} \kappa(s, t, x(t)) d t=f(s), \quad s \in[0,1] \tag{1.2}
\end{equation*}
$$

where $f \in \mathcal{X}$ is given and $x$ is the unknown to be determined. We assume that $\varphi$ is an isolated solution of the above equation and consider its numerical approximations.

We are interested in approximate solution which converges to $\varphi$ uniformly. We consider some projection methods associated with a sequence of orthogonal projections converging to the Identity operator point-wise.

In this paper, we consider the case when the kernel $\kappa$ of the integral operator $\mathcal{K}$, is of the type of Green's function in its domain. We allow the partial derivatives of the kernel $\kappa$ to have jump discontinuities along the diagonal $s=t$. For $r \geq 1$, let $\mathscr{X}_{n}$ be a space of piecewise polynomials of degree $\leq r-1$ with respect to a uniform partition of $[0,1]$ with $n$ subintervals each of length $h=\frac{1}{n}$. Let $\pi_{n}$ be the restriction to $L^{\infty}[0,1]$ of the orthogonal projection from $L^{2}[0,1]$ onto $\mathscr{X}_{n}$. Galerkin method is a classical projection method for the approximate solution of an integral equation. In this method, (1.2) is approximated by

$$
\begin{equation*}
x_{n}^{G}-\pi_{n} \mathcal{K}\left(x_{n}^{G}\right)=\pi_{n} f \tag{1.3}
\end{equation*}
$$

The above projection method has been studied extensively in the research literature. See Krasnoselsii [6], Krasnoselskii, Vainikko et al [7] and Krasnoselskii and Zabreiko [8] for details.
The iterated Galerkin solution is defined by

$$
x_{n}^{S}=\mathcal{K}\left(x_{n}^{G}\right)+f
$$

Note that

$$
x_{n}^{G}=\pi_{n} x_{n}^{S}
$$

and then the iterated Galerkin solution satisfies the following equation:

$$
\begin{equation*}
x_{n}^{S}-\mathcal{K}\left(\pi_{n} x_{n}^{S}\right)=f \tag{1.4}
\end{equation*}
$$

From Atkinson-Potra [3], we quote the following orders of convergence:
If $r=1$, then

$$
\begin{equation*}
\left\|x_{n}^{G}-\varphi\right\|_{\infty}=O(h), \quad\left\|x_{n}^{S}-\varphi\right\|_{\infty}=O\left(h^{2}\right), \tag{1.5}
\end{equation*}
$$

whereas if $r \geq 2$, then

$$
\begin{equation*}
\left\|x_{n}^{G}-\varphi\right\|_{\infty}=O\left(h^{r}\right), \quad\left\|x_{n}^{S}-\varphi\right\|_{\infty}=O\left(h^{r+2}\right) . \tag{1.6}
\end{equation*}
$$

Asymptotic error analysis and extrapolation methods are classical topics in numerical analysis. Richardson extrapolation is the popular one. In Ford et al. [5], Hammerstein integral equation with Green's function type of kernel is considered. Composite trapezoidal quadrature method is used to approximate the integral operator, and then an asymptotic error expansion is obtained for the approximate solution at the node points. Richardson extrapolation is applied to improve the orders of convergence. In Kulkarni-Rane [12], the authors have defined the Nyström operator based on the composite midpoint and the composite modified Simpson rules to approximate the integral operator of a Hammerstein integral equation with Green's function type kernel. Asymptotic expansions for the approximate solution at the node points as well as at the partition points, are obtained and Richardson extrapolation is used to obtain approximate solutions with higher orders of convergence. Hammerstein integral equation is a special case of Urysohn integral equation. The case when the kernel of the Urysohn integral equation is sufficiently smooth, asymptotic error analysis are investigated for various projection methods in Kulkarni-Nidhin [10]. In the case of a linear integral equation of the second kind with smooth kernel, asymptotic series expansion for the iterated Galerkin solution is proved by McLean [14. The case of asymptotic expansion for approximate solution of integral equations with Green's kernel, at the partition points in the case of Nyström method with midpoint rule, modified Simpsons rule and iterated collocation method is treated in Kulkarni-Rane [11].

In Rakshit-Rane [15], we considered a Fredholm integral equation with kernel of the type of Green's function and then asymptotic error analysis is investigated for the iterated Galerkin solution at the partition points. Richardson extrapolation is applied to obtain an approximate solution with higher rate of convergence.

In this paper we shall analyze asymptotic expansion for the iterated Galerkin solution of Urysohn integral equation with Green's function type kernel. We will use Richardson extrapolation to improve the order of convergence.

The paper is organized as follows. In Section 2, notation is set and some preliminary results are proved for later use. In Section 3, asymptotic error analysis for the iterated Galerkin solution at the partition points are investigated. Numerical illustration is given in Section 4.

## 2 Preliminaries

In this section we describe the Urysohn integral operator with Green's function type kernel, its Fréchet derivatives and related preliminary results. We introduce the following notations.

For an integer $\alpha \geq 0$, let $C^{\alpha}[0,1]$ denotes the space of all real valued $\alpha$-times continuously differentiable functions on $[0,1]$ with the following norm.

$$
\|x\|_{\alpha, \infty}=\max _{0 \leq j \leq \alpha}\left\|x^{(j)}\right\|_{\infty},
$$

where $x^{(j)}$ is the $j^{\text {th }}$ derivative of the function $x$ and

$$
\left\|x^{(j)}\right\|_{\infty}=\sup _{0 \leq t \leq 1}\left|x^{(j)}(t)\right|
$$

Define

$$
\|\kappa\|_{\alpha, \infty}=\max _{0 \leq i+j+k \leq \alpha}\left\|D^{(i, j, k)} \kappa(s, t, u)\right\|_{\infty}
$$

where

$$
D^{(i, j, k)} \kappa(s, t, u)=\frac{\partial^{i+j+k} \kappa}{\partial s^{i} \partial t^{j} \partial u^{k}}(s, t, u) .
$$

### 2.1 Properties of the kernel (Green's function type)

Let $r \geq 1$ be an integer and assume that the kernel $\kappa$ has the following properties.

1. For $i=1,2,3,4$, the functions $\kappa, \frac{\partial^{i} \kappa}{\partial u^{i}} \in C(\Omega)$, where $C(\Omega)$ denotes the space of all real valued continuous function on $\Omega=[0,1] \times[0,1] \times \mathbb{R}$.
2. Let
$\Omega_{1}=\{(s, t, u): 0 \leq t \leq s \leq 1, u \in \mathbb{R}\}, \quad \Omega_{2}=\{(s, t, u): 0 \leq s \leq t \leq 1, u \in \mathbb{R}\}$.
There are two functions $\kappa_{j} \in C^{r}\left(\Omega_{j}\right), j=1,2$, such that

$$
\kappa(s, t, u)= \begin{cases}\kappa_{1}(s, t, u), & (s, t, u) \in \Omega_{1} \\ \kappa_{2}(s, t, u), & (s, t, u) \in \Omega_{2}\end{cases}
$$

3. Denote $\ell(s, t, u)=\frac{\partial \kappa}{\partial u}(s, t, u)$ and $q(s, t, u)=\frac{\partial^{2} \kappa}{\partial u^{2}}(s, t, u)$, for all $(s, t, u) \in \Omega$. The partial derivatives of $\ell(s, t, u)$ and $q(s, t, u)$ with respect to $s$ and $t$ have jump discontinuities on $s=t$.
4. There are functions $\ell_{j}, q_{j} \in C^{r}\left(\Omega_{j}\right), j=1,2$, with

$$
\begin{aligned}
& \ell(s, t, u)= \begin{cases}\ell_{1}(s, t, u), & (s, t, u) \in \Omega_{1}, \\
\ell_{2}(s, t, u), & (s, t, u) \in \Omega_{2}\end{cases} \\
& q(s, t, u)= \begin{cases}q_{1}(s, t, u), & (s, t, u) \in \Omega_{1}, \\
q_{2}(s, t, u), & (s, t, u) \in \Omega_{2}\end{cases}
\end{aligned}
$$

Following Atkinson-Potra [3], if the kernel $\kappa$ satisfies the above conditions, then we say that $\kappa$ is of class $\mathscr{G}_{4}(r, 0)$.

Under the above assumptions, the operator $\mathcal{K}$ is four times Fréchet differentiable and its Fréchet derivatives at $x \in \mathcal{X}$ are given by

$$
\mathcal{K}^{\prime}(x) v_{1}(s)=\int_{0}^{1} \frac{\partial \kappa}{\partial u}(s, t, x(t)) v_{1}(t) d t
$$

and

$$
\begin{equation*}
\mathcal{K}^{(i)}(x)\left(v_{1}, \ldots, v_{i}\right)(s)=\int_{0}^{1} \frac{\partial^{i} \kappa}{\partial u^{i}}(s, t, x(t)) v_{1}(t) \cdots v_{i}(t) d t, \quad i=2,3,4 \tag{2.1}
\end{equation*}
$$

where

$$
\frac{\partial^{i} \kappa}{\partial u^{i}}(s, t, x(t))=\left.\frac{\partial^{i} \kappa}{\partial u^{i}}(s, t, u)\right|_{u=x(t)}, \quad i=1,2,3,4
$$

and $v_{1}, v_{2}, v_{3}, v_{4} \in \mathcal{X}$. Note that $\mathcal{K}^{\prime}(x): \mathcal{X} \rightarrow \mathcal{X}$ is linear and $\mathcal{K}^{(i)}(x): \mathcal{X}^{i} \rightarrow \mathcal{X}$ are multi-linear operators, where $\mathcal{X}^{i}$ is the cartesian product of $i$ copies of $\mathcal{X}$. See Rall [16]. We define

$$
\left\|\mathcal{K}^{(i)}(x)\right\|=\sup _{\substack{\left\|v_{j}\right\|_{\infty} \leq 1 \\ j=1, \ldots, i}}\left\|\mathcal{K}^{(i)}(x)\left(v_{1}, \ldots, v_{i}\right)\right\|_{\infty}, \quad i=1,2,3,4
$$

It follows that

$$
\left\|\mathcal{K}^{(i)}(x)\right\| \leq \sup _{0 \leq s, t \leq 1}\left|\frac{\partial^{i} \kappa}{\partial u^{i}}(s, t, x(t))\right|, \quad i=1,2,3,4 .
$$

We rewrite the equation (1.2) as

$$
x-\mathcal{K}(x)=f, \quad x \in \mathcal{X} .
$$

Let

$$
\begin{equation*}
\mathcal{T}(x)=\mathcal{K}(x)+f, \quad x \in \mathcal{X} . \tag{2.2}
\end{equation*}
$$

Assume that $\varphi$ is a fixed point of $\mathcal{T}$. Since $\mathcal{K}$ is compact, $\mathcal{K}^{\prime}(\varphi)$ is a compact linear operator. See Krasnoselskii [6]. Assume that 1 is not an eigenvalue of $\mathcal{K}^{\prime}(\varphi)$. Then, $\varphi$ is an isolated solution of (1.2). Let $f \in C^{\alpha}[0,1]$, then by the Corollary 3.2 of Atkinson-Potra [3], it follows that $\varphi \in C^{\alpha}[0,1]$.

### 2.2 Approximating Space and Projection operator

Let $n \in \mathbb{N}$ and consider the following uniform partition of $[0,1]$ :

$$
\begin{equation*}
\Delta: 0<\frac{1}{n}<\cdots<\frac{n-1}{n}<1 . \tag{2.3}
\end{equation*}
$$

Define

$$
\begin{equation*}
t_{j}=\frac{j}{n}, \quad j=0, \ldots, n . \tag{2.4}
\end{equation*}
$$

Let

$$
\Delta_{j}=\left[t_{j-1}, t_{j}\right] \quad \text { and } \quad h=t_{j}-t_{j-1}=\frac{1}{n}, \quad j=1, \ldots, n
$$

Consider a finite dimensional approximating space as

$$
\mathscr{X}_{n}=\left\{g \in L^{\infty}[0,1]: g \text { is a polynomial of degree } \leq r-1 \text { on } \Delta_{j}, j=1,2, \ldots, n\right\} .
$$

As no continuity conditions are imposed at the partition points, the dimension of $\mathscr{X}_{n}$ is $n r$ and $\mathscr{X}_{n} \subset L^{\infty}[0,1]$.

Let $\pi_{n}$ be the restriction to $L^{\infty}[0,1]$ of the orthogonal projection from $L^{2}[0,1]$ onto $\mathscr{X}_{n}$, which converges to the Identity operator pointwise. Then

$$
\begin{equation*}
\sup _{n \geq 1}\left\|\pi_{n}\right\|_{L^{\infty}[0,1] \rightarrow L^{\infty}[0,1]}<\infty \tag{2.5}
\end{equation*}
$$

If $x \in C^{\alpha}[0,1]$, it is well-known that

$$
\begin{equation*}
\left\|\left(I-\pi_{n}\right) x\right\|_{\infty} \leq C_{1}\left\|x^{(\beta)}\right\|_{\infty} h^{\beta} \tag{2.6}
\end{equation*}
$$

where $\beta=\min \{\alpha, r\}$ and $C_{1}$ is a constant independent of $h$. See Atkinson [1], ChatelinLebbar [4]. Denote

$$
\pi_{n, j} x=\left.\pi_{n} x\right|_{\Delta_{j}}, \quad j=1,2, \ldots, n
$$

For $x \in C^{\alpha}\left(\Delta_{j}\right)$, we have

$$
\begin{equation*}
\left\|\left(I-\pi_{n, j}\right) x\right\|_{\Delta_{j}, \infty} \leq C_{2}\left\|x^{(\beta)}\right\|_{\Delta_{j}, \infty} h^{\beta}, \tag{2.7}
\end{equation*}
$$

where $\beta=\min \{\alpha, r\}$ and $C_{2}$ is a constant independent of $h$. See Atkinson-Potra 3, Corollary 4.3].

### 2.3 Asymptotic Expansions and the higher order terms

Let $\varphi \in C^{2 r+2}[0,1]$. For $\delta>0$, let

$$
\mathcal{B}(\varphi, \delta)=\left\{x \in \mathcal{X}:\|x-\varphi\|_{\infty} \leq \delta\right\}
$$

denote the closed $\delta$-neighbourhood of $\varphi$. Without loss of generality, we assume that the Galerkin solution $x_{n}^{G}$ and the iterated Galerkin solution $x_{n}^{S}$ belong to the above neighbourhood.
Denote

$$
\begin{aligned}
\ell_{*}(s, t)=\frac{\partial \kappa}{\partial u}(s, t, \varphi(t)), \quad s, t \in[0,1] \\
q_{*}(s, t)=\frac{\partial^{2} \kappa}{\partial u^{2}}(s, t, \varphi(t)), \quad s, t \in[0,1] .
\end{aligned}
$$

It follows that

$$
\begin{gather*}
\mathcal{K}^{\prime}(\varphi) v(s)=\int_{0}^{1} \ell_{*}(s, t) v(t) d t, \quad v \in \mathcal{X}, s \in[0,1]  \tag{2.8}\\
\mathcal{K}^{\prime \prime}(\varphi)\left(v_{1}, v_{2}\right)(s)=\int_{0}^{1} q_{*}(s, t) v_{1}(t) v_{2}(t) d t, \quad v_{1}, v_{2} \in \mathcal{X}, s \in[0,1],
\end{gather*}
$$

where the kernels $\ell_{*}(\cdot, \cdot), q_{*}(\cdot, \cdot) \in C[0,1] \times C[0,1]$ are of the type of Green's function as mentioned in the section 2.1, Then

$$
\begin{align*}
& \left\|\mathcal{K}^{\prime}(\varphi)\right\| \leq \sup _{0 \leq t \leq 1} \int_{0}^{1}\left|\ell_{*}(s, t)\right| d s \\
& \left\|\mathcal{K}^{\prime \prime}(\varphi)\right\| \leq \sup _{0 \leq t \leq 1} \int_{0}^{1}\left|q_{*}(s, t)\right| d s \tag{2.9}
\end{align*}
$$

See Atkinson [1].
By assumption, $I-\mathcal{K}^{\prime}(\varphi)$ is invertible. Let

$$
\begin{aligned}
\mathcal{M} & =\left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1} \mathcal{K}^{\prime}(\varphi) \\
\mathcal{M}_{2} & =\left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1} \mathcal{K}^{\prime \prime}(\varphi) \\
\mathcal{M}_{3} & =\left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1} \mathcal{K}^{(3)}(\varphi)
\end{aligned}
$$

Then $\mathcal{M}, \mathcal{M}_{2}$ and $\mathcal{M}_{3}$ are respectively compact linear, bi-linear and tri-linear integral operators. See Riesz-Nagy [17]. For $v \in \mathcal{X}$, let

$$
\mathcal{M} v(s)=\int_{0}^{1} m(s, t) v(t) d t, \quad s \in[0,1]
$$

Note that the kernels of $\mathcal{M}, \mathcal{M}_{2}$ and $\mathcal{M}_{3}$ inherit the same smoothness properties as the kernels of $\mathcal{K}^{\prime}(\varphi), \mathcal{K}^{\prime \prime}(\varphi)$ and $\mathcal{K}^{(3)}(\varphi)$ respectively. See Atkinson-Potra [Lemma 5.1][3]. Hence, the kernels of the above three operators are of the type of Green's function as mentioned in section 2.1.

We quote the following result from Rakshit-Rane [15].

$$
\begin{equation*}
\mathcal{M} \varphi\left(t_{i}\right)=\mathcal{M} \pi_{n} \varphi\left(t_{i}\right)+\left(\mathcal{A}_{2 r} \varphi\right)\left(t_{i}\right) h^{2 r}+O\left(h^{2 r+2}\right), \quad i=0,1, \ldots, n \tag{2.10}
\end{equation*}
$$

where

$$
\begin{aligned}
\left(\mathcal{A}_{2 r} \varphi\right)\left(t_{i}\right)=\bar{b}_{2 r, 2 r} \int_{0}^{1} m\left(t_{i}, t\right)(t) \varphi^{(2 r)}(t) d t & \\
& +\sum_{p=1}^{2 r-1} \bar{b}_{2 r, p}\left\{\left[\left(\frac{\partial}{\partial t}\right)^{2 r-p-1}\left(m\left(t_{i}, t\right) \varphi^{(p)}(t)\right)\right]_{t=0}^{t=1}\right. \\
& \left.-\left[\left(\frac{\partial}{\partial t}\right)^{2 r-p-1}\left(m\left(t_{i}, t\right) \varphi^{(p)}(t)\right)\right]_{t=t_{i}-}^{t=t_{i}+}\right\}
\end{aligned}
$$

with

$$
\bar{b}_{2 r, p}=\int_{0}^{1} \int_{0}^{1} \Lambda_{r}(\sigma, \tau) \frac{(\sigma-\tau)^{p}}{p!} \frac{B_{2 r-p}(\tau)}{(2 r-p)!} d \sigma d \tau
$$

$\Lambda_{r}(\sigma, \tau)=\sum_{q=0}^{r-1} e_{q}(\sigma) e_{q}(\tau),\left\{e_{0}, e_{1}, e_{2}, \ldots\right\}$ is the sequence of orthonormal polynomials in $L^{2}[0,1]$ and $B_{k}$ is the Bernoulli polynomial of degree $k$.

As in Kulkarni-Nidhin [10, Lemma 2.4], it can be shown that for all $s \in[0,1]$,

$$
\begin{equation*}
\mathcal{M}_{2}\left(\pi_{n} \varphi-\varphi\right)^{2}(s)=\mathcal{V}_{1}(\varphi)(s) h^{2 r}+O\left(h^{2 r+2}\right) \tag{2.11}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{M}_{3}\left(\pi_{n} \varphi-\varphi\right)^{3}(s)=\mathcal{V}_{2}(\varphi)(s) h^{3 r}+O\left(h^{3 r+1}\right) \tag{2.12}
\end{equation*}
$$

where

$$
\begin{gathered}
\mathcal{V}_{1}(\varphi)=\left(\int_{0}^{1}\left[\chi_{r}(t)\right]^{2} d t\right) \mathcal{M}_{2}\left(\varphi^{(r)}\right)^{2} \\
\mathcal{V}_{2}(\varphi)=\left(\int_{0}^{1}\left[\chi_{r}(t)\right]^{3} d t\right) \mathcal{M}_{3}\left(\varphi^{(r)}\right)^{3} \text { and } \mathcal{V}_{2}(\varphi)=0 \text { for } r=1
\end{gathered}
$$

with

$$
\chi_{r}(t)=\int_{0}^{1} \Lambda_{r}(\sigma, t) \frac{(\sigma-t)^{r}}{r!} d \sigma
$$

are independent of $h$. In the proof of Lemma 2.4 in Kulkarni-Nidhin [10], the authors used Euler-McLaurin expansion for smooth kernel. Since, the kernels of $\mathcal{M}_{2}$ and $\mathcal{M}_{3}$ are of the type of Green's function, we use the extended Euler-McLaurin summation formula from Kulkarni-Rane [11]. It follows that

$$
\begin{equation*}
\left\|\mathcal{M}_{3}\left(\pi_{n} \varphi-\varphi\right)^{3}\right\|_{\infty}=O\left(h^{4}\right) \quad \text { for } r=1 \tag{2.13}
\end{equation*}
$$

Let

$$
\begin{gathered}
C_{3}=\max \left\{\sup _{\substack{0 \leq t \leq s \leq 1 \\
|u| \leq| | \varphi \| \infty}}\left|D^{(1,0,0)} q_{1}(s, t, u)\right|, \sup _{\substack{0 \leq s \leq t \leq 1 \\
|u| \leq\|\varphi\|_{\infty}+\delta}}\left|D^{(1,0,0)} q_{2}(s, t, u)\right|\right\}, \\
C_{4}=\max \left\{\sup _{0 \leq t \leq s \leq 1}\left|D^{(1,0)} \ell_{*, 1}(s, t)\right|, \sup _{0 \leq s \leq t \leq 1}\left|D^{(1,0)} \ell_{*, 2}(s, t)\right|\right\}
\end{gathered}
$$

We first prove the following preliminary result which is needed later on.
Lemma 2.1. Let $x \in \mathcal{B}(\varphi, \delta)$. Then, for $v_{1}, v_{2} \in \mathcal{X}$,

$$
\left\|\left(\mathcal{K}^{\prime \prime}(x)\left(v_{1}, v_{2}\right)\right)^{\prime}\right\|_{\infty} \leq C_{3}\left\|v_{1}\right\|_{\infty}\left\|v_{2}\right\|_{\infty}
$$

Proof. Let $s \in[0,1]$. Then

$$
\begin{aligned}
\mathcal{K}^{\prime \prime}(x)\left(v_{1}, v_{2}\right)(s) & =\int_{0}^{1} q(s, t, x(t)) v_{1}(t) v_{2}(t) d t \\
& =\int_{0}^{s} q_{1}(s, t, x(t)) v_{1}(t) v_{2}(t) d t+\int_{s}^{1} q_{2}(s, t, x(t)) v_{1}(t) v_{2}(t) d t
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left(\mathcal{K}^{\prime \prime}(x)\left(v_{1}, v_{2}\right)\right)^{\prime}(s)= & \int_{0}^{s} \frac{\partial q_{1}}{\partial s}(s, t, x(t)) v_{1}(t) v_{2}(t) d t+q_{1}(s, s, x(s)) v_{1}(s) v_{2}(s) \\
& +\int_{s}^{1} \frac{\partial q_{2}}{\partial s}(s, t, x(t)) v_{1}(t) v_{2}(t) d t-q_{2}(s, s, x(s)) v_{1}(s) v_{2}(s)
\end{aligned}
$$

Since $q$ is continuous on $\Omega$,

$$
\left(\mathcal{K}^{\prime \prime}(x)\left(v_{1}, v_{2}\right)\right)^{\prime}(s)=\int_{0}^{s} \frac{\partial q_{1}}{\partial s}(s, t, x(t)) v_{1}(t) v_{2}(t) d t+\int_{s}^{1} \frac{\partial q_{2}}{\partial s}(s, t, x(t)) v_{1}(t) v_{2}(t) d t
$$

Hence,

$$
\left\|\left(\mathcal{K}^{\prime \prime}(x)\left(v_{1}, v_{2}\right)\right)^{\prime}\right\|_{\infty} \leq C_{3}\left\|v_{1}\right\|_{\infty}\left\|v_{2}\right\|_{\infty}
$$

This completes the proof.

Let $x \in \mathcal{B}(\varphi, \delta)$. Then by the above lemma, we obtain

$$
\begin{equation*}
\left\|\left(I-\pi_{n}\right)\left(\mathcal{K}^{\prime \prime}(x)\left(v_{1}, v_{2}\right)\right)\right\|_{\infty} \leq C_{1} C_{3}\left\|v_{1}\right\|_{\infty}\left\|v_{2}\right\|_{\infty} h \tag{2.14}
\end{equation*}
$$

Similarly, for any $v \in \mathcal{X}$, it can be shown that the function $\mathcal{K}^{\prime}(\varphi) v$ is differentiable on $[0,1]$ and

$$
\begin{equation*}
\left\|\left(\mathcal{K}^{\prime}(\varphi) v\right)^{\prime}\right\|_{\infty} \leq C_{4}\|v\|_{\infty} \tag{2.15}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
\left\|\left(I-\pi_{n}\right)\left(\mathcal{K}^{\prime}(\varphi) v\right)\right\|_{\infty} \leq C_{1} C_{4}\|v\|_{\infty} h \tag{2.16}
\end{equation*}
$$

The following crucial estimate

$$
\left\|\mathcal{K}^{\prime}(\varphi)\left(I-\pi_{n}\right) \varphi\right\|_{\infty}= \begin{cases}O\left(h^{2}\right), & r=1  \tag{2.17}\\ O\left(h^{r+2}\right), & r \geq 2\end{cases}
$$

follows from Lemma 9 of Chatelin-Lebbar [4]. From Theorem 3.1 of Kulkarni [9], we have

$$
\begin{equation*}
\left\|\left(I-\pi_{n}\right) \mathcal{K}^{\prime}(\varphi)\left(I-\pi_{n}\right) \varphi\right\|_{\infty}=O\left(h^{r+2}\right), \quad r \geq 1 \tag{2.18}
\end{equation*}
$$

In order to prove our main result, we need to establish the following lemmas and propositions. Note that

$$
\begin{equation*}
x_{n}^{G}-\varphi=\pi_{n}\left(x_{n}^{S}-\varphi\right)-\left(I-\pi_{n}\right) \varphi . \tag{2.19}
\end{equation*}
$$

We use the above relation between Galerkin and iterated Galerkin solution several times in the following lemmas and propositions.
Lemma 2.2. Let $x_{n}^{G}$ be the Galerkin solution defined by the equation (1.3). Then for $r \geq 1$,

$$
\begin{equation*}
\left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1} \mathcal{K}^{\prime \prime}(\varphi)\left(x_{n}^{G}-\varphi\right)^{2}(s)=\left(\mathcal{V}_{1}(\varphi)\right)(s) h^{2 r}+O\left(h^{2 r+2}\right), \quad s \in[0,1] \tag{2.20}
\end{equation*}
$$

where $\mathcal{V}_{1}$ is defined by (2.11).
Proof. Using (2.19), we write

$$
\begin{aligned}
\mathcal{K}^{\prime \prime}(\varphi)\left(x_{n}^{G}-\varphi\right)^{2}= & \mathcal{K}^{\prime \prime}(\varphi)\left[\pi_{n}\left(x_{n}^{S}-\varphi\right)-\left(I-\pi_{n}\right) \varphi\right]^{2} \\
= & \mathcal{K}^{\prime \prime}(\varphi)\left(\pi_{n}\left(x_{n}^{S}-\varphi\right)\right)^{2} \\
& -2 \mathcal{K}^{\prime \prime}(\varphi)\left(\pi_{n}\left(x_{n}^{S}-\varphi\right),\left(I-\pi_{n}\right) \varphi\right) \\
& +\mathcal{K}^{\prime \prime}(\varphi)\left(\left(I-\pi_{n}\right) \varphi\right)^{2}
\end{aligned}
$$

It follows that

$$
\begin{align*}
\left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1} \mathcal{K}^{\prime \prime}(\varphi)\left(x_{n}^{G}-\varphi\right)^{2}= & \mathcal{M}_{2}\left(\pi_{n}\left(x_{n}^{S}-\varphi\right)\right)^{2} \\
& -2\left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1} \mathcal{K}^{\prime \prime}(\varphi)\left(\pi_{n}\left(x_{n}^{S}-\varphi\right),\left(I-\pi_{n}\right) \varphi\right) \\
& +\mathcal{M}_{2}\left(\left(I-\pi_{n}\right) \varphi\right)^{2} \tag{2.21}
\end{align*}
$$

Since $\left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1} \mathcal{K}^{(3)}(\varphi)$ is bounded, from (1.6), (2.5), (2.6) and (2.9), it is easy to see that

$$
\left\|\mathcal{M}_{2}\left(\pi_{n}\left(x_{n}^{S}-\varphi\right)\right)^{2}\right\|_{\infty}= \begin{cases}O\left(h^{4}\right), & r=1  \tag{2.22}\\ O\left(h^{2 r+4}\right), & r \geq 2\end{cases}
$$

and

$$
\left\|\mathcal{K}^{\prime \prime}(\varphi)\left(\pi_{n}\left(x_{n}^{S}-\varphi\right),\left(I-\pi_{n}\right) \varphi\right)\right\|_{\infty}= \begin{cases}O\left(h^{3}\right), & r=1 \\ O\left(h^{2 r+2}\right), & r \geq 2\end{cases}
$$

When $r=1$, that is, when $\mathscr{X}_{n}$ is the space of piecewise constant functions, the order of the term $\left\|\mathcal{K}^{\prime \prime}(\varphi)\left(\pi_{n}\left(x_{n}^{S}-\varphi\right),\left(I-\pi_{n}\right) \varphi\right)\right\|_{\infty}$ can be improved to $h^{4}$ in the following way. Note that

$$
\begin{aligned}
\mathcal{K}^{\prime \prime}(\varphi)\left(\pi_{n}\left(x_{n}^{S}-\varphi\right),\left(I-\pi_{n}\right) \varphi\right)(s) & =\int_{0}^{1} q_{*}(s, t)\left(\pi_{n}\left(x_{n}^{S}-\varphi\right)\right)(t)\left(I-\pi_{n}\right) \varphi(t) d t \\
& =\sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}} q_{*}(s, t)\left(\pi_{n, j}\left(x_{n}^{S}-\varphi\right)\right)(t)\left(I-\pi_{n, j}\right) \varphi(t) d t
\end{aligned}
$$

Since, the range of $\pi_{n}$ is $\mathscr{X}_{n}, \pi_{n, j}\left(x_{n}^{S}-\varphi\right)$ is a constant on $\left[t_{j-1}, t_{j}\right]$. It follows that

$$
\begin{aligned}
& \mathcal{K}^{\prime \prime}(\varphi)\left(\pi_{n}\left(x_{n}^{S}-\varphi\right),\left(I-\pi_{n}\right) \varphi\right)(s)= \\
& \qquad \sum_{j=1}^{n}\left(\pi_{n, j}\left(x_{n}^{S}-\varphi\right)\right)\left(\frac{t_{j-1}+t_{j}}{2}\right) \int_{t_{j-1}}^{t_{j}} q_{*}(s, t)\left(I-\pi_{n, j}\right) \varphi(t) d t .
\end{aligned}
$$

As in the proof of Lemma 9 of Chatelin-Lebbar [4], it can be shown that

$$
\left|\int_{t_{j-1}}^{t_{j}} q_{*}(s, t)\left(I-\pi_{n, j}\right) \varphi(t) d t\right|=O\left(h^{3}\right) .
$$

Since $\left\{\pi_{n, j}\right\}$ is uniformly bounded and $\left\|x_{n}^{S}-\varphi\right\|_{\infty}=O\left(h^{2}\right)$, from the above estimate it follows that

$$
\left\|\mathcal{K}^{\prime \prime}(\varphi)\left(\pi_{n}\left(x_{n}^{S}-\varphi\right),\left(I-\pi_{n}\right) \varphi\right)\right\|_{\infty}=O\left(h^{4}\right), \quad r=1
$$

Hence,

$$
\left\|\mathcal{K}^{\prime \prime}(\varphi)\left(\pi_{n}\left(x_{n}^{S}-\varphi\right),\left(I-\pi_{n}\right) \varphi\right)\right\|_{\infty}= \begin{cases}O\left(h^{4}\right), & r=1  \tag{2.23}\\ O\left(h^{2 r+2}\right), & r \geq 2\end{cases}
$$

Hence, (2.20) follows from (2.11), (2.21) and (2.23), the proof of the proposition is complete.

Lemma 2.3. Let $s \in[0,1]$ and $r \geq 1$. Then

$$
\left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1} \mathcal{K}^{(3)}(\varphi)\left(x_{n}^{G}-\varphi\right)^{3}(s)= \begin{cases}O\left(h^{4}\right), & r=1  \tag{2.24}\\ \left(\mathcal{V}_{2}(\varphi)\right)(s) h^{3 r}+O\left(h^{3 r+1}\right), & r \geq 2\end{cases}
$$

where $\mathcal{V}_{2}$ is defined by (2.12).
Proof. We write

$$
\begin{align*}
\left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1} \mathcal{K}^{(3)}(\varphi)\left(x_{n}^{G}-\varphi\right)^{3}= & \left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1} \mathcal{K}^{(3)}(\varphi)\left[\pi_{n}\left(x_{n}^{S}-\varphi\right)-\left(I-\pi_{n}\right) \varphi\right]^{3} \\
= & \mathcal{M}_{3}\left(\pi_{n}\left(x_{n}^{S}-\varphi\right)\right)^{3} \\
& -3 \mathcal{M}_{3}\left(\pi_{n}\left(x_{n}^{S}-\varphi\right), \pi_{n}\left(x_{n}^{S}-\varphi\right),\left(I-\pi_{n}\right) \varphi\right) \\
& +3 \mathcal{M}_{3}\left(\pi_{n}\left(x_{n}^{S}-\varphi\right),\left(I-\pi_{n}\right) \varphi,\left(I-\pi_{n}\right) \varphi\right) \\
& -\mathcal{M}_{3}\left(\left(I-\pi_{n}\right) \varphi\right)^{3} \tag{2.25}
\end{align*}
$$

Since $\mathcal{M}_{3}=\left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1} \mathcal{K}^{(3)}(\varphi)$ is bounded, (1.6), (2.5), (2.6) we have

$$
\begin{array}{r}
\left\|\mathcal{M}_{3}\left(\pi_{n}\left(x_{n}^{S}-\varphi\right)\right)^{3}\right\|_{\infty}= \begin{cases}O\left(h^{6}\right), & r=1, \\
O\left(h^{3 r+6}\right), & r \geq 2\end{cases} \\
\left\|\mathcal{M}_{3}\left(\pi_{n}\left(x_{n}^{S}-\varphi\right), \pi_{n}\left(x_{n}^{S}-\varphi\right),\left(I-\pi_{n}\right) \varphi\right)\right\|_{\infty}= \begin{cases}O\left(h^{5}\right), & r=1 \\
O\left(h^{3 r+4}\right), & r \geq 2\end{cases} \\
\left\|\mathcal{M}_{3}\left(\pi_{n}\left(x_{n}^{S}-\varphi\right),\left(I-\pi_{n}\right) \varphi,\left(I-\pi_{n}\right) \varphi\right)\right\|_{\infty}= \begin{cases}O\left(h^{4}\right), & r=1 \\
O\left(h^{3 r+2}\right), & r \geq 2\end{cases} \tag{2.28}
\end{array}
$$

On the other hand, from (2.12) we have

$$
\mathcal{M}_{3}\left(\left(I-\pi_{n}\right) \varphi\right)^{3}(s)= \begin{cases}O\left(h^{4}\right), & r=1 \\ \left(\mathcal{V}_{2}(\varphi)\right)(s) h^{3 r}+O\left(h^{3 r+1}\right), & r \geq 2\end{cases}
$$

Hence, (2.24) follows from (2.25), (2.26), (2.27), (2.28) and the above equation.

## 3 The Main Result

Recall that the Iterated Galerkin solution is defined by

$$
x_{n}^{S}-\mathcal{K}\left(\pi_{n} x_{n}^{S}\right)=f
$$

and the exact solution as

$$
\varphi-\mathcal{K}(\varphi)=f
$$

In this section, we prove our main result about the asymptotic series expansion for the iterated Galerkin solution $x_{n}^{S}$ at the partition points $t_{i}, i=0,1, \ldots, n$. That is, we will show the following.

$$
\begin{equation*}
\varphi\left(t_{i}\right)-x_{n}^{S}\left(t_{i}\right)=\mathcal{A}_{2 r}\left(t_{i}\right) h^{2 r}+O\left(h^{2 r+2}\right), \tag{3.1}
\end{equation*}
$$

where $\mathcal{A}_{2 r}$ is a function independent of $n$.
Then, we can apply Richardson extrapolation to obtain an approximation of $\varphi$ with higher order at the partition points. From Ford et al [5, Section 5], it can be shown that a continuous function can be reconstructed from the extrapolated discrete values at the partition points and it approximates the exact solution $\varphi$ to higher order in the uniform norm. We will not discuss this thing here. Our main aim is to prove (3.1).

Recall that

$$
\mathcal{M}=\left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1} \mathcal{K}^{\prime}(\varphi)
$$

with

$$
\begin{equation*}
(\mathcal{M} x)(s)=\int_{0}^{1} m(s, t) x(t) d t, \quad s \in[0,1], x \in \mathcal{X} \tag{3.2}
\end{equation*}
$$

where the kernel $m$ is of the type of Green's function.
We quote the following expression for the error in the iterated Galerkin solution from Atkinson et al [2, equation (2.28)]:

$$
\begin{align*}
x_{n}^{S}-\varphi= & \left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1}\left\{\left[\mathcal{K}\left(x_{n}^{G}\right)-\mathcal{K}(\varphi)-\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\right]\right\} \\
& -\mathcal{M}\left(I-\pi_{n}\right)\left[\mathcal{K}\left(x_{n}^{G}\right)-\mathcal{K}(\varphi)-\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\right] \\
& -\mathcal{M}\left(I-\pi_{n}\right) \mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right) \\
& -\mathcal{M}\left(I-\pi_{n}\right) \varphi \tag{3.3}
\end{align*}
$$

By the following propositions, we will prove that the second and the third terms on the right hand side of the above equation are of the order $2 r+2$ or higher, and the first and the last term has an asymptotic expansion at the partition points.

Let

$$
C_{5}=\max _{0 \leq i \leq 4}\left(\sup _{\substack{s, t[0,1] \\|u| \leq\left\lfloor\mid \varphi \|_{\infty}+\delta\right.}}\left|\frac{\partial^{i} \kappa}{\partial u^{i}}(s, t, u)\right|\right) .
$$

Let us investigate the first term on the right hand side of the equation (2.26) for an asymptotic expansion.
Proposition 3.1. Let $x_{n}^{G}$ be the Galerkin solution defined by the equation (1.3) and $s \in[0,1]$. Then for $r \geq 1$,

$$
\left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1}\left[\mathcal{K}\left(x_{n}^{G}\right)-\mathcal{K}(\varphi)-\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\right](s)=\mathcal{V}_{1}(\varphi)(s) h^{2 r}+O\left(h^{2 r+2}\right),
$$

where $\mathcal{V}_{1}$ is defined by (2.11).
Proof. Using the generalized Taylor's series expansion (see Linz [13]) in the neighbour$\operatorname{hood} \mathcal{B}(\varphi, \delta)$, we obtain

$$
\begin{aligned}
& \mathcal{K}\left(x_{n}^{G}\right)-\mathcal{K}(\varphi)-\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right) \\
&=\frac{1}{2} \mathcal{K}^{\prime \prime}(\varphi)\left(x_{n}^{G}-\varphi\right)^{2}+\frac{1}{6} \mathcal{K}^{(3)}(\varphi)\left(x_{n}^{G}-\varphi\right)^{3}+\mathcal{R}_{4}\left(x_{n}^{G}, \varphi\right)
\end{aligned}
$$

where

$$
\mathcal{R}_{4}\left(x_{n}^{G}, \varphi\right)=\frac{1}{6} \int_{0}^{1} \mathcal{K}^{(4)}\left(\varphi+\theta\left(x_{n}^{G}-\varphi\right)\right)\left(x_{n}^{G}-\varphi\right)^{4}(1-\theta)^{3} d \theta
$$

By (2.1), we have
$\mathcal{K}^{(4)}\left(\varphi+\theta\left(x_{n}^{G}-\varphi\right)\right)\left(x_{n}^{G}-\varphi\right)^{4}(s)=\int_{0}^{1} \frac{\partial^{4} \kappa}{\partial u^{4}}\left(s, t, \varphi(t)+\theta\left(x_{n}^{G}-\varphi\right)(t)\right)\left(x_{n}^{G}-\varphi\right)^{4}(t) d t$.
Since $\left\|x_{n}^{G}-\varphi\right\|_{\infty} \rightarrow 0$ as $n \rightarrow \infty$ and $\theta \in(0,1)$,

$$
\varphi+\theta\left(x_{n}^{G}-\varphi\right) \in \mathcal{B}(\varphi, \delta)
$$

It follows that

$$
\left\|\mathcal{K}^{(4)}\left(\varphi+\theta\left(x_{n}^{G}-\varphi\right)\right)\left(x_{n}^{G}-\varphi\right)^{4}\right\|_{\infty} \leq C_{5}\left\|x_{n}^{G}-\varphi\right\|_{\infty}^{4}
$$

Using (1.6) and the above estimate, we obtain

$$
\begin{equation*}
\mathcal{R}_{4}\left(x_{n}^{G}, \varphi\right)=O\left(h^{4 r}\right) \tag{3.4}
\end{equation*}
$$

Note that

$$
\begin{aligned}
\left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1} & {\left[\mathcal{K}\left(x_{n}^{G}\right)-\mathcal{K}(\varphi)-\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\right] } \\
= & \frac{1}{2}\left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1} \mathcal{K}^{\prime \prime}(\varphi)\left(x_{n}^{G}-\varphi\right)^{2}+\frac{1}{6}\left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1} \mathcal{K}^{(3)}(\varphi)\left(x_{n}^{G}-\varphi\right)^{3} \\
& +\left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1} \mathcal{R}_{4}\left(x_{n}^{G}, \varphi\right) .
\end{aligned}
$$

Hence, by Lemma [2.2, Lemma [2.3, equation (3.4) and and the above estimate, we obtain the followings.
For $r=1$,

$$
\left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1}\left[\mathcal{K}\left(x_{n}^{G}\right)-\mathcal{K}(\varphi)-\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\right](s)=\mathcal{V}_{1}(\varphi)(s) h^{2 r}+O\left(h^{4}\right),
$$

for $r \geq 2$,

$$
\begin{aligned}
\left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1}\left[\mathcal{K}\left(x_{n}^{G}\right)-\mathcal{K}(\varphi)-\mathcal{K}^{\prime}(\varphi)\right. & \left.\left(x_{n}^{G}-\varphi\right)\right](s) \\
& =\mathcal{V}_{1}(\varphi)(s) h^{2 r}+\mathcal{V}_{2}(\varphi)(s) h^{3 r}+O\left(h^{2 r+2}\right)
\end{aligned}
$$

Since $3 r \geq 2 r+2$ for $r \geq 2$,
$\left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1}\left[\mathcal{K}\left(x_{n}^{G}\right)-\mathcal{K}(\varphi)-\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\right](s)=\mathcal{V}_{1}(\varphi)(s) h^{2 r}+O\left(h^{2 r+2}\right), \quad r \geq 2$.
This completes the proof.
Now we investigate the second term on the R.H.S. of the equation (3.3).
Proposition 3.2. Let $\left\{t_{i}: i=0,1, \ldots, n\right\}$ be the set of all partition points defined by (2.4). Then for $r \geq 1$,

$$
\mathcal{M}\left(I-\pi_{n}\right)\left[\mathcal{K}\left(x_{n}^{G}\right)-\mathcal{K}(\varphi)-\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\right]\left(t_{i}\right)=O\left(h^{2 r+2}\right) .
$$

Proof. By the generalized Taylor's theorem we obtain

$$
\mathcal{K}\left(x_{n}^{G}\right)-\mathcal{K}(\varphi)-\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)=\int_{0}^{1} \mathcal{K}^{\prime \prime}\left(\varphi+\theta\left(x_{n}^{G}-\varphi\right)\right)\left(x_{n}^{G}-\varphi\right)^{2}(1-\theta) d \theta
$$

Let

$$
\begin{equation*}
\mathcal{R}_{2}\left(x_{n}^{G}, \varphi\right)=\int_{0}^{1} \mathcal{K}^{\prime \prime}\left(\varphi+\theta\left(x_{n}^{G}-\varphi\right)\right)\left(x_{n}^{G}-\varphi\right)^{2}(1-\theta) d \theta \tag{3.5}
\end{equation*}
$$

Note that
$\mathcal{K}^{\prime \prime}\left(\varphi+\theta\left(x_{n}^{G}-\varphi\right)\right)\left(x_{n}^{G}-\varphi\right)^{2}(s)=\int_{0}^{1} \frac{\partial^{2} \kappa}{\partial u^{2}}\left(s, t, \varphi(t)+\theta\left(x_{n}^{G}-\varphi\right)(t)\right)\left(x_{n}^{G}-\varphi\right)^{2}(t) d t$.

It follows that

$$
\left\|\mathcal{K}^{\prime \prime}\left(\varphi+\theta\left(x_{n}^{G}-\varphi\right)\right)\left(x_{n}^{G}-\varphi\right)^{2}\right\|_{\infty} \leq C_{3}\left\|x_{n}^{G}-\varphi\right\|_{\infty}^{2} .
$$

Since $\left\|x_{n}^{G}-\varphi\right\|_{\infty}=O\left(h^{r}\right)$,

$$
\begin{equation*}
\left\|\mathcal{K}^{\prime \prime}\left(\varphi+\theta\left(x_{n}^{G}-\varphi\right)\right)\left(x_{n}^{G}-\varphi\right)^{2}\right\|_{\infty}=O\left(h^{2 r}\right) . \tag{3.6}
\end{equation*}
$$

Let $s \in[0,1]$ be fixed and $m_{s}(t)=m(s, t), t \in[0,1]$, then

$$
\begin{aligned}
\mathcal{M}\left(I-\pi_{n}\right)\left[\mathcal{K}\left(x_{n}^{G}\right)-\mathcal{K}(\varphi)-\right. & \left.\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\right](s) \\
& =\left\langle m_{s},\left(I-\pi_{n}\right)\left[\mathcal{K}\left(x_{n}^{G}\right)-\mathcal{K}(\varphi)-\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\right]\right\rangle
\end{aligned}
$$

where $\langle\cdot, \cdot\rangle$ is the usual inner product in $L^{2}[0,1]$, i.e.,

$$
\langle x, y\rangle=\int_{0}^{1} x(t) y(t) d t, \quad x, y \in L^{2}[0,1]
$$

Since $I-\pi_{n}$ is self-adjoint,

$$
\begin{align*}
& \mathcal{M}\left(I-\pi_{n}\right)\left[\mathcal{K}\left(x_{n}^{G}\right)-\mathcal{K}(\varphi)-\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\right]\left(t_{i}\right) \\
& \quad=\left\langle\left(I-\pi_{n}\right) m_{t_{i}},\left(I-\pi_{n}\right)\left[\mathcal{K}\left(x_{n}^{G}\right)-\mathcal{K}(\varphi)-\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\right]\right\rangle \tag{3.7}
\end{align*}
$$

Note that $m_{t_{i}}$ is continuous on $\left[t_{j-1}, t_{j}\right]$ and $r$ times continuously differentiable on $\left(t_{j-1}, t_{j}\right)$ for all $j=1,2, \ldots, n$. Therefore by (2.7)

$$
\begin{equation*}
\left\|\left(I-\pi_{n, j}\right) m_{t_{i}}\right\|_{\Delta_{j}, \infty}=O\left(h^{r}\right) \tag{3.8}
\end{equation*}
$$

Using (2.5), (3.6), (3.7) and the above estimate, we obtain

$$
\begin{equation*}
\mathcal{M}\left(I-\pi_{n}\right)\left[\mathcal{K}\left(x_{n}^{G}\right)-\mathcal{K}(\varphi)-\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\right]\left(t_{i}\right)=O\left(h^{3 r}\right), \quad r \geq 2 \tag{3.9}
\end{equation*}
$$

Consider the case $\mathbf{r}=1$.
From (3.5), it is easy to see that if $\mathcal{K}^{\prime \prime}\left(\varphi+\theta\left(x_{n}^{G}-\varphi\right)\right)\left(x_{n}^{G}-\varphi\right)^{2}$ is differentiable, then $\mathcal{R}_{2}\left(x_{n}^{G}, \varphi\right)$ is differentiable. Since $\varphi+\theta\left(x_{n}^{G}-\varphi\right) \in \mathcal{B}(\varphi, \delta)$, from Lemma 2.1 we have $\mathcal{R}_{2}\left(x_{n}^{G}, \varphi\right)$ is differentiable. Thus, from (3.5)

$$
\left(\mathcal{R}_{2}\left(x_{n}^{G}, \varphi\right)\right)^{\prime}=\int_{0}^{1}\left(\mathcal{K}^{\prime \prime}\left(\varphi+\theta\left(x_{n}^{G}-\varphi\right)\right)\left(x_{n}^{G}-\varphi\right)^{2}\right)^{\prime}(1-\theta) d \theta
$$

This implies,

$$
\left\|\left(\mathcal{R}_{2}\left(x_{n}^{G}, \varphi\right)\right)^{\prime}\right\|_{\infty} \leq \frac{1}{2}\left\|\left(\mathcal{K}^{\prime \prime}\left(\varphi+\theta\left(x_{n}^{G}-\varphi\right)\right)\left(x_{n}^{G}-\varphi\right)^{2}\right)^{\prime}\right\|_{\infty}, \quad 0<\theta<1
$$

Using Lemma 2.1,

$$
\left\|\left(\mathcal{R}_{2}\left(x_{n}^{G}, \varphi\right)\right)^{\prime}\right\|_{\infty} \leq C_{3}\left\|x_{n}^{G}-\varphi\right\|^{2}
$$

From (2.6), it follows that

$$
\begin{aligned}
\left\|\left(I-\pi_{n}\right)\left[\mathcal{K}\left(x_{n}^{G}\right)-\mathcal{K}(\varphi)-\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\right]\right\|_{\infty} & =\left\|\left(I-\pi_{n}\right) \mathcal{R}_{2}\left(x_{n}^{G}, \varphi\right)\right\|_{\infty} \\
& \leq C_{1} C_{3}\left\|x_{n}^{G}-\varphi\right\|^{2} h
\end{aligned}
$$

By (1.5), (3.7), (3.8) and the above estimate, we obtain

$$
\begin{equation*}
\mathcal{M}\left(I-\pi_{n}\right)\left[\mathcal{K}\left(x_{n}^{G}\right)-\mathcal{K}(\varphi)-\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\right]\left(t_{i}\right)=O\left(h^{4}\right), \quad r=1 \tag{3.10}
\end{equation*}
$$

Hence, the required result follows from (3.9) and (3.10).
Next we investigate the third term on the R.H.S. of the equation (3.3).
Proposition 3.3. Let $\left\{t_{i}: i=0,1, \ldots, n\right\}$ be the set of all partition points defined by (2.4). Then

$$
\mathcal{M}\left(I-\pi_{n}\right) \mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\left(t_{i}\right)=O\left(h^{2 r+2}\right), \quad \text { for } r \geq 1
$$

Proof. From (3.2) we have

$$
\begin{align*}
\mathcal{M}\left(I-\pi_{n}\right) \mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\left(t_{i}\right) & =\int_{0}^{1} m\left(t_{i}, t\right)\left(I-\pi_{n}\right) \mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)(t) d t \\
& =\left\langle m_{t_{i}},\left(I-\pi_{n}\right) \mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\right\rangle \\
& =\left\langle\left(I-\pi_{n}\right) m_{t_{i}},\left(I-\pi_{n}\right) \mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\right\rangle . \tag{3.11}
\end{align*}
$$

It is easy to see that, $m_{t_{i}}$ is continuous on $\left[t_{j-1}, t_{j}\right]$ and $r$ times continuously differentiable on $\left(t_{j-1}, t_{j}\right)$ for all $j=1,2, \ldots, n$. Therefore by (2.7)

$$
\begin{equation*}
\left\|\left(I-\pi_{n, j}\right) m_{t_{i}}\right\|_{\Delta_{j}, \infty}=O\left(h^{r}\right) \tag{3.12}
\end{equation*}
$$

Note that

$$
\begin{equation*}
\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)=\mathcal{K}^{\prime}(\varphi)\left(\pi_{n}\left(x_{n}^{S}-\varphi\right)\right)-\mathcal{K}^{\prime}(\varphi)\left(I-\pi_{n}\right) \varphi \tag{3.13}
\end{equation*}
$$

Thus, by (1.6), (2.5) and (2.17) we obtain,

$$
\left\|\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\right\|_{\infty}=O\left(h^{r+2}\right), \quad r \geq 2
$$

Hence, from (3.11), (3.12) and the above estimate, we obtain

$$
\begin{equation*}
\mathcal{M}\left(I-\pi_{n}\right) \mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\left(t_{i}\right)=O\left(h^{2 r+2}\right), \quad r \geq 2 \tag{3.14}
\end{equation*}
$$

When $\mathbf{r}=\mathbf{1}$, that is, when $\mathscr{X}_{n}$ is the space of piecewise constant functions, it is easy to see from (1.5), (2.17) and (3.13) that

$$
\left\|\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\right\|_{\infty}=O\left(h^{2}\right),
$$

which is not equal to $O\left(h^{2 r+2}\right)$ with $r=1$. We consider this case separately.
Note that,

$$
\begin{equation*}
\left(I-\pi_{n}\right) \mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)=\left(I-\pi_{n}\right) \mathcal{K}^{\prime}(\varphi)\left(\pi_{n}\left(x_{n}^{S}-\varphi\right)\right)-\left(I-\pi_{n}\right) \mathcal{K}^{\prime}(\varphi)\left(I-\pi_{n}\right) \varphi . \tag{3.15}
\end{equation*}
$$

From (2.16), we have

$$
\left\|\left(I-\pi_{n}\right) \mathcal{K}^{\prime}(\varphi)\left(\pi_{n}\left(x_{n}^{S}-\varphi\right)\right)\right\|_{\infty} \leq C_{1} C_{4}\left\|\pi_{n}\right\|\left\|x_{n}^{S}-\varphi\right\|_{\infty} h
$$

From (1.5) and (2.5), it follows that

$$
\left\|\left(I-\pi_{n}\right) \mathcal{K}^{\prime}(\varphi)\left(\pi_{n}\left(x_{n}^{S}-\varphi\right)\right)\right\|_{\infty}=O\left(h^{3}\right)
$$

By (2.18), (3.11), (3.12), (3.15) and the above estimate, we obtain

$$
\begin{equation*}
\mathcal{M}\left(I-\pi_{n}\right) \mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\left(t_{i}\right)=O\left(h^{4}\right), \quad r=1 \tag{3.16}
\end{equation*}
$$

Hence, the required result follows from (3.14) and (3.16).
Now, we prove our main theorem.
Theorem 3.4. Let $f \in C^{2 r}[0,1]$, and the kernel of the Urysohn integral operator (1.1) be of class $\mathscr{G}_{4}(r, 0)$. Let $\varphi$ be a fixed point of the operator $\mathcal{T}$ defined by (2.2), with 1 not an eigenvalue of $\mathcal{K}^{\prime}(\varphi)$. For $r \geq 1$, let $\mathscr{X}_{n}$ be the space of piecewise polynomials of degree $\leq r-1$ with respect to the partition (2.3) and $\pi_{n}$ be the orthogonal projection defined by (2.5) -(2.6). Let $x_{n}^{S}$ be the iterated Galerkin solution defined by (1.4). Then, for $i=0,1, \ldots, n$,

$$
x_{n}^{S}\left(t_{i}\right)-\varphi\left(t_{i}\right)=-\zeta_{2 r}\left(t_{i}\right) h^{2 r}+O\left(h^{2 r+2}\right),
$$

where $\zeta_{2 r}$ is a function bounded by a constant independent of $h$.
Proof. From the equation (3.3)

$$
\begin{aligned}
x_{n}^{S}\left(t_{i}\right)-\varphi\left(t_{i}\right)= & \left(I-\mathcal{K}^{\prime}(\varphi)\right)^{-1}\left[\mathcal{K}\left(x_{n}^{G}\right)-\mathcal{K}(\varphi)-\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\right]\left(t_{i}\right) \\
& -\mathcal{M}\left(I-\pi_{n}\right)\left[\mathcal{K}\left(x_{n}^{G}\right)-\mathcal{K}(\varphi)-\mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\right]\left(t_{i}\right) \\
& -\mathcal{M}\left(I-\pi_{n}\right) \mathcal{K}^{\prime}(\varphi)\left(x_{n}^{G}-\varphi\right)\left(t_{i}\right) \\
& -\mathcal{M}\left(I-\pi_{n}\right) \varphi\left(t_{i}\right) .
\end{aligned}
$$

Let

$$
\zeta_{2 r}=\mathcal{A}_{2 r}+\mathcal{V}_{1}(\varphi)
$$

Hence, the proof of this theorem follows from the equation (2.10), Proposition 3.1, Proposition 3.2 and Proposition 3.3.

We can now apply one step of Richardson extrapolation and obtain an approximations of $\varphi$ of order $h^{2 r+2}$ at the partition points.

Define

$$
x_{n}^{E X}=\frac{2^{4 r} x_{2 n}^{S}-x_{n}^{S}}{2^{4 r}-1}
$$

Then under the assumptions of Theorem 3.4, we have the following result

$$
\begin{equation*}
x_{n}^{E X}\left(t_{i}\right)-\varphi\left(t_{i}\right)=O\left(h^{2 r+2}\right), \quad i=0,1,2, \ldots, n \tag{3.17}
\end{equation*}
$$

## 4 Numerical Illustration

For the sake of numerical illustration, we consider the following example of a non-linear Hammerstein integral equation from Kulkarni-Rane [12]. Consider

$$
\begin{equation*}
\varphi(s)-\int_{0}^{1} \kappa(s, t)[\psi(t, \varphi(t))] d t=f(s), \quad 0 \leq s \leq 1 \tag{4.1}
\end{equation*}
$$

where

$$
\kappa(s, t)=\frac{1}{\gamma \sinh \gamma} \begin{cases}\sinh \gamma s \sinh \gamma(1-t), & 0 \leq t \leq s \leq 1 \\ \gamma(1-s) \sinh \gamma t, & 0 \leq s \leq t \leq 1\end{cases}
$$

with $\gamma=\sqrt{12}$, and

$$
\psi(t, \varphi(t))=\gamma^{2} \varphi(t)-2(\varphi(t))^{3}, \quad t \in[0,1]
$$

We have $f(s)=\frac{1}{\gamma \sinh \gamma}\left\{2 \sinh \gamma(1-s)+\frac{2}{3} \sinh \gamma s\right\}$. The exact solution of (4.1) is given by

$$
\varphi(s)=\frac{2}{2 s+1}, \quad s \in[0,1]
$$

Let $\mathscr{X}_{n}$ be the space of piecewise constant polynomials with respect to the uniform partition (2.3) of the interval $[0,1]$ considered before. Let $\pi_{n}: L^{\infty}[0,1] \rightarrow \mathscr{X}_{n}$ be the
orthogonal projection defined by (2.5) $-(2.6)$.
In this case, it is given by

$$
\left(\pi_{n} \varphi\right)(s)=\frac{1}{h} \int_{(i-1) h}^{i h} \varphi(t) d t, s \in[(i-1) h, i h]
$$

where $h=\frac{1}{n}$. In the definition of projection operator defined above, if we replace the integral by the right hand rule, then $\left(\pi_{n} \varphi\right)(i h-)=\varphi(i h-)$. Let $x_{n}^{S}$ be the Sloan solution defined by (1.4). Then $x_{n}^{G}(i h)=\frac{\pi_{n} x_{n}^{S}(i h-)+\pi_{n} x_{n}^{S}(i h+)}{2}$ at the partition points is obtained by solving the approximate system of non-linear equations which gives the values of $\pi_{n} x_{n}^{S}(i h-)$ and $\pi_{n} x_{n}^{S}(i h+)$. The system is as follows:

$$
\begin{equation*}
\alpha_{j}=h \sum_{l=1}^{n} \kappa\left(s_{j}, s_{l}\right)\left[\gamma^{2} \alpha_{l}-2 \frac{\alpha_{l}^{3}}{h}\right]+\frac{f\left(s_{j}\right)}{\sqrt{h}}, \quad j=1,2, \ldots, n, \tag{4.2}
\end{equation*}
$$

where $\alpha_{l}=\pi_{n} x_{n}^{G}\left(s_{l}\right)$ and $s_{l}=\left(l-\frac{1}{2}\right) h$ for $l=1,2, \ldots, n$. The above system is obtained by replacing all the integrals by numerical integration formula.

We have used Picard's iteration to solve the above system of non-linear equations.
Let $t_{i}=(i-1) / 20, i=1,2, \ldots, 21$ be the partition points with step size $h=\frac{1}{20}$. It is easy to see that

$$
E_{1}^{n}\left(t_{i}\right)=\left|\varphi\left(t_{i}\right)-x_{n}^{S}\left(t_{i}\right)\right|=O\left(h^{2}\right) .
$$

We define

$$
x_{n}^{E X}\left(t_{i}\right)=\frac{4 x_{2 n}^{S}\left(t_{i}\right)-x_{n}^{S}\left(t_{i}\right)}{3}
$$

Then

$$
E_{2}^{n}\left(t_{i}\right)=\left|\varphi\left(t_{i}\right)-x_{n}^{E X}\left(t_{i}\right)\right|=O\left(h^{4}\right) .
$$

The orders of convergence are calculated using the formula:

$$
\begin{aligned}
\alpha_{1} & =\frac{\log \left(E_{1}^{n}\left(t_{i}\right) / E_{1}^{2 n}\left(t_{i}\right)\right)}{\log (2)}, \\
\beta & =\frac{\log \left(E_{2}^{n}\left(t_{i}\right) / E_{2}^{2 n}\left(t_{i}\right)\right)}{\log (2)}, \\
\alpha_{2} & =\frac{\log \left(E_{1}^{n}\left(t_{i}\right) / E_{1}^{2 n}\left(t_{i}\right)\right)}{\log (2)}, \quad n=40
\end{aligned}
$$

We expect $\alpha_{1}=\alpha_{2}=2$ and $\beta=4$.

Table 4.1

| $t_{i}$ | $E_{1}^{n}\left(t_{i}\right): n=20$ | $E_{1}^{n}\left(t_{i}\right): n=40$ | $E_{1}^{n}\left(t_{i}\right): n=80$ | $\alpha_{1}$ | $\alpha_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.05 | $8.6 \times 10^{-3}$ | $2.15 \times 10^{-3}$ | $5.37 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.1 | $7.56 \times 10^{-3}$ | $1.89 \times 10^{-3}$ | $4.72 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.15 | $6.79 \times 10^{-3}$ | $1.7 \times 10^{-3}$ | $4.24 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.2 | $6.22 \times 10^{-3}$ | $1.55 \times 10^{-3}$ | $3.89 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.25 | $5.78 \times 10^{-3}$ | $1.44 \times 10^{-3}$ | $3.61 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.3 | $5.45 \times 10^{-3}$ | $1.36 \times 10^{-3}$ | $3.4 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.35 | $5.19 \times 10^{-3}$ | $1.3 \times 10^{-3}$ | $3.24 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.4 | $4.98 \times 10^{-3}$ | $1.25 \times 10^{-3}$ | $3.11 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.45 | $4.82 \times 10^{-3}$ | $1.2 \times 10^{-3}$ | $3.01 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.5 | $4.68 \times 10^{-3}$ | $1.17 \times 10^{-3}$ | $2.92 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.55 | $4.55 \times 10^{-3}$ | $1.14 \times 10^{-3}$ | $2.84 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.6 | $4.44 \times 10^{-3}$ | $1.11 \times 10^{-3}$ | $2.77 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.65 | $4.33 \times 10^{-3}$ | $1.08 \times 10^{-3}$ | $2.7 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.7 | $4.22 \times 10^{-3}$ | $1.05 \times 10^{-3}$ | $2.64 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.75 | $4.10 \times 10^{-3}$ | $1.02 \times 10^{-3}$ | $2.56 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.8 | $3.98 \times 10^{-3}$ | $9.94 \times 10^{-4}$ | $2.48 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.85 | $3.84 \times 10^{-3}$ | $9.6 \times 10^{-4}$ | $2.4 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.9 | $3.69 \times 10^{-3}$ | $9.22 \times 10^{-4}$ | $2.3 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.95 | $3.52 \times 10^{-3}$ | $8.8 \times 10^{-4}$ | $2.2 \times 10^{-4}$ | 2.00 | 2.00 |

Table 4.2

| $t_{i}$ | $E_{2}^{n}\left(t_{i}\right): n=20$ | $E_{2}^{n}\left(t_{i}\right): n=40$ | $\beta$ |
| :---: | :--- | :--- | ---: |
| 0.05 | $2.98 \times 10^{-6}$ | $1.87 \times 10^{-7}$ | 3.99 |
| 0.1 | $2.23 \times 10^{-6}$ | $1.41 \times 10^{-7}$ | 3.99 |
| 0.15 | $1.59 \times 10^{-6}$ | $1.01 \times 10^{-7}$ | 3.99 |
| 0.2 | $1.09 \times 10^{-6}$ | $6.94 \times 10^{-8}$ | 3.97 |
| 0.25 | $7.13 \times 10^{-7}$ | $4.58 \times 10^{-8}$ | 3.96 |
| 0.3 | $4.46 \times 10^{-7}$ | $2.91 \times 10^{-8}$ | 3.94 |
| 0.35 | $2.7 \times 10^{-7}$ | $4.58 \times 10^{-8}$ | 3.91 |
| 0.4 | $1.69 \times 10^{-7}$ | $4.58 \times 10^{-8}$ | 3.86 |
| 0.45 | $1.3 \times 10^{-7}$ | $4.58 \times 10^{-8}$ | 3.83 |
| 0.5 | $1.41 \times 10^{-7}$ | $4.58 \times 10^{-8}$ | 3.85 |
| 0.55 | $1.91 \times 10^{-7}$ | $4.58 \times 10^{-8}$ | 3.89 |
| 0.6 | $2.72 \times 10^{-7}$ | $4.58 \times 10^{-8}$ | 3.93 |
| 0.65 | $3.75 \times 10^{-7}$ | $4.58 \times 10^{-8}$ | 3.95 |
| 0.7 | $4.95 \times 10^{-7}$ | $4.58 \times 10^{-8}$ | 3.97 |
| 0.75 | $6.26 \times 10^{-7}$ | $4.58 \times 10^{-8}$ | 3.98 |
| 0.8 | $7.6 \times 10^{-7}$ | $4.58 \times 10^{-8}$ | 3.99 |
| 0.85 | $8.94 \times 10^{-7}$ | $4.58 \times 10^{-8}$ | 3.99 |
| 0.9 | $1.02 \times 10^{-6}$ | $4.58 \times 10^{-8}$ | 3.99 |
| 0.95 | $1.14 \times 10^{-6}$ | $4.58 \times 10^{-8}$ | 4 |

This verifies the result (3.17).

## Acknowledgment

The author Akshay S. Rane would like to thank UGC faculty recharge program, India for their support.

## References

[1] K. E. Atkinson, The Numerical Solutions of Integral Equations of the Second Kind, Cambridge University Press, Cambridge, 1997.
[2] K. E. Atkinson, I. Graham and I. H. Sloan, Piecewise continuous collocation for integral equations, SIAM J. Numer. Anal. 20 (1983), 172-186.
[3] K.E. Atkinson and F.A. Potra, Projection and iterated projection methods for nonlinear integral equations, SIAM J. Numer. Anal. 24 (1987), 1352-1373.
[4] F. Chatelin and R. Lebbar, Superconvergence results for the iterated projection method applied to a Fredholm integral equation of the second kind and the corresponding eigenvalue problem, J. Integral Equations, 6 (1984), pp. 71-91.
[5] W. F. Ford, J. A. Pennline, Y. Xu and Y. Zhao, Asymptotic error analysis of a quadrature method for integral equations with Green's function kernels, J. Integr. Eq. Appl. 12 (2000), 349-384.
[6] M. A. Krasnoselskii, Topological Methods in the Theory of Nonlinear Integral Equations, Pergamon Press, London, 1964.
[7] M. A. Krasnoselskii, G. M. Vainikko, P. P. Zabreiko, Ya. B. Rutitskii and V. Ya Stetsenko, Approximate Solution of Operator Equations, P. Noordhoff, Groningen, 1972.
[8] M. A. Krasnoselskii and P. P. Zabreiko, Geometrical Methods of Nonlinear Analysis, Springer-Verlag, Berlin, 1984.
[9] R. P. Kulkarni, On improvement of the iterated Galerkin solution of the second kind integral equations, J. Numer. Math. 13(3) (2005), 205-218.
[10] R. P. Kulkarni and T. J. Nidhin, Asymptotic error analysis of projection and modified projection methods for nonlinear integral equations, J. Integr. Eq. Appl. 27(1) (2015), 67-101.
[11] R. P. Kulkarni and A. S. Rane, Asymptotic expansions for approximate solutions of Fredholm integral equations with Green's function type kernels, J. Integr. Eq. Appl. 24 (2012), 39-79.
[12] R. P. Kulkarni and A. S. Rane, Asymptotic expansions for approximate solutions of Hammerstein integral equations with Green's function Type Kernels, Mathematical Modelling and Analysis 19(1) (2014), 127-143.
[13] P. Linz, Theoretical numerical analysis, Courier Dover Publications, 2019.
[14] W. McLean, Asymptotic error expansions for numerical solution of integral equations, IMA J. of Numer. Anal. 9 (1989), 373-384.
[15] G. Rakshit and A. S. Rane, Asymptotic expansion of iterated Galerkin solution of second kind Fredholm integral equations with Green's kernels, J. Integr. Eq. Appl. (2020), To appear.
[16] L. B. Rall, Computational solution of nonlinear operator equations, Wiley New York, 1969.
[17] F. Riesz and B. S. Nagy, Functional Analysis, Frederick Ungar Pub., New York, 1955.


[^0]:    *School of Mathematical Sciences, UM-DAE Centre for Excellence in Basic Sciences, University of Mumbai Campus, Kalina, Mumbai 400 098, India, email : gobinda.rakshit@cbs.ac.in
    ${ }^{\dagger}$ Department of Mathematics, Institute of Chemical Technology, Nathalal Parekh Marg, Matunga, Mumbai 400 019, India, email : as.rane@ictmumbai.edu.in
    ${ }^{\ddagger}$ Department of Mathematics, Institute of Chemical Technology, Nathalal Parekh Marg, Matunga, Mumbai 400 019, India, email : kshitijanandpatil@gmail.com

