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A numerical technique for solving a second-order nonlinear Neumann problem is presented. The authors' approach is based on semi-orthogonal B-spline wavelets. Two test problems are presented and numerical results are tabulated to show the efficiency of the proposed technique for the studied problem.

Keywords: Nonlinear Neumann problem; Semi-orthogonal B-spline wavelets

1. Introduction

In this paper we solve the second-order nonlinear Neumann problem of the form

$$-\ddot{x}(t) = f(t, x(t)), \quad t \in [0, 1], \tag{1}$$

$$\dot{x}(0) = A, \quad \dot{x}(1) = B. \tag{2}$$

The existence of a solution to equation (1) with Neumann boundary conditions is studied in [5] using the quasi-linearization method. In the present paper we apply compactly supported linear semi-orthogonal (SO) B-spline wavelets, specially constructed for the bounded interval, to solve the nonlinear second-order Neumann problem of the form (1).

In recent years, the application of methods based on wavelets has influenced many areas of applied mathematics. In areas such as numerical analysis and differential equations, wavelets are recognized as a powerful tool. Another area in which the wavelet is gaining considerable attention is the study of integral equations. Wavelets can be separated into two distinct types, orthogonal and semi-orthogonal [1]. Publications on integral equation methods have shown a marked preference for orthogonal wavelets [2]. This is probably because the original wavelets, which were widely used for signal processing, were primarily orthogonal. In signal processing applications, unlike integral equation methods, the wavelet itself is never constructed since only its scaling function and coefficients are needed. However, orthogonal wavelets either have infinite support or an asymmetric, and in some cases fractal, nature. These properties can make them a poor choice for characterization of a function. In contrast, semi-orthogonal wavelets have finite support, both even and odd symmetry and simple analytical expressions – ideal attributes for a basis function [3].

The use of SO compactly supported spline wavelets is justified by their interesting properties [3]. Our method consists of reducing expressions (1) and (2) to a set of algebraic equations by expanding the unknown function as linear B-spline wavelets with unknown coefficients. The properties of linear B-spline scaling functions and wavelets are then utilized to evaluate the unknown coefficients.

This paper is organized as follows. In section 2, we describe the formulation of the B-spline scaling functions and wavelets on [0,1], and derive the operational matrices of derivatives required for our subsequent development. In section 3, the proposed method is used to approximate the solution of the second-order nonlinear Neumann problem. In section 4, we report our computational results and demonstrate the accuracy of the proposed numerical scheme by presenting numerical examples. Section 5 ends the paper with a brief conclusion.

2. B-spline scaling functions and wavelets on [0, 1]

When semi-orthogonal wavelets are constructed from B-splines of order m, the lowest octave level $j = j_0$ is determined in [4] by

$$2^{j_0} \geq 2m - 1, \tag{3}$$

to give a minimum of one complete wavelet on the interval [0, 1]. In this paper we will use a wavelet generated by a linear spline ($m = 2$) cardinal B-spline function. From equation (2), the second-order B-spline lowest level, which must be an integer, is determined to be $j_0 = 2$. This constrains all octave levels to $j \geq 2$.

As is the case with all semi-orthogonal wavelets, the second-order B-splines also serve as scaling functions. The second-order B-splines scaling functions are given by

$$\phi_{j,k}(t) = \begin{cases} t_j - k & \text{for } k \leq t_j \leq k+1, \\ 2 - (t_j - k) & \text{for } k+1 \leq t_j \leq k+2, \quad k = 0, \ldots, 2^j - 2, 0 \\ & \text{otherwise,} \end{cases} \tag{4}$$

with the left- and right-hand side boundary scaling functions

$$\phi_{j,k}(t) = \begin{cases} 2 - (t_j - k) & \text{for } 0 \leq t_j \leq 1, \quad k = -1, \\ 0 & \qquad \text{otherwise,} \end{cases} \tag{5}$$

and

$$\phi_{j,k}(t) = \begin{cases} t_j - k & \text{for } \leq t_j \leq k+1, \ k = 2^j - 1, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

respectively. The actual coordinate position t is related to t_j, according to $t_j = 2^j t$. The second-order B-spline wavelets are given by

$$\psi_{j,k}(t) = \frac{1}{6}\begin{cases} t_j - k, & \text{for } k \le t_j \le k + 1/2, \\ 4 - 7(t_j - k), & \text{for } k + 1/2 \le t_j \le k + 1, \\ -19 + 16(t_j - k), & \text{for } k + 1 \le t_j \le k + 3/2, \\ 29 - 16(t_j - k), & \text{for } k + 3/2 \le t_j \le k + 2, \quad k = 0, \ldots, 2^j - 3, \\ -17 + 7(t_j - k), & \text{for } k + 2 \le t_j \le k + 5/2, \\ 3 - (t_j - k), & \text{for } k + 5/2 \le t_j \le k + 3, \\ 0 & \text{otherwise}, \end{cases} \tag{7}$$

with the left- and right- side boundary wavelets

$$\psi_{j,k}(t) = \frac{1}{6}\begin{cases} -6 + 23t_j, & \text{for } 0 \le t_j \le 1/2, \\ 14 - 17t_j, & \text{for } 1/2 \le t_j \le 1, \\ -10 + 7t_j, & \text{for } 1 \le t_j \le 3/2, \quad k = -1, \\ 2 - t_j, & \text{for } 3/2 \le t_j \le 2, \\ 0 & \text{otherwise}, \end{cases} \tag{8}$$

$$\psi_{j,k}(t) = \frac{1}{6}\begin{cases} 2 - (k + 2 - t_j), & \text{for } k \le t_j \le k + 1/2, \\ -10 + 7(k + 2 - t_j), & \text{for } k + 1/2 \le t_j \le k + 1, \\ 14 - 17(k + 2 - t_j), & \text{for } k + 1 \le t_j \le k + 3/2, \quad k = 2^j - 2, \\ -6 + 23(k + 2 - t_j), & \text{for } k + 3/2 \le t_j \le k + 2, \\ 0 & \text{otherwise}, \end{cases} \tag{9}$$

respectively. From expressions (4)–(9) we get

$$\begin{cases} \phi_{i,j}(t) = \frac{1}{2}\phi_{i+1,2j}(t) \\ \qquad + \phi_{i+1,2j+1}(t) + \frac{1}{2}\phi_{i+1,2j+2}(t), \quad i = 2, 3, \ldots, \quad j = 0, 1, \ldots, 2^i - 2, \\ \phi_{i,-1}(t) = \phi_{i+1,-1}(t) + \dfrac{1}{2}\phi_{i+1,0}(t), \quad i = 2, 3, \ldots \\ \phi_{i,2^i-1}(t) = \phi_{i+1,2^{i+1}-1}(t) + \dfrac{1}{2}\phi_{i+1,2^{i+1}-2}(t), \quad i = 2, 3, \ldots \end{cases} \tag{10}$$

$$\begin{cases}\psi_{i,j}(t) = \dfrac{1}{12}\phi_{i+1,2j}(t) - \dfrac{1}{2}\phi_{i+1,2j+1}(t) + \dfrac{5}{6}\phi_{i+1,2j+2}(t) - \dfrac{1}{2}\phi_{i+1,2j+3}(t) \\ \quad + \dfrac{1}{12}\phi_{i+1,2j+4}(t), \quad i = 2, 3\dots \quad j = 0, 1, 2, \dots, 2^i - 3, \\ \psi_{i,-1}(t) = -\phi_{i+1,-1}(t) + \dfrac{11}{12}\phi_{i+1,0}(t) \\ \quad - \dfrac{1}{2}\phi_{i+1,1}(t) + \dfrac{1}{12}\phi_{i+1,2}(t), \quad i = 2, 3, \dots \\ \psi_{i,2^i-2}(t) = -\phi_{i+1,2^{i+1}-2}(t) + \dfrac{11}{12}\phi_{i+1,2^{i+1}-3}(t) \\ \quad - \dfrac{1}{2}\phi_{i+1,2^{i+1}-4}(t) + \dfrac{1}{12}\phi_{i+1,2^{i+1}-5}(t), \quad i = 2, 3, \dots \end{cases} \tag{11}$$

2.1 *Function approximation*

A function $f(t)$ defined over [0,1] may be represented by B-spline scaling functions as

$$f(t) = \sum_{k=-1}^{2^{M+1}-1} s_k \phi_{M+1,k} = S^{\mathrm{T}}\Phi_{M+1},$$

where

$$\left.\begin{aligned} S &= [s_{-1}, s_0, \dots, s_{2^{M+1}-1}]^{\mathrm{T}} \\ \Phi_{M+1} &= [\phi_{M+1,-1}, \phi_{M+1,0}, \dots, \phi_{M+1,2^{M+1}-1}]^{\mathrm{T}} \end{aligned}\right\} \tag{12}$$

with

$$s_k = \int_0^1 f(t)\tilde{\phi}_{M+1,k}(t)\,\mathrm{d}t, \quad k = -1, 0, \dots, 2^{M+1} - 1,$$

in which $\tilde{\phi}_{M+1,k}(t)$ are dual functions of $\phi_{M+1,k}(t)$. These can be obtained by linear combinations of $\phi_{M+1,k}(t)$, $k = -1, \dots, 2^{M+1} - 1$, as follows. Let $\tilde{\Phi}_{M+1}$ be the dual functions of Φ_{M+1} given by

$$\tilde{\Phi}_{M+1} = \left[\tilde{\phi}_{M+1,-1}, \tilde{\phi}_{M+1,0}, \dots, \tilde{\phi}_{M+1,2^{M+1}-1}\right]^{\mathrm{T}}. \tag{13}$$

Equations (12) and (13) give

$$\int_0^1 \tilde{\Phi}_{M+1}\Phi_{M+1}^{\mathrm{T}}\,\mathrm{d}t = I_1, \tag{14}$$

where I_1 is the $(2^{(M+1)} + 1) \times (2^{(M+1)} + 1)$ identity matrix. Using expressions (4)–(6) and (12) we have

$$\int_0^1 \Phi_{M+1}\Phi_{M+1}^{\mathrm{T}}\,\mathrm{d}t = P_{M+1} = \frac{1}{2^{M-1}}\begin{bmatrix} \frac{1}{12} & \frac{1}{24} & 0 & \cdots & 0 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{24} & \frac{1}{6} & \frac{1}{24} \\ 0 & \cdots & 0 & \frac{1}{24} & \frac{1}{12} \end{bmatrix}, \tag{15}$$

where P_{M+1} is a symmetric $(2^{M+1}+1)\times(2^{M+1}+1)$ matrix. From (14) and (15) we get

$$\tilde{\Phi}_{M+1} = (P_{M+1})^{-1}\Phi_{M+1}.$$

Furthermore, a function $f(x)$ defined over [0,1] may be represented by B-spline wavelets as

$$f(t) = \sum_{k=-1}^{3} c_k\phi_{2,k}(t) + \sum_{i=2}^{M}\sum_{j=-1}^{2^i-2} d_{i,j}\psi_{i,j}(t) = \alpha^{\mathrm{T}}\Psi(t), \tag{16}$$

where $\phi_{2,k}$ and $\psi_{i,j}$ are scaling and wavelets functions, respectively, and α and Ψ are $(2^{(M+1)}+1)\times 1$ vectors given by

$$\alpha = [c_{-1}, c_0, \dots, c_3, d_{2,-1}, \dots, d_{2,2}, d_{3,-1}, \dots, d_{3,6}, \dots, d_{M,-1}, \dots, d_{M,2^M-2}]^{\mathrm{T}}, \tag{17}$$

$$\Psi = [\phi_{2,-1}, \phi_{2,0}, \dots, \phi_{2,3}, \psi_{2,-1}, \dots, \psi_{2,2}, \psi_{3,-1}, \dots, \psi_{3,6}, \dots, \psi_{M,-1}, \dots, \psi_{M,2^M-2}]^{\mathrm{T}}. \tag{18}$$

2.2 *The operational matrices of derivatives*

The differentiation of vectors Φ_{M+1} and Ψ in (12) and (18) can be expressed as

$$\Phi'_{M+1} = D_\Phi\Phi_{M+1}, \tag{19}$$

$$\Psi' = D_\Psi\Psi, \tag{20}$$

where D_Φ and D_Ψ are $(2^{M+1}+1)\times(2^{M+1}+1)$ operational matrices of derivatives for B-spline scaling functions and wavelets, respectively. The matrix D_Φ can be obtained by considering

$$D_\Phi = \int_0^1 \Phi'_{M+1}(t)\tilde{\Phi}^{\mathrm{T}}_{M+1}(t)\,\mathrm{d}t = \left(\int_0^1 \Phi'_{M+1}(t)\Phi^{\mathrm{T}}_{M+1}(t)\,\mathrm{d}t\right)(P_{M+1})^{-1}, \tag{21}$$

where

$$E = \int_0^1 \Phi'_{M+1}(t)\Phi^{\mathrm{T}}_{M+1}(t)\,\mathrm{d}t. \tag{22}$$

In (22), E is a $(2^{M+1}+1)\times(2^{M+1}+1)$ matrix given by

$$E = \begin{bmatrix} \int_0^1 \phi'_{M+1,-1}(t)\phi_{M+1,-1}(t)\,\mathrm{d}t & \dots & \int_0^1 \phi'_{M+1,-1}(t)\phi_{M+1,2^{M+1}-1}(t)\,\mathrm{d}t \\ \vdots & \ddots & \vdots \\ \int_0^1 \phi'_{M+1,2^{M+1}-1}(t)\phi_{M+1,-1}(t)\,\mathrm{d}t & \dots & \int_0^1 \phi'_{M+1,2^{M+1}-1}(t)\phi_{M+1,2^{M+1}-1}(t)\,\mathrm{d}t \end{bmatrix}.$$

For any entries of $E_{j,k}$ we have

$$\begin{aligned} E_{j,k} &= \int_0^1 \phi'_{M+1,j}(t)\phi_{M+1,k}(t)\,\mathrm{d}t \\ &= \int_{j/2^{M+1}}^{(j+1)/2^{M+1}} 2^{M+1}\phi_{M+1,k}(t)\,\mathrm{d}t + \int_{(j+1)/2^{M+1}}^{(j+2)/2^{M+1}} 2^{M+1}\phi_{M+1,k}(t)\,\mathrm{d}t. \end{aligned} \tag{23}$$

From (23) we get

$$E = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & & & \\ \frac{1}{2} & 0 & -\frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{2} & 0 & -\frac{1}{2} \\ & & & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

The matrix D_Ψ can be obtained by considering

$$\Psi = G\Phi_{M+1}, \tag{24}$$

where G is a $(2^{M+1}+1) \times (2^{M+1}+1)$ matrix, which can be calculated as follows. Let

$$\Phi_j = [\phi_{j,-1}, \phi_{j,0}, \ldots, \phi_{j,2^j-1}]^{\mathrm{T}}, \tag{25}$$

$$\Psi_j = [\psi_{j,-1}, \psi_{j,0}, \ldots, \psi_{j,2^j-2}]^{\mathrm{T}}. \tag{26}$$

Equations (10) and (25) give

$$\Phi_j = \beta_j \Phi_{j+1}, \tag{27}$$

with

$$\beta_j = \begin{bmatrix} 1 & \frac{1}{2} & & & & & & \\ & \frac{1}{2} & 1 & \frac{1}{2} & & & & \\ & & & \frac{1}{2} & 1 & \frac{1}{2} & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & \frac{1}{2} & 1 & \frac{1}{2} \\ & & & & & & & \frac{1}{2} & 1 \end{bmatrix},$$

where β_j, $j = 2, 3, \ldots,$ is a $(2^j+1) \times (2^{j+1}+1)$ matrix. From (11) and (20) we have

$$\Psi_j = L_j \Phi_{j+1}, \tag{28}$$

with

$L_j =$

$$\begin{bmatrix} -1 & \frac{11}{12} & -\frac{1}{2} & \frac{1}{12} & & & & & & & & & & \\ & \frac{1}{12} & -\frac{1}{2} & \frac{5}{6} & -\frac{1}{2} & \frac{1}{12} & & & & & & & & \\ & & & \frac{1}{12} & -\frac{1}{2} & \frac{5}{6} & -\frac{1}{2} & \frac{1}{12} & & & & & & \\ & & & & & \frac{1}{12} & -\frac{1}{2} & \frac{5}{6} & -\frac{1}{2} & \frac{1}{12} & & & & \\ & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & & & & \frac{1}{12} & -\frac{1}{2} & \frac{5}{6} & -\frac{1}{2} & \frac{1}{12} & & \\ & & & & & & & & & \frac{1}{12} & -\frac{1}{2} & \frac{5}{6} & -\frac{1}{2} & \frac{1}{12} \\ & & & & & & & & & & & \frac{1}{12} & -\frac{1}{2} & \frac{11}{12} & -1 \end{bmatrix},$$

where L_j, $j = 2, 3, \dots,$ is a $2^j \times (2^{j+1} + 1)$ matrix. Using expressions (24), (27) and (28), we get

$$G = \begin{bmatrix} \beta_2 \times \beta_3 \times \cdots \times \beta_M \\ \hline L_2 \times \beta_3 \times \cdots \times \beta_M \\ \hline \vdots \\ L_{M-2} \times \beta_{M-1} \times \beta_M \\ \hline L_{M-1}\beta_M \\ \hline L_M \end{bmatrix}. \tag{29}$$

Using expressions (19), (21), (24) and (29) we have

$$\begin{aligned} \Psi' &= G\Phi'_{M+1} = GD_\Phi \Phi_{M+1} = GE(P_{M+1})^{(-1)}\Phi_{M+1} \\ &= GE(P_{M+1})^{(-1)}G^{-1}\Psi = D_\Psi \Psi, \end{aligned} \tag{30}$$

where

$$D_\Psi = GE(P_{M+1})^{-1}G^{-1}.$$

3. The second-order nonlinear differential equation

In this section we solve a second-order nonlinear Neumann problem of the form (1) by using B-spline wavelets. For this purpose, we use equation (16) to approximate $x(t)$ and $f(t, x(t))$

as

$$x(t) = C^{\mathrm{T}}\Psi(t), \tag{31}$$

$$f(t, x(t)) = F^{\mathrm{T}}\Psi(t), \tag{32}$$

where $\Psi(t)$ is defined in (18) and C and F are $(2^{(M+1)}+1)\times 1$ unknown vectors defined similar to α in (17). Using (30) and (31) gives

$$\dot{x}(t) = C^{\mathrm{T}}\Psi'(t) = C^{\mathrm{T}}D_{\Psi}\Psi(t), \tag{33}$$

$$\ddot{x}(t) = C^{\mathrm{T}}D_{\Psi}^{2}\Psi(t). \tag{34}$$

From expressions (1), (32) and (33) we have

$$-C^{\mathrm{T}}D_{w}^{2}\Psi(t) = F^{\mathrm{T}}\Psi(T). \tag{35}$$

The entries of vector $\Psi(t)$ are independent, so using (35) we get

$$-C^{\mathrm{T}}D_{w}^{2} = F^{\mathrm{T}}. \tag{36}$$

Expressions (31) and (32) yield

$$f(t, C^{\mathrm{T}}\Psi(t)) = F^{\mathrm{T}}\Psi(t). \tag{37}$$

Also using boundary values in expressions (2) and (33) we have

$$C^{\mathrm{T}}D_{w}\Psi(0) = A, \tag{38}$$

$$C^{\mathrm{T}}D_{w}\Psi(1) = B. \tag{39}$$

To find the solution in (1) we first collocate (37) in $t_i = (2i-1)/(2^{M+2}-2), i = 1, 2, \dots, 2^{M+1}-1$. The resulting equation generates $2^{M+1}-1$ algebraic equations. The total unknowns for vector C in (31) and F in (32) are $2\times(2^{M+1}+1)$. These unknowns can be obtained by using expressions (36)–(39).

4. Numerical examples

To support our theoretical discussion, we applied the method presented in this paper to several examples. To the best of our knowledge, the problem studied in this paper has not been investigated previously. Thus we could not compare the results obtained for our four test problems with others.

4.1 *Example 1*

Consider the second-order linear Neumann problem

$$-\ddot{x}(t) = (2-4t^2)x(t), \quad t \in [0, 1],$$
$$\dot{x}(0) = 0, \quad \dot{x}(1) = -2/\mathrm{e}.$$

The exact solution of this problem is e^{-t^2}. Table 1 shows the absolute values of errors of the solution at some points using the method proposed in this paper.

Table 1. Absolute values of errors.

t_i	$M = 5$	$M = 6$
0.0	3.9×10^{-7}	1.3×10^{-7}
0.1	2.5×10^{-5}	5.9×10^{-6}
0.2	1.8×10^{-5}	5.6×10^{-6}
0.3	1.4×10^{-5}	5.2×10^{-6}
0.4	1.7×10^{-5}	2.2×10^{-6}
0.5	1.8×10^{-6}	4.4×10^{-7}
0.6	1.1×10^{-5}	4.0×10^{-7}
0.7	7.4×10^{-6}	1.8×10^{-6}
0.8	1.1×10^{-5}	1.1×10^{-6}
0.9	1.5×10^{-6}	4.0×10^{-6}
1.0	9.4×10^{-6}	2.3×10^{-6}

4.2 *Example 2*

As the second test problem, consider the nonlinear Neumann problem

$$-\ddot{x}(t) = 2x^3(t), \quad t \in [0, 1], \tag{40}$$

$$\dot{x}(0) = -1, \quad \dot{x}(1) = -1/4. \tag{41}$$

The exact solution of this problem is $1/(t+1)$. Table 2 shows the absolute values of errors of the solution at some points using the technique developed in this work.

4.3 *Example 3*

Consider the Neumann problem

$$-\ddot{x}(t) = 4x(t) - 2, \quad t \in [0, 1],$$

$$\dot{x}(0) = 0, \quad \dot{x}(1) = \sin(2).$$

The exact solution of this problem is $\sin^2(t)$. Table 3 shows the absolute values of errors of the solution at some points using the method proposed in this paper.

Table 2. Absolute values of errors.

t_i	$M = 4$	$M = 6$
0.0	2.5×10^{-5}	5.6×10^{-6}
0.1	8.5×10^{-5}	2.6×10^{-5}
0.2	9.0×10^{-5}	1.7×10^{-5}
0.3	6.9×10^{-5}	1.6×10^{-5}
0.4	5.5×10^{-5}	1.4×10^{-5}
0.5	5.3×10^{-5}	1.2×10^{-5}
0.6	4.1×10^{-5}	1.0×10^{-5}
0.7	3.6×10^{-5}	7.2×10^{-6}
0.8	2.5×10^{-5}	5.3×10^{-6}
0.9	7.6×10^{-6}	5.5×10^{-6}
1.0	6.2×10^{-6}	1.6×10^{-6}

Table 3. Absolute values of errors.

t_i	$M = 4$	$M = 5$	$M = 6$
0.0	1.8×10^{-6}	1.5×10^{-7}	1.2×10^{-8}
0.2	9.4×10^{-5}	2.0×10^{-5}	5.6×10^{-6}
0.4	4.5×10^{-5}	1.6×10^{-5}	3.1×10^{-6}
0.6	1.8×10^{-5}	4.9×10^{-6}	9.2×10^{-7}
0.8	1.2×10^{-5}	4.3×10^{-6}	4.5×10^{-7}
1.0	2.7×10^{-6}	2.2×10^{-7}	1.8×10^{-8}

Table 4. Absolute values of errors.

t_i	$M = 4$	$M = 5$	$M = 6$
0.0	3.6×10^{-7}	2.8×10^{-8}	2.3×10^{-9}
0.2	5.3×10^{-5}	1.4×10^{-5}	3.4×10^{-6}
0.4	6.3×10^{-5}	1.5×10^{-5}	3.8×10^{-6}
0.6	6.8×10^{-5}	1.6×10^{-5}	4.2×10^{-6}
0.8	7.2×10^{-5}	1.9×10^{-5}	4.4×10^{-6}
1.0	6.7×10^{-7}	5.4×10^{-8}	4.6×10^{-9}

4.4 *Example 4*

Consider the Neumann problem

$$\ddot{x}(t) = x(t), \quad t \in [0, 1],$$
$$\dot{x}(0) = 0, \quad \dot{x}(1) = \sinh(1).$$

The exact solution of this problem is $\cosh(t)$. Table 4 shows the absolute values of errors of the solution at some points using the method proposed in this paper.

5. Conclusion

In this paper a second-order differential equation with Neumann boundary conditions was investigated. The semi-orthogonal B-spline wavelet was proposed for solving this problem. The proposed technique was tested on some examples. The results obtained show the efficiency of the method presented.

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