# REMARKS ON SYMMETRIES OF 2 D-QUASICRYSTALS (SI - CMMSE - 2006)

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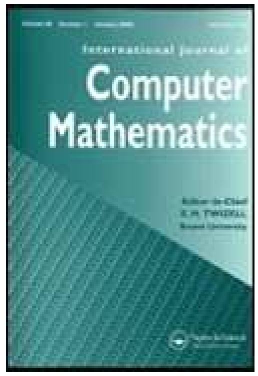




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# REMARKS ON SYMMETRIES OF 2 D-QUASICRYSTALS (SI - CMMSE - 2006)

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Abstract. The paper presents a strict mathematical proof of the classifications of finite groups of symmetries of 2-dimensional quasicrystals given in [4].

## Introduction

There are various approaches to a construction of mathematical models of quasicrystals and a definition of their symmetries, see for example [1], [2, Chapter 6], [4], [5], [9]. We shall adopt the following model which is usually call *cut and project scheme* [9]. Nowadays the common definition of this method is the following one. Let $V$ be an additive locally compact topological Abelian group, $U$ an additive group of a real *physical* or *internal* vector space of dimension $d$ and $M$ a discrete subgroup in $E = U \oplus V$ such that $E/M$ is compact and $M \cap V$ consists only of zero vector. The group $E$ is often called a *hyperspace* or *embedding space*. Consider the diagram of projections of groups

$$\begin{array}{ccc} U \xleftarrow[\pi]{\qquad} & E = U \oplus V & \xrightarrow{\;\rho\;} V \\ & \bigcup & \\ & M & \end{array}$$

Since the intersection **of** $V$ and $M$ consists only of zero element the projection $\pi$ is injective on $M$. It is assumed that $\rho(M)$ is dense is in $V$.

Let us take a convex compact subset $W$ in $V$ called a *window*. Let us put $Q = \rho^{-1}(W) \cap M$. Then the set $\pi(Q)$ is called a *model set* or a *cut and project set*. Note that the set $Q$ under the projection $\pi$ is mapping injectively into $U$.

It is necessary to mention [8, Chapter 2, §3.2] that any locally compact Abelian group $V$ is an inverse limit of direct product of additive groups $\mathbb{R}^m \times \mathbb{T}^s \times D$ where $\mathbb{T}$ is a one-dimensional torus and $D$ is a

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discrete group. So the case when $V$ is a real vector space is in fact one of the most important.

Throughout the paper we shall consider the special case of the model when $V$ is a normed real vector space of a finite dimension $n - d > 0$. Then $E$ is a vector space of dimension $n > d$ and $M$ is a lattice in $E$. It means that $M$ is discrete and $E = \mathbb{R} \otimes_{\mathbb{Z}} M$. Equivalently the rank of the free Abelian group $M$ is equal to $n$. We shall assume that $Q$ contains a basis of $M$ as a free Abelian group.

Since all norms in $E$ are equivalent without loss of generality we can assume that $E$ is an Euclidean space.

According to [1] and [2, Chapter 6] a symmetry of the model $Q$ in this case is the group of all affine transformations of $E$ which map $Q$ bijectively onto itself. Since $Q$ contains a basis of $M$ each symmetry leaves the lattice $M$ invariant. We shall now call all these symmetries *proper*. **It is necessary to mention that there exist non-proper symmetries. For example for Fibonacci tilings powers of the matrix**

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

**show examples of non-proper symmetries.**

Suppose that $M$ has also zero intersection with $V$. If $f$ is a symmetry of $Q$ and 0 is the origin then $f(0) \in M \cap U = 0$. Hence each symmetry $f$ is in fact a linear operator. As in [1] we can show that $U$ is invariant under an action of $f$.

Generalizing this idea we shall denote by $\operatorname{Sym} Q$ the subgroup in the group $\mathrm{GL}(E)$ of all invertible linear operators in $E$ such that both $U$ and $M$ are invariant under an action of each element from $\operatorname{Sym} Q$. The group of proper symmetries is a subgroup of $\operatorname{Sym} Q$. Each element of the group $\operatorname{Sym} Q$ in some special basis of $E$ has the lower left zero block from (1). Moreover there exists an invertible matrix $Z \in \mathrm{GL}(n, \mathbb{R})$ such that $Z^{-1} \operatorname{Sym} Q\, Z \subset \mathrm{GL}(n, \mathbb{Z})$. These two conditions are equivalent to the definition of the group $\operatorname{Sym} Q$.

In the paper we shall consider the case when $\dim U = \dim V = 2$ and $G$ is a finite subgroup of $\operatorname{Sym} Q$. The main results of the paper are in Theorems 2.5, 2.11, 2.12, 2.13, 2.14, 2.15. The list of these finite groups is given in [4] without proof. Here we justify this classification and expose strict mathematical proofs based on group theory approach. The main results of the present paper are complementary to those in [4], [3] where realizations of some symmetries are presented. We give a list of group and in also in §3 their representation in 4-dimensional hyperspace.

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## 1. Matrix representations of symmetries

Suppose that $e_1, \dots, e_n$ is a basis of the lattice $M$. Then these elements form a basis of $E$. If $\mathcal{A} \in \operatorname{Sym} Q$ and $A$ is a matrix of $\mathcal{A}$ is the basis $e_1, \dots, e_n$. Since $M$ is invariant under $\mathcal{A}$ we can conclude that $A$ is an integer matrix, $A \in \mathrm{GL}(n, \mathbb{Z})$.

Let $u_1, \dots, u_d$ be a basis of $U$ and $u_{d+1}, \dots, u_n$ a basis of $V$. Suppose that

$$(u_1, \dots, u_n) = (e_1, \dots, e_n)C, \quad C \in \mathrm{GL}(n, \mathbb{R}).$$

Since $U$ is invariant under $\mathcal{A}$ we have

$$\begin{gathered} C^{-1}AC = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \\ A_1 \in \mathrm{GL}(d, \mathbb{R}),\ A_3 \in \mathrm{GL}(n-d, \mathbb{R}),\ A_2 \in \mathrm{Mat}(d \times (n-d), \mathbb{R}). \end{gathered} \tag{1}$$

We shall use throughout the paper these properties of matrices representing elements from $\operatorname{Sym} Q$.

## 2. 2D symmetries

In this section we shall assume that $n = 4,\ d = 2$ and $G$ is a finite subgroup in $\operatorname{Sym} Q$. Recall that a dihedral group $D_k$ is a group of orthogonal symmetries of a regular $k$-gon [2]. $D_k$ is a subgroup of the group of orthogonal matrices $\mathrm{O}(2, \mathbb{R})$ generated by two matrices

$$a = \begin{pmatrix} \cos\frac{2\pi}{k} & -\sin\frac{2\pi}{k} \\ \sin\frac{2\pi}{k} & \cos\frac{2\pi}{k} \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

It is known [2] that $D_k$ consists of $2k$ elements

$$1,\ a, \dots, a^{k-1},\ b,\ ba,\ \dots, ba^{k-1},$$

where $a^k = (ba^l)^2 = 1$ for any $l$. In particular $ba^l = a^{-l}b$. Note that

$$a^l = \begin{pmatrix} \cos\frac{2\pi l}{k} & -\sin\frac{2\pi l}{k} \\ \sin\frac{2\pi l}{k} & \cos\frac{2\pi l}{k} \end{pmatrix}, \quad ba^l = \begin{pmatrix} \cos\frac{2\pi l}{k} & \sin\frac{2\pi l}{k} \\ \sin\frac{2\pi l}{k} & -\cos\frac{2\pi l}{k} \end{pmatrix} \tag{3}$$

In order to classify finite subgroups in $\operatorname{Sym} Q$ we need some auxiliary statements. We exclude some proofs of known facts.

**Proposition 2.1.** *Let $H$ be a finite subgroup of $\mathrm{O}(2, \mathbb{R})$. If $H \subset \mathrm{SO}(2, \mathbb{R})$ then $H$ is a cyclic group. If $H \not\subseteq \mathrm{SO}(2, \mathbb{R})$, then $H = D_k$ for some $k$.*

**Proposition 2.2.** *Suppose that a finite group $G$ is a subgroup of a direct product of two cyclic groups $H = \langle a_1 \rangle_{k_1} \times \langle a_2 \rangle_{k_2}$ of orders $k_1$ and $k_2$ respectively. Denote by $\nu_i : G \to \langle a_i \rangle_{k_i}$ the natural projection and suppose that $\nu_1, \nu_2$ are surjective. Then $(a_1^{s_1}, a_2)$, $(a_1, a_2^{s_2}) \in G$ for some $s_i \in \mathbb{Z}$ and therefore $a_i^{s_1 s_2 - 1} \in G \cap \langle a_i \rangle_{k_i}$. If $d = (k_1, k_2)$ is the greatest common divisor of $k_1, k_2$ then $a_i^d \in G \cap \langle a_i \rangle_{k_i}$ for $i = 1, 2$. In particular $G \cap \langle a_i \rangle_{k_i}$ is a cyclic group $\langle a_i^{l_i} \rangle$, where $l_i \mid d$ and $l_i \mid (s_1 s_2 - 1)$.*

*Proof.* Since $\nu_1$ is surjective, then there exists an element $(a_1, b) \in G$ for some $b \in \langle a_2 \rangle$. Then $(a_1, b)^{k_2} = (a_1^{k_2}, b^{k_2}) = (a_1^{k_2}, 1) \in G \cap \langle a_1 \rangle$. Since the order of $a_1$ is $k_1$ by [2, Theorem 1.6, § 1.4] we can conclude that $a_1^d \in G \cap \langle a_1 \rangle$. Apply [2, Theorem 1.5, § 1.4]. Similarly if $(a_1^{s_1}, a_2), (a_1, a_2^{s_2}) \in G$, then

$$(a_1^{s_1}, a_2)^{s_2}(a_1, a_2^{s_2})^{-1} = (a_1^{s_1 s_2 - 1}, 1) \in G \cap \langle a_1 \rangle .$$

Similarly $a_2^{s_1 s_2 - 1} \in G \cap \langle a_2 \rangle$. □

Recall that the *Euler* function $\phi(m)$, $m \in \mathbb{Z}$, is the number of all integers $1 \leqslant k < m$ which are coprime with $m$. Recall that if $\zeta = \exp \frac{2\pi i}{m}$ then $\phi(m)$ coincides with the degree of the minimal integer cyclotomic polynomial $\Phi_m(X) \in \mathbb{Z}[X]$ such that $\Phi_m(\zeta) = 0$ and with the order of the Galois group $\mathrm{Gal}\,(\mathbb{Q}[\xi]/\mathbb{Q})$.

**Proposition 2.3.** *Let $m$ be a positive integer such that $\phi(m) \leqslant 4$. Then $m$ is one of the numbers*

| $m$ | 10 | 5 | 12 | 6 | 3 | 8 | 4 | 2 | 1 |
|---|---|---|---|---|---|---|---|---|---|
| $\phi(m)$ | 4 | 4 | 4 | 2 | 2 | 4 | 2 | 1 | 1 |

**Proposition 2.4** ([2], Theorem 2.16, § 2.4)**.** *Let $\Delta$ be a finite subgroup in $\mathrm{SO}(2, \mathbb{R})$. Then $\Delta = \langle a \rangle_k$ is a cyclic group of order $k$, where*

$$a = \begin{pmatrix} \cos\frac{2\pi}{k} & -\sin\frac{2\pi}{k} \\ \sin\frac{2\pi}{k} & \cos\frac{2\pi}{k} \end{pmatrix}.$$

*Moreover, the following are equivalent*

**1)** *$\Delta$ is conjugate in $\mathrm{GL}(2, \mathbb{R})$ to a subgroup of $\mathrm{SL}(2, \mathbb{Z})$;*
**2)** *the trace $\mathrm{tr}\, a = 2\cos\frac{2\pi}{k} \in \mathbb{Z}$;*
**3)** *$k = 1, 2, 3, 4, 6$,*
**4)** *$\phi(k) \leqslant 2$.*

Recall that a subgroup $G$ of a direct product of groups $G_1 \times G_2$ is a *subdirect* product of $G_1, G_2$ if each projection $G \to G_i$, $i = 1, 2$, is surjective. In this case we also say that $G$ is a subdirect subgroup of $G_1 \times G_2$.

**Theorem 2.5.** *Let $G$ be a finite subgroup in $\mathrm{Sym}\, Q$ where $\dim U = 2 = \dim V$. Then $G$ is a subdirect product of either $C_{k_1} \times C_{k_2}$, $C_{k_1} \times D_{k_2}$, $D_{k_1} \times D_{k_2}$ ($C_k$: cyclic group of order $k$) The integers $k_1, k_2$ satisfy one of the following conditions*

**1)** *$k_1, k_2 = 1, 2, 3, 4, 6$;*
**2)** *$(k_1, k_2)$* ***is either (5,10) or (10,5)***
**3)** *$k_1 = k_2 = 12$.*
**4)** *$k_1 = k_2 = 8$.*

*Proof.* By [2, Theorem 3.1, § 3.1] there exists a scalar product $[x, y]$ in $E$ such that each linear operator from $G$ is orthogonal with respect to the new product $[x, y]$. The physical space $U$ is $G$-invariant. Let

$W$ be an orthogonal complement of $U$ with respect to $[x, y]$. Then $W$ is also $G$-invariant. it means that there exists an invertible matrix $C$ such that (1) holds with $A_2 = 0$. In this case $G$ is a subgroup of a direct product $G \subseteq \nu_1(G) \times \nu_2(G)$, where $\nu_1, \nu_2$ are projections and each $\nu_i(G)$ is a finite subgroup of $\mathrm{O}(2, R)$.

According to Proposition 2.1 each $\nu_j(G)$ is either a cyclic group $\langle a_j \rangle_{k_j}$ generated by the matrix $a_j$ of order $k_j$ from (2) or a dihedral group $D_{k_j}$ generated by matrices $a_j, b$ from (2). A subgroup $G \cap \langle a_j \rangle$ of a cyclic group $\langle a_j \rangle$ is cyclic generated by the element $a_j^{l_j}$ where $l_j$ is the least exponent of elements from $G \cap \langle a_j \rangle$. Note that if, say $k_1 = 1$, then $G = G_2$ and applying Proposition 2.4 we deduce the statement of the theorem. Hence in what follows we can assume that $k_1, k_2 > 1$. Let $d = (k_1, k_2)$ be the greatest common divisor of $k_1$ and $k_2$.

Without loss of generality we can assume that $\nu_1, \nu_2$ are surjective that is $G$ is a subdirect subgroup of $\nu_1(G) \times \nu_2(G)$ and one of the required cases takes place.

**Lemma 2.6.** $\phi(k_1)$, $\phi(k_2) \leqslant 4$.

*Proof.* Since $G$ is a subdirect subgroup it contains a matrix

$$F = \begin{pmatrix} a_1 & 0 \\ 0 & b_2^i a_2^j \end{pmatrix}, \quad j \in \mathbb{Z},\ i = 0, 1.$$

where $a_1, b_2^i a_2^j$ are from (2), (3). Since $F$ is conjugate in $\mathrm{GL}(4, \mathbb{R})$ to an integer matrix the characteristic polynomial $\det(F - tE)$ has integer coefficients.

Let $\zeta = \exp \frac{2\pi i}{k_1} \in \mathbb{C}$. Then $\zeta$ is a root of the integer polynomial $\det(F - tE)$ of degree 4 because

$$\det(F - tE) = \det(a_1 - tE) \det(b_2^i a_2^j - tE).$$

The minimal integer polynomial of $\exp \frac{2\pi i}{k_1}$ is the cyclotomic integral polynomial $\Phi_{k_1}(t)$ of degree $\phi(k_1)$. Hence $\Phi_{k_1}(t)$ divides $\det(F - tE)$ and therefore $\phi(k_1) \leqslant 4$. The case of $k_2$ is similar. $\square$

**Lemma 2.7.** *If $G$ contains a matrix*

$$F = \begin{pmatrix} a_1 & 0 \\ 0 & b_2 a_1^j \end{pmatrix}, \quad j \in \mathbb{Z}, \tag{4}$$

*then $\phi(k_1) \leqslant 2$. Similarly if*

$$F_1 = \begin{pmatrix} b_1 a_1^j & 0 \\ 0 & a_1 \end{pmatrix} \in G, \quad j \in \mathbb{Z},$$

*then $\phi(k_2) \leqslant 2$.*

*Proof.* Note that $\mathrm{tr}\, F = \mathrm{tr}\, a_1 + \mathrm{tr}(b_2 a_1^j) = \mathrm{tr}\, a_1 \in \mathbb{Z}$ by (3). Apply Proposition 2.4. $\square$

**Corollary 2.8.** *One of the cases takes place:*

**1)** $\phi(k_1)$, $\phi(k_2) \leqslant 2$;
**2)** $\phi(k_1) = \phi(k_2) = 4$.

*Proof.* Suppose that $\phi(k_2) \leqslant 2$. If $G$ contains a matrix $F_1$ then we apply Lemma 2.7. Suppose that $G$ contains a matrix

$$B = \begin{pmatrix} a_1 & 0 \\ 0 & a_2^{s_2} \end{pmatrix} = \begin{pmatrix} \cos\frac{2\pi}{k_1} & -\sin\frac{2\pi}{k_1} & 0 & 0 \\ \sin\frac{2\pi}{k_1} & \cos\frac{2\pi}{k_1} & 0 & \\ 0 & 0 & \cos\frac{2\pi s_2}{k_2} & -\sin\frac{2\pi s_2}{k_2} \\ 0 & 0 & \sin\frac{2\pi s_2}{k_2} & \cos\frac{2\pi s_2}{k_2} \end{pmatrix} \in G \subseteq G_1 \times G_2. \quad (5)$$

Then $\operatorname{tr} B = 2\left(\cos\frac{2\pi}{k_1} + \cos\frac{2\pi s_2}{k_2}\right) \in \mathbb{Z}$. By the assumption $2\cos\frac{2\pi}{k_1} \in \mathbb{Z}$. Apply Proposition 2.4. □

Starting from now we shall assume that $\phi(k_1) = \phi(k_2) = 4$. By Proposition 2.3 it means that $k_1, k_2 = 10,\ 5,\ 12,\ 8$ and by Lemma 2.7 the group $G$ contains the matrix $B$ a nd a matrix

$$B_2 = \begin{pmatrix} a_1^{s_1} & 0 \\ 0 & a_2 \end{pmatrix}. \quad (6)$$

Let $\xi_n = \exp\frac{2\pi i}{n} \in \mathbb{C}$. As in the proof of Lemma 2.6 we know that $\Phi_4(t) = \det(B - tE)$. Hence the roots of $\det(B - tE)$ are $\xi_{k_1}^r$, where $1 \leqslant r < k_1$ and $(r, k_1) = 1$. Similarly the roots of $\det(B - tE)$ are $\xi_{k_2}^{s_2 t}$ where $1 \leqslant t < \frac{k_2}{(s,k_2)}$ and $(t, \frac{k_2}{(s,k_2)}) = 1$ Hence $\xi_{k_2}^{s_2} = \xi_{k_1}^r$, $\xi_{k_1} = \xi_{k_2}^{s_2 t}$. In particular $\mathbb{Z}[\xi_{k_1}] = \mathbb{Z}[\xi_{k_2}^{s_2}] \subseteq \mathbb{Z}[\xi_{k_2}]$. Similarly $\mathbb{Z}[\xi_{k_2}] = \mathbb{Z}[\xi_{k_1}^{s_1}] \subseteq \mathbb{Z}[\xi_{k_1}]$. Hence

$$\mathbb{Z}[\xi_{k_1}] = \mathbb{Z}[\xi_{k_1}^{s_1}] = \mathbb{Z}[\xi_{k_2}] = \mathbb{Z}[\xi_{k_2}^{s_2}]. \quad (7)$$

**Lemma 2.9.** *The following cases take place*

**a)** $k_1,\ k_2 = 10,\ 5$ *and* $s_1 s_2 \equiv 1 \mod 5$;
**b)** $k_1 = k_2 = 12$ *and* $s_1, s_2 = 1,\ 5,\ 7,\ 11$;
**c)** $k_1 = k_2 = 8$ *and* $s_1, s_2 = 1,\ 3,\ 5,\ 7$.

*Proof.* Recall that $\mathbb{Z}[\xi_{10}] = \mathbb{Z}[\xi_5]$ and

$$\mathbb{Z}[\xi_{10}] \neq \mathbb{Z}[\xi_{12}] \neq \mathbb{Z}[\xi_8] \neq \mathbb{Z}[\xi_{10}].$$

Apply (7). Note also that $a_i^{s_1 s_2 - 1} \in G$ by Proposition 2.2. Hence by Proposition 2.4 $s_1 s_2 - 1$ in the first case should be divisible by 5. □

By Lemma 2.7 we complete the proof of Theorem 2.5. □

In order to present more detailed classification of groups in Theorem 2.5 we need

**Proposition 2.10.** *Let $G, k_1, k_2$ be from Theorem 2.5 and $\phi(k_1) = \phi(k_2) = 4$. Suppose that the group $G$ contains a matrix*

$$F = \begin{pmatrix} a_1^t & 0 \\ 0 & b_2 \end{pmatrix}.$$

*Then $G$ contains*

$$a_1^2 = \begin{pmatrix} a_1^2 & 0 \\ 0 & 1 \end{pmatrix}$$

*and a matrix $F$ with $t = 0$, that is $b_2$. In this case $G \cap (\langle a_1 \rangle \times D_{k_2})$ is a semidirect product of a normal subgroup $\langle B \rangle \times \langle a_2^2 \rangle$ by a cyclic group $\langle F \rangle$ of order 2 with $t = 0$, where $B$ is from (5) with $s_2 = 1$.*

*Proof.* We know that $G$ contains a matrix $B$ from (5). Then

$$U = BFBF^{-1} = a_1^2 \in G.$$

Multiplying $F$ and $B_2$ from (6) by powers of $U$ we can assume that $t = 0, 1$ and $s_1 = 1$. The case $t = 1$ is impossible because otherwise $\operatorname{tr} a_1^2 \in \mathbb{Z}$ which contradicts Proposition 2.4. Thus we can assume that $B_2 = B$ and $t = 0$.

Note that $G_1 = \langle B \rangle \times \langle a_2^2 \rangle$ is a subgroup in $G$ containing $a_1^2$ and therefore $a_2^2$. Hence the index of $G_1$ in $H = \langle a_1 \rangle \times \langle a_2 \rangle$ is dividing 4. Since $a_1, a_2 \notin G$ we can conclude that $G_1 = G \cap H$. Now the proof follows. □

**Theorem 2.11.** *Let a group $G$ be a subdirect product from Theorem 2.5 where $k_1 = k_2 = 10$. Then $G$ is one of the following groups.*

**a)** *If $G$ is a subdirect product of two cyclic groups $\langle a_1 \rangle_{10} \times \langle a_2 \rangle_{10}$ then $G$ is direct product of two cyclic groups*

$$\langle B \rangle \times \langle a_1^l \rangle \tag{8}$$

*where $B$ is from (5) with $1 \leqslant s_2 \leqslant 9$, $s_2 \neq 5$ and $l = 0, 5$. If $s_2$ is even then $l = 5$.*

**b)** *$G$ is a subdirect subgroup of $D_{10} \times D_{10}$ and a semidirect product of the normal group (8) by a cyclic group $\langle W \rangle$ of order 2, where*

$$W = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 a_2^j \end{pmatrix}, \quad j = 0, \dots, 9. \tag{9}$$

*Proof.* Case **a)**. We know that some matrices $B$, $B_2$ from (5), (6) belong to $G$ where $s_i$ are from Lemma 2.9, case a). The group $G$ contains the cyclic group $\langle B \rangle$ of order 10 which projects onto $\langle a_1 \rangle_{10}$. Thus $G = \langle B \rangle \times (G \cap \langle a_2 \rangle_{10})$. But by Proposition 2.4 the order of the group $G \cap \langle a_2 \rangle_{10}$ is equal to 1,2,3,4 or 6. Hence $G \cap \langle a_2 \rangle_{10} = \langle a_1^l \rangle$ where $l = 0$ or 5. Suppose that $s_2$ is even and $l = 0$. Then the cyclic group $\langle B \rangle$ does not contain $B_2$ which is impossible.

Suppose that $G$ is a subdirect product of $\langle a_1 \rangle_{10} \times D_{10}$. Then $G \cap \langle a_1 \rangle_{10} \times \langle a_2 \rangle_{10}$ is a group (8). Also $G$ contains the matrix $F$ from

Proposition 2.10 which contradicts Proposition 2.4. So this case is impossible.

Case **b)**. Using previous argument we can show that $G$ does not contain the matrix $F$ from Proposition 2.10. Thus $G \cap (\langle a_1 \rangle_{10} \times \langle a_2 \rangle_{10})$ is a subdirect product and therefore as in the case **a)** it has the form (8). As above $G$ does not contain the matrix $F$ from Proposition 2.10. Hence $G$ contains a matrix $W \in G$ as in (9). Then $WgW^{-1} = g^{-1}$ for all $g \in \langle a_1 \rangle_{10} \times \langle a_2 \rangle_{10}$ and $W^2 = E$. Thus $G$ has the required form. □

**Theorem 2.12.** *Let $k_1 = 10$, $k_2 = 5$. Then there are only two cases.*

**a)** *If $G$ is a subdirect product $\langle a_1 \rangle_{10} \times \langle a_2 \rangle_5$ then $G = \langle B \rangle$ where $s_2 = 1, 2, 3, 4$.*

**b)** *If $G$ is a subdirect product of $D_{10} \times D_5$ then $G$ is generated by $B$ and by $W$ from (9), where $j = 0, 1, 2, 3, 4$. Thus $G \simeq D_{10}$.*

*Proof.* The group $G$ contains a matrix $B$ from (5) where $s_2$ is invertible modulo 5 by Lemma 2.9. In the case **a)** either $G$ is generated by $B$ or $G$ has a nontrivial intersection with $\langle a_2 \rangle$ which is impossible by Proposition 2.4.

The case when $G$ is a subdirect product $\langle a_1 \rangle_{10} \times D_5$ is impossible by Proposition 2.10.

The proof in the case **b)** is similar to those in the previous Theorem. □

The proof of the next Theorem is similar.

**Theorem 2.13.** *Let $k_1 = 5$, $k_2 = 10$. Then there are only two cases.*

**a)** *If $G$ is a subdirect product $\langle a_1 \rangle_5 \times \langle a_2 \rangle_{10}$ then $G = \langle B_2 \rangle$ from (6) where $s_1 = 1, 2, 3, 4$.*

**b)** *If $G$ is a subdirect product of $D_5 \times D_{10}$ then $G$ is generated by $B_2$ and by $W$ from (9). Thus $G \simeq D_{10}$.*

**Theorem 2.14.** *Let group $G$ be a subdirect product from Theorem 2.5, where $k_1 = k_2 = 8$.*

**A)** *If $G$ is a subdirect product of cyclic groups $\langle a_1 \rangle_8 \times \langle a_2 \rangle_8$, then $G$ is a direct product of two cyclic groups (8), where $l = 0, 2, 4$, and $s_2$ being odd in $B$ from (5).*

**B)** *Suppose that $G$ is a subdirect product of $\langle a_1 \rangle \times D_8$. Then $G$ is a semidirect product of the normal subgroup (8) and a cyclic group $\langle b_2 \rangle$ of order 2, where $l = 0, 2, 4$, and $s_2$ being odd in $B$ from (5).*

**C)** *Suppose that $G$ is a subdirect subgroup of $D_8 \times D_8$. Then $G$ is one of the groups*

- **a)** *a semidirect product of a normal subgroup (8) with $l = 2$ and $s_2$ odd by a direct product $\langle b_1 \rangle_2 \times \langle b_2 \rangle_2$.*
- **b)** *$G$ is semidirect product of the normal subgroup (8) and a cyclic group $\langle W \rangle_2$, where $l = 0, 2, 4$, $s_2$ being odd in $B$ from (5) and $W$ is from (9).*

*Proof.* In the case **A)** the group $G$ contains a matrix $B$ from (5) with odd $s_2$. If $G \neq \langle B \rangle$ then $G = \langle B \rangle \times (G \cap \langle a_2 \rangle)$, where $G \cap \langle a_2 \rangle = \langle a_2^l \rangle$ and $l = 0,\, 2,\, 4$ by Proposition 2.4.

The case **B)** follows from Proposition 2.10.

In the case **C)** by Lemma 2.9 the group $G$ contains a matrix $B$ from (5) with odd $s_2$. Hence $G \cap (\langle a_1 \rangle \times \langle a_2 \rangle)$ is a subdirect product from the case **A)**. If $F$ from Proposition 2.10 belongs to $G$ then $t = 0$ and

$$G \cap (\langle a_1 \rangle \times \langle a_2 \rangle) = \langle B \rangle \times \langle a_2^2 \rangle .$$

Symmetrically $G$ contains a matrix

$$b_1 = \begin{pmatrix} b_1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $G$ has the form Ca).

Suppose that $U, U' \notin G$. Then $G$ contains a matrix $W$ from (9) where $j = 0, \ldots, 8$. Also

$$G \cap (\langle a_1 \rangle \times \langle a_2 \rangle) = \langle B \rangle \times \langle a_2^l \rangle, \quad l = 0,\, 2,\, 4.$$

Then $G$ has the form **Cb)**. □

**Note that the case B) with $l = 0$ and $s_2 = 3$ corresponds to perfect 8-fold dihedral symmetry in a certain member of the LI-class of Ammann-Beenker tilings [4]. The generating matrices of the group are**

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Using the same argument one can prove

**Theorem 2.15.** *Let $k_1 = k_2 = 12$. Then $G$ is one of the following groups.*

**A)** *If $G$ is subdirect product of $\langle a_1 \rangle_{12} \times \langle a_2 \rangle_{12}$ then $G$ is the group*

$$G = \langle B \rangle \times \langle a_2^l \rangle \tag{10}$$

*where $B$ is from* (5) ***with*** $s_2 = 1, 5, 7, 11$ *and* $l = 0,\, 2,\, 3,\, 4,\, 6$.

**B)** *If $G$ is a subdirect product of $\langle a_1 \rangle_{12} \times D_{12}$ then $G$ is a semidirect product of the normal subgroup* (10) *with $l = 2$ and $\langle b_2 \rangle_2$.*

**C)** *If $G$ is a subdirect product of $D_{12} \times D_{12}$ then $G$ is one of groups:*

- **a)** *a semidirect product of a normal subgroup* (10) *with $l = 2$ and $s_2$ odd by a direct product $\langle b_1 \rangle_2 \times \langle b_2 \rangle_2$.*
- **b)** *a semidirect product of the normal subgroup* (10) *and a cyclic group $\langle W \rangle_2$, where $l = 0,\, 2,\, 4$, $s_2 = 1,\, 5, 7,\, 11$ in $B$ from* (5) *and where $W$ is from* (9).

**Applying previous results to the window $W$ which is a closure of a projection of $A$ into $V$ we obtain**

**Corollary 2.16.** *The symmetry group of the window of any plane module set with* 2 dim *internal space is one of the following:* $C_i$ *or* $D_i$, $i = 1, 2, 3, 4, 5, 6, 8, 10, 12$.

## 3. Realization

In this section we shall show that two exceptional cases 2), 3) in Theorem 2.5 can be realized. The idea of this realization comes from [10], [7], [4]. Let $m \geqslant 3$ be a positive integer. Take the real vector space $E = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[\xi]$ of dimension $\phi(m)$ where $\xi = \exp\left(\frac{2\pi i}{m}\right)$. Then **basis** of $E$ consists of $\phi(m)$ vectors $e_j = 1 \otimes \xi^j$, $0 \leqslant j < \phi(m)$, [6, Chapter IV, § 1]. Define in in $E$ two linear operators

$$a(1 \otimes \xi^j) = 1 \otimes \xi^{j+1}, \quad b(1 \otimes \xi^j) = 1 \otimes \xi^{-j}$$

for all $j \in \mathbb{Z}$. These definitions are correct since $\mathbb{Z}[\xi]$ is a left $\mathbb{Z}[\xi]$-module and the map $b$ is a Galois automorphism of $\mathbb{Z}[\xi]$. It is easy to see that $b^2 = (ba)^2 = 1$. Hence the group of invertible operators generated by $a, b$ is the dihedral group $D_m$ and we have a representation of $D_m$ in $E$ of dimension $\phi(m)$. It is decomposed into direct sum of irreducible ones. Since dimensions of irreducible representations of dihedral groups are 1 or 2 [2, chapter 3], it suffices to show that $\pm 1$ are not eigenvalues of operators $a$ and $b$. Note that the characteristic polynomial of the operator $a$ is equal to $\Phi_m(t)$, and **in fact** $\pm 1$ are not roots of $\Phi_m(t)$, provided $m \geqslant 3$.

We have proved

**Theorem 3.1.** *Let $m \geqslant 3$ be an integer. Then the dihedral group $D_m$ is a subgroup of a symmetry group* Sym $Q$ *of some 2D-quasicrystal $Q$ which can be constructed by a cut and project method using a hyperspace $E$ of dimension $\phi(m) \geqslant 2$.*

There is another way of constructing symmetries of quasicrystals based on algebraic integers. Let $\xi = \exp\left(\frac{2\pi i}{m}\right)$ be as above. Then again $\mathbb{Z}[\xi]$ is a left $\mathbb{Z}[\xi]$-module and therefore the Abelian group $\mathbb{Z}[\xi]^*$ of invertible elements of the ring $\mathbb{Z}[\xi]$ acts on $E = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[\xi]$. By [2, Theorem 3.3, Chapter3] the complex space $\mathbb{C} \otimes_{\mathbb{R}} E$ contains one-dimensional subspace which is invariant under the action of the **Abelian** group $\mathbb{Z}[\xi]^*$. Hence $E$ contains 2-dimensional real invariant subspace $U$. So taking $U$ as a physical space and the lattice $\mathbb{Z}[\xi]$ in $E$ as $M$ we can construct a quasicrystal $Q$ in which the symmetry group Sym $Q$ contains $\mathbb{Z}[\xi]^*$ as a subgroup. By [3, page 561] the Abelian group $\mathbb{Z}[\xi]^*$ has rank $\frac{\phi(m)}{2} - 1$. Applying Proposition 2.3 as in [7] we obtain

**Theorem 3.2.** *Let $m$ be an integer and either $m = 5$ or $m \geqslant 7$. Consider 2D-quasicrystal $Q$ constructed by a cut and project method* *using a hyperspace $E$ of dimension $\phi(m) \geqslant 2$ and $M = \mathbb{Z}[\xi]$ as a lattice. The symmetry group* $\operatorname{Sym} Q$ *contains an Abelian group isomorphic to* $\mathbb{Z}[\xi]^*$ *and having rank* $\frac{\phi(m)}{2} - 1 \geqslant 1$. *In particular* $\operatorname{Sym} Q$ *contains elements of infinite order.*

For example take $E = \mathbb{R} \otimes \mathbb{Z}[\xi]$, $\xi = \frac{1+i}{\sqrt{2}}$. Then $\dim E = \phi(8) = 4$ and $E$ is a left module over $\mathbb{Z}[\xi]$. Therefore the group $\mathbb{Z}[\xi]^*$ of invertible elements of the ring $\mathbb{Z}[\xi]$ acts on $E$. The rank of the group $\mathbb{Z}[\xi]^*$ is equal to $\frac{\phi(8)}{2} - 1 = 1$ [3, page 561] and therefore it contains an element $g$ of infinite order.

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