Periodic and solitary waves in systems of coherently coupled nonlinear envelope equations

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ABSTRACT

Exact solutions for two classes of coherently coupled nonlinear envelope equations are derived in terms of products of Jacobi elliptic functions. Physical applications are illustrated in the context of nonlinear optics, namely, polarization of light beams and quadratic (or parametric) solitons. Stabilities of these double humped solitary pulses are studied by direct numerical simulations. The usage of computer is crucial, both in terms of symbolic manipulation in the derivation process and in the implementation of numerical schemes in stability consideration.

1. Introduction

Vector solitons in multiple waveguides have attracted tremendous attention recently. While solitons can propagate over a long distance from a balance of dispersion and nonlinearity, coupling in two or more waveguides permits additional flexibility which generates a rich variety of phenomena. The main goal of this work is to present *exact*, periodic wave patterns for two classes of 'coherently coupled' envelope equations in terms of the classical elliptic functions. In a 'coherently coupled' system, the phase factors of wave motion in each individual waveguide are dictated by some constraints. In case where they are identical to each other, such systems will be termed 'phase locked'. Nonlinear optics will be chosen as the potential field of application, and the examples cited will be described in that context.

Two types of nonlinearities will be treated, namely cubic and quadratic ones. For cubic nonlinearity, coupled systems of nonlinear Schrödinger (NLS) equations, which are well established to be relevant in biology, hydrodynamics, optics and plasma physics, will be considered. For quadratic nonlinearity, parametric solitons in optical media where the second order susceptibility tensor is the dominant feature will be studied.

(A) Cubic nonlinearity

An intensively studied system of coupled nonlinear Schrödinger equations is

$$i\frac{\partial A}{\partial t} \pm \frac{\partial^2 A}{\partial x^2} + (AA^* + \sigma BB^*)A = 0,$$

$$i\frac{\partial B}{\partial t} \pm \frac{\partial^2 B}{\partial x^2} + (\sigma AA^* + BB^*)B = 0. \tag{1}$$

When the coefficient of cross phase modulation, σ, is unity, the resulting NLS system is integrable, and is then known as the Manakov model [1 – 10]. The '+' and '–' signs correspond to the anomalous dispersion (bright soliton) and normal dispersion (dark soliton) regimes respectively. Analytically the Manakov model is a system of *incoherently* coupled NLS equations as the phase factor of each component is invariant up to an addition or subtraction of a constant. Nevertheless, the Manakov model possesses many special noteworthy features, a good example being the presence of *symbiotic solitons*. When two or more channels or waveguides are present, nonlinear coupling may lead to modes which are otherwise forbidden in a single, isolated waveguide. A simple example is a pair of bright solitons propagating in a two–waveguide system consisting of one anomalous component and one normal component. These pairs will be termed 'symbiotic' here. One of our goals is to demonstrate that a 'bright – dark' soliton pair can exist in a coherently coupled system when both waveguides are in the normal dispersion (dark soliton) regime.

Coherent coupling can arise in the polarization of light beams, as the relative phase factors of the interacting electric fields there play a crucial role in the four–wave mixing process. In the terminology of optics, this coupling will arise if the medium is weakly anisotropic or birefringent. The resulting model is a system of coherently coupled nonlinear Schrödinger equations,

$$\begin{aligned} i\frac{\partial A}{\partial t} - \delta\frac{\partial^2 A}{\partial x^2} - \gamma A + \left[|A|^2 + \sigma|B|^2\right]A + \lambda B^2 A^* = 0, \\ i\frac{\partial B}{\partial t} - \delta\frac{\partial^2 B}{\partial x^2} + \gamma B + \left[|B|^2 + \sigma|A|^2\right]B + \lambda A^2 B^* = 0, \end{aligned} \tag{2}$$

where A, B are slowly varying envelopes of the electric field associated with two orthogonally polarized components [11 – 19]. The second derivative terms are associated with group velocity dispersion in the temporal domains but represent diffraction in the spatial context. Attention is placed here on the comparatively less intensively studied regime of normal dispersion ($\delta > 0$ and negative sign in front of the second derivative), thus effectively concentrating on the temporal case. The difference in group velocities is assumed to be negligible. The terms involving γ and σ are related to the degree of birefringence and cross phase modulation respectively. The self phase modulation coefficients are scaled to be unity in both equations, but will deviate from one in waveguides / media with anisotropic properties [16, 17]. The coefficient of the four–wave mixing term, λ, will be selected to be $\lambda = 1 - \sigma$ in the present optical setting.

For special values of the parameters ($\lambda = 0$, $\sigma = 1$, $\gamma = 0$), the system (2) reduces to (1). Indeed Manakov solitons can be observed in special configurations of the appropriate semi–conductor waveguide [19].

In many calculations involving incoherently coupled NLS systems, the angular frequencies of each mode must be determined as part of the dynamics. The phase locked structures of these modes appear to reduce the degree of freedom by one in this system of two equations. However, the birefringence parameter (γ) participates in a nonlinear manner and supplies the missing link, i.e. γ will now be coupled to the amplitude and frequency relation of these nonlinear waves.

(B) Quadratic nonlinearity

The second example is the second harmonic generation in optical materials with $\chi^{(2)}$ (or roughly, quadratic) nonlinearity [20]. Although rigorous mathematical proof of existence has been given [21], exact solutions will be calculated explicitly here in a simple manner by employing products of elliptic functions. Previous works place the focus mostly on asymptotic, variational, numerical or other approximation methods [22 – 25].

The plan of the paper can now be explained. Double humped solitary pulses for coherently coupled, cubic NLS equations are derived in the normal dispersion regime (Section 2). The stability properties are studied by numerical simulations (Section 3). Similar reasoning is applied to the case of quadratic, parametric solitons (Section 4).

The present work will thus illustrate a very productive use of computers in mathematics, firstly in the symbolic manipulation involving elliptic functions, and secondly in utilizing efficient numerical schemes in studying stability of nonlinear waves.

2. Double humped solitary pulses

Exact analytic solutions in terms of products of Jacobi elliptic functions [26, 27] can be derived by either direct manipulation with computer algebra or a ‘tri–linear’ version of the Hirota method [6, 7]. One possible ‘dark – bright’ pair for the present configuration is

$$A = \left(\frac{kr\sqrt{6\delta}}{[2c\sqrt{1-k^2}-1]^{1/2}} \right) \mathrm{sn}(rx)\,\mathrm{dn}(rx)\exp(-i\Omega t), \tag{3}$$

$$B = \frac{r\sqrt{6\delta}}{[2c\sqrt{1-k^2}-1]^{1/2}} [c\sqrt{1-k^2} - \mathrm{dn}^2(rx)]\exp(-i\Omega t), \tag{4}$$

where c is the numerically larger, positive root of the equation

$$3c^2 - 2c\left(\sqrt{1-k^2} + \frac{1}{\sqrt{1-k^2}} \right) + 1 = 0. \tag{5}$$

Simple manipulations show the appropriate limits for c to be

$$k \to 0, \quad c \sim 1, \qquad \text{and} \qquad k \to 1, \quad c \sim \frac{2}{3\sqrt{1-k^2}}, \tag{6}$$

with the latter especially useful in the long wave limit. The ‘phase–locked’ angular frequency, Ω, and the degree of birefringence, γ, must be given by

$$\Omega = \delta r^2 \left(\frac{13}{2} - 4k^2 \right) - \frac{3\delta r^2 c(1-k^2)^{1/2}[4c\sqrt{1-k^2}-1]}{[2c\sqrt{1-k^2}-1]}, \tag{7}$$

and

$$\gamma = -\frac{3\delta r^2}{2}[2c\sqrt{1-k^2} - 1], \tag{8}$$

respectively, with k being the modulus of the Jacobi elliptic functions. This set (equations (3 – 8)) is a family of exact solutions with two free parameters, r and k, which are roughly related to the amplitude and period of the wave respectively. A simple plot of the intensity of the profiles is illustrated in Figure 1.

The long wave limit (k tending to unity) is

$$A = 3\sqrt{2\delta}r \tanh(rx)\operatorname{sech}(rx)\exp(-i\Omega t), \tag{9}$$

$$B = 3\sqrt{2\delta}r\left[\frac{2}{3} - \operatorname{sech}^2(rx)\right]\exp(-i\Omega t), \tag{10}$$

where the corresponding angular frequency and birefringence parameter are

$$\Omega = -\frac{15\delta r^2}{2}, \tag{11}$$

$$\gamma = -\frac{\delta r^2}{2}, \tag{12}$$

since sn, dn tend to tanh, sech respectively in this limit.

The waveguide A will display a double–humped (bright soliton) structure, while B will display characters of dark solitons. The intensity of B ($|B|^2$) tends to a constant in the far field, but has two local minima in the near field. The waveguide A, B thus constitute a symbiotic pair. Some properties of the wave profiles relevant to stability consideration in Section 3 will be described more precisely here. The distance between the two peaks of $|A|^2$ is

$$\frac{2\,\mathrm{sech}^{-1}\left(\frac{1}{\sqrt{2}}\right)}{r},$$

and thus the peaks of A are more closely packed as r increases.

3. Stability

The stability of these solitary pulses (9 – 12) is now tested by direct numerical simulations. The exact solutions (9 – 12) plus a random noise of a few percent in amplitude will be used as the initial condition. Numerical methods for NLS systems have been studied earlier in the literature [28 – 30]. For the present problem, marching forward in time for system (2) will be performed with the split step Fourier method [31]. The spatial domain is chosen to be sufficiently large, i.e. roughly ten times the 'full width half maximum' of the pulses.

Although σ does not appear explicitly in (9 – 12), the time evolution of the solitary waves will depend on σ through the evolution system (2).

(a) σ fixed, r varies: The parameter σ will be chosen to be 2/3, pertaining to the physically important case of linear polarization. These double humped pulses are more stable if the amplitudes are smaller, or equivalently, their peaks (equation (9)) are further apart, since that distance scales as $1/r$. Indeed for a relatively small value of r (r = 0.09), stable propagation can be observed up to around t = 300 (Figure 2). In contrast, on increasing r to 0.15, disintegration of the structure starts to appear around t = 200 (Figure 3).

(b) r fixed, σ varies: In more complicated configurations, e.g. presence of anisotropy [16, 17], the self and cross phase modulation coefficients may change with the composition or orientation of the waveguide. No comprehensive study will be attempted here. Instead, as a simple example of the complexity, the change of the instability properties on varying the cross phase modulation parameter (σ) alone will be discussed. In general, the pulses are more stable as σ increases from zero to unity. For σ = 0.9, instability already starts to creep in at t = 200 (Figure 4). For σ = 0.75, the destruction of the pulses occurs much sooner (Figure 5).

4. Solitons in a quadratic medium

In media where second harmonic resonance, or quadratic nonlinearity, is the dominant wave guiding mechanism, evolution equations with second order nonlinearity will arise. Derivations from the fundamental physical principles, such as the Maxwell's equations, can be found in monographs on optical solitons [20]. A typical system is

$$i\frac{\partial\phi}{\partial t}+\lambda\frac{\partial^2\phi}{\partial x^2}+\psi\phi^*=0\,, \tag{13}$$

$$i\frac{\partial\psi}{\partial t}+\mu\frac{\partial^2\psi}{\partial x^2}-\alpha\psi+\frac{\phi^2}{2}=0\ , \tag{14}$$

where roughly ψ is the second harmonic relative to the fundamental mode. We can thus proceed directly to the governing model of such 'parametric solitons':

$$\lambda \frac{d^2 v}{dx^2} - v + w v^* = 0, \tag{15}$$

$$\mu \frac{d^2 w}{dx^2} - \alpha w + \frac{v^2}{2} = 0. \tag{16}$$

In terms of optical physics, v and w are the slowly varying electric fields. The second derivative terms involving λ, μ again represent diffraction in the spatial context, but measure group velocity dispersion in the temporal domain. In the former case, both λ, μ will be positive while they can be of either sign in the temporal (optical fiber) situation. The quantity α represents a phase mismatch, and any difference in group velocity (or 'walk off' effect) has been ignored. The asterisk (*) denotes the complex conjugate and the system is thus driven by quadratic rather than cubic nonlinearity. Various isolated, exact solutions for solitary pulses in terms of hyperbolic functions can be found earlier in the literature [20], and are supplemented by various asymptotic and variation approximations.

The contribution here is to demonstrate that products of elliptic functions are again applicable in calculating exact solutions for (15, 16). More precisely, one particular solution is:

$$v = 6(\sqrt{-2\lambda\mu}) k r^2 \,\mathrm{sn}(rx)\,\mathrm{dn}(rx) \;, \tag{17}$$

$$w = 1 + \lambda r^2 (1 + 4k^2) - 6\lambda k^2 r^2 \,\mathrm{sn}^2(rx), \tag{18}$$

where sn, dn are the Jacobi elliptic functions with modulus k. This is a family of solution with two free parameters, r and k, which are related to the amplitude and period of the wave.

The parameter λ is constrained to be

$$\lambda = \frac{2k^2 - 1}{(1 + 8k^2 - 8k^4)r^2}, \tag{19}$$

while the phase mismatch parameter α is given by

$$\alpha = 2\mu r^2 (1 - 2k^2), \tag{20}$$

and thus will change sign at $k^2 = 1/2$.

Obviously this formulation demands $\lambda\mu < 0$, and thus dispersion coefficients of opposite signs. Families of solutions for the regime $\lambda\mu > 0$ will be reported in a future work. Furthermore, stability of these pulses is a delicate issue [20 – 24] which must be addressed using direct numerical simulations in a another future study.

5. Conclusions

Two classes of coherently coupled nonlinear Schrödinger equations, with either cubic or quadratic nonlinearity, have been solved in terms of products of the classical Jacobi elliptic functions. Hyperbolic, trigonometric and other special functions have been employed extensively in the exact and numerical solutions of nonlinear equations [32 – 34]. Nonlinear optical configurations, more precisely, polarization of light beams and solitons in quadratic media, have been employed here as illustrative examples. The usage of computer is important in the symbolic manipulation and the numerical schemes in stability studies. Whether this same set of techniques involving elliptic functions can be applied to other coherently coupled envelope equations [35, 36] will be subjects of future research.

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Figures captions

(1) Figure 1: Intensities of the waveguides A and B in (3) and (4), $|A|^2$ and $|B|^2$ versus x and t, $r = 0.2$, $\delta = 1/2$, $\sigma = 2/3$ (linear polarization), $k = 0.5$.

(2) Figure 2: Intensities of the double humped bright and dark pulses in waveguides A and B in (9) and (10), $|A|^2$ and $|B|^2$, versus x and t, $r = 0.09$, $\delta = 1/2$, $\sigma = 2/3$.

(3) Figure 3: Intensities of the double humped bright and dark pulses in waveguides A and B in (9) and (10), $|A|^2$ and $|B|^2$, versus x and t, $r = 0.15$, $\delta = 1/2$, $\sigma = 2/3$.

(4) Figure 4: Intensities of the double humped bright and dark pulses in waveguides A and B in (9) and (10), $|A|^2$ and $|B|^2$, versus x and t, $r = 0.2$, $\delta = 1/2$, $\sigma = 9/10$.

(5) Figure 5: Intensities of the double humped bright and dark pulses in waveguides A and B in (9) and (10), $|A|^2$ and $|B|^2$, versus x and t, $r = 0.2$, $\delta = 1/2$, $\sigma = 3/4$.

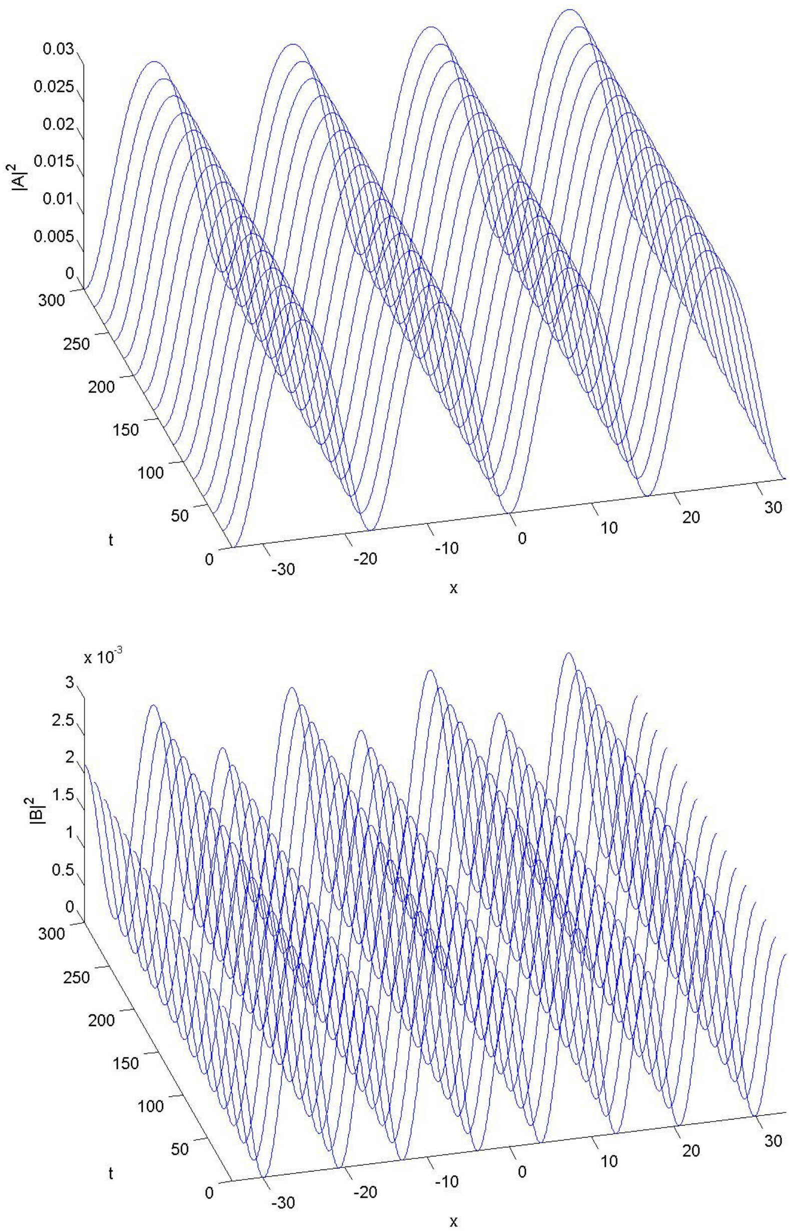

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Figure 1

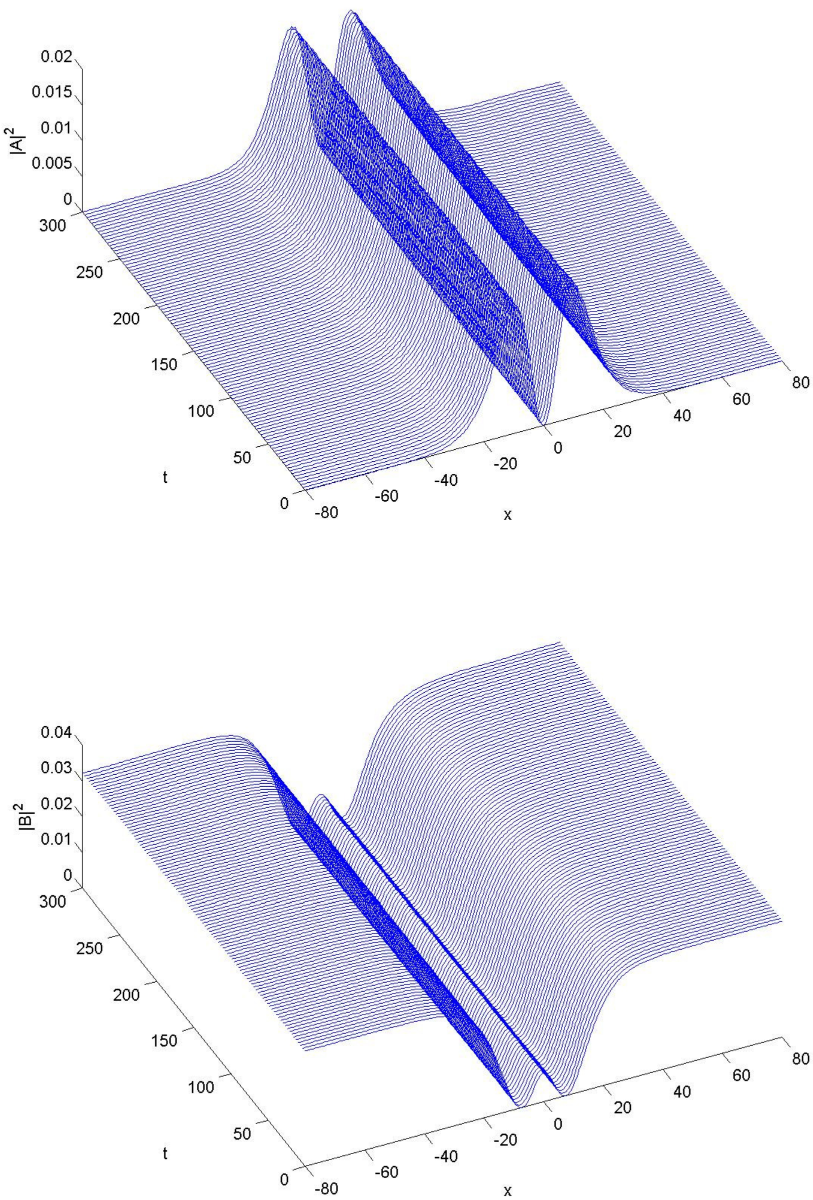

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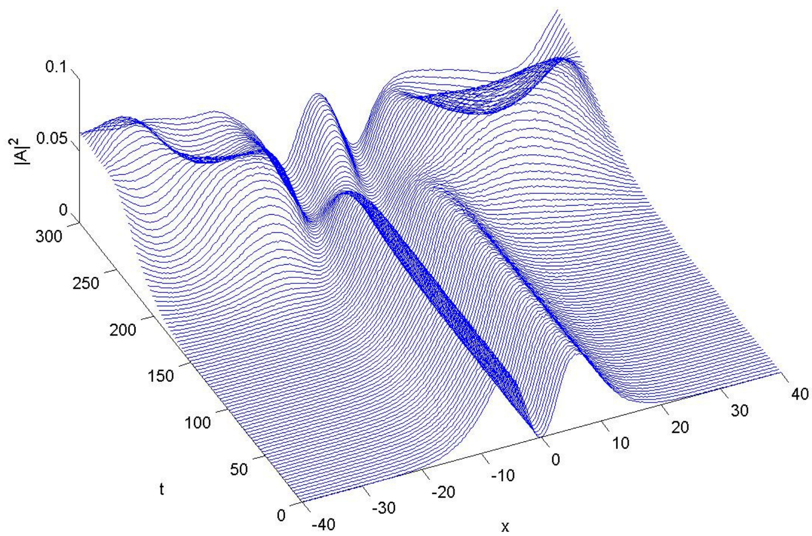
Figure 2



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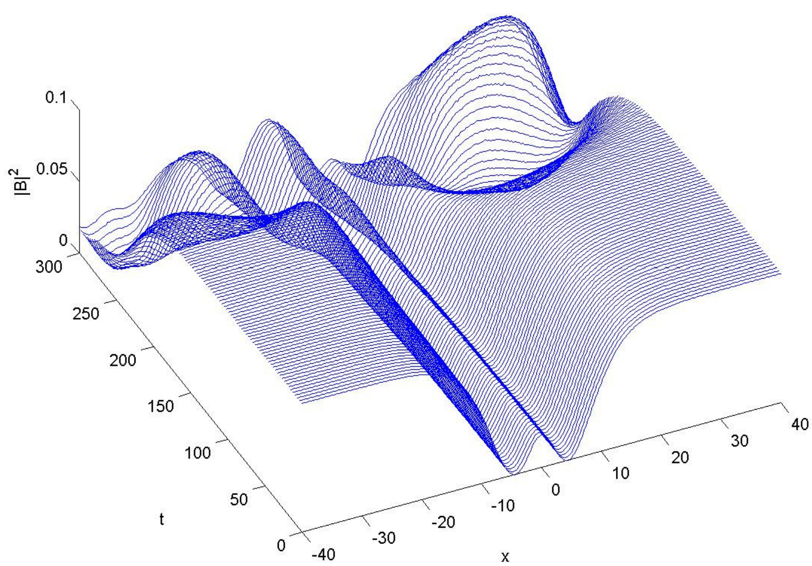

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Figure 3

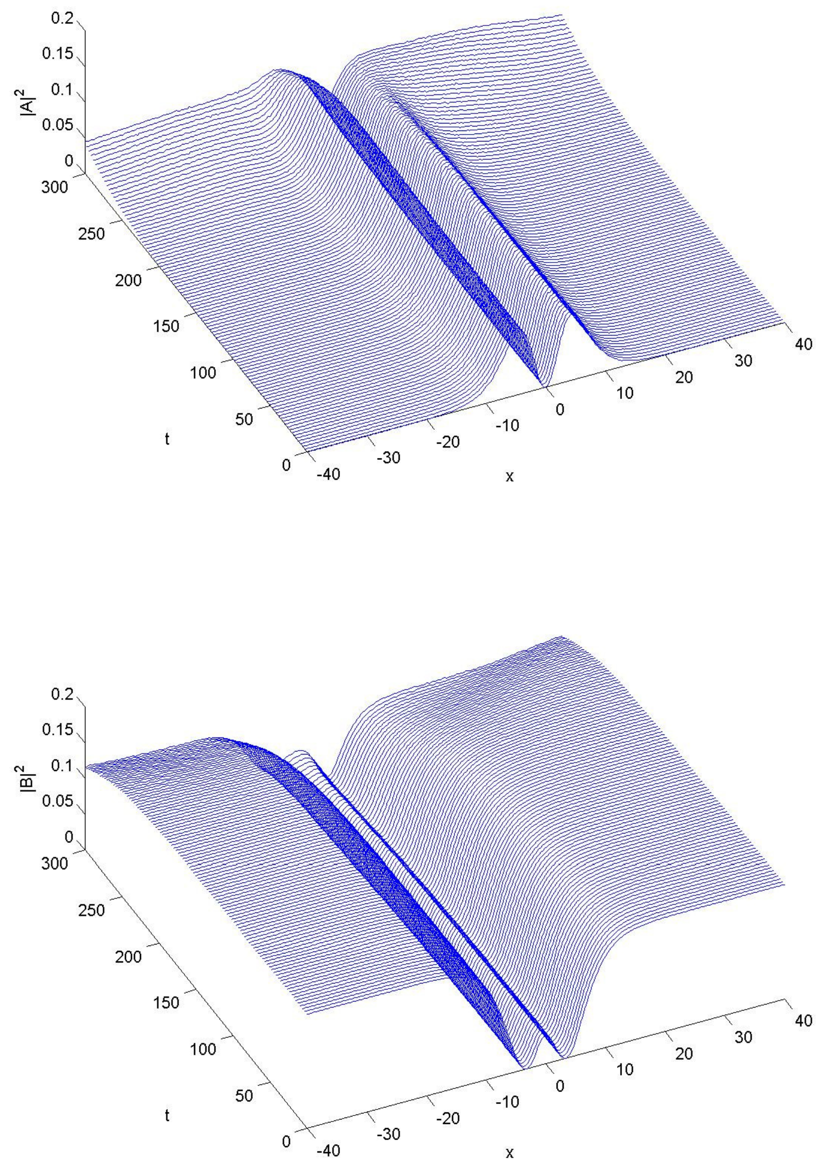

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Figure 4

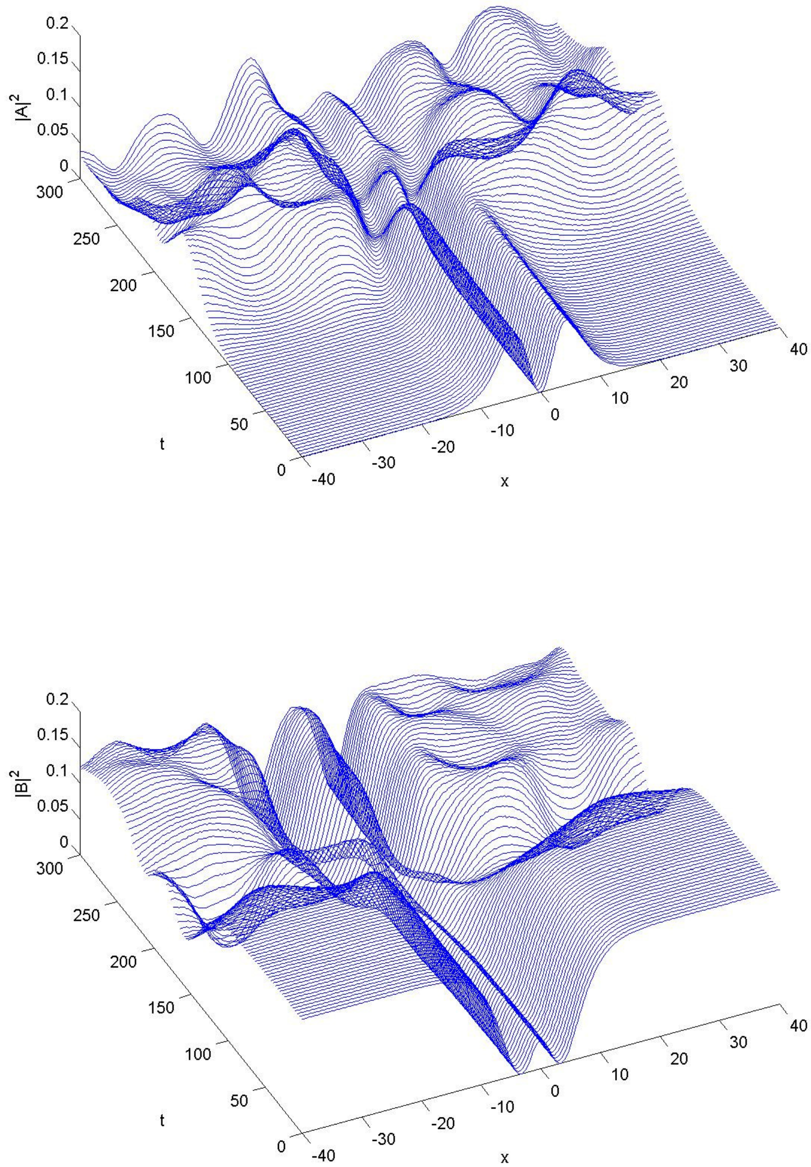
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Figure 5