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Continuous-time non-minimal state-space design

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A continuous time non-minimal state-space (NMSS) representation is shown to be explicitly related to the underlying minimal state-space observer/state feedback design method and, moreover, the corresponding state feedback gains are explicitly related. This result provides a starting point for NMSS methods in the continuous-time domain. Numerical examples are given which illustrate the underlying relationship.

1. Introduction

The state-space approach to control system design (see, for example, the textbooks of Kwakernaak and Sivan (1972) and Kailath (1980)) has a long history. In essence, a minimal state-space model of a system is used to design an output feedback controller with two parts: an observer which estimates the current system state and a (non-dynamic) state-feedback which gives the control signal in terms of the current state estimate. Both observer and state-feedback can be designed in a number of ways, including pole-placement and linear-quadratic optimization.

An alternative approach, introduced by Young *et al.* (1987) uses a particular non-minimal state-space (NMSS) representation which has the property that the state can be directly measured thus avoiding the use of an observer. The NMSS approach has been shown to give an observer-free implementation of model-based predictive control by Wang and Young (2006). Related NMSS approaches have been used by Jiang *et al.* (1996), Daams and Polderman (2002) and Hoagg and Bernstein (2004). Although much of this work has been in the discrete-time domain, both delta-operator (Young *et al.* 1991, 1998, Chotai *et al.* 1998) and continuous-time versions (Taylor *et al.* 1998) exist. The NMSS approach has been successfully applied to diverse practical problems (Ghavipanjeh *et al.* 2001, Quanten *et al.* 2003, Taylor and Shaban 2006).

The state-variable filter (SVF) concept has long been used in continuous-time system identification (Young 1965, 1966, 1981) and in this context corresponds to a NMSS representation. Moreover, continuous-time self-tuning control (Gawthrop 1982, 1987a,b) is based on the concept of an emulator (Gawthrop *et al.* 1996) which uses the SVF and thus is also another form of non-minimal state representation.

Taylor *et al.* (2000) give the relationship between the NMSS and conventional minimal state-space representation in the discrete-time case. The purpose of this paper is to investigate the same relationship, but in the continuous-time domain. This is not a trivial rewrite of the discrete-time result as, unlike the discrete-time case, filters are required to avoid pure derivatives. In particular, we show that conventional minimal-state observer/state feedback control for SISO systems is equivalent to a special case of the continuous-time NMSS of Taylor *et al.* (1998).

The outline of the paper follows. Section 2 provides some preliminary background and notation. Section 3 derives the NMSS representation from any minimal state-space representation using a sequence of state transformations. Section 4 considers state-feedback control and derives an explicit formula for the NMSS

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feedback gain vector in terms of the corresponding minimal-state feedback vector. Section 5 gives two numerical examples to illustrate the approach. Section 6 draws some conclusions and suggests future directions.

2. Preliminaries

This paper considers nth order SISO LTI systems with a transfer-function representation:

$$y = \frac{b(s)}{a(s)}u + \frac{c(s)}{a(s)}\xi \tag{1}$$

where y is the system output, u the control input and ξ a disturbance. The polynomials $a(s) - c(s)$ are of the form

$$a(s) = s^n + \sum_{k=1}^{n} a_k s^{n-k} \tag{2}$$

$$b(s) = \sum_{k=1}^{n} b_k s^{n-k} \tag{3}$$

$$c(s) = s^n + \sum_{k=1}^{n} c_k s^{n-k} \tag{4}$$

and it is assumed that $a(s)$ and $b(s)$ are coprime. In terms of these parameters, it is convenient to define the n-dimensional row vectors

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \tag{5}$$

$$\mathbf{b} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} \tag{6}$$

$$\mathbf{c} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \tag{7}$$

$$\mathbf{f} = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \end{bmatrix} = \mathbf{c} - \mathbf{a}. \tag{8}$$

There are two interpretations of the disturbance ξ. Firstly, it can be regarded as white noise and thus (1) represents a standard stochastic setup and, in particular $c(s)$ is part of the noise model. Secondly, $c(s)$ is chosen as an NMSS design parameter (called $T(s)$ in Taylor *et al.* (1998)) and thus ξ is the corresponding disturbance signal with no particular stochastic interpretation. The latter approach is more pragmatic and is discussed by Taylor *et al.* (1998), Gawthrop *et al.* (1996) and Gawthrop (1987a).

Defining

$$\hat{y} = y - \xi \tag{9}$$

equation (1) can be rewritten as

$$\hat{y} = \frac{b(s)}{a(s)}u + \frac{f(s)}{a(s)}\xi. \tag{10}$$

Equation (10) can be further rewritten as

$$\frac{c(s)}{a(s)}\hat{y} = \frac{b(s)}{a(s)}u + \frac{f(s)}{a(s)}y, \tag{11}$$

where

$$f(s) = c(s) - a(s). \tag{12}$$

Combining (9) and (11) gives

$$\begin{cases} \hat{y} = \dfrac{b(s)}{c(s)}u + \dfrac{f(s)}{c(s)}y \\ y = \hat{y} + \xi. \end{cases} \tag{13}$$

Equation (13) is an alternative way of writing (1); in the particular case that ξ is white noise, it is related to the innovations (Kailath 1970) representation of the system (1).

In fact, it is possible to rewrite the representation (13) in state-variable filter (SVF) form as

$$\hat{\mathbf{x}}_{sy} = \begin{bmatrix} s^{n-1} \\ s^{n-2} \\ \cdots \\ 1 \end{bmatrix} y_f \tag{14}$$

$$\hat{\mathbf{x}}_{su} = \begin{bmatrix} s^{n-1} \\ s^{n-2} \\ \cdots \\ 1 \end{bmatrix} u_f, \tag{15}$$

where

$$y_f = \frac{1}{c(s)}y \tag{16}$$

$$u_f = \frac{1}{c(s)}u. \tag{17}$$

That is $\hat{\mathbf{x}}_{sy}$ contains filtered derivatives of y and $\hat{\mathbf{x}}_{su}$ contains filtered derivatives of u where the filter is defined by $c(s)$. Equations (14) and (15) correspond to the standard state-variable filter (SVF) and together, the $2n$ SVF states form the NMSS representation. In particular, (13) can be rewritten as

$$\hat{y} = \mathbf{f}\hat{\mathbf{x}}_{sy} + \mathbf{b}\hat{\mathbf{x}}_{su}. \tag{18}$$

Equation (1) has many minimal state-space representations of the form

$$\begin{cases} \frac{d\mathbf{x}}{dt}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{B}_\xi \xi \\ y(t) = \mathbf{C}\mathbf{x}(t) + \xi \\ \mathbf{x}(0) = \mathbf{x}_0, \end{cases} \tag{19}$$

where the state x has dimension n and x_0 is the initial state.

The standard state observer (Kwakernaak and Sivan 1972, Kailath 1980, Goodwin *et al.* 2001) for such a system is

$$\begin{cases} \frac{d\hat{\mathbf{x}}}{dt}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}[y(t) - \hat{y}(t)] \\ \hat{y}(t) = \mathbf{C}\hat{\mathbf{x}}(t) \\ \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0, \end{cases} \tag{20}$$

where $\mathbf{L}$ is the observer gain. $\mathbf{L}$ determines the observer poles (the eigenvalues of $\mathbf{A} - \mathbf{LC}$) and can be chosen in a number of ways including pole-placement and LQG optimization (Kwakernaak and Sivan 1972, Kailath 1980, Goodwin *et al.* 2001).

3. The NMSS representation

The NMSS representation is constructively derived from the minimum-state representation in three stages.

(i) The state of the system in minimal-state-space form (19) is transformed into observer canonical form and, using superposition, split into two separate states (§ 3.1).
(ii) Each of the new states is separately transformed into controller canonical form and seen to be equivalent to the two SVFs of (14) and (15) (§ 3.2).
(iii) The two states are combined in to a single $2n$ dimensional non-minimal state which is shown to arise from the system equations of Taylor *et al.* (1998) (§ 3.3).

3.1 *Observer canonical form*

Following Kailath (1980, § 2.1), consider the system (19) in observer canonical form

$$\begin{cases} \frac{d\mathbf{x}_o}{dt}(t) = \mathbf{A}_o\mathbf{x}_o(t) + \mathbf{B}_o u(t) + \mathbf{B}_{o\xi}\xi \\ y(t) = \mathbf{C}_o\mathbf{x}_o(t) + \xi \\ \mathbf{x}_o(0) = \mathbf{x}_{o0} \end{cases} \tag{21}$$

where

$$\mathbf{A}_o = \begin{bmatrix} & | & \mathbf{I} \\ -\mathbf{a}^T & | & \vdots \\ & | & \mathbf{z} \end{bmatrix} \tag{22}$$

$$= \begin{bmatrix} -a_1 & | & 1 & 0 & \cdots & 0 \\ -a_2 & | & 0 & 1 & \cdots & 0 \\ \vdots & | & \vdots & \vdots & \vdots & \vdots \\ -a_{n-1} & | & 0 & 0 & \cdots & 1 \\ -a_n & | & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{23}$$

$$\mathbf{B}_o = \mathbf{b}^T = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{24}$$

$$\mathbf{B}_{o\xi} = \mathbf{f}^T = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \tag{25}$$

$$\mathbf{C}_o = \mathbf{o} \tag{26}$$

and $\mathbf{I}$ is the $(n-1) \times (n-1)$ unit matrix, $\mathbf{z}$ is the $(n-1)$ dimensional zero row vector and $\mathbf{o}$ is the the n-dimensional row vector

$$\mathbf{o} = \begin{bmatrix} 1 & | & \mathbf{z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{27}$$

The initial condition $\mathbf{x}_o$ is given by

$$\mathbf{x}_o = \mathcal{T}_o \mathbf{x}, \tag{28}$$

where the transformation matrix $\mathcal{T}_o$ is given by

$$\mathcal{T}_o = \mathcal{C}_o \mathcal{C}^{-1} \tag{29}$$

where the two $n \times n$ controllability matrices $\mathcal{C}_o$ and $\mathcal{C}$ are given by

$$\mathcal{C}_o = \begin{bmatrix} \mathbf{B}_o & | & \mathbf{A}_o\mathbf{B}_o & | & \cdots & | & \mathbf{A}_o^{n-1}\mathbf{B}_o \end{bmatrix} \tag{30}$$

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & | & \mathbf{AB} & | & \cdots & | & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}. \tag{31}$$

It is a standard result (Kwakernaak and Sivan 1972, Kailath 1980), based on the special structure of (21), that the corresponding state observer, with initial state $\hat{\mathbf{x}}_{o0}$ and poles corresponding to the roots of $c(s)$ can be written as

$$\begin{cases} \frac{d\hat{\mathbf{x}}_o}{dt}(t) = \boldsymbol{\Gamma}_o\hat{\mathbf{x}}_o(t) + \mathbf{B}_o u(t) + \mathbf{f}^T y(t) \\ \hat{y}(t) = \mathbf{C}_o\hat{\mathbf{x}}_o(t) \\ \hat{\mathbf{x}}_o(0) = \hat{\mathbf{x}}_{o0}, \end{cases} \tag{32}$$

where

$$\boldsymbol{\Gamma}_o = \mathbf{A}_o - \mathbf{f}^T\mathbf{C}_o \tag{33}$$

$$= \left[\begin{array}{c|c} & \mathbf{I} \\ -\mathbf{c}^T & \vdots \\ & \mathbf{z} \end{array}\right] \tag{34}$$

$$= \left[\begin{array}{c|cccc} -c_1 & 1 & 0 & \cdots & 0 \\ -c_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -c_{n-1} & 0 & 0 & \cdots & 1 \\ -c_n & 0 & 0 & \cdots & 0 \end{array}\right] \tag{35}$$

and the initial state $\hat{\mathbf{x}}_{o0}$ is chosen to correspond to $\hat{\mathbf{x}}_0$ by choosing

$$\hat{\mathbf{x}}_{o0} = \mathcal{T}_o\hat{\mathbf{x}}_0. \tag{36}$$

Using superposition (the initial state has been assigned to $\hat{\mathbf{x}}_{oy}$, it could equally well have been assigned to $\hat{\mathbf{x}}_{ou}$), (32) can be rewritten in non-minimal state-space form as

$$\begin{cases} \dfrac{\mathrm{d}\hat{\mathbf{x}}_{oy}}{\mathrm{d}t}(t) = \boldsymbol{\Gamma}_o\hat{\mathbf{x}}_{oy}(t) + \mathbf{f}^T y(t) \\ \hat{y}_{oy}(t) = \mathbf{C}_o\hat{\mathbf{x}}_{oy}(t) \\ \hat{\mathbf{x}}_{oy}(0) = \hat{\mathbf{x}}_{o0} \end{cases} \tag{37}$$

$$\begin{cases} \dfrac{\mathrm{d}\hat{\mathbf{x}}_{ou}}{\mathrm{d}t}(t) = \boldsymbol{\Gamma}_o\hat{\mathbf{x}}_{ou}(t) + \mathbf{B}_o u(t) \\ \hat{y}_{ou}(t) = \mathbf{C}_o\hat{\mathbf{x}}_{ou}(t) \\ \hat{\mathbf{x}}_{ou}(0) = \mathbf{0}, \end{cases} \tag{38}$$

where $\mathbf{0}$ is the n dimensional zero column vector and the non-minimal state is the $2n$-dimensional column vector

$$\hat{\mathbf{x}}_{on} = \begin{bmatrix} \hat{\mathbf{x}}_{oy} \\ \hat{\mathbf{x}}_{ou} \end{bmatrix}. \tag{39}$$

It is clear that (with correct initial conditions)

$$\hat{\mathbf{x}}_o = \hat{\mathbf{x}}_{oy} + \hat{\mathbf{x}}_{ou} \tag{40}$$

$$\hat{y}_o = \hat{y}_{oy} + \hat{y}_{ou}. \tag{41}$$

Equations (37) and (38) have no particular interest in themselves; but form the basis for a NMSS in controller form in the next section.

3.2 *Controller canonical form*

Following Kailath (1980), the dual of the observer canonical form is the controller canonical form. Dualizing the two equations (37) and (38) of the NMSS observer separately gives

$$\begin{cases} \dfrac{\mathrm{d}\hat{\mathbf{x}}_{cy}}{\mathrm{d}t}(t) = \boldsymbol{\Gamma}_c\hat{\mathbf{x}}_{cy}(t) + \mathbf{B}_c y(t) \\ \hat{y}_{cy}(t) = \mathbf{f}\hat{\mathbf{x}}_{cy}(t) \\ \hat{\mathbf{x}}_{cy}(0) = \hat{\mathbf{x}}_{c0} \end{cases} \tag{42}$$

$$\begin{cases} \dfrac{\mathrm{d}\hat{\mathbf{x}}_{cu}}{\mathrm{d}t}(t) = \boldsymbol{\Gamma}_c\hat{\mathbf{x}}_{cu}(t) + \mathbf{B}_c u(t) \\ \hat{y}_{cu}(t) = \mathbf{b}\hat{\mathbf{x}}_{cu}(t) \\ \hat{\mathbf{x}}_{cu}(0) = \mathbf{0} \end{cases} \tag{43}$$

$$\hat{y}_c(t) = \hat{y}_{cy}(t) + \hat{y}_{cu}(t), \tag{44}$$

where

$$\boldsymbol{\Gamma}_c = \boldsymbol{\Gamma}_o^T \tag{45}$$

$$= \begin{bmatrix} & -\mathbf{c} & \\ \cdots & \cdots & \cdots \\ \mathbf{I} & | & \mathbf{z}^T \end{bmatrix} \tag{46}$$

$$= \begin{bmatrix} -c_1 & -c_2 & \cdots & -c_{n-1} & -c_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{47}$$

$$\mathbf{B}_c = \mathbf{C}_o^T = \mathbf{o}^T = \begin{bmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix}. \tag{48}$$

Equations (42) and (43) are the state-space realizations of the SVF equations (14) and (15) respectively.

Once again, following Kailath (1980), the states in (42) and (43) can be expressed as linear transformations of the states of (37) and (38) respectively as

$$\hat{\mathbf{x}}_{cy} = \mathcal{T}_y\hat{\mathbf{x}}_{oy} \tag{49}$$

$$\hat{\mathbf{x}}_{cu} = \mathcal{T}_u\hat{\mathbf{x}}_{ou}, \tag{50}$$

where the transformation matrices $\mathcal{T}_y$ and $\mathcal{T}_u$ are respectively given by

$$\mathcal{T}_y = \mathcal{C}_c\mathcal{C}_{oy}^{-1} \tag{51}$$

$$\mathcal{T}_u = \mathcal{C}_c\mathcal{C}_{ou}^{-1}, \tag{52}$$

where the three $n \times n$ controllability matrices $\mathcal{C}_c$–$\mathcal{C}_{ou}$ are given by

$$\mathcal{C}_c = \begin{bmatrix} \mathbf{o}^T & | & \boldsymbol{\Gamma}_c \mathbf{o}^T & | & \cdots & | & \boldsymbol{\Gamma}_c^{n-1} \mathbf{o}^T \end{bmatrix} \tag{53}$$

$$\mathcal{C}_{oy} = \begin{bmatrix} \mathbf{f}^T & | & \boldsymbol{\Gamma}_o \mathbf{f}^T & | & \cdots & | & \boldsymbol{\Gamma}_o^{n-1} \mathbf{f}^T \end{bmatrix} \tag{54}$$

$$\mathcal{C}_{ou} = \begin{bmatrix} \mathbf{b}^T & | & \boldsymbol{\Gamma}_o \mathbf{b}^T & | & \cdots & | & \boldsymbol{\Gamma}_o^{n-1} \mathbf{b}^T \end{bmatrix}. \tag{55}$$

In particular, the initial state $\hat{\mathbf{x}}_{c0}$ can be written as

$$\hat{\mathbf{x}}_{c0} = \mathcal{T}_y \hat{\mathbf{x}}_{o0}. \tag{56}$$

Equations (42) and (43) are an alternative NMSS representation and the corresponding state is

$$\mathbf{X} = \begin{bmatrix} \hat{\mathbf{x}}_{cy} \\ \hat{\mathbf{x}}_{cu} \end{bmatrix}. \tag{57}$$

Unlike the state of (37) and (38), the state $\mathbf{X}$ is independent of system parameters.

Theorem 1: *The NMSS state* $\mathbf{X}$ of (57) *is related to the minimal state* $\mathbf{x}$ *of* (19) *by*

$$\mathbf{X} = \begin{bmatrix} \mathcal{T}_y \\ \mathcal{T}_u \end{bmatrix} \mathcal{T}_o \mathbf{x}. \tag{58}$$

Proof: Equation (58) follows from (28), (49) and (50). □

3.3 *Non-minimal state-space representation*

Equations (42) and (43) give a non-minimal state-space representation of the the system (19). However, due to the presence of the system output y on the right-hand side of (42), these equations cannot be directly used for conventional state feedback design. However, $y = \hat{y}_c$ can be eliminated from the right-hand side of (42) using (44) to give

$$\frac{\mathrm{d}\hat{\mathbf{x}}_{cy}}{\mathrm{d}t}(t) = \boldsymbol{\Gamma}_c \hat{\mathbf{x}}_{cy}(t) + \mathbf{o}^T \big(\mathbf{f}\hat{\mathbf{x}}_{cy}(t) + \mathbf{b}\hat{\mathbf{x}}_{cu}(t)\big) + \mathbf{o}^T \xi(t) \tag{59}$$

$$= \mathbf{A}_c \hat{\mathbf{x}}_{cy}(t) + \mathbf{o}^T \mathbf{b} \hat{\mathbf{x}}_{cu}(t) + \mathbf{o}^T \xi(t). \tag{60}$$

Combining (59) and (43) gives

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}(t) = \mathbf{F}\mathbf{X}(t) + \mathbf{g}u(t) + \mathbf{O}^T \xi(t) \\ y(t) = \mathbf{h}\mathbf{X}(t) + \xi(t) \\ \mathbf{X}(0) = \hat{\mathbf{x}}_{cn0} = \begin{pmatrix} \hat{\mathbf{x}}_{cy0} \\ \mathbf{0} \end{pmatrix} \end{cases} \tag{61}$$

where

$$\mathbf{F} = \begin{bmatrix} \mathbf{A}_c & \mathbf{o}^T \mathbf{b} \\ 0 & \boldsymbol{\Gamma}_c \end{bmatrix} \tag{62}$$

$$= \begin{bmatrix}
-a_1 & -a_2 & \cdots & -a_{n-1} & -a_n & b_1 & b_2 & \cdots & b_{n-1} & b_n \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & -c_1 & -c_2 & \cdots & -c_{n-1} & -c_n \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0
\end{bmatrix} \tag{63}$$

$$\mathbf{g} = \begin{bmatrix} \mathbf{0} \\ \mathbf{z} \\ \mathbf{o} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{64}$$

and $\mathbf{O} = \begin{bmatrix} \mathbf{o} & \mathbf{0} \end{bmatrix}$ and $\mathbf{h} = \begin{bmatrix} \mathbf{f} & \mathbf{b} \end{bmatrix}$.

Equation (61) corresponds to Taylor *et al.* (1998, (3) and (4)). It is a non-minimal system state representation of (1) which has the important property that the state $\mathbf{X}$ is directly measurable.

4. State-feedback control

The control signal u_m using standard state feedback with minimal realization is

$$u_m(t) = -\mathbf{k}\hat{\mathbf{x}}(t), \tag{65}$$

where $\mathbf{k}$ is the $1 \times n$ state-feedback matrix and $\hat{\mathbf{x}}$ the state estimate generated from observer (20). The corresponding NMSS feedback control signal u_n is

$$u_n(t) = -\mathbf{K}\mathbf{X}(t), \tag{66}$$

where $\mathbf{K}$ is the $1 \times 2n$ state-feedback matrix and $\mathbf{X}$ the NMSS state (57).

The following theorem gives the conditions under which u_m (65) and u_n (66) are the same.

Theorem 2: *If the NMSS feedback gain vector is given by*

$$\mathbf{K} = \mathbf{k}\mathcal{T}_o^{-1}\begin{bmatrix} \mathcal{T}_y^{-1} & \mathcal{T}_u^{-1} \end{bmatrix} \tag{67}$$

and the initial NMSS state is

$$X(0) = X_0 = \begin{bmatrix} \mathcal{T}_y\mathcal{T}_o\hat{\mathbf{x}}_0 \\ \mathbf{0} \end{bmatrix}, \tag{68}$$

where the transformation matrices $\mathcal{T}_y$, $\mathcal{T}_u$ and $\mathcal{T}_o$ are given by equations (51), (52) *and* (29) *respectively*, *then* $u_m = u_n$.

Proof: Using the results of § 3, (68) ensures that

$$\mathbf{X} = \begin{bmatrix} \hat{\mathbf{x}}_{cy} \\ \hat{\mathbf{x}}_{cu} \end{bmatrix} = \begin{bmatrix} \mathcal{T}_y\hat{\mathbf{x}}_{oy} \\ \mathcal{T}_u\hat{\mathbf{x}}_{ou} \end{bmatrix} = \begin{bmatrix} \mathcal{T}_y\mathcal{T}_o\hat{\mathbf{x}} \\ \mathcal{T}_u\mathcal{T}_o\hat{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \mathcal{T}_y \\ \mathcal{T}_u \end{bmatrix}\mathcal{T}_o\,\hat{\mathbf{x}}. \tag{69}$$

Moreover (67), (49), (50) and (28) imply that:

$$\begin{aligned} \mathbf{K}\mathbf{X} &= \mathbf{k}\mathcal{T}_o^{-1}\begin{bmatrix} \mathcal{T}_y^{-1}\mathcal{T}_u^{-1} \end{bmatrix}\mathbf{X} \\ &= \mathbf{k}\mathcal{T}_o^{-1}\begin{bmatrix} \mathcal{T}_y^{-1} & \mathcal{T}_u^{-1} \end{bmatrix}\begin{bmatrix} \mathcal{T}_y & \mathcal{T}_u \end{bmatrix}\mathcal{T}_o\mathbf{x} = \mathbf{k}\mathbf{x}. \end{aligned} \tag{70}$$

□

5. Examples

The main results of this paper are illustrated by applying LQ control to two example systems. In each case, the minimal-state-feedback vector $\mathbf{k}$ is obtained from the minimization of the standard infinite horizon LQ regulation cost function with output weighting:

$$J_m = \int_{t=0}^{\infty} y^2(t) + \lambda u^2(t)\mathrm{d}t = \int_{t=0}^{\infty} \mathbf{x}(t)^T\mathbf{C}^T\mathbf{C}\mathbf{x}(t) + \lambda u^2(t)\mathrm{d}t \tag{71}$$

and the NMSS state-feedback vector $\mathbf{K}_{lq}$ from the minimization of

$$J_n = \int_{t=0}^{\infty} \mathbf{X}^T(t)\mathbf{h}^T\mathbf{h}\mathbf{X}(t) + \lambda u^2(t)\mathrm{d}t \tag{72}$$

As the two cost functions are the same, it must be true that $\mathbf{K}_{lq} = \mathbf{K}$ where $\mathbf{K}$ is obtained from $\mathbf{k}$ and (67).

5.1 *Example 1: Simple integrator*

Consider the simple integrator written in the form of (1) with

$$a(s) = s \tag{73}$$

$$b(s) = 1 \tag{74}$$

$$c(s) = s + 1. \tag{75}$$

The state-space representation is in the form of (19) with

$$\mathbf{A} = 0;\ \ \mathbf{B} = 1;\ \ \mathbf{C} = 1. \tag{76}$$

In this case, the transformation matrices of (29), (49), and (50) become:

$$\mathcal{T}_o = \mathcal{T}_y = \mathcal{T}_u = 1. \tag{77}$$

It follows from Theorem 2 that

$$\mathbf{K} = \begin{bmatrix} \mathbf{k} & \mathbf{k} \end{bmatrix}. \tag{78}$$

The minimal-state LQ optimization (71) was solved with the control weight $\lambda = 0.01$ to give the optimum gain $\mathbf{k} = 10$. The NMSS LQ optimization (72) was solved with the same weight and gave $\mathbf{K} = [10 \quad 10]$ thus confirming (78).

The closed-loop pole for the minimal state-feedback controller (eigenvalue of $\mathbf{A} - \mathbf{Bk}$) is $s = -10$, the closed-loop poles for the NMSS state-feedback controller (eigenvalues of $\mathbf{F} - \mathbf{gK}$) are at $s = -10$, $s = -1$; the first corresponds to the minimal state-feedback controller, the latter to the root of $c(s) = 0$.

5.2 *Example 2: Third-order system*

Consider the third-order integrated oscillator written in the form of (1) with

$$a(s) = s^3 + s = s(s + j)(s - j) \tag{79}$$

$$b(s) = 1 \tag{80}$$

$$c(s) = s^3 + 3s^2 + 3s + 1 = (s+1)^3. \tag{81}$$

The particular state-space representation

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \tag{82}$$

was chosen. In this case, the transformation matrices of (29), (49), and (50) become

$$\mathcal{T}_o = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \tag{83}$$

$$\mathcal{T}_y = \begin{bmatrix} 0.25 & 0.25 & -0.25 \\ 0.25 & -0.25 & -0.25 \\ -0.25 & -0.25 & 1.25 \end{bmatrix} \tag{84}$$

$$\mathcal{T}_u = \begin{bmatrix} 6 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{85}$$

The minimal-state LQ optimization (71) was solved with the control weight $\lambda = 0.01$ to give the optimum gain

$$\mathbf{k} = \begin{bmatrix} 4 & 8 & 10 \end{bmatrix} \tag{86}$$

It follows from Theorem 2 that

$$\mathbf{K} = \begin{bmatrix} 38 & -4 & 10 & 4 & 20 & 42 \end{bmatrix}. \tag{87}$$

The NMSS LQ optimization (72) was solved with the same weight to give a value of **K** identical to (87).

The closed-loop poles for the minimal state-feedback controller (eigenvalue of $\mathbf{A} - \mathbf{Bk}$) are $s = -2$, $s = -1 \pm 2j$, there are six closed-loop poles for the NMSS state-feedback controller (eigenvalues of $\mathbf{F} - \mathbf{gK}$), three corresponding to the minimal state-feedback controller and three at $s = -1$ corresponding to the roots of $c(s) = 0$.

6. Conclusion

The paper has shown that a particular continuous-time non-minimal state-space (NMSS) representation can be explicitly expressed in terms of a minimal state-space representation and thus, in this sense, the NMSS and minimal-state representations are equivalent.

The practical implication of this result is that, in the circumstances considered in this paper, the choice of implementation is a matter of convenience rather than performance. An example of this is self-tuning control where the NMSS is required for linear-in-the-parameters system identification.

There are, however, NMSS representations not considered in this paper. One example involves the case where the s^n term does not appear in (4) so that the degree of $c(s)$ is $n-1$ rather than n. This case is easily handled in the NMSS setup and yields controller with a direct link from y to u. On the other hand, the minimal state-space approach is more complicated and requires the construction of a reduced-order observer. It is an open question as to whether the resulting controllers are again equivalent. This question,and those relating to other versions of NMSS, are the subject of ongoing research.

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