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Luenberger boundary observer synthesis for Sturm–Liouville systems

D. Vries[ab], K.J. Keesman[a*] and H. Zwart[c]

[a]*Systems and Control Group, Wageningen University, PO Box 17, 6700 AA Wageningen, The Netherlands;*
[b]*Water Technology Group, KWR Watercycle Research Institute, Groningenhaven 7, 3433 PE Nieuwegein, The Netherlands;*
[c]*Faculty of EEMCS, Department of Applied Mathematics, University of Twente, PO Box 217, 7500 AE Enschede, The Netherlands*

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A static Luenberger observer of a system with Sturm–Liouville operator is synthesised with the aid of a boundary control formulation. To this aim, approximate observability, detectability and stability of the system is studied and design results are worked out for a typical biochemical case study.

Keywords: boundary control system; observer synthesis; Luenberger observer; distributed parameter system; Sturm–Liouville system

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1. Introduction

In this article, the observer synthesis of a typical bilinear system with *point* measurements and *boundary control* actions is studied. As far as the authors know, the literature about observers using point measurements is scarce.

The motivation is to treat the observer design of a convection–diffusion–reaction (CDR) system with infinite-dimensional system theory concepts. Since the observer correction will be formulated at the boundary, the theoretical framework developed for boundary control (Fattorini 1968; Emirsajlow and Townley 2000) suffices and provides an elegant and mathematically simple approach for observer design as well. As far as we know, there has been little attention to apply this theory to CDR type of problems where boundary or point measurements are used for observations. Instead, one usually considers a control or observation on a small interval $[0, w]$ for the analysis of CDR systems (see e.g. Xu, Ligarius, and Gauthier 1995; Winkin, Dochain, and Ligarius 2000; Delattre, Dochain, and Winkin 2004).

With point measurements and boundary control in mind, we make the following choice regarding the observer design: *the output estimation error* $(\hat{y} - y)$ *is manipulated by the observer at the boundary of the error system*. Such an observer will be referred to as a *boundary* observer.

For the moment, the boundary observer design problem with given inputs is considered. To clarify the idea, the bilinear system and its observer is introduced in abstract boundary control form:

$$\Sigma := \begin{cases} \dot{z}(t) = \mathfrak{A}z(t) - q(t)z(t); \quad z(0) = z_0 \\ \mathfrak{B}z(t) = u(t) + v_1(t) \\ \mathfrak{C}z(t) = y(t) + v_2(t). \end{cases} \tag{1}$$

And, similar to Equation (1), we define the observer system as

$$\Sigma^{\text{obs}} := \begin{cases} \dot{\hat{z}}(t) = \mathfrak{A}\hat{z}(t) - q(t)\hat{z}(t); \quad \hat{z}(\eta, 0) = \hat{z}_0, \\ \mathfrak{B}\hat{z}(t) = u(t) + \mathbf{L}\mathfrak{C}(z(t) - \hat{z}(t)), \\ \mathfrak{C}\hat{z}(t) = \hat{y}(t), \end{cases} \tag{2}$$

with $z, \hat{z}$ in the Hilbert space Z, differential operator $\mathfrak{A}$ with $\mathfrak{A} : D(\mathfrak{A}) \subset Z \mapsto Z$ and $D(\cdot)$ denoting the domain of an operator. Furthermore, we have the integrable scalar (possibly manipulated) variable $q \in \mathbb{R}$, and also the vector of boundary inputs $u \in U$. For simplicity, the problem is considered on a one-dimensional spatial domain, i.e. $U \subset \partial\Omega = \mathbb{R}^2$ where $\partial\Omega$ denotes the boundary of the spatial domain Ω. Here, $\Omega = [\eta_1, \eta_2]$ with spatial coordinates $\eta_1, \eta_2 \in \mathbb{R}$. The boundary control operator $\mathfrak{B}$ and boundary observation operator $\mathfrak{C}$ should be interpreted in the sense of Definition 3.3.2 in Curtain and Zwart (1995), where $\mathfrak{B} : D(\mathfrak{B}) \subset Z \mapsto U$ satisfies $D(\mathfrak{A}) \subset D(\mathfrak{B})$ and $\mathfrak{C} : D(\mathfrak{C}) \subset Z \mapsto \mathbb{R}^q$, $q \geq 1$. To comply with the condition in their definition, it is further assumed that A is given by

$$Az = \mathfrak{A}z \quad \text{for } z \in D(A) = D(\mathfrak{A}) \cap \ker(\mathfrak{B}) \tag{3}$$

*Corresponding author. Email: karel.keesman@wur.nl

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and generates a C_0-semigroup.[1] The specification of $\mathbf{L}$ is postponed to Section 3.

This article is organised as follows. In Section 2, some characteristics with respect to Σ as defined in Equation (1) are specified, including the characterisation of A as a Sturm–Liouville (S-L)-type system operator. The concept of approximate observability and detectability for this class of boundary control systems is introduced in Section 3. In Section 4, the results are illustrated by an observer design for a UV disinfection process modelled by a CDR partial differential equation with boundary measurements and under boundary control action. Some final remarks and conclusions are given in Section 5.

2. System preliminaries

The following theorem shows that Σ, as in Equation (1), has a mild solution.

2.1 *Mild solution*

Theorem 2.1: *For Equation* (1), *where A as in Equation* (3), *there exists a mild solution, with mild solution operator $U(t,s)z_0 = T(t-s)e^{\int_s^t b_1 u_1(\tau)\mathrm{d}\tau} z_0$, where T is the C_0-semigroup generated by* $(A, D(A))$.

Proof: The proof originates from the work of Jean Bernoulli on ordinary differential equations for the scalar case. First, let $\mathcal{L}(Z)$ be a shorthand notation for a bounded linear operator from Z to Z. Furthermore, write $z = vw$, then with $v = T(t-s)v_0$ and with z subject to $\dot{z} - Az = b_1 u_1 z$, we get $v\dot{w} = b_1 u_1 vw$. It follows that $w = w_0(\exp \int_s^t b_1 u_1(\tau)\mathrm{d}\tau)$. Substituting $z_0 = v_0 w_0$ gives the result. Further, according to Definition 3.2.4 of Curtain and Zwart (1995), $U(t,s): \Lambda(\tau) \to \mathcal{L}(Z)$ is a mild solution operator with $\Lambda(\tau) = \{(t,s); 0 \le s \le t \le \tau\}$, since:

(a) $U(s,s) = I$, $s \in [0,\tau]$ holds,

(b) A is an infinitesimal generator of a C_0-semigroup, hence:

$$\begin{aligned} U(t,r)U(r,s)z_0 &= T(t-r)e^{\int_r^t b_1 u_1(\tau)\mathrm{d}\tau} T(r-s)e^{\int_s^r b_1 u_1(\tau)\mathrm{d}\tau} z_0 \\ &= T(t,s)e^{\int_r^t b_1 u_1(\tau)\mathrm{d}\tau + \int_s^r b_1 u_1(\tau)\mathrm{d}\tau} z_0, \end{aligned}$$

which equals $U(t,s)$, $0 \le s \le t \le \tau$,

(c) it is standard to show that $U(\cdot,s)$ is strongly continuous on $[s,\tau]$ and that $U(t,\cdot)$ is strongly continuous on $[0,t]$. □

2.2 *The operator A*

A in Σ is defined as an S-L operator and we summarise some properties of A. As a consequence, $\Sigma(A)$ with A, a S-L operator, will be denoted as $\Sigma_{S.L.}(A)$.

As also pointed out in Delattre, Dochain, and Winkin (2003), in many physical systems (e.g. vibration/diffusion problems or convection–dispersion in chemical reactor models) A or $-A$ is an S-L operator. As such, we are motivated to inspect the properties of A being an S-L type.

The differential operator in Equation (1) is written as

$$\mathfrak{A}z = \frac{1}{w}\left(\frac{\mathrm{d}}{\mathrm{d}\eta}\left(p\frac{\mathrm{d}z}{\mathrm{d}\eta}\right) - qz\right), \tag{4a}$$

with

$$p(\eta), w(\eta) \in \mathbb{R}_+, \text{ both } C^1 \text{ functions} \tag{4b}$$

and

$$q(\eta) \in \mathbb{R} \quad \text{on } [\eta_1, \eta_2]. \tag{4c}$$

In the following, the state space is considered as

$$Z = L_2(\eta_1, \eta_2) \tag{5a}$$

under the weighted inner product $\langle \cdot,\cdot \rangle_w$ with the properties of $w(\eta)$ as given in Equation (4), i.e.

$$\langle z_1, z_2 \rangle_w = \int_{\eta_1}^{\eta_2} z_1(\eta)\overline{z_2(\eta)}w(\eta)\mathrm{d}\eta. \tag{5b}$$

Furthermore, the domain $D(\mathfrak{A})$ is given by

$$D(\mathfrak{A}) = \left\{ z \in Z \mid z, \frac{\mathrm{d}z}{\mathrm{d}\eta} \text{ absolutely continuous}, \frac{\mathrm{d}^2}{\mathrm{d}\eta} z \in Z \right\} \tag{6a}$$

and the boundary (control) operator by

$$\mathfrak{B}z = \begin{pmatrix} \beta_1 z(\eta_1) + \gamma_1 \dfrac{\mathrm{d}z}{\mathrm{d}\eta}(\eta_1) \\ \beta_2 z(\eta_2) + \gamma_2 \dfrac{\mathrm{d}z}{\mathrm{d}\eta}(\eta_2) \end{pmatrix} := u_2, \tag{6b}$$

with β_i, γ_j real constants satisfying $|\beta_1| + |\gamma_1| > 0$ and $|\beta_2| + |\gamma_2| > 0$. The boundary observations are written as

$$\mathfrak{C}z = \begin{pmatrix} z(\eta_i) \\ \dfrac{\mathrm{d}z}{\mathrm{d}\eta}(\eta_i) \end{pmatrix} := y, \quad i = 1,2. \tag{7}$$

Now, we turn to some characteristics of the S-L system. Recognise from Equation (4) that $-(A, D(A))$ with $Az = \mathfrak{A}z$ for $z \in D(\mathfrak{A}) \cap \ker(\mathfrak{B})$ and $\mathfrak{A}$, $D(\mathfrak{A})$ and $\mathfrak{B}$ as in Equations (4) and (6), respectively, is an S-L operator, self-adjoint in a weighted inner product $\langle \cdot,\cdot \rangle_w$ and closed on Z, as in Equation (5) (see also Curtain and Zwart (1995), Exercise 2.10).

We mention the following result from Delattre et al. (2003).

Lemma 2.2: *Let A be the negative part of an S-L operator defined on its domain $D(A) \in Z$ given by*

Equation (3). *Then*,

(i) *A is a Riesz spectral operator*,[2]
(ii) *A is the infinitesimal generator of a* C_0*-semigroup of bounded linear operators on* Z *and on* $L_2(\eta_1, \eta_2)$,
(iii) *A has compact resolvent*.

As a consequence of Lemma 2.2, $\Sigma_{S.L.}(A)$ and properties of $\mathfrak{A}$ and $\mathfrak{B}$ given as before, $\Sigma_{S.L}$ has a mild solution. See Theorem 2.1 for details of this solution.

As will be shown in Section 4, it is convenient to check whether A in $\Sigma_{S.L.}$ is negative, which is investigated in the next lemma. Recall that the negativity of the generator of a C_0-semigroup implies the stability of the semigroup (see also Definition 5.1.1 and Theorem 5.1.3 in Curtain and Zwart (1995)).

Lemma 2.3 (Positivity of operator $-A$)**:** *The S-L operator* $-A$, *with* A *as in Equations* (3), (4) *and* (6) *and with positive real-valued continuous functions* $p(\eta)$, $w(\eta)$ *and* $q(\eta)$, *is positive*[3] *on* Z, *as in Equation* (5), *for* $z \neq 0$:

$$\text{if } \frac{\beta_2}{\gamma_2} \geq 0, \ \frac{\beta_1}{\gamma_1} \leq 0 \ \text{ and } \ |\beta_1| + |\beta_2| > 0 \ \text{ for } \gamma_1, \gamma_2 \neq 0,$$
$$\text{if } \gamma_1 = 0\text{: } \frac{\beta_2}{\gamma_2} \geq 0 \ \text{ and } \ \gamma_2 \neq 0,$$
$$\text{if } \gamma_2 = 0\text{: } \frac{\beta_1}{\gamma_1} \leq 0 \ \text{ and } \ \gamma_1 \neq 0,$$
$$\text{if } \gamma_1 = 0 = \gamma_2.$$

Proof: It is sufficient to check the time derivative of the weighted norm of z, $\frac{\mathrm{d}}{\mathrm{d}t}\|z\|_w^2 = \langle z, -Az\rangle_w \geq 0$, using the inner product Equation (5b). It follows that

$$\begin{aligned}
&2\frac{\mathrm{d}}{\mathrm{d}t}\|z(\eta,t)\|_w^2 \\
&= \int_{\eta_1}^{\eta_2} -\left(\frac{\mathrm{d}}{\mathrm{d}\eta}\left(p\frac{\mathrm{d}z}{\mathrm{d}\eta}\right) + qz\right)\cdot z\,\mathrm{d}\eta \\
&= -p(\eta)\frac{\mathrm{d}z(\eta,\cdot)}{\mathrm{d}\eta}z(\eta,\cdot)\Big|_{\eta_1}^{\eta_2} + \int_{\eta_1}^{\eta_2} p(\eta)\left(\frac{\mathrm{d}z(\eta,\cdot)}{\mathrm{d}\eta}\right)^2 \\
&\quad + q(\eta)z(\eta,\cdot)^2\,\mathrm{d}\eta \\
&= -\left[p(\eta_2)\frac{\mathrm{d}z}{\mathrm{d}\eta}(\eta_2,\cdot)z(\eta_2,\cdot) - p(\eta_1)\frac{\mathrm{d}z}{\mathrm{d}\eta}(\eta_1,\cdot)z(\eta_1,\cdot)\right] + \cdots \\
&\quad + \int_{\eta_1}^{\eta_2} p(\eta)\left|\frac{\mathrm{d}z(\eta,\cdot)}{\mathrm{d}\eta}\right|^2 + q(\eta)z(\eta,\cdot)^2\,\mathrm{d}\eta \\
&= -\left[p(\eta_2)\left(-\frac{\beta_2}{\gamma_2}z(\eta_2,\cdot)^2\right) - p(\eta_1)\left(-\frac{\beta_1}{\gamma_1}z(\eta_1,\cdot)^2\right)\right] + \cdots \\
&\quad + \int_{\eta_1}^{\eta_2} p(\eta)\left|\frac{\mathrm{d}z(\eta,\cdot)}{\mathrm{d}\eta}\right|^2 + q(\eta)z(\eta,\cdot)^2\,\mathrm{d}\eta.
\end{aligned}$$

Hence, given p, $w > 0$ and $q \geq 0$, the (sufficient) conditions directly follow. □

3. Approximate observability and detectability

In Curtain and Zwart (1995), controllability and observability results are derived for bounded B and C operators. In an earlier work, stabilisability and detectability results are obtained for parabolic distributed systems in the case of unbounded B and C operators using a modal approach (Curtain 1982). Stability and observability results, again in the case where B and C are bounded operators, are deduced for the S-L class of systems (Winkin et al. 2000; Delattre et al. 2004).

In this section, a generalisation with respect to the (approximate) observability of Σ with A, as in Equation (3), and $\mathfrak{B}$ and $\mathfrak{C}$, as in Equation (1) is presented. Instead of depending heavily on technical notions of admissibility and regularity (Weiss and Curtain 1997; Bounit and Hammouri 1997; Bounit and Idrissi 2005), we present an approximate observability result which closely resembles the results in Curtain and Zwart (1995). In a subsequent section on the observer design, we deal with the detectability and stability of S-L systems, typically encountered in CDR processes. For the observability result, we need the following concepts.

Definition 3.1 (Semigroup invariance)**:** Let V be a subspace of the Hilbert space Z and let $T(t)$ be a C_0-semigroup on Z. We say that V is $T(t)$-invariant if for all $t \geq 0$: $T(t)V \subset V$.

Definition 3.2 (Admissibility)**:** Let $\mathfrak{C} : D(A) \mapsto Y$. Then $\mathfrak{C}$ is *admissible* if $\exists t_1 > 0$, $\exists m(t_1) \geq 0$ and $\forall z \in D(A)$,

$$\int_0^{t_1} \|\mathfrak{C}T(t)z\|^2\,\mathrm{d}t \leq m(t_1)\|z\|^2.$$

Definition 3.3 (Approximate observability)**:** Let a system $\Sigma(A, -, \mathfrak{C})$, as in Equation (1), be defined with A, as in Equation (3), an infinitesimal generator of a C_0-semigroup $T(t)$, $q = 0$ and $\mathfrak{C}$ admissible. The *observability map* of $\Sigma(A, -, \mathfrak{C})$ on $[0, \tau]$, $\tau < \infty$, is the bounded linear map $\mathcal{C}^\tau : Z \to L_2([0, \tau], Y)$ defined by:

$$\mathcal{C}^\tau z := \mathfrak{C}T(\cdot)z.$$

The *non-observable* subspace of $\Sigma(A, -, \mathfrak{C})$ is the subspace of all initial states producing a zero output for almost all $t \geq 0$:

$$\begin{aligned}
\mathcal{N} &:= \{z \in Z \mid \mathfrak{C}T(t)z = 0 \text{ for almost all } t \geq 0\} \\
&= \bigcap_{\tau > 0} \ker \mathcal{C}^\tau.
\end{aligned} \tag{8}$$

$\Sigma(A, -, \mathfrak{C})$ is *approximately observable* if the only initial state producing the output zero on $[0, \infty)$ is the zero state, i.e. if $\mathcal{N} = \{0\}$.

We now characterise $\mathcal{N}$ with respect to our system Σ.

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Lemma 3.4 (Properties non-observable subspace): *The non-observable subspace $\mathcal{N}$ has the following characterisation with respect to $\Sigma(A, -, \mathfrak{C})$, as in Equation* (1):

(a) $\mathcal{N}$ *of* $\Sigma(A, -, \mathfrak{C})$ *is the largest closed* $T(t)$*-invariant subspace contained in* $\ker \mathfrak{C}(rI - A)^{-1}$, *with* $r > \omega_0$ *and* ω_0 *the* growth bound[4] *on* $T(t)$, *i.e.* $\forall \omega > \omega_0$, $\exists M$ *such that* $\forall t \geq 0$, $\|T(t)\| \leq Me^{\omega t}$.

(b) $\mathcal{N} = \overline{Span_{n\in\mathbb{J}}}\{\phi_n\}$ *for* $\mathbb{J} = \{n \in \mathbb{N} \mid \mathfrak{C}\phi_n = 0\}$.

Proof: First, let the operator A be invertible.

(a) We start with a simple but important equality. For $z_1 := A^{-1}z$, we have $\mathfrak{C}T(t)A^{-1}z = \mathfrak{C}A^{-1}T(t)z$, and so

$$\mathfrak{C}A^{-1}T(t)z = 0 \Longleftrightarrow \mathfrak{C}T(t)z_1 = 0. \tag{9}$$

Next, we define

$$\mathcal{N}_1 = \{z \mid \mathfrak{C}A^{-1}T(t)z = 0 \ \forall t\}.$$

We now prove that $\mathcal{N} = \mathcal{N}_1$ and we begin by showing that $\mathcal{N}_1 \subset \mathcal{N}$. Suppose $z \in \mathcal{N}_1$, then by the above equality we have $A^{-1}z \in \mathcal{N}$. Furthermore, since $z \in Z$, we have that $A^{-1}z \in \mathcal{N} \cap D(A)$. Since $\mathcal{N}$ is $T(t)$-invariant, $A(\mathcal{N} \cap D(A)) \subset \mathcal{N}$ (see Exercise 2.31 in Curtain and Zwart (1995)). Hence, $z = A(A^{-1})z \in \mathcal{N}$, i.e. $\mathcal{N}_1 \subset \mathcal{N}$.

Now take $z \in \mathcal{N}$ and consider $\mathfrak{C}A^{-1}T(t)z = \mathfrak{C}T(t)A^{-1}z = \mathfrak{C}T(t)z_1$. Since $z \in \mathcal{N}$, $A^{-1}z \in \mathcal{N}$ by Lemma 2.5.6 in Curtain and Zwart (1995). In other words, $\mathcal{N}$ is closed and A^{-1}-invariant. We also have $\mathfrak{C}T(t)A^{-1}z = 0$ and $z \in \mathcal{N}_1$. Consequently, $\mathcal{N} \subset \mathcal{N}_1$.

Hence $\mathcal{N} = \mathcal{N}_1$. By Lemma 4.1.18 of Curtain and Zwart (1995) we have that $\mathcal{N}_1$ is the largest closed $T(t)$-invariant subspace contained in $\ker \mathfrak{C}(A)^{-1}$. Since $\mathcal{N} = \mathcal{N}_1$, the first part is proved.

(b) Since A is a Riesz-spectral operator, we have the closed $T(t)$-invariant subspace $\mathcal{N}$ of the form

$$\mathcal{N} = \overline{\underset{n\in\mathbb{J}}{Span}}\{\phi_n\}$$

for some index set $\mathbb{J}$ (Curtain and Zwart 1995, Lemma 2.5.8). Since $\mathcal{N}$ is contained in $\ker \mathfrak{C}A^{-1}$, we must have that $\mathfrak{C}A^{-1}\phi_n = 0$ for all $n \in \mathbb{J}$. However, since $\mathcal{N}$ is the largest closed $T(t)$-invariant subspace contained in $\ker \mathfrak{C}(A)^{-1}$, we see that for $n \notin \mathbb{J}$, there holds that $\mathfrak{C}A^{-1}\phi_n \neq 0$. Consequently,

$$\begin{aligned}\mathbb{J} &= \{n \in \mathbb{N} \mid \mathfrak{C}A^{-1}\phi_n = 0\}\\ &= \{n \in \mathbb{N} \mid \mathfrak{C}\frac{\phi_n}{\lambda_n} = 0\}\\ &= \{n \in \mathbb{N} \mid \mathfrak{C}\phi_n = 0\}.\end{aligned}$$

This concludes the proof of part (b).

If the operator A is not invertible, then replace in the above A by $A - rI$ where $r > \omega_0$. □

3.1 *Detectability and stability*

Let us now inspect the estimation error system $\Sigma^{\varepsilon}(A, -, \mathfrak{C})$, with A as in Equations (4) and (6a) and $\mathfrak{C}$ as in Equation (7), with $\gamma_1 = 0 = \gamma_2$. To comply with the S-L framework, we describe the dynamics of the estimation error $\varepsilon := z - \hat{z}$ by

$$\Sigma^{\varepsilon} := \begin{cases} \dot{\varepsilon} = \mathfrak{A}\varepsilon - b_1 u_1 \varepsilon, \quad \varepsilon(0) = \varepsilon_0 \\ \mathfrak{B}\varepsilon = \begin{pmatrix} \beta_1\varepsilon(\eta_1) + \gamma_1 \dfrac{d\varepsilon}{d\eta}(\eta_1) \\ \beta_2\varepsilon(\eta_2) + \gamma_2 \dfrac{d\varepsilon}{d\eta}(\eta_2) \end{pmatrix} \\ \quad = \mathbf{L}\mathfrak{C}\varepsilon(\eta), \end{cases} \tag{10a}$$

with $\mathfrak{C}$, as in Equation (7), mapping the states to point observations at η_1^* and η_2^* and

$$\mathbf{L}\mathfrak{C}\varepsilon = \begin{pmatrix} L_{11}\varepsilon(\eta_1) + L_{12}\dfrac{d\varepsilon}{d\eta}(\eta_1) + L_{13}\varepsilon(\eta_2) + L_{14}\dfrac{d\varepsilon}{d\eta}(\eta_2) \\ L_{21}\varepsilon(\eta_2) + L_{22}\dfrac{d\varepsilon}{d\eta}(\eta_2) + L_{23}\varepsilon(\eta_1) + L_{24}\dfrac{d\varepsilon}{d\eta}(\eta_1) \end{pmatrix}. \tag{10b}$$

Note that it is not straightforward to arrive at guaranteed stability results for the closed-loop observer configuration in case we have relaxed the assumption of having an observation at the boundary to one at a point anywhere along the spatial domain, i.e. $y(t) = z(\eta^*)$, $\eta^* \in [0, 1]$.

For simplicity (but without loss of generality), we furthermore let $L_{i3} = 0 = L_{i4}$, $i \in \{1, 2\}$. In what follows in the conditions for detectability and stability, it is convenient to introduce the operator $\mathfrak{B}^L$ and for notational convenience 'renew' the definition of $\mathbf{L}$:

$$\mathfrak{B}^L\varepsilon := (\mathfrak{B} - \mathbf{L}\mathfrak{C})\varepsilon = \begin{pmatrix} \beta_1^L\varepsilon(\eta_1) + \gamma_1^L \dfrac{d\varepsilon}{d\eta}(\eta_1) \\ \beta_2^L\varepsilon(\eta_2) + \gamma_2^L \dfrac{d\varepsilon}{d\eta}(\eta_2) \end{pmatrix}, \tag{11}$$

where

$$\mathbf{L} := \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}, \quad \beta^L := \begin{pmatrix} \beta_1 - L_{11} \\ \beta_2 - L_{21} \end{pmatrix} \quad \text{and}$$

$$\gamma^L := \begin{pmatrix} \gamma_1 - L_{12} \\ \gamma_2 - L_{22} \end{pmatrix}. \tag{12}$$

Finally, we let

$$A^L = \mathfrak{A}, \quad D(A^L) = D(A) \cap \ker \mathfrak{B}^L. \tag{13}$$

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Definition 3.5 (Detectability)**:** Whenever there exists an $\mathbf{L} \in \mathcal{L}(\mathbb{R}^q, \mathbb{R}^m)$, such that A^L as in Equation (13) generates an exponentially stable C_0-semigroup; then we say that $\Sigma^\varepsilon(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is detectable.

Lemma 3.6 (Existence boundary static observer)**:** *There exists an* $\mathbf{L} \in \mathcal{L}(\mathbb{R}^q, \mathbb{R}^m)$ *such that* A^L, *as in Equation* (13), *generates an exponentially stable* C_0*-semigroup. Thus, the system* $\Sigma(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *is detectable*.

Proof: Consider the estimation error $\varepsilon = z - \hat{z}$. From the proof of Theorem 2.3, it follows that it is sufficient to check whether $-A^L$ is a positive S-L operator, i.e. $\langle \varepsilon, -A^L \varepsilon \rangle > 0$. Since $\beta^L \varepsilon + \gamma^L \frac{\mathrm{d}\varepsilon}{\mathrm{d}\eta} = 0$, we can always choose a suitable L such that $|\beta^L| + |\gamma^L| > 0$ and the boundary conditions in $D(A^L)$ are such that $-A^L$ is positive. Since $-A^L$ is a S-L operator, it has compact resolvent and therefore generates an exponentially stable semigroup (Delattre et al. 2004). □

The following corollary immediately follows.

Corollary 3.7 (Static observer design)**:** *The system* $\Sigma^\varepsilon(A^L, \mathfrak{B}^L)$ *is exponentially stable*

$$\text{if } \frac{\beta_2^L}{\gamma_2^L} \geq 0, \ \frac{\beta_1^L}{\gamma_1^L} \leq 0 \text{ and } |\beta_1^L| + |\beta_2^L| > 0 \text{ for } \gamma_1^L, \gamma_2^L \neq 0,$$

$$\text{if } \gamma_1^L = 0\text{: } \frac{\beta_2^L}{\gamma_2^L} \geq 0 \text{ and } \gamma_2^L \neq 0,$$

$$\text{if } \gamma_2^L = 0\text{: } \frac{\beta_1^L}{\gamma_1^L} \leq 0 \text{ and } \gamma_1^L \neq 0,$$

$$\text{if } \gamma_1^L = 0 = \gamma_2^L.$$

For many processes, it is not practical to implement an observer where $\gamma^L \neq 0 \neq \gamma$, since a calculation or measurement of the spatial derivative of $z(\eta_i)$ is needed. Hence, if possible, a boundary observer matrix $\mathbf{L}$ should be chosen so that derivative terms of y in the error system are cancelled.

4. Observer design for CDR example

4.1 *Model*

UV disinfection is a practical example of a CDR system where, typically, sensors and actuators are placed at pre-specified points or at the boundary. UV light is, amongst others, applied in fluid (water/juice) treatment processes to deactivate (pathogenic) micro-organisms, in the food process industry, in (waste)water treatment and in greenhouse technology industries (see some examples in Duse, da Silva, and Zietsman 2003; Guerrero-Beltran and Barbosa-Canovas 2004; Lazarova, Savoye, Janex, Blatchley, and Pommepuy 1999; Mavrov, Fahnrich, and Chmiel 1997). UV disinfection techniques have gained more attention since they do not leave traces of chemical reagents, in contrast to e.g. water disinfection by chlorination.

However, current control practice is rather conservative and indirect, since only transmittance of the fluid to be treated is measured. Transmittance cannot be related to the actual *active* pathogenic biomass and therefore one relies on the off-line laboratory analysis of measurement samples. In order to efficiently cut lamp energy costs, we would like to implement an observer that uses one or more direct biomass measurements. If properly designed, such an observer allows us to monitor the (most resistant) pathogen concentration at any point in the reactor.

The inputs (control variables) and outputs (measurements) of our annular UV system model are specified at the boundaries. After normalising and making the variables non-dimensional, the UV disinfection process reads (see Chapter 2 in Vries (2008) for modelling details):

$$\Sigma_{UV} = \Sigma(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}) \text{ with } \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \text{ as in Equations (4), (6) and (7), respectively,} \tag{14}$$

and where

$$p(\eta) = e^{-\mathrm{p_e}\eta}, \quad w(\eta) = \mathrm{p_e} e^{-\mathrm{p_e}\eta} \quad \text{and} \quad q = 0 \tag{15}$$

and boundary conditions specified with

$$\beta_1 = 1, \quad \gamma_1 = -1/\mathrm{p_e}, \quad \beta_2 = 0 \quad \text{and} \quad \gamma_2 = 1. \tag{16}$$

4.2 *Design of L*

In the UV disinfection process, it is desired to have a good estimate of the concentration at the outlet of the reactor. It is now illustrated how to tune $\mathbf{L}$ by the aid of eigenvalue analysis.

The calculation of eigenvalues of the system dynamics is less straightforward (and is numerically more involved) with the current Danckwerts boundary conditions compared to a setup with e.g. Dirichlet and Neumann conditions. Hence, for demonstration purposes and the ease of design, we impose a Dirichlet-type condition at η_1 for the eigenvalue calculations of the *error system*. To this aim, the first row of $\mathbf{L}$ is set to $\mathbf{L}_{1j} = (0 \ \ \gamma_1)$ such that the derivative term in the boundary of the error system cancels. Furthermore, from an engineering point of view we prefer to tune our observer with just one scalar gain, so we let $L_{11} = 0 = L_{22}$.

These prerequisites lead to

$$\mathbf{L} = \begin{pmatrix} 0 & \gamma_1 \\ L_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\dfrac{1}{\mathrm{p_e}} \\ L_{21} & 0 \end{pmatrix} \quad \text{with} \quad L_{21} \neq \beta_2 \text{ to be chosen,} \tag{17}$$

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with γ and β as in Equation (16). Consequently,

$$\beta^L = \begin{pmatrix} 1 \\ -L_{21} \end{pmatrix} \quad \text{and} \quad \gamma^L = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The error system Σ_I^ε for this example with L, as in Equation (17), then reads

$$\Sigma_{UV}^\varepsilon := \begin{cases} \dot{\varepsilon} = \mathfrak{A}\varepsilon - b_1 u_1 \varepsilon \\ \quad = \dfrac{1}{\mathrm{p_e}}\dfrac{\mathrm{d}^2}{\mathrm{d}\eta}\varepsilon - \dfrac{\mathrm{d}\varepsilon}{\mathrm{d}\eta}, \quad \varepsilon(0) = \varepsilon_0, \\ \mathfrak{B}^L\varepsilon = \begin{pmatrix} \varepsilon(0) \\ -L_{21}\varepsilon(1) + \dfrac{\mathrm{d}\varepsilon}{\mathrm{d}\eta}(1) \end{pmatrix}. \end{cases} \tag{18}$$

For observer design, L_{21} can be tuned by eigenvalue placement of the error system. By Theorem 3.7 we also know the following remark.

Remark 1 (Condition for L_{21} in the UV disinfection process)**:** Σ_{UV}^ε, as in Equation (18), is exponentially stable, whenever the sufficient condition $\frac{\beta_2^L}{\gamma_2^L} \geq 0$ holds, i.e. whenever $L_{21} \leq 0$.

4.3 *Eigenvalue analysis*

We would like to know how L_{21} influences the error dynamics. Therefore, we calculate the eigenvalues λ of A^L, where $A^L\varepsilon = \mathfrak{A}\varepsilon$ for $\varepsilon \in D(\mathfrak{A}) \cap \ker(\mathfrak{B}^L)$.

For $\Sigma_{UV}^\varepsilon(A^L)$ we obtain the following lemma.

Lemma 4.1: *Suppose there exists an $L_{21} \leq 0$ such that A^L generates an exponentially stable semigroup. Then the spectrum of the operator A^L consists of isolated eigenvalues with finite multiplicities given by*

$$\sigma(A^L) = \sigma_p(A^L) = \{\lambda_k^L : k \geq 0\} \subset (-\infty, 0),$$

where $\sigma_p(A^L)$ denotes the point spectrum of A^L. The eigenvalues λ_k^L, $k \geq 0$ are simple, real and given by

$$\lambda_k^L = -\frac{1}{\mathrm{p_e}}\left(\varsigma_k^L\right)^2 - \frac{1}{4}\mathrm{p_e}, \tag{19}$$

where ς_k^L, $k \geq 0$ is the set of all the solutions to the resolvent equation

$$\tan(\varsigma_k^L) = -\frac{\varsigma_k^L}{\frac{1}{2}\mathrm{p_e} - L_{21}} \quad \text{and} \quad \varsigma_k^L > 0 \tag{20}$$

such that

$$0 < \varsigma_k^L < \varsigma_{k+1}^L \quad \forall k > 0. \tag{21}$$

Hence, $\lambda_k^L < \frac{1}{4}\mathrm{p_e} < 0$, $\lambda_k^L \to -\infty$ as $k \to \infty$ and $\left|\lambda_{k+1}^L - \lambda_k^L\right| \to \infty$ as $k \to \infty$. The associated eigenvectors $\phi_k^L \in D(A^L)$, $k \geq 0$ are given for all $\eta \in [0, 1]$ and for all $k \geq 1$ by

$$\phi_k(\eta) = \exp\left(\frac{1}{2}\mathrm{p_e}\eta\right)\sin(\varsigma_k^L\eta). \tag{22}$$

Proof: The stability condition follows from Corollary 3.7 and Remark 1. The derivation of the spectral properties of Σ_{UV}^ε are shown in Appendix A. □

As a consequence of Theorem 4.1, ς^L behaves like $\pm\pi k$ for $k \to \infty$, and for k finite, we have to obtain ς_k^L numerically. To illustrate, Figure 1 shows some intersection points ς_k^L, $k = \{1, 2, 3\}$ and their corresponding eigenvalues λ_k^L have been calculated for $\mathrm{p_e} = 0.5$ and different values of L_{21}.

Remark 2 (Case where $\mathrm{d}q/\mathrm{d}t = 0$)**:** In the case that the control u_1 is given and constant, the operator A^L changes. In that case, A^L is specified with constant $q \geq 0$, i.e. $\mathrm{d}q/\mathrm{d}t = 0$ and the eigenvalues change accordingly. Here, we only give the result, since the calculation is similar as in Theorem 4.1. We impose the stability of Σ_{UV}^ε with $L_{21} \leq 0$. Then, the spectrum of A^L with q assumed constant consists of isolated eigenvalues λ_k^q with finite multiplicities given by $\sigma(A^L) = \sigma_{p,q}(A^L) = \{\lambda_k^q : k \geq 0\} \subset (-\infty, 0)$, where $\sigma_{p,q}(A^L)$ denotes the

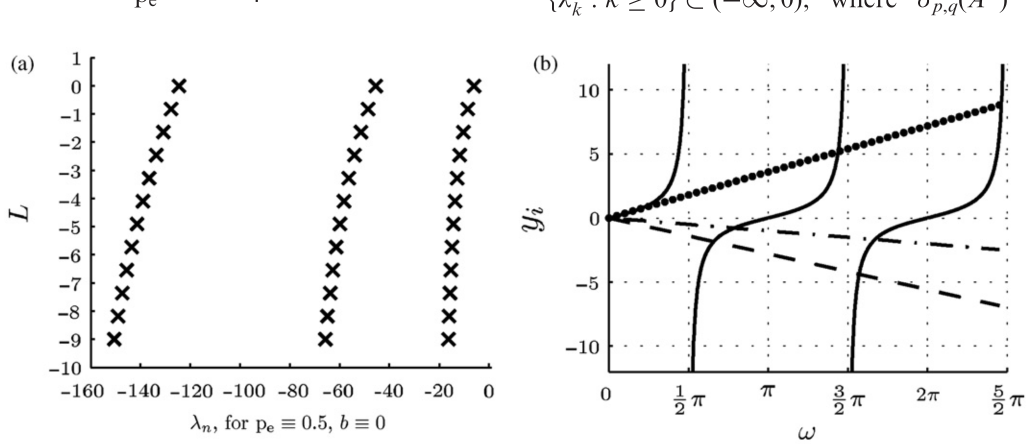

Figure 1. (a) λ_k^L, from left to right: λ_2^L, λ_1^L, λ_0^L for varying L_{21} and (b) the intersection between f and the tangent function in Equation (20) for different values of L_{21}: $\tan(\varsigma^L)$ [—], $f(\varsigma^L, 1)$ [•], $f(\varsigma^L, -1)$ [– – –], $f(\varsigma^L, -3)$, [— •] with $f(\varsigma^L, L_{21}) := \varsigma^L/(L_{21} - 0.5)$.

point spectrum of A^L. The eigenvalues λ_k^q, $k \geq 0$ are simple, real and given by

$$\lambda_k^q = -\frac{1}{\mathrm{p_e}}(\varsigma_k^L)^2 - \frac{1}{4}\mathrm{p_e} - q < \frac{1}{4}\mathrm{p_e} - q < 0, \tag{23}$$

where ς_k^L, $k \geq 0$ are all the solutions to the resolvent equation

$$\tan \varsigma_k^L = -\frac{\varsigma_k^L}{\frac{1}{2}\mathrm{p_e} - L_{21}}, \quad \varsigma_k^q > 0 \tag{24}$$

such that

$$0 < \varsigma_k^L < \varsigma_{k+1}^L \quad \forall k > 0. \tag{25}$$

In the time-varying case we have that the evolution operator $U(t,s)$, as in Theorem 2.1, is bounded from above by

$$\|U(t,s)\| \leq \exp((\omega_0 - q_{\min})(t-s))$$

with ω_0 the growth bound[5] on $T(t)$ and $q \geq q_{\min} \geq 0$ the lower bound on the lamp strength.

For reference, the eigenvalues λ and associated eigenvectors ϕ_k of Σ_{UV} are given here as well (calculation goes similar to the calculation of λ^L and ϕ^L):

$$\lambda_k = -\frac{1}{4}\mathrm{p_e} - \frac{1}{\mathrm{p_e}}\varsigma_k^2, \tag{26}$$

with ς_k, $k \geq 0$, the set of all solutions to the resolvent equation

$$\tan(\varsigma_k) = \frac{2\mathrm{p_e}\varsigma_k}{\varsigma_k^2 - \left(\frac{1}{2}\mathrm{p_e}\right)^2} \tag{27}$$

and orthonormal associated eigenvectors

$$\phi_k = C_0 \exp\left(\frac{1}{2}\mathrm{p_e}\eta\right)\left[\frac{\mathrm{p_e}}{\varsigma_k}\sin(\varsigma_k\eta) + 2\cos(\varsigma_k\eta)\right], \quad C_0 > 0. \tag{28}$$

Note that the solution of this eigenvalue problem is numerically more involved due to the presence of the Danckwerts conditions.

Equipped with the design conditions for a Luenberger observer for system Σ_{UV}, we can now study the influence of the observer gain L_{21} on the eigenvalues of the error system. Using the results obtained in Sections 2 and 3, we also obtain some remarks on the observability of $\Sigma_{UV}^{\varepsilon}$ and the solution of the whole system follows.

4.4 *Performance bounds*

Remark 3: From Equation (20), it follows that

- for fixed k, $\lim_{L \to -\infty} \varsigma_k^L = \pm k\pi$, and therefore $\lim_{L \to \infty} \lambda_k^L = -\frac{1}{4}k^2\pi^2/\mathrm{p_e}$;
- for fixed $L \in [-\infty, 0]$, we get $\varsigma_k \in \left[(k-\frac{1}{2})\pi, k\pi\right]$, and therefore the eigenvalues $\lambda_k^L \in -\frac{1}{\mathrm{p_e}}\left[(k-\frac{1}{2})^2\pi^2, k^2\pi^2\right]$.

Indeed, Remark 1 reveals what would be suspected from Figure 1, i.e. the magnitude of the distance $|\omega_k - \omega_{k-1}|$ if $L_{21} \to -\infty$ or if $k \to \infty$.

Furthermore, note that the difference between the growth bounds of $T^{\varepsilon}(t)$ is dependent on L_{21}:

$$\Delta_\lambda := \lambda_0 - \lambda_0^L, \quad \text{with growth bound } \lambda_0 := \sup_{k \in \mathbb{N}} \lambda_k. \tag{29}$$

The difference Δ_λ will be referred to as the *performance increase* of the observer. For the UV disinfection case, Remark 1 and the eigenvalues λ^L, as in Equation (19), tells us that $-\frac{1}{4}\mathrm{p_e}^2 - \pi^2 < \mathrm{p_e}\lambda_0^L < -\frac{1}{4}\mathrm{p_e}^2$ for all L_{21}, hence there is a maximal performance increase $\Delta_\lambda^{\max} = \pi^2/\mathrm{p_e}$. Similarly, if we only allow $L_{21} \leq 0$, then $-\frac{1}{4}\mathrm{p_e}^2 - \pi^2 < \mathrm{p_e}\lambda_0 < -\frac{1}{4}\mathrm{p_e}^2 - \frac{1}{4}\pi^2$ and the performance increase can maximally be $\frac{3}{4}\pi^2/\mathrm{p_e}$.

4.5 *Observability*

Given system $\Sigma_{UV}(A, -, \mathfrak{C})$, as in Equations (14) and (7), we can calculate when the UV system is approximately observable. We propose the following proposition.

Proposition 4.2: *Given system $\Sigma_{UV}(A, -, \mathfrak{C})$, as in Equations (14) and (7), i.e. with observations on the interval $\eta^* \in [0,1]$, then*

(i) *by considering $y = z(\eta^*)$ as the only observation, Σ_{UV} is approximately observable if*

$$\varsigma_k \neq -\frac{1}{2}\mathrm{p_e}\eta^* \tan(\varsigma_k\eta^*), \quad k \geq 0 \tag{30}$$

(ii) *by considering $y = \frac{\mathrm{d}z}{\mathrm{d}\eta}(\eta^*)$ as the only observation, Σ_{UV} is approximately observable if*

$$\tan(\varsigma_k\eta^*) \neq -\frac{\varsigma}{\mathrm{p_e}}, \quad k \geq 0 \quad \textit{and} \quad \eta^* \neq 1. \tag{31}$$

Proof: From Theorem 4.2, we should have that $\mathfrak{C}\phi_k(\eta^*) \neq 0$ $k \geq 0$, and $\mathfrak{C}$ admissible to obtain approximate observability. First admissibility is checked. Introduce the shorthand notation. Consider

$$\begin{aligned} y(t) = \mathfrak{C}T(t)z_0 &= \mathfrak{C}e^{At}\sum_k z_k\phi_k \\ &= \mathfrak{C}\sum_k^{\infty} z_k e^{\lambda_k t}\phi_k = \sum_k z_k e^{\lambda_k t}\underbrace{\mathfrak{C}\phi_k}_{c_k}. \end{aligned}$$

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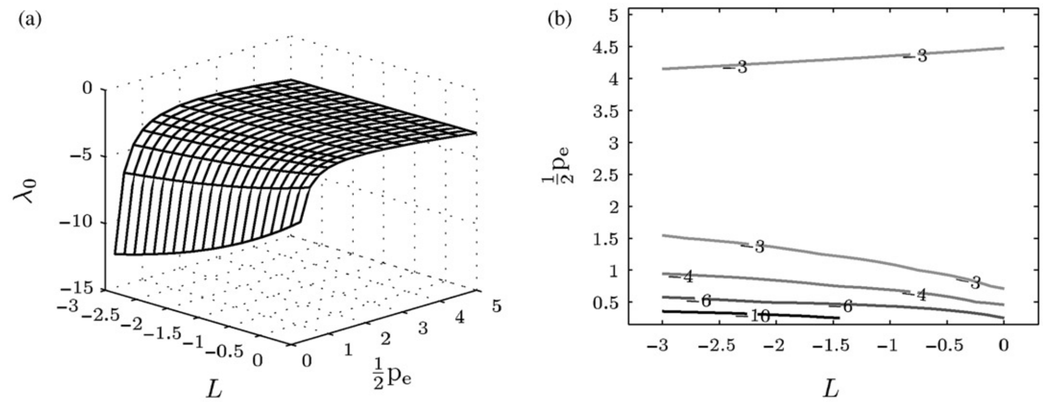

Figure 2. λ_0 versus $-3<L<0$ and $0<\mathrm{p_e}\le 10$. (a) 2D Graph of λ_0 and (b) contour lines of λ_0.

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By Cauchy–Schwarz and orthonormality of the eigenvectors of A (straightforward calculation, see also Appendix D2 in Vries (2008)), we have

$$\|y(t)\|^2 \le \sum_{k=1} \left|c_k e^{\lambda_k t}\right|^2 \sum_k |z_k|^2 = \sum_k e^{2\lambda_k t}|c_k|^2 \|z_0\|^2.$$

By Theorem 3.2, $\int_0^\infty \|y\|^2\,\mathrm{d}t \le m\|z_0\|^2$ should hold. The above leads to

$$\int_0^\infty \|y(t)\|^2 = \int_0^\infty \left|\sum_k c_k e^{\lambda_k t}\right|^2 \mathrm{d}t \le \int_0^\infty \sum_k \left|c_k e^{\lambda_k t}\right|^2 \sum_k |z_k|^2\,\mathrm{d}t$$
$$\le \left(\int_0^\infty \sum_k e^{2\lambda_k t}|c_k|^2\,\mathrm{d}t\right)\|z_0\|^2$$
$$\Leftrightarrow \sum_k \int_0^\infty \frac{c_k^2}{-2\lambda_k}\,\mathrm{d}t\|z_0\|^2 \le m\|z_0\|^2$$

with m, an arbitrary positive constant. Now, consider

(i) $y = z(\eta^*)$, thus $\mathfrak{C}\phi = \phi(\eta^*)$, $\eta^* \in [0,1]$ and the eigenvalues of A, i.e. λ_k in Equation (26) with their associated eigenvectors ϕ_k in Equation (28). By the above $\mathfrak{C}$ is admissible, since for $k \to \infty$, λ_k behaves like $k^2\pi^2/(4\mathrm{p_e})$ so that there exists indeed a value of m which makes the inequality true.

(ii) $y = \dot{z}(\eta^*)$, thus $\mathfrak{C}\phi = \dot{\phi}(\eta^*) \ne 0$, $k \ge 0$. Hence, if $\eta^* = 1$, $\dot{\phi}_k(1) = 0$ so the system is not (approximately) observable. Again, the eigenvectors ϕ_k read as Equation (28), thus we obtain the condition that $\dot{\phi}_k(\eta) = \mathrm{p_e}\sin(\varsigma_k\eta) + \varsigma\cos(\varsigma_k\eta) \ne 0$ for $\eta^* < 1$. The admissibility check goes analogously to the proof of (i). $\square$

As a consequence of Theorem 4.2, Σ_{UV} is always approximately observable for $\mathfrak{C}z = \mathfrak{C}^b z := z(1)$, since we get the condition that $\varsigma \ne \pm\imath\frac{\sqrt{3}}{2}\mathrm{p_e}$ which is always true since $\varsigma \in \mathbb{R}_+$. It is easy to see that for observations $y = z(0)$, the system is not observable. For a point observation in the interval $(0,1)$, the approximate observability has to be checked by Equation (30).

It has already been mentioned that only the estimate $\hat{y} = \hat{z}(1)$ is desired. The non-observability for $\Sigma_{UV}(\cdot,\cdot,\mathfrak{C}^b)$ when $\frac{\mathrm{d}}{\mathrm{d}\eta}z(1)$ or $z(0)$ is involved is not a problem if only estimates of $z(1)$ are needed due to the degrees of freedom in the choice of $\mathbf{L}$.

4.6 *Mild solution in Riesz bases*

The mild solution of the system Σ_{UV} with $-A$, an S-L operator, and the error system Σ^ε, as in Equation (18), can be directly written in orthogonal Riesz bases. By Theorem 2.1 we have

$$z(\cdot,t) = U(t,0)z_0(\cdot,t) = \sum_{k=1}^\infty e^{\lambda_k t}\phi_k\langle z_0,\phi_k\rangle e^{-\int_0^t q(t)\mathrm{d}\tau},$$

with λ_k given in Equation (28) satisfying Equation (27) and with associated eigenvectors ϕ_k, as in Equation (28), for the UV disinfection model Σ_{UV}. Similarly, for the error system Σ^ε_{UV}, λ^L_k is given in Equation (19) satisfying Equation (20) for all $L_{21} \le 0$, and has associated eigenvectors ϕ_k, as in Equation (22), for the error dynamics system Σ^ε_{UV}.

4.7 *Performance evaluation*

With slight abuse of notation, we omit the sub- and superscript of L in the figures.

Figure 2(a) and its equivalent contour plot (Figure 2(b)) show the behaviour of the growth bound λ_0 for $\mathrm{p_e} \in (0,10]$ and $L \in [-3,0]$. Indeed, the larger the $\mathrm{p_e}$-numbers, the larger the growth bound and the lesser the effect of the observer gain. Notice also that for smaller L, the growth bound tends to zero and the stability margin becomes smaller.

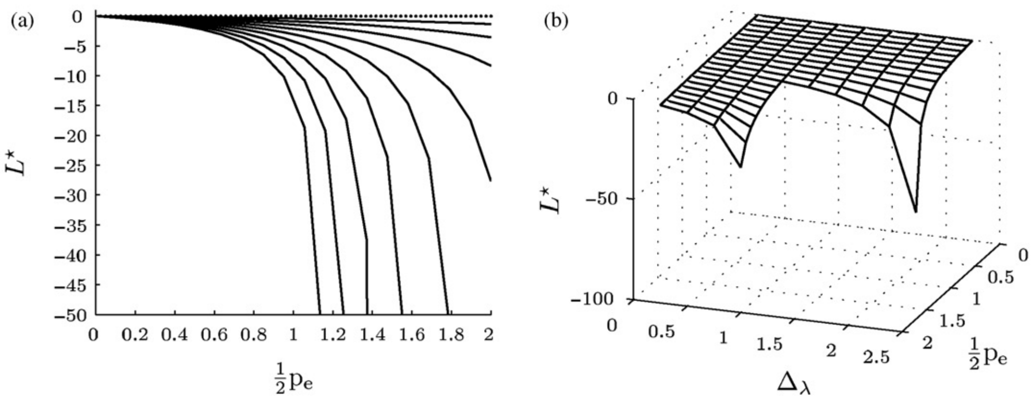

Figure 3. $L^\star$ versus Δ_λ and $\mathrm{p_e}$; (a) [$L=0$ (dashed) and $L^\star$ for $0<\mathrm{p_e}\leq 4$ and varying Δ_λ, i.e. from upper to lowest line $\Delta_\lambda = 0.25$ to 2.25 (solid) with steps of 0.25 and (b) 2D plot of $L^\star$ versus $\mathrm{p_e}$ and Δ_λ. The values of $\mathrm{p_e}$ for which the performance gain Δ_λ, is not feasible (i.e. no solutions exist), is depicted by empty space.

For (control) engineering applications, it may be more interesting to find out the magnitude of L_{21} for a given performance *increase* $\Delta_\lambda^{\max} := \lambda_0 - \lambda_0^L$. In Equation (29), bounds on Δ_λ are given with the aid of Remark 3. In addition, we now calculate $L^\star := L_{21}$ for several *Péclet*-numbers at which a given Δ_λ is obtained. The results are depicted in Figure 3. We see that, for increasing $\mathrm{p_e}$, $L^\star$ increases rapidly for some Δ_λ in Figure 3(a). For reference, the line $L=0$ is also shown in Figure 3(b). Notice from Figure 3 that a particular value of the performance gain Δ_λ can only be achieved for a certain range of Péclet-numbers.

5. Conclusions

Inspired by CDR processes in food and water treatment industry, we analysed the approximate observability, detectability and stability for distributed parameter systems with a differential operator belonging to the S-L class and the assumption that there are only boundary measurements available. Conditions on detectability and stability have been derived for the design of a static, Luenberger boundary observer. With the aid of eigenvalue placement, the performance of an observer for a UV disinfection process case has been assessed and tested by numerical calculations.

In the example case, we come to the following conclusions:

- From the eigenvalue analysis and numerical calculations, it follows that for mild Péclet-numbers ($\mathrm{p_e} \ll 1$, hence a low convection–diffusion ratio), there is more room to obtain a performance gain with a suitable observer gain L_{21}. For large Péclet numbers, fast process dynamics already push the estimation error to zero. In this case one may decide to choose a small positive $L_{21} > \frac{1}{2}\mathrm{p_e}$ as a smoothing filter.
- The growth bound of the error system is pushed to higher absolute magnitudes whenever the lamp strength is stronger, i.e. for increasing $b_1 u_1$.

The presented observer design approach gives a good impression how the error system will behave, *independent* of some choice of discretisation or approximation method.

Notes

1. The semigroup is defined as in Definition 2.1.8, Chapter 2 of Curtain and Zwart (1995).
2. The definition of a Riesz spectral operator is found in Theorem 2.3.5 in Curtain and Zwart (1995).
3. Operator Q is said to be positive if $\langle Az, z\rangle > 0$ for all nonzero $z \in D(Q)$.
4. The growth bound of a semigroup T is given by $\omega_0 = \inf_{t>0}\left(\frac{1}{t}\log\|T\|\right)$.
5. The definition of a growth bound ω_0 of a Riesz spectral operator is $\omega_0 = \inf_{t>0}\left(\frac{1}{t}\log\|T\|\right) = \sup_{n\geq 1}\mathrm{Re}\,(\lambda_n)$.

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Appendix A. Proof of Theorem 4.1

A^L is an S-L operator and therefore it has a spectrum with isolated eigenvalues with finite multiplicities (Theorem 2.2). Furthermore, it is self-adjoint under the inner product $\langle\cdot,\cdot\rangle_w$ and negative for $L_{21} \leq 0$ (Remark 1). Consequently, $\lambda^L < 0$.

For this S-L-type problem, we write

$$\phi^L = C_1^L \phi_1^L + C_2^L \phi_2^L. \tag{A1}$$

Furthermore, ϕ^L should satisfy the boundary conditions

$$\mathfrak{B}_1^L \phi^L := \phi^L(0) = 0 \tag{A2}$$

$$\mathfrak{B}_2^L \phi^L := \dot{\phi}^L(1) - L_{21}\phi^L(1) = 0. \tag{A3}$$

The eigenvalues λ^L can be found from $A^L\phi^L = \lambda^L\phi$. To this aim, consider the following three cases.

Case 1a: Let $\lambda^L > \frac{1}{4}\mathrm{p_e}$, i.e. $\phi_1^L = e^{\mu_1(\lambda^L)\eta}$ and $\phi_2^L = e^{\mu_1(\lambda^L)\eta}$ with

$$\mu_1(\lambda^L) = \frac{\mathrm{p_e}}{2} - \varsigma_a^L,\ \ \mu_2(\lambda^L) = \frac{\mathrm{p_e}}{2} + \varsigma_a^L \ \text{ and } \ \varsigma_a^L = \sqrt{\left(\frac{\mathrm{p_e}}{2}\right)^2 + \mathrm{p_e}\lambda^L}. \tag{A4}$$

Case 1b: Let $\lambda < \frac{1}{4}\mathrm{p_e}$, i.e. $\phi_1^L = e^{\mu_1^{\lambda L}\eta}$ and $\phi_2^L = e^{\mu_1(\lambda^L)\eta}$ with

$$\mu_1(\lambda^L) = \frac{\mathrm{p_e}}{2} - \varsigma_b^L, \quad \mu_2(\lambda^L) = \frac{\mathrm{p_e}}{2} + \varsigma_b^L \quad \text{and}$$
$$\varsigma_b^L = \imath\varsigma_L = \sqrt{\left(\frac{\mathrm{p_e}}{2}\right)^2 + \mathrm{p_e}\lambda^L}. \tag{A5}$$

Case 2: $\lambda^L = (\frac{1}{2}\mathrm{p_e})^2$, i.e. let $\phi_1^L = e^{\frac{1}{2}\mathrm{p_e}\eta}$ and $\phi_2^L = \eta e^{\frac{1}{2}\mathrm{p_e}\eta}$.

The eigenvalues λ^L of A^L exist if and only if the determinant $\Delta(\lambda^L)$ of the system of boundary equations (A3) is zero (see Exercise 2.10b in Curtain and Zwart (1995)), i.e.

$$\Delta(\lambda^L) := \det\begin{pmatrix} \mathfrak{B}_1^L\phi_1^L & \mathfrak{B}_1^L\phi_2^L \\ \mathfrak{B}_2^L\phi_1^L & \mathfrak{B}_2^L\phi_2^L \end{pmatrix} = 0.$$

Hence, case by case we get the following.

Case 1a:

$$\delta(\lambda^L) = \det\begin{pmatrix} 1 & 1 \\ (\mu_1(\lambda^L) - L_{21})e^{\mu_1(\lambda^L)} & (\mu_2(\lambda^L) - L_{21})e^{\mu_2(\lambda^L)} \end{pmatrix}.$$

Hence, $\Delta(\lambda^L) = 1\cdot(\mu_2(\lambda^L) - L_{21})e^{\mu_2(\lambda L)} - 1\cdot(\mu_1(\lambda^L) - L_{21})e^{\mu_1(\lambda L)}$. Since for all L_{21} and for $\mathrm{p_e} > 0$, $\varsigma_L^a > 0$, this leads to

$$e^{\varsigma_L^a} > e^{-\varsigma_a^L} > 0 \quad \text{and}$$
$$\left(\frac{1}{2}\mathrm{p_e} + \varsigma_a^L - L_{21}\right) > \left(\frac{1}{2}\mathrm{p_e} - \varsigma_a^L - L_{21}\right) > 0.$$

Consequently, $\Delta(\lambda) > 0$ and no solution for the eigenvalues λ^L can be found.

Case 1b: Analogous to Case 1a, we get

$$\Delta(\lambda^L) = 1\cdot\left(\mu_2(\lambda^L) - L_{21}\right)e^{\mu_2(\lambda^L)} - 1\cdot\left(\mu_1(\lambda^L) - L_{21}\right)e^{\mu_1(\lambda^L)}.$$

However, for $\lambda^L < 0$ this reduces to

$$\Delta(\lambda) = e^{\frac{\mathrm{p_e}}{2}}\left[\left(\frac{1}{2}\mathrm{p_e} + \imath\varsigma^L - L_{21}\right)e^{\imath\varsigma^L} - \left(\frac{1}{2}\mathrm{p_e} - \imath\varsigma^L - L_{21}\right)e^{-\imath\varsigma_L}\right]$$
$$= e^{\frac{\mathrm{p_e}}{2}}\left[2\imath\varsigma^L\cos(\varsigma^L) + 2\imath\left(\frac{1}{2}\mathrm{p_e} - L_{21}\right)\sin(\varsigma^L)\right].$$

Hence for $\Delta(\lambda^L) = 0$, ς_k^L, $k \geq 0$ is the set of all solutions to the resolvent equation:

$$\tan(\varsigma_k^L) = -\frac{\varsigma_k^L}{\frac{1}{2}\mathrm{p_e} - L_{21}}. \tag{A6}$$

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Case 2:

$$\Delta(\lambda^L) = \det\begin{pmatrix} 1 & 0 \\ \left(\frac{1}{2}\mathrm{p_e} - L_{21}\right)e^{\frac{1}{2}\mathrm{p_e}} & \left(\frac{1}{2}\mathrm{p_e} + 1 - L_{21}\right)e^{\frac{1}{2}\mathrm{p_e}} \end{pmatrix}.$$

Hence, $\Delta(\lambda^L) = 0$ if $L_{21} = \frac{1}{2}\mathrm{p_e} + 1$, since $\mathrm{p_e} > 0$.

The eigenvalues λ^L follow from Equation (A5), Case 1b or the rather exceptional Case 2. For a detectable system $\Sigma_{UV}^{\varepsilon}$ with $L_{21} \leq 0$, Case 2 does not occur. Furthermore, recognise that $\lambda_k^L \to -\infty$ as $k \to \infty$ and $\left|\lambda_{k+1}^L - \lambda_k^L\right| \to \infty$ as $k \to \infty$.

From $\mathfrak{B}_1\phi$, i.e. the Dirichlet condition at $\eta_1 = 0$ and λ^L as in Equation (A5), we obtain for the eigenvectors ϕ, as in Equation (A1),

$$\mu_1(\lambda^L)C_1^L + \mu_2(\lambda^L)C_2^L = 0, \quad C_1^L \neq 0 \neq C_1^L.$$

Hence, the associated eigenvectors of A^L, i.e. $\phi_k^L \in D(A^L)$, $k \geq 0$ are given for all $\eta \in [0, 1]$ and for all $k \geq 1$ by

$$\phi_k^L = C_0\left[\left(\frac{1}{2}\mathrm{p_e} - \imath\varsigma_k^L\right)\exp\left(\frac{1}{2}\mathrm{p_e} - \imath\varsigma_k^L\right) - \left(\frac{1}{2}\mathrm{p_e} - \imath\varsigma^L\right)\right.$$
$$\left.\times \exp\left(\frac{1}{2}\mathrm{p_e} + \imath\varsigma_k^L\right)\right]$$

$$\Leftrightarrow \phi_k^L(\cdot) = C_0 \exp\left(\frac{1}{2}\mathrm{p_e}\eta\right)\sin(\varsigma_k^L\eta) \quad \text{with} \quad C_0^L := C_1^L = -C_2^L.$$

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