

## RESEARCH ARTICLE

### Fixed-order $H_\infty$ controller design for systems with ellipsoidal parametric uncertainty

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In this paper, fixed-order robust  $H_\infty$  controller design for systems with ellipsoidal parametric uncertainty based on parameter dependent Lyapunov functions is studied. Using the concept of Strictly Positive Realness (SPRness) of transfer functions, a fixed-order robust control design method in terms of solution to a set of Linear Matrix Inequalities (LMIs) is proposed. Since, controller parameters are decision variables, any controller structure, such as PID, can be considered. The weighted infinity-norm of closed loop sensitivity functions are considered as performance specification in the synthesis problem. The simulation results show the effectiveness of the proposed method.

#### 1 Introduction

This paper studies the robust control design problem for systems with ellipsoidal parametric uncertainty, deduced from the classical prediction error identification methods. The development of the identification theory in the control literature followed on the development of model-based controller design [Gevers (2006)]. Models resulted from system identification are uncertain due to measurement noise. When the true system belongs to the set of parameterized models, the modeling error can be represented by parametric uncertainty. The identified parameters are known to belong, with a probability level, to ellipsoidal areas, characterized by the covariance matrix of the parameters [Ljung (1999)].

Models with real parametric uncertainty allow more accurate representation of some systems with respect to unstructured uncertainty. The structured singular value ( $\mu$ ) was introduced to study structured uncertainty in linear models. It is now well recognized that generally such  $\mu$ -synthesis problems are NP-hard and hence computationally intractable. In the control literature, several fundamentally different approaches for solving these problems are presented.

In Peaucelle and Arzelier (1998), the problem of quadratic stability in the presence of parametric uncertainty is considered. Both state feedback and output feedback controller design are presented. Static output feedback controller design for ellipsoidal uncertain systems based on quadratic separation concept is investigated in Peaucelle and Arzelier (2005). In Ballamudi and Crisalle (1995), it is shown that the robust pole-placement problem for systems with ellipsoidal parametric uncertainty can be expressed as a convex min-max optimization problem. Using the ellipsoidal inner approximation of the stability domain in the space of polynomial coefficients [Henrion et al. (2003)], a fixed-order stabilizing controller design for systems with ellipsoidal parametric uncertainty is proposed in Henrion et al. (2001). In Raynaud et al. (2000) the design specification is robust stabilization. The objective is to find the controller that maximizes the volume of the ellipsoidal model set. The proposed synthesis methods in Peaucelle and Arzelier (1998), Ballamudi and Crisalle (1995), Henrion et al. (2001), and Raynaud et al. (2000) only consider the stabilizing controller design.

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The problem of  $H_\infty$  state feedback controller design for systems with ellipsoidal parametric uncertainty is presented in Barenthin and Hjalmarsson (2008). They assume that only one of the matrices  $A$  and  $C$  in the state space realization depends on the uncertain parameters. Moreover, a common Lyapunov matrix for all systems in the model set is used that is known to be too conservative. The paper by Bombois et al. (2002) presents a recursive procedure for controller design for systems with ellipsoidal parametric uncertainty. At the first step, without considering the parametric uncertainty, a set of controllers is designed for nominal system such that the nominal performance is somewhat better than the desired performance. Then, in the second step, the robust stability and performance of the closed-loop system is analyzed. If the results are not satisfactory, a new iteration should be carried out with tighter performance specification for the nominal model. In Rantzer and Megretski (1994), a synthesis method is presented that is based on infinite dimensional Youla-Kucera parameterization. Robust performance problem is stated in terms of a quasi-convex optimization. Uncertain real parameters are assumed to appear linearly in the closed loop characteristic polynomial (rank-one problem). This synthesis approach cannot handle fixed-order controller design with an order less than that of the plant model. Since the order of the plant may be high, full-order controllers cause significant drawbacks in practical applications.

In this contribution, we consider the problem of fixed-order  $H_\infty$  controller design for SISO systems with ellipsoidal parametric uncertainty. In addition to stabilization, an upper bound on the  $H_\infty$  norm of weighted closed loop transfer functions is minimized. We concentrate on dynamic output feedback controller structure.

It is well known that fixed-order controller design leads to either a nonconvex rank constraint or bilinear matrix inequalities. Since, in general, these problems are computationally intractable, several researchers focused on devising simpler but conservative design methods, e.g. see Henrion et al. (2003) and Khatibi et al. (2008). In these papers, a convex set of stabilizing controllers is parameterized such that the closed-loop characteristic polynomial divided by a so called central polynomial is a strictly positive real (SPR) transfer function. This convex set is an inner approximation of the nonconvex set of all fixed-order stabilizing controllers and the quality of this approximation is related to the choice of the central polynomial. In the present contribution, instead of a fixed central polynomial, we consider an affine parameter dependent central polynomial, which will reduce the conservatism of the approach. For  $H_\infty$  fixed-order controller design a recently developed LMI formulation is used in Khatibi and Karimi (2009). This formulation is based on the strict positive realness of two transfer functions with the same Lyapunov matrix in the matrix inequality of the Kalman-Yakubovic-Popov Lemma. Using this approach, a convex set of fixed-order  $H_\infty$  controllers for a SISO system is given in terms of two LMIs. We use these LMIs for all systems in an ellipsoidal set to ensure the robust performance satisfaction for all models in the model set. Consequently, an infinite dimensional convex optimization problem is resulted. The key to developing a tractable solution for this problem is to use a parameter dependent Lyapunov matrix. We propose to look for a parameter dependent Lyapunov matrix in which, each element is a quadratic function of the uncertain parameters. This way, a fixed-order robust controller design method for systems with ellipsoidal parametric uncertainty in terms of solution to a finite set of linear matrix inequalities is obtained. To the best of our knowledge, no fixed-order  $H_\infty$  control design approach for systems with ellipsoidal parametric uncertainty is available in the literature.

The rest of the paper is structured as follows. In the next section, robust control design problem is formulated. Section 3, shows how to design  $H_\infty$  controller based on an infinite dimensional convex optimization problem. A technique to recast the optimization problem of previous section to semidefinite programming (SDP) is developed in Section 4. Section 5 is devoted to simulation example. Finally, some conclusions are drawn in the last section.

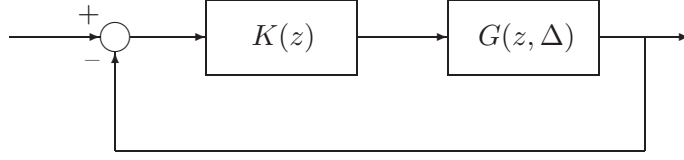


Figure 1. Standard negative feedback configuration

## 2 Problem Formulation and Preliminaries

Consider the transfer function of an uncertain discrete-time linear time-invariant SISO system

$$G(z, \theta) = \frac{N(z, \theta)}{M(z, \theta)} = \frac{\theta_0 z^p + \theta_1 z^{p-1} + \cdots + \theta_p}{z^q + \theta_{p+1} z^{q-1} + \cdots + \theta_{n-1}},$$

where  $N(z, \theta)$  and  $M(z, \theta)$  are polynomials and parameter  $\theta = [\theta_0 \ \theta_1 \ \cdots \ \theta_{n-1}]^T \in \mathbb{R}^n$  is a vector that parameterizes  $G$ . The model uncertainty will be represented by ellipsoidal parametric uncertainty. Therefore, the parameters lie in an ellipsoid centered on a nominal estimate  $\bar{\theta}$ . The quantities  $\bar{\theta}$  and  $R_e = R_e^T \in \mathbb{R}^{n \times n}$ , which define the ellipsoid can for example originate from system identification using Prediction Error Method (PEM) [Ljung (1999)]. The ellipsoid is described by

$$U = \{\theta | (\theta - \bar{\theta})^T R_e (\theta - \bar{\theta}) \leq 1\}. \quad (1)$$

To simplify notation, we define a new variable as

$$\Delta = \begin{pmatrix} \theta \\ 1 \end{pmatrix}. \quad (2)$$

Therefore, the ellipsoidal uncertainty set can be given by

$$U = \{\Delta | \Delta^T U_0 \Delta \geq 0\}, \quad (3)$$

where

$$U_0 = \begin{pmatrix} -R_e & R_e \bar{\theta} \\ \bar{\theta}^T R_e & 1 - \bar{\theta}^T R_e \bar{\theta} \end{pmatrix}. \quad (4)$$

We consider a standard negative feedback configuration shown in Fig. 1. The goal is to design a fixed-order controller

$$K(z) = \frac{X(z)}{Y(z)} = \frac{x_0 z^m + x_1 z^{m-1} + \cdots + x_m}{z^m + y_1 z^{m-1} + \cdots + y_m}, \quad (5)$$

that for all admissible uncertainty, the closed-loop system achieves the following  $H_\infty$  performance

$$\|W(z)H(z, \theta)\|_\infty < \gamma, \quad (6)$$

where  $H(z, \theta)$  can be any of the closed-loop transfer functions such as sensitivity function, complementary sensitivity function or input sensitivity function and  $W(z)$  is an appropriate weighting filter [Zhou et al. (1995)]. Recently, a convex minimization problem has been developed

for fixed-order  $H_\infty$  controller design, based on Strictly Positive Realness (SPRness) of transfer functions by Khatibi and Karimi (2009). In the following, the main idea of this approach is explained. We will use this method for robust control design for systems with ellipsoidal parametric uncertainty in the next sections.

*Lemma 2.1* (KYP Lemma [Landau et al. (1997)]): The transfer function  $H(z) = C(zI - A)^{-1}B + D$  is SPR if and only if there exists matrix  $P = P^T > 0$  such that

$$\begin{bmatrix} A^T P A - P & A^T P B - C^T \\ B^T P A - C & B^T P B - D - D^T \end{bmatrix} < 0. \quad (7)$$

*Definition 2.2* [Khatibi and Karimi (2009)]: Two equal order SPR transfer functions  $H_1(z)$  and  $H_2(z)$  with controllable canonical state space realization  $(A_1, B, C_1, D_1)$  and  $(A_2, B, C_2, D_2)$  are called Common Lyapunov Strictly Positive Real, or CL-SPR, if both satisfy the inequality of the KYP lemma (Inequality (7)) with the same Lyapunov matrix  $P$ .

*Lemma 2.3* [Khatibi and Karimi (2009)]: Given a polynomial  $S(z)$  and two equal order stable polynomials  $L(z)$  and  $E(z)$  (central polynomial), then  $\|S(z)/L(z)\|_\infty < \gamma$  if transfer functions  $H_1(z) = (L(z) - \gamma^{-1}S(z))/E(z)$  and  $H_2(z) = (L(z) + \gamma^{-1}S(z))/E(z)$  are CL-SPR.

In Lemma 2.3,  $S(z)/L(z)$  can be any weighted closed loop sensitivity function. Suppose that we would like to design a fixed-order controller for a nominal system  $G(z)$ , then the controller parameters appear linearly in the both polynomials  $S(z)$  and  $L(z)$ . Therefore  $L(z) - \gamma^{-1}S(z)$  and  $L(z) + \gamma^{-1}S(z)$  depend linearly on the controller parameters. In the controllable canonical state space realizations of  $H_1(z)$  and  $H_2(z)$ , the controller parameters appear linearly in  $C_1$  and  $C_2$ . Based on this fact, the inequality (7) is an LMI with respect to the controller parameters and the parameters of  $P$  when a central polynomial is given (the parameters of central polynomial appear in  $A$ ). Therefore, Lemma 2.3 provides a fixed-order  $H_\infty$  controller design method for a nominal system  $G(z)$ . For more details see Khatibi and Karimi (2009).

### 3 $H_\infty$ Controller Design

$H_\infty$  control theory addresses the issue of worst case controller design for linear systems to ensure disturbance attenuation and some robustness with respect to unstructured uncertainty. Besides, the performance for a closed-loop system is often defined via the modulus of the frequency response of different closed loop sensitivity functions [Zhou et al. (1995)]. Usually, it is needed to shape these sensitivity functions to achieve certain performance specifications. In this paper, for simplicity of presentation, the control objective is to guarantee a certain upper bound  $\gamma$  on the  $H_\infty$  norm for only one weighted transfer function  $H(z, \Delta)$ . However, the results can straightforwardly be applied when more than one sensitivity function should be shaped. In this case, the LMIs developed in this paper should be duplicated for each sensitivity function and only the controller parameters are common in the LMIs.

Therefore, performance specification will be formulated as:

$$\|W(z)H(z, \Delta)\|_\infty = \left\| \frac{S(z, \Delta)}{L(z, \Delta)} \right\|_\infty < \gamma, \quad (8)$$

for all  $\Delta \in U$ , where  $W(z)$  is the weighting filter.

The following definitions and lemma are required to proceed.

*Definition 3.1* Polynomial  $R(z, \Delta) = \Delta^T r_t z^t + \Delta^T r_{t-1} z^{t-1} + \dots + \Delta^T r_0$  that all its coefficients are real affine functions of the parameter vector  $\Delta \in U$  is called Linearly Parameter Dependent (LPD) polynomial of  $U$ , where vectors  $r_0, r_1, \dots, r_t \in \mathbb{R}^{n+1}$ .

*Definition 3.2* A linear time-invariant uncertain system with transfer function

$$G(z, \Delta) = \frac{N(z, \Delta)}{M(z, \Delta)}$$

is called LPD system of  $U$  if both  $N(z, \Delta)$  and  $M(z, \Delta)$  are LPD polynomials of  $U$ .

*Lemma 3.3* Let  $X_1(\Delta) \in \mathbb{R}^{m \times m}$ ,  $X_2(\Delta) \in \mathbb{R}^{m \times q}$  be dependent on the uncertain parameter  $\Delta \in U$ . Then, the following statements are equivalent:

- (1)  $\exists \varepsilon(\Delta) \in \mathbb{R} : X_1(\Delta) + \varepsilon(\Delta)X_2(\Delta)X_2^T(\Delta) > 0, \quad \forall \Delta \in U.$
- (2)  $\exists \varepsilon \in \mathbb{R} : X_1(\Delta) + \varepsilon X_2(\Delta)X_2^T(\Delta) > 0, \quad \forall \Delta \in U.$

*Proof* Suppose that (1) is satisfied, we define

$$\varepsilon = \max_{\Delta \in U} \varepsilon(\Delta),$$

therefore,

$$(\varepsilon - \varepsilon(\Delta)) X_2(\Delta)X_2^T(\Delta) \geq 0.$$

Adding this inequality to Statement 1, Statement 2 is obtained. Now, suppose that inequality in Statement 2 is satisfied, defining  $\varepsilon(\Delta) = \varepsilon$  leads to Statement 1, which ends the proof.  $\square$

One of the key points in robust fixed-order controller design by convex optimization is the existence of a so called central polynomial. This polynomial should be chosen such that if divided by the characteristic closed-loop polynomial, the resulting transfer function is SPR. Choice of the central polynomial is the main source of conservatism for fixed-order controller design. In fixed-order robust control design for systems with polytopic uncertainty, a common central polynomial should be considered for all the vertices that causes more conservatism. The following example shows the importance of this choice.

**Example 3.4** Consider the following LPD polynomial of  $U$

$$R(z, \delta) = z^3 + (2.1 - 6\delta)z^2 + 1.47z + (0.343 - 0.98\delta),$$

where,

$$U = \{\delta \mid 0.6592 \leq \delta \leq 0.7038, \quad -0.0038 \leq \delta \leq 0.0408\}.$$

$R(z, \delta)$  is Schur stable for all values of  $\delta \in U$ . For two members of the above set, related to  $\delta = 0$  and  $\delta = 0.7$ , we have  $R(z, 0) = (z + 0.7)^3$  and  $R(z, 0.7) = (z - 0.7)^3$ . It can be easily verified that the phase difference between these two members exceeds  $\pi$  for some values of  $0 \leq \omega \leq \pi$ . Therefore, there exists no fixed central polynomial  $E(z)$  such that  $R(z, \delta)/E(z)$  is SPR for all values of  $\delta \in U$  [Khatibi et al. (2008)]. However, the choice  $E(z, \delta) = R(z, \delta)$  which is a Schur stable LPD polynomial of  $U$ , makes the rational function  $R(z, \delta)/E(z, \delta) = 1$  obviously SPR.

The above example emphasizes the importance of the choice of the central polynomial. It is obvious that there exists always an LPD SPR-maker polynomial  $E(z, \Delta)$  for a stable LPD polynomial  $R(z, \Delta)$  (choosing  $E(z, \Delta) = R(z, \Delta)$ ). This fact motivates us to consider LPD central polynomial for fixed-order robust controller design. In the literature, it is proposed that the desired closed loop characteristic polynomial would be a suitable choice of the central polynomial [Henrion et al. (2003), Yang et al. (2007)]. This means that one way to find the central polynomial is to consider a nominal controller and then compute the characteristic polynomial and use it as the central polynomial. However, a controller for an uncertain LPD system results

in a set of closed loop characteristic polynomials in form of an LPD polynomial. Therefore, it is reasonable to consider this LPD polynomial as the central polynomial.

The following proposition gives necessary and sufficient condition for ensuring the SPRness of an LPD system. Perhaps the most interesting aspect of this condition is that it can be easily extended to a semidefinite program for ellipsoidal uncertain systems.

*Proposition 3.5* An LPD system

$$\frac{R(z, \Delta)}{E(z, \Delta)} = \frac{\Delta^T r_t z^t + \Delta^T r_{t-1} z^{t-1} + \dots + \Delta^T r_0}{z^t + \Delta^T e_{t-1} z^{t-1} + \dots + \Delta^T e_0}$$

is SPR, for all  $\Delta \in U$ , if and only if there exist a symmetric positive matrix  $P(\Delta) = P(\Delta)^T > 0$  and a scalar  $\varepsilon \in \mathbb{R}$  such that

$$\text{Spr}(P(\Delta), \varepsilon, R(z, \Delta), E(z, \Delta)) \triangleq \left[ \begin{array}{cc|c} P(\Delta) & R^T \Delta & 0_{t \times t} \\ \Delta^T R & 0 & \Psi(\Delta) \\ \hline 0_{t \times t} & \Psi(\Delta)^T & -P(\Delta) \end{array} \right] + \varepsilon \left[ \begin{array}{c} \Pi + \Upsilon(\Delta) \\ \hline -I_t \end{array} \right] \left[ \begin{array}{c} \Pi^T + \Upsilon(\Delta)^T \\ \hline -I_t \end{array} \right] > 0, \quad (9)$$

$$\text{where } \Psi(\Delta) = \begin{bmatrix} 0_{1 \times (t-1)} & \Delta^T r_t \end{bmatrix}, \Pi^T = \begin{bmatrix} 0_{t \times 1} \\ I_t \end{bmatrix}, \text{ and } \Upsilon(\Delta) = \begin{bmatrix} 0_{t \times (t-1)} & -E^T \Delta \\ 0_{1 \times (t-1)} & 0 \end{bmatrix},$$

$$R = \begin{bmatrix} r_0 & r_1 & \dots & r_{t-1} \end{bmatrix}, E = \begin{bmatrix} e_0 & e_1 & \dots & e_{t-1} \end{bmatrix}.$$

*Proof* According to Lemma 2.1,  $R(z, \Delta)/E(z, \Delta)$  is SPR if and only if condition (7) is satisfied, for all  $\Delta \in U$ . It is easy to see that the controllable canonical realization of transfer function  $R(z, \Delta)/E(z, \Delta)$  is given by

$$\begin{aligned} A(\Delta) &= \begin{bmatrix} 0_{(t-1) \times 1} & I_{t-1} \\ \hline -\Delta^T E & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0_{(t-1) \times 1} \\ 1 \end{bmatrix}, \\ C(\Delta) &= \Delta^T R - (\Delta^T r_t) \Delta^T E, \quad D(\Delta) = \Delta^T r_t. \end{aligned} \quad (10)$$

Condition (7) for all  $\Delta \in U$  can be written as

$$\left[ \begin{array}{c} I_{t+1} \\ \hline A(\Delta) \ B \end{array} \right]^T \left[ \begin{array}{cc|c} P(\Delta) & R^T \Delta & 0_{t \times t} \\ \Delta^T R & 0 & \Psi(\Delta) \\ \hline 0_{t \times t} & \Psi(\Delta)^T & -P(\Delta) \end{array} \right] \left[ \begin{array}{c} I_{t+1} \\ \hline A(\Delta) \ B \end{array} \right] > 0. \quad (11)$$

Using Finsler's lemma [Skelton et al. (1998)], (11) is equivalent to the existence of  $P(\Delta)$  and a scalar function  $\varepsilon(\Delta)$  such that

$$\left[ \begin{array}{cc|c} P(\Delta) & R^T \Delta & 0_{t \times t} \\ \Delta^T R & 0 & \Psi(\Delta) \\ \hline 0_{t \times t} & \Psi(\Delta)^T & -P(\Delta) \end{array} \right] + \varepsilon(\Delta) \left[ \begin{array}{c} A(\Delta)^T \\ B^T \\ \hline -I_t \end{array} \right] \left[ \begin{array}{c} A(\Delta) \ B \\ \hline -I_t \end{array} \right] > 0.$$

Taking into account Lemma 3.3, this last inequality can be expressed as (9) and this ends the proof.  $\square$

The LMIs for fixed-order  $H_\infty$  controller design for LPD systems is given in the following theorem.

**Theorem 3.6** Given an LPD central polynomial  $E(z, \Delta)$ , a fixed-order controller given by (5), stabilizes the uncertain LPD system  $G(z, \Delta) = N(z, \Delta)/M(z, \Delta)$  of  $U$  and satisfies the  $H_\infty$  performance (8) for all  $\Delta \in U$ , if there exist two real scalar  $\varepsilon_1$  and  $\varepsilon_2$  and a symmetric positive matrix  $P(\Delta) = P(\Delta)^T > 0$  such that

$$\begin{aligned} \text{Spr}(P(\Delta), \varepsilon_1, L(z, \Delta) + \gamma^{-1}S(z, \Delta), E(z, \Delta)) &> 0, \\ \text{Spr}(P(\Delta), \varepsilon_2, L(z, \Delta) - \gamma^{-1}S(z, \Delta), E(z, \Delta)) &> 0. \end{aligned} \quad (12)$$

*Proof* The proof is a direct application of Lemma 2.3 and Proposition 3.5.  $\square$

It is clear that the choice of  $E(z, \Delta)$ , central polynomial, plays an important role on the conservatism of the approach. In what follows, this choice is discussed:

- (1) The simplest choice is a fixed central polynomial (not a function of  $\Delta$ ). This choice leads to conservative results as will be seen in the simulation example of Section 5.
- (2) The conservatism can be reduced if a fixed-order controller  $K_0(z) = X_0(z)/Y_0(z)$  is available which stabilizes the LPD system for all  $\Delta \in U$  (without any specific performance). This controller can be computed e.g. by the approach in Henrion et al. (2001) or can be the conservative controller computed by a fixed central polynomial (the simplest choice explained above). Then a better choice for the central polynomial is :

$$E(z, \Delta) = X_0(z)N(z, \Delta) + Y_0(z)M(z, \Delta) \quad (13)$$

This is an LPD polynomial which is stable for all  $\Delta \in U$ .

**Remark:** It should be mentioned that the LPD central polynomial can also be employed for robust fixed-order controller design using the LMIs developed in Henrion (2003) and Yang et al. (2007).

#### 4 Finite Dimensional Convex Parameterization

The robust control design problem for LPD systems was studied in the previous section. Consequently, fixed-order  $H_\infty$  controller design for systems with ellipsoidal parametric uncertainty can also be presented as an infinite dimensional convex optimization problem. At this level of generality, this problem is hardly solvable. In order to simplify the problem, one possibility is to consider a common Lyapunov matrix for all models in the model set, i.e.,  $\forall \Delta \in U, P = P(\Delta)$ . This constraint causes too conservatism. A modified approach consists in searching for parameter dependent Lyapunov matrices that leads to less conservative results, see e.g. Gahinet et al. (1996), Feron et al. (1996), de Oliveira et al. (1999), Oliveira and Peres (2006) and Chesi et al. (2005). We concentrate on the following type of parameter dependent Lyapunov matrix:

$$P(\Delta) = (I_t \otimes \Delta^T)P_q(I_t \otimes \Delta). \quad (14)$$

This type of parameter dependent Lyapunov matrix has been used in Peaucelle and Arzelier (2001) for robust performance analysis for systems with real parametric uncertainty.

Theorem 3.6 is constructed based on the matrix inequality (9). For systems with ellipsoidal parametric uncertainty, we can introduce a finite dimensional approximation of (9). Accordingly, matrices  $\Pi$ ,  $\Upsilon(\Delta)$ ,  $I_t$ ,  $R$ , and  $\Psi(\Delta)$  can be factorized as follows:

$$\Pi = (I_{t+1} \otimes \Delta^T)\Pi_q, \quad \Upsilon(\Delta) = (I_{t+1} \otimes \Delta^T)\Upsilon_q, \quad I_t = (I_t \otimes \Delta^T)\Lambda_q,$$



$$\Psi(\Delta) = \Delta^T \Psi_q(I_t \otimes \Delta), \quad R = R_q(I_t \otimes \Delta).$$

Note that matrices  $\Pi$ ,  $I_t$  and  $R$  are not really functions of  $\Delta$ . However, this type of factorization will help us to further simplify the equations. Matrices  $\Pi_q$ ,  $\Upsilon_q$ ,  $\Lambda_q$ ,  $R_q$ , and  $\Psi_q$  are given below:

$$R_q = \left[ \begin{array}{c} (0_{(n+1) \times n} \ r_0) \ (0_{(n+1) \times n} \ r_1) \ \cdots \ (0_{(n+1) \times n} \ r_{t-1}) \end{array} \right],$$

$$\Lambda_q = \left[ \begin{array}{ccc} \begin{pmatrix} 0_{n \times 1} \\ 1 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} 0_{n \times 1} \\ 1 \end{pmatrix} \end{array} \right], \quad \Upsilon_q = \left[ \begin{array}{c|c} 0_{((n+1)(t+1)) \times (t-1)} & \begin{matrix} -e_0 \\ -e_1 \\ \vdots \\ -e_{t-1} \end{matrix} \\ \hline 0_{(n+1) \times 1} & \end{array} \right].$$

$$\Pi_q = \left[ \begin{array}{c} 0_{(n+1) \times t} \\ \hline \Lambda_q \end{array} \right], \quad \Psi_q = \left[ 0_{(n+1) \times (nt+t-1)} \ r_t \right],$$

Using these notations, (9) can be written as

$$(I_{2t+1} \otimes \Delta^T) H_q(P, \varepsilon, R, E, r_t) (I_{2t+1} \otimes \Delta) > 0, \quad (15)$$

where,

$$H_q(P, \varepsilon, R, E, r_t) = \left[ \begin{array}{ccc} P_q & R_q^T & \mathbf{0} \\ R_q & \mathbf{0} & \Psi_q \\ \mathbf{0} & \Psi_q^T & -P_q \end{array} \right] + \varepsilon \left[ \begin{array}{c} \Pi_q + \Upsilon_q \\ \hline -\Lambda_q \end{array} \right] \left[ \Pi_q^T + \Upsilon_q^T \mid -\Lambda_q^T \right]. \quad (16)$$

Since, we have imposed a special structure for Lyapunov function  $P(\Delta)$ , the satisfaction of (15) is only a sufficient condition for inequality (9). Now, the inequality (15) should be held for all parameters such that  $\Delta^T U_0 \Delta \geq 0$ . This problem is frequently encountered in robust control and is treated with the so-called *S*-Procedure Boyd et al. (1994). Similar to the papers by Henrion et al. (2003) and Henrion et al. (2001), *S*-procedure can be applied to our problem as follows.

*Lemma 4.1* Statement (2) implies statement (1).

- (1)  $(I_f \otimes \Delta^T) X_q(I_f \otimes \Delta) > 0$  is satisfied for all vectors  $\Delta$  such that  $\Delta^T U_0 \Delta \geq 0$ .
- (2) There exist a scalar  $\tau > 0 \in \mathbb{R}$  and symmetric block matrix

$$T = \left[ \begin{array}{cccc} \mathbf{0} & T_{21}^T & \cdots & T_{f1}^T \\ T_{21} & \mathbf{0} & & T_{f2}^T \\ \vdots & & \ddots & \vdots \\ T_{f1} & T_{f2} & \cdots & \mathbf{0} \end{array} \right], \quad (17)$$

made up of skew-symmetric matrices  $T_{ij} = -T_{ij}^T \in \mathbb{R}^{(n+1) \times (n+1)}$  satisfying

$$X_q - \tau(I_f \otimes U_0) + T > 0. \quad (18)$$

*Proof* For any vector  $\Delta$ , since  $I_f \otimes \Delta^T$  is full row rank, (18) implies that

$$(I_f \otimes \Delta^T)(X_q - \tau(I_f \otimes U_0) + T)(I_f \otimes \Delta) > 0.$$



It can be written as

$$(I_f \otimes \Delta^T)X_q(I_f \otimes \Delta) > \tau(I_f \otimes \Delta^T)(I_f \otimes U_0)(I_f \otimes \Delta). \quad (19)$$

Additionally, it is easy to see that  $\Delta^T U_0 \Delta \geq 0$  is equivalent to  $(I_f \otimes \Delta^T)(I_f \otimes U_0)(I_f \otimes \Delta) \geq 0$ . This means that for all vectors  $\Delta$  such that  $\Delta^T U_0 \Delta \geq 0$  is satisfied, the right hand side of (19) is positive and this results in the statement (1), which ends the proof. A more general treatment for this kind of  $S$ -procedure application is given in Henrion et al. (2003).  $\square$

To conclude this section, we should mention that the infinite dimensional condition  $Spr(P(\Delta), \varepsilon, R(z, \Delta), E(z, \Delta)) > 0$ , given by (9), should be replaced by the existence of a  $\tau_{spr} > 0$  and a matrix  $T_{spr}$  such that

$$H_q(P, \varepsilon, R, E, r_t) - \tau_{spr}(I_{2t+1} \otimes U_0) + T_{spr} > 0, \quad (20)$$

where  $T_{spr}$  is given by (17) with  $f = 2t + 1$ . We had in Theorem 3.6 that  $P(\Delta)$  should be a positive matrix. Taking into account (14) and Lemma 4.1,  $P(\Delta) > 0$ , for all  $\Delta \in U$  can be replaced by

$$P_q - \tau_p(I_t \otimes U_0) + T_p > 0, \quad (21)$$

where  $\tau_p$  is a positive real variable and  $T_p$  is defined by (17) with  $f = t$ . It should be mentioned that  $r_i$  for  $i = 1, \dots, t$  are affine functions of controller parameters.

## 5 Numerical Illustration

In order to illustrate our results, we consider the uncertain system of the simulation example studied in Henrion et al. (2001). The identification of the nominal model, using PE framework, results in the following second order discrete time model

$$P(z) = \frac{\theta_0 z + \theta_1}{z^2 + \theta_2 z + \theta_3}$$

with nominal parameter vector  $\bar{\theta} = [0.0028 \ 0.0038 \ -1.1871 \ 0.2087]^T$ , and the ellipsoidal parametric uncertainty defined by (1). Where

$$R_e = 10^5 \begin{pmatrix} 2.4121 & 0.0568 & 0.0062 & 0.0045 \\ 0.0568 & 2.4179 & 0 & 0.0069 \\ 0.0062 & 0 & 0.0015 & 0.0014 \\ 0.0045 & 0.0069 & 0.0014 & 0.0015 \end{pmatrix}.$$

The objective is to design a first order controller

$$K = \frac{x_0 z + x_1}{z + y_1}$$

such that the upper bound  $\gamma$  on the  $H_\infty$  norm of the closed-loop sensitivity function is minimized,

$$\left\| \frac{1}{1 + K(z)P(z)} \right\|_\infty = \left\| \frac{S(z, \Delta)}{L(z, \Delta)} \right\|_\infty < \gamma.$$

Therefore, we have

$$S(z, \Delta) = \Delta^T s_3 z^3 + \Delta^T s_2 z^2 + \Delta^T s_1 z + \Delta^T s_0$$

$$L(z, \Delta) = \Delta^T l_3 z^3 + \Delta^T l_2 z^2 + \Delta^T l_1 z + \Delta^T l_0$$

and  $l_3 = s_3 = (0 \ 0 \ 0 \ 0 \ 1)^T$ , where

$$S = (s_0 \ s_1 \ s_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & y_1 & 1 \\ y_1 & 1 & 0 \\ 0 & 0 & y_1 \end{pmatrix} \quad L = (l_0 \ l_1 \ l_2) = \begin{pmatrix} 0 & x_1 & x_0 \\ x_1 & x_0 & 0 \\ 0 & y_1 & 1 \\ y_1 & 1 & 0 \\ 0 & 0 & y_1 \end{pmatrix}$$

To convexify the fixed-order robust control design problem, we consider an LPD central polynomial. Based on the approach proposed in Khatibi and Karimi (2009), the following controller is designed for nominal model without considering the parametric uncertainty.

$$K_0 = \frac{50.5540z - 11.1954}{z + 0.6801}$$

Accordingly, we can consider the following LPD central polynomial,

$$E(z, \Delta) = \Delta^T e_3 z^3 + \Delta^T e_2 z^2 + \Delta^T e_1 z + \Delta^T e_0 = L(z, \Delta)|_{K=K_0}.$$

Consequently,  $e_3 = (0 \ 0 \ 0 \ 0 \ 1)^T$  and

$$E_{LPD} = (e_0 \ e_1 \ e_2) = L|_{K=K_0} = \begin{pmatrix} 0 & -11.1954 & 50.5540 \\ -11.1954 & 50.5540 & 0 \\ 0 & 0.6801 & 1 \\ 0.6801 & 1 & 0 \\ 0 & 0 & 0.6801 \end{pmatrix}.$$

Taking into account Theorem 3.6 and the results of Section 4, robust control design problem is presented by the following optimization program:

$$\min_{x_0, x_1, y_1, \gamma, \tau_1, \tau_2, \tau_p, \varepsilon_1, \varepsilon_2, T_{spr1}, T_{spr2}, T_{spr3}, P} \gamma \quad (22)$$

subject to:

$$H_q(P, \varepsilon_1, L - \gamma^{-1}S, E_{LPD}, l_3 - \gamma^{-1}s_3) - \tau_1(I_7 \otimes U_0) + T_{spr1} > 0,$$

$$H_q(P, \varepsilon_2, L + \gamma^{-1}S, E_{LPD}, l_3 + \gamma^{-1}s_3) - \tau_2(I_7 \otimes U_0) + T_{spr2} > 0,$$

$$P_q - \tau_p(I_3 \otimes U_0) + T_{spr3} > 0,$$

$$\tau_1, \tau_2, \tau_p > 0,$$

where  $T_{spr1}$ ,  $T_{spr2}$ , and  $T_{spr3}$  are given by (17) and  $U_0$  is defined by (4).

The smallest feasible  $\gamma$ , which is obtained by a line search, is the optimal solution to this optimization problem. The use of YALMIP, Löfberg (2004), with SDPT3, Toh et al. (1999),

	Fixed central Poly.	LPD Central Poly.
Quadratically dep. Lya. Fun.	1.295	1.173
Linearly dep. Lya. Fun.	1.42	1.175
Common Lya. Fun.	1.715	1.455

Table 1.  $\gamma_{opt}$  for different choices of central polynomial and Lyapunov function structure

results in the optimal value of  $\gamma_{opt} = 1.173$  with the following controller

$$K_{LPD} = \frac{39.5981z - 8.4412}{z + 0.7310}$$

This problem has been also solved with a fixed central polynomial equal to

$$E(z, \Delta)|_{K=K_0, \Delta=(\bar{\theta} \ 1)^T} = z^3 - 0.3655z^2 - 0.4378z + 0.09939,$$

Therefore,

$$E_{FIX} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.09939 & -0.4378 & -0.3655 \end{pmatrix}.$$

Using this fixed central polynomial, the convex optimization in (22) results in  $\gamma_{opt} = 1.295$ .

We have enforced LMI structural constraints on Lyapunov function to study the impacts of the structure of Lyapunov function on the results. Therefore, both of linearly dependent Lyapunov function approach and common Lyapunov function approach for two cases of LPD central polynomial and fixed central polynomial are considered. The results are summarized in Table 1. The first column shows the results related to the fixed central polynomial, whereas the second column represents the results for the LPD central polynomial. It is obvious that the LPD central polynomial leads to significant improvement of performance.

## 6 Conclusions

Fixed-order robust  $H_\infty$  controller design for systems with ellipsoidal parametric uncertainty is considered. The results can also be used when uncertain real parameters appear linearly in the characteristic polynomial. A convex semi-infinite program for robust  $H_\infty$  synthesis problem is proposed thanks to a new concept of common Lyapunov Strictly Positive Realness. The semi-infinite program is transformed into a semidefinite program for systems with ellipsoidal parametric uncertainty using parameter dependent Lyapunov functions. The nonconvexity of fixed-order robust controller design is captured into a tunable parameter dependent central polynomial. The simulation results show that this kind of parameter dependent central polynomial reduces the conservatism of the robust control design problem in comparison with methods that consider fixed central polynomials.

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