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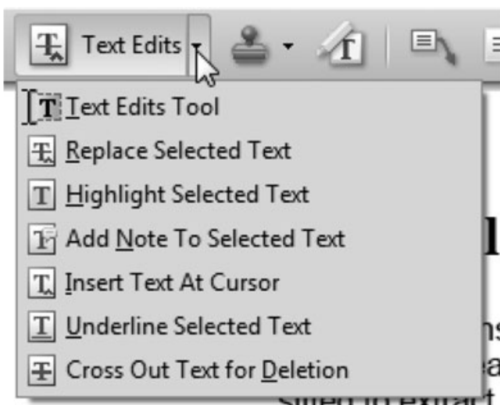

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Bounds and invariant sets for a class of discrete-time switching systems with perturbations

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We present a novel method to compute componentwise ultimate bounds and invariant regions for a class of switching discrete-time linear systems with perturbation bounds that may depend nonlinearly on a delayed state. The method has the important advantage that it allows each component of the perturbation vector to have an independent bound and that the bounds and sets obtained are also given componentwise. This componentwise method does not employ a standard norm for bounding either the perturbation or state vectors, and thus may avoid conservativeness due to different perturbation or state vector components having substantially different bounds. We also establish the relationship between the class of switching linear systems to which the proposed method can be applied and those that admit a common quadratic Lyapunov function. We illustrate the application of our method via numerical examples, including the fault tolerance analysis of the feedback control of a winding machine.

Keywords: switching systems; ultimate bounds; invariant sets; componentwise methods; practical stability; fault tolerant control

1. Introduction

Switched systems are systems whose dynamics change between a finite number of individual dynamics according to a switching rule. The stability of switched systems is a topic of current research interest (see, e.g. Liberzon, 2003; Lin & Antsaklis, 2009; Margaliot, 2006). A particular type of problem is that of stability under 'arbitrary switching', which refers to problems where the stability properties of interest hold for every possible switching signal. Switched systems undergoing arbitrary switching are referred to as *switching systems*. In the present paper, we will focus on a specific class of switching systems, and we consider the 'practical stability' problem of analysing the existence and computation of invariant sets and ultimate bounds for the system-state trajectories. This type of stability is important in every practical setting where non-vanishing perturbations may act on the system (Khalil, 2002, Chapter 9). The class of discrete-time switching systems considered is that having a switching linear nominal (unperturbed) system for which a matrix constructed from the subsystems' A matrices is Schur stable, affected by perturbations that may be non-vanishing and depend nonlinearly on a delayed state.

Standard methods for the computation of bounds and invariant sets make use of Lyapunov functions (Khalil, 2002). Lyapunov function-based methods are very powerful and widely applicable, although finding a suitable Lyapunov function is a difficult problem in general. When the nominal system is linear, however, a quadratic Lyapunov function can easily be computed as the solution to a Lyapunov equation. Likewise, for switching systems with a switching linear nominal system, a quadratic Lyapunov function common to all linear subsystems can be computed via linear matrix inequalities (LMIs) in case one exists (see, e.g. Lin & Antsaklis, 2009, and the references therein). State bounds computed by means of a quadratic Lyapunov function are given as a bound on the norm, typically the 2-norm, of the state vector and usually require a bound on the norm of the perturbation vector. Substantial conservativeness may thus be introduced since the information on the different bounds for each component of the perturbation vector is lost when taking its norm; in addition, the bounds corresponding to different components of the state vector may be largely dissimilar and hence its 2-norm will not yield tight bounds.

In this paper, we propose a methodology based on *componentwise analysis* which differs from the one just described in that the use of either a norm of the state or a Lyapunov function can be avoided. Moreover, this componentwise methodology can be easily combined with Lyapunov analysis, and/or other methods, such as those based on set-theoretic tools (Ghaemi, Kolmanovsky, & Sun, 2011; Olaru, De Doná, Seron, & Stoican, 2010), in order to possibly improve on the results of either method applied individually. The current paper builds upon and extends to discrete-time switching systems with delayed state-dependent perturbations previous results of Kofman, Haimovich, and Seron (2007); Kofman, Seron, and

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Haimovich (2008); Haimovich and Seron (2009, 2010), and it contains the discrete-time counterpart to Haimovich and Seron (2013). In our initial work (Kofman et al., 2007), a method to compute componentwise ultimate bounds for perturbed (non-switching) linear systems is given for perturbation bounds that may depend nonlinearly on the system state. The case of perturbation bounds that have affine dependence on a delayed system state is treated in Section 3 of Kofman et al. (2008), where a sufficient condition for practical stability is also provided. In Haimovich and Seron (2009, 2010), a method to derive componentwise transient and ultimate bounds was proposed for a class of switching linear systems with constant perturbation bounds. It was shown in Haimovich and Seron (2009, 2010) that the proposed method can be applied when the switching linear system is close to being simultaneously triangularisable (see Definition 2.4 in Section 2.3 for the definition of simultaneous triangularisation). In such a case, a common quadratic Lyapunov function (CQLF) exists for the switching system. However, the precise relationship between the class of switching linear systems to which the proposed method can be applied and those that admit a CQLF was left as an open question.

The present paper derives results analogous to those of Haimovich and Seron (2013) but for discrete-time systems, and hence its main contribution is to extend the aforementioned previous results (Haimovich & Seron, 2009, 2010; Kofman et al., 2007, 2008) by providing ultimate bounds and invariant regions based on componentwise analysis for a class of discrete-time switching linear systems with perturbation bounds that may depend nonlinearly on a delayed state. This kind of setting can describe, for example, switching linear systems with uncertainty in the state evolution matrix, switching linear systems with an uncertain time delay and, more generally, switching nonlinear systems expressed as their switching linear approximation perturbed by an additive disturbance with a bound depending nonlinearly on the system state. A second contribution of the current paper is to show that the class of discrete-time switching linear systems to which our componentwise bound and invariant set method can be applied is strictly contained in the class of switching linear systems that admit a CQLF, although the switching linear system need not be close to simultaneously triangularisable. This relationship between the class of systems considered and those that admit a CQLF was actually reported in Mori, Mori, and Kuroe (2001) but the proof was not given. Moreover, our method yields one admissible CQLF. In addition, both the componentwise method and the Lyapunov technique can be combined to obtain tighter bounds than could be obtained by either methodology applied individually. Some of the results in the current paper have been presented in Haimovich and Seron (2011a, 2011b).

The remainder of the paper proceeds as follows. We conclude this introductory section with a notation summary. Section 2 motivates and presents the problem formulation, together with some preliminary definitions and properties. Section 3 contains the main results of the paper, and is organised into three subsections presenting, respectively, the new results for the case of nonlinear perturbation bounds, the connection between the latter results and the existence of a CQLF and the new results for the special case of affine perturbation bounds, including the connection with CQLF when no delay is present. Section 4 illustrates the results by means of academic numerical examples. Application of our results to the analysis of fault tolerance of a real winding machine control system is provided in Section 5. Conclusions and directions for future work are given in Section 6. To ease readability, some of the proofs are provided in the Appendix.

Notation. $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of integer, real and complex numbers, respectively, and 0 denotes the zero scalar, vector or matrix, depending on the context. $\mathbb{R}_+$ and $\mathbb{R}_{+0}$ denote the positive and non-negative real numbers, respectively, and similarly for $\mathbb{Z}_+$ and $\mathbb{Z}_{+0}$. If M is a matrix, then M' denotes its transpose, M^* its conjugate transpose and $|M|$ is the matrix whose entries are the magnitude of the corresponding entries in M. If P is a square matrix, then $\rho(P)$ denotes its spectral radius, and $P > 0$ ($P < 0$) means that P is positive (negative) definite. If $x(t)$ is a vector-valued function, then $\limsup_{t\to\infty} x(t)$ denotes the vector obtained by taking $\limsup_{t\to\infty}$ of each component of $x(t)$. Similarly, 'lim' and 'max' denote componentwise operations on a vector or a matrix. The expression $x \preceq y$ ($x \prec y$) denotes the set of componentwise inequalities $x_i \le y_i$ ($x_i < y_i$) between the elements of the real vectors x and y, and similarly for $x \succeq y$ ($x \succ y$) and in the case when x and y are matrices. If $T : \mathbb{R}^n_{+0} \to \mathbb{R}^n_{+0}$, then T^k denotes the iteration of T, that is the maps defined by $T^1(x) = T(x)$ and $T^{k+1}(x) = T(T^k(x))$. The index set $\{1, 2, \ldots, N\}$ is denoted $\underline{N}$ and $\mathbf{i}$ denotes $\sqrt{-1}$. Employing this notation, note that $P \succ 0$ means that every entry of P is positive and $P > 0$ that P is positive definite.

2. Problem formulation

In this section, we formulate the problem to be addressed. We begin by motivating the setting considered with an example of a real system. The example also illustrates the applicability of ultimate bounds and invariant sets in fault detection and fault tolerant control; see also Stoican, Olaru, Seron, and De Doná (2010), Olaru et al. (2010) and Seron and De Doná (2010).

2.1 Motivating example

Consider the winding machine application presented in Noura, Theilliol, Ponsart, and Chamseddine (2009, Chapter 3), consisting of three reels driven by direct current motors: the unwinding reel motor M_1, the traction reel motor M_2 and the rewinding reel motor M_3. The state $x \triangleq (x^1, x^2, x^3) = (T_1, \Omega_2, T_3)$ of the system consists of the strip tensions between the reels (T_1 and T_3) and the angular velocity Ω_2 of motor M_2. The control input $u = (I_1, U_2, I_3)$ consists of the current set points (I_1 and I_3) of local torque controllers used for motors M_1 and M_3 and the input voltage U_2 of motor M_2. The nominal linearised (incremental) model around the operating point x_0, u_0, discretised with a sampling period t_s has the form (Noura et al., 2009)

$$x(t+1) = A_0 x(t) + B_0 u(t), \tag{1}$$

where $x(t) \in \mathbb{R}^3$ and $u(t) \in \mathbb{R}^3$. Suppose that a state-feedback controller $u(t) = K_0 x(t)$ has been designed for the above system (e.g. to achieve some desired closed-loop performance specifications), but the states are measured via sensors that can be prone to errors. Hence, the actual control applied to the system has the form,

$$u(t) = K_0(x(t) + w(t)), \tag{2}$$

where $w(t) \in \mathbb{R}^3$ models sensor measurement errors. In addition to sensor measurement errors, we consider the possible occurrence, at arbitrary times, of outages of any one of the actuators, so that the system models (1) and (2) are modified as

$$\begin{aligned} x(t+1) &= A_0 x(t) + B_{\sigma(t)} u(t) \\ &= (A_0 + B_{\sigma(t)} K_0) x(t) + B_{\sigma(t)} K_0 w(t) \qquad (3) \\ &= A_{\sigma(t)} x(t) + H_{\sigma(t)} w(t), \qquad (4) \end{aligned}$$

where $\sigma(t) \in \{1, 2, 3, 4\}$, $B_4 = B_0$ is the 'fault-free' input matrix, and each B_i, $i = 1, 2, 3$, models the outage of the ith actuator and is obtained from B_0 by setting to zero its ith column. Considering, for example, that the measurement of x^1 is affected by drift, that of x^2 by bounded noise and that of x^3 by an uncompensated sinusoidal nonlinearity, the sensor measurement error vector $w(t)$ is assumed to have the componentwise bound,

$$|w(t)| \preceq \begin{bmatrix} \alpha |x^1(t)| \\ \nu \\ \beta |\sin x^3(t)| \end{bmatrix}, \tag{5}$$

where α, β and ν are some positive real numbers. Note that Equations (4) and (5) constitute a discrete-time switching linear system with a perturbation having a componentwise bound that depends nonlinearly on the system state. The setting that we consider in the current paper corresponds to a slight extension of that motivated by this example. This example will be revisited in Section 5, where we will apply the tools developed in the current paper to the analysis of the tolerance of the winding machine control to faults in the actuators.

2.2 Problem statement

We consider switching discrete-time perturbed systems of the form,

$$x(t+1) = A_{\sigma(t)} x(t) + H_{\sigma(t)} w_{\sigma(t)}(t), \tag{6}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $\sigma(t) \in \underline{N} \triangleq \{1, 2, \ldots, N\}$ is the switching function, $A_i \in \mathbb{R}^{n\times n}$, $H_i \in \mathbb{R}^{n\times k_i}$ for $i \in \underline{N}$ and the perturbation vectors $w_i(t) \in \mathbb{R}^{k_i}$ satisfy the componentwise bound,

$$|w_i(t)| \preceq \delta_i(\theta(t)) \ \text{ for all } t \geq 0, \quad \text{for } i \in \underline{N}, \tag{7}$$

with continuous bounding functions $\delta_i : \mathbb{R}^n_{+0} \to \mathbb{R}^{k_i}_{+0}$ and $\theta(t) \in \mathbb{R}^n_{+0}$ defined as

$$\theta(t) \triangleq \max_{t-\bar{\tau} \leq \tau \leq t} |x(\tau)|, \tag{8}$$

where $\bar{\tau} \geq 0$ and the maximum is taken componentwise. Note that for each $i \in \underline{N}$, Equation (7) expresses a bound for each one of the k_i components of the perturbation vector $w_i(t)$, and that the maximum in Equation (8) denotes a componentwise operation.

The settings (6)–(8) extend the setting motivated by Equations (4) and (5) in the following two directions: (1) it allows the dimension of the perturbation vector to be different for every switching mode of the system (hence this vector is denoted $w_{\sigma(t)}(t)$ and not just $w(t)$) and (2) it allows the perturbation bound to depend on previous values of the state. These two extensions are included because they require only minor variations in our derivations. In addition, the settings (6)–(8) can describe, inter alia, the following situations.

- Uncertainty in the system evolution matrix, where $x(t+1) = (A_{\sigma(t)} + \Delta A_{\sigma(t)}(t))x(t)$, and $|\Delta A_i(t)| \preceq \overline{\Delta A_i}$, for all $t \geq 0$ and $i \in \underline{N}$; in this case, we can take $H_i = \mathrm{I}$ in Equation (6), $\delta_i(\theta) = \overline{\Delta A}\theta$ in Equation (7), and $\bar{\tau} = 0$ in Equation (8).
- Uncertain time delays, where $w_i(t) = F_i x(t - \tau_i)$, and $0 \leq \tau_i \leq \tau_{\max}$; in this case, we can take $\delta_i(\theta) = |F_i|\theta$ in Equation (7), and $\bar{\tau} = \tau_{\max}$ in Equation (8).
- Disturbances with constant bounds: $\delta_i(\theta) = \mathbf{w}_i$ in Equation (7).
- Switching nonlinear systems where $x(t+1) = f_{\sigma(t)}(x(t))$; in this case, we may take $A_i = \frac{\partial f_i}{\partial x}(x_0)$, $H_i = \mathrm{I}$, $\bar{\tau} = 0$, $\delta_i(\theta) = \max_{x:\,|x| \preceq \theta} |f_i(x) - A_i x|$.

The problem of interest is to derive transient bounds, ultimate bounds and invariant sets for switching systems of the form (6) with perturbations bounded as in Equations (7) and (8). This will be addressed in Section 3. In the next subsection, we give some definitions and preliminary results.

2.3 *Definitions and properties*

Definition 2.1: A non-negative vector function $f : \mathbb{R}^n_{+0} \to \mathbb{R}^m_{+0}$ is said to be *componentwise non-increasing* (CNI) if, whenever $x_1, x_2 \in \mathbb{R}^n_{+0}$ and $x_1 \preceq x_2$, then $f(x_1) \preceq f(x_2)$.

Remark 1: Every continuous function $\hat{f} : \mathbb{R}^n_{+0} \to \mathbb{R}^m_{+0}$ can be overbounded by a continuous CNI function. In particular, the tightest continuous CNI overbound of $\hat{f}$ is the function $f : \mathbb{R}^n_{+0} \to \mathbb{R}^m_{+0}$ given by

$$f(x) = \max_{0 \preceq y \preceq x} \hat{f}(y). \tag{9}$$

The following two lemmas provide properties of CNI functions that are required throughout the paper. These results and their proofs appear in Haimovich and Seron (2013). For the sake of completeness, we include the corresponding proofs in the Appendix at the end of the current paper.

Lemma 2.2: *Let $f : \mathbb{R}^n_{+0} \to \mathbb{R}^n_{+0}$ be a continuous CNI function and suppose that there exists $\beta \in \mathbb{R}^n_{+0}$ satisfying $f(\beta) \preceq \beta$. Then,*

(*i*) *for every $k \in \mathbb{Z}_+$, $f^{k+1}(\beta) \preceq f^k(\beta)$ and*

$$\lim_{k\to\infty} f^k(\beta) = b \succeq 0. \tag{10}$$

(*ii*) *For every $\epsilon \in \mathbb{R}^n_+$, there exist $k = k(\epsilon) \in \mathbb{Z}_+$ and $\gamma = \gamma(\epsilon) \in \mathbb{R}^n_+$, such that $f^k_\gamma(\beta) \prec b + \epsilon$, where b is as in Equation* (10) *and $f_\gamma(x) \triangleq f(x) + \gamma$, $\forall x \in \mathbb{R}^n_{+0}$.*

Lemma 2.3: *Consider the affine function $\ell(x) \triangleq Rx + r$, where $r \in \mathbb{R}^n_{+0}$ and $R \in \mathbb{R}^{n\times n}_{+0}$ is such that $\boldsymbol{\rho}(R) < 1$. Then,*

(*i*) *The function $\ell : \mathbb{R}^n_{+0} \to \mathbb{R}^n_{+0}$ is CNI.*
(*ii*) *For all $\beta \in \mathbb{R}^n_{+0}$, $\lim_{k\to\infty} \ell^k(\beta) = \tilde{b} = \ell(\tilde{b})$, where*

$$\tilde{b} = (I - R)^{-1} r. \tag{11}$$

(*iii*) *For every $v \in \mathbb{R}^n_{+0}$, there exists $\beta \in \mathbb{R}^n_+$ satisfying*

$$\ell(\beta) + v \prec \beta. \tag{12}$$

(*iv*) *Let $f : \mathbb{R}^n_{+0} \to \mathbb{R}^n_{+0}$ be a continuous CNI function satisfying $f(x) \preceq \ell(x)$ for all $x \in \mathbb{R}^n_{+0}$. Let $\beta \in \mathbb{R}^n_+$ be such that Equation* (12) *holds for some $v \in \mathbb{R}^n_{+0}$, and let $\tilde{b}$ be as in Equation* (11). *Then, Equation* (10) *holds and, in addition,*

$$b = \lim_{k\to\infty} f^k(\tilde{b}) \preceq \tilde{b}. \tag{13}$$

For completeness, we include the following definition.

Definition 2.4: A set of square matrices $\{A_i \in \mathbb{C}^{n\times n} : i = 1, 2, \ldots, N\}$ is said to be simultaneously triangularisable if an invertible $V \in \mathbb{C}^{n\times n}$ exists such that $V^{-1}A_iV$ is upper triangular for all $i = 1, 2, \ldots, N$.

3. Bounds and invariant sets

In this section, we present the main results of the paper. We provide novel transient bounds, ultimate bounds and invariant sets for a class of discrete-time switching linear systems with perturbations bounded by a possibly nonlinear function of a delayed state. We also establish the link between the applicability of the proposed results and that of the CQLF.

3.1 *Delayed state-dependent perturbation bounds*

In this section, we consider system (6) with perturbation bound of the forms (7) and (8), where the bounding functions δ_i are CNI satisfying some mild properties. Theorem 3.1 below gives invariant sets, transient and ultimate bounds. The proof is given in Section A.2 in the Appendix.

Theorem 3.1: *Consider the system* (6) *with perturbation bound given by Equations* (7) *and* (8), *where the bounding functions δ_i are CNI. Let $V \in \mathbb{C}^{n\times n}$ be invertible and consider*

$$\Lambda \triangleq \max_{i\in\underline{N}} M_i, \quad M_i \triangleq |\Lambda_i|, \quad \Lambda_i \triangleq V^{-1}A_iV. \tag{14}$$

Let $\psi : \mathbb{R}^n_{+0} \to \mathbb{R}^n_{+0}$ be defined by Equation (15), *let $\delta : \mathbb{R}^n_{+0} \to \mathbb{R}^n_{+0}$ be continuous, CNI and satisfy Equation* (16), *and consider the transformation $T : \mathbb{R}^n_{+0} \to \mathbb{R}^n_{+0}$ given by Equation* (17).

$$\psi(x) = \max_{i\in\underline{N}} \left[\max_{w:|w|\preceq\delta_i(|V|x)} |V^{-1}H_iw| \right], \tag{15}$$

$$\delta(x) \succeq \psi(x), \tag{16}$$

$$T(x) = \Lambda x + \delta(x). \tag{17}$$

Suppose that there exists $\beta \in \mathbb{R}^n_{+0}$ such that $T(\beta) \preceq \beta$. Then,

(*a*) *$T^{k+1}(\beta) \preceq T^k(\beta)$ for all $k \geq 0$ and $\lim_{k\to\infty} T^k(\beta) = b \succeq 0$.*

(*b*) *Invariance. For every $k \geq 0$ and for all $t \geq -\bar{\tau}$, the state is bounded as*

$$|V^{-1}x(t)| \preceq T^k(\beta), \tag{18}$$

provided $|V^{-1}x(t)| \preceq T^k(\beta)$ for all $-\bar{\tau} \leq t \leq 0$.

(*c*) *Transient bounds. For every $k \geq 0$,*

$$|V^{-1}x(t)| \preceq T^k(\beta), \quad \textit{for all } t \geq (k-1)(\bar{\tau}+1)+1.$$

provided $|V^{-1}x(t)| \preceq \beta$ for all $-\bar{\tau} \leq t \leq 0$.

(*d*) *Ultimate bounds. The state is ultimately bounded as*

$$\limsup_{t\to\infty} |V^{-1}x(t)| \preceq b, \tag{19}$$

whenever $|V^{-1}x(t)| \preceq \beta$ for all $-\bar{\tau} \leq t \leq 0$.

Theorem 3.1 requires a non-negative vector β satisfying $T(\beta) \preceq \beta$. It is thus sufficient to find β such that $T(\beta) = \beta$. If such a vector exists, then it can be computed by iterating T starting from 0, as shown in Theorem 5 of Kofman et al. (2007). Alternatively, we may seek β satisfying $T(\beta) \prec \beta$ by means of Algorithm 1 and Theorem 3 of Kofman et al. (2007). Also, note that a non-negative vector β satisfying $T(\beta) \preceq \beta$ but so that $T(\beta) \neq \beta$ and $T(\beta) \not\prec \beta$ may also exist.

Theorem 3.1(a) establishes that the iteration of the map T on the vector β constitutes a componentwise non-increasing sequence and converges to a non-negative vector b. Theorem 3.1(b) establishes that any iteration $T^k(\beta)$, for $k \geq 0$, defines a set with bounds given by Equation (18) and having the invariance property that, if the state has remained in the set for the previous $\bar{\tau}+1$ time steps, then it will remain in the set thereafter. Theorem 3.1(c) gives state bounds that are valid at every time instant. In addition, Theorem 3.1(d) provides local ultimate bounds, i.e. ultimate bounds that are valid only when the state has remained in a specific region during $\bar{\tau}+1$ steps.

The current discrete-time results of Theorem 3.1, though analogous to the continuous-time case addressed in Theorem 4 of Haimovich and Seron (2013), have three main differences with respect to it. In the continuous-time case, the matrix Λ is also constructed from the transformation V and the subsystems' matrices A_i, but the elementwise magnitude on Λ_i in Equation (14) is replaced by another elementwise operation (cf. Equations (4) and (7) of Haimovich and Seron (2013)). Then, the first main difference is that in the continuous-time case the matrix Λ is required to be (Hurwitz) stable but in the current discrete-time case the matrix Λ need not be (Schur) stable. The second difference is that the vector β required by Theorem 3.1 needs only satisfy $T(\beta) \preceq \beta$, whereas in Theorem 4 of Haimovich and Seron (2013) such a vector needs to satisfy the more stringent condition $T(\beta) \prec \beta$. The third main difference is that for discrete time the ultimate bounds corresponding to constant perturbation bounds can be derived as a special case of application of Theorem 3.1, whereas in continuous time the results for constant perturbation bounds are needed in order to derive those for the more general perturbation bounds.

The ultimate bounds corresponding to constant perturbation bounds are derived in the following corollary. Note that this corollary does require the matrix Λ to be (Schur) stable and that the given ultimate bounds are global, i.e. valid for every initial conditions.

Corollary 3.2: *Consider the system* (6) *with componentwise constant perturbation bounds*

$$|w_i(t)| \preceq \mathbf{w}_i, \tag{20}$$

and $\mathbf{w}_i \in \mathbb{R}^{k_i}_{+0}$. Let $V \in \mathbb{C}^{n\times n}$ be invertible, and consider the matrix Λ as in Equation (14)*. Suppose that $\boldsymbol{\rho}(\Lambda) < 1$ and define*

$$\mathbf{z} \triangleq \max_{i\in\underline{N}} \left[\max_{w:|w|\preceq \mathbf{w}_i} |V^{-1}H_i w| \right], \tag{21}$$

$$b \triangleq (\mathrm{I}-\Lambda)^{-1}\mathbf{z}. \tag{22}$$

Then,

(*i*) *Transient bounds. There exists $\eta \in \mathbb{R}^n_{+0}$ satisfying $|V^{-1}x(0)| \preceq b + \eta$ and $\Lambda\eta \preceq \eta$. For every such η and for all $t \geq 0$,*

$$|V^{-1}x(t)| \preceq b + \Lambda^t \eta. \tag{23}$$

(*ii*) *Global ultimate bounds.*

$$\limsup_{t\to\infty} |V^{-1}x(t)| \preceq b, \tag{24}$$

Proof: Defining $\delta_i : \mathbb{R}^n_{+0} \to \mathbb{R}^n_{+0}$ by $\delta_i(x) = \mathbf{w}_i$ for all $x \in \mathbb{R}^n_{+0}$, then Equation (7) is satisfied and we may arbitrarily select $\bar{\tau} = 0$ in Equation (8). The function ψ as defined in Equation (15) then satisfies $\psi(x) = \mathbf{z}$ for all $x \in \mathbb{R}^n_{+0}$, with $\mathbf{z}$ as in Equation (21). Consider $\delta(x) \triangleq \psi(x)$ and $T(x)$ as defined in Equation (17). Note that δ is CNI, Equation (16) is satisfied, and $T(x) = \Lambda x + \mathbf{z}$.

(i) We first show that η as required exists. Apply Lemma 2.3(iii) with $\ell = T$ and $v = |V^{-1}x(0)|$, to obtain $\beta \in \mathbb{R}^n_+$ satisfying $T(\beta) + v \prec \beta$, which implies that $T(\beta) \prec \beta$ and $|V^{-1}x(0)| \preceq \beta$. By Theorem 3.1(a) and Lemma 2.3(ii), we have $b = \lim_{k\to\infty} T^k(\beta) = (\mathrm{I}-\Lambda)^{-1}\mathbf{z} \preceq \beta$. Write $\beta = b +\ \eta$ with $\eta \succeq 0$. We have $|V^{-1}x(0)| \preceq \beta = b + \eta$. Since $T(b) = b$, it follows that $T(\beta) = T(b+\eta) = b + \Lambda\eta \prec \beta = b + \eta$. This implies that $\Lambda\eta \prec \eta$. We have thus established that η exists as required.

Next, let $\eta \in \mathbb{R}^n_{+0}$ satisfy $|V^{-1}x(0)| \preceq b+\eta$ and $\Lambda\eta \preceq \eta$. Let $\beta = b+\eta$ and note that $T(\beta) = b+\Lambda\eta \preceq \beta$. The application of Theorem 3.1(c) yields $|V^{-1}x(t)| \preceq T^k(\beta)$ for all $t \geq k$, for every $k \geq 0$. In particular, we have $|V^{-1}x(t)| \preceq T^t(\beta)$ for all $t \geq 0$. By direct computation, it follows that $T^t(b+\eta) = b+\Lambda^t\eta$.

(ii) This follows by taking $\limsup_{t\to\infty}$ on Equation (23) and using $\boldsymbol{\rho}(\Lambda) < 1$. □

Remark 2: The computation of the maximum between square brackets in Equation (21) requires, for each i=1, ..., N, solving the n optimisation problems $\max_{w:|w|\preceq \mathbf{w}_i} |V^{-1}H_i w|$ (one optimisation problem per component). The solution to these problems can be easily obtained (see Haimovich and Seron (2009) and Haimovich, Kofman, & Seron (2008) for details).

Remark 3: A region of the form $\{x \in \mathbb{R}^n : |V^{-1}x| \preceq \bar{z}\}$, with $\bar{z} \succeq 0$ as given by Equation (18), (19), (23) or (24) has polyhedral shape if the entries of V are real, and a combined ellipsoidal/polyhedral shape if V has some complex entries (see Haimovich et al., 2008, for more details). Every (componentwise) bound $|V^{-1}x| \preceq \bar{z}$ yields a corresponding componentwise bound $|x| \preceq |V|\bar{z}$, since

$$|x| = |VV^{-1}x| \preceq |V||V^{-1}x| \preceq |V|\bar{z}.$$

3.2 Relationship to CQLF

The following theorem uses properties of non-negative Schur matrices to establish that the class of switching systems for which Corollary 3.2 can be applied, that is those for which the matrix V satisfying Equation (14) is such that $\boldsymbol{\rho}(\Lambda) < 1$, admit a common quadratic Lyapunov function. This relationship was reported in Mori et al. (2001), but the proof was not given. Here, we provide a proof and, in addition, extend it so that it becomes useful for the obtention of a CQLF for the case of affine perturbation bounds in the next subsection.

Theorem 3.3: *Let $\bar{\Lambda}, \Lambda \in \mathbb{R}^{n\times n}_{+0}$ and suppose that $\bar{\Lambda} \succeq \Lambda$ and $\boldsymbol{\rho}(\bar{\Lambda}) < 1$. Then,*

(*a*) *There exists a diagonal and positive definite matrix $D > 0$ satisfying*

$$\bar{\Lambda}' D \bar{\Lambda} - D < 0. \tag{25}$$

(*b*) $\boldsymbol{\rho}(\Lambda) < 1$.

(*c*) *If Λ satisfies Equation (14) for some invertible $V \in \mathbb{C}^{n\times n}$ and $A_i \in \mathbb{R}^{n\times n}$, then for each D as in (a) above, the corresponding symmetric and positive definite matrix $P = \mathbb{R}\mathrm{e}\{(V^{-1})^* D V^{-1}\}$ satisfies*

$$A_i' P A_i - P < 0. \tag{26}$$

Proof: (a) Since $\bar{\Lambda}$ has non-negative entries and satisfies $\boldsymbol{\rho}(\bar{\Lambda}) < 1$, then a diagonal (discrete-time) Lyapunov function exists.

(b) We have $0 \preceq \Lambda \preceq \bar{\Lambda}$. Therefore, $\boldsymbol{\rho}(\Lambda) \leq \boldsymbol{\rho}(\bar{\Lambda}) < 1$ (see, e.g. Theorem 8.1.18 of Horn and Johnson (1985)).

(c) By Equation (14) and the assumptions, we have

$$0 \preceq M_i \preceq \Lambda \preceq \bar{\Lambda}, \tag{27}$$

for all $i \in \underline{N}$. Consequently,

$$|z|' M_i' D M_i |z| \leq |z|' \bar{\Lambda}' D \bar{\Lambda} |z|, \tag{28}$$

for all $z \in \mathbb{C}^n$ and all $i \in \underline{N}$. Moreover,

$$z^* \Lambda_i^* D \Lambda_i z \leq |z^*||\Lambda_i^*|\, D\, |\Lambda_i||z| = |z|' M_i' D M_i |z|, \tag{29}$$

where the inequality in Equation (29) follows from the application and properties of componentwise absolute value and the equality follows from Equation (14). Combining Equations (28) and (29), it follows that

$$z^* \Lambda_i^* D \Lambda_i z \leq |z|' \bar{\Lambda}' D \bar{\Lambda} |z|, \tag{30}$$

for all $z \in \mathbb{C}^n$ and all $i \in \underline{N}$. Since D is diagonal, then $z^* D z = |z|' D |z|$ for all $z \in \mathbb{C}^n$. Subtracting $z^* D z = |z|' D |z|$ from each side of the inequality, Equation (30) yields

$$z^*(\Lambda_i^* D \Lambda_i - D) z \leq |z|'(\bar{\Lambda}' D \bar{\Lambda} - D)|z|. \tag{31}$$

Recalling Equation (25), then Equation (31) implies that $\Lambda_i^* D \Lambda_i - D < 0$. Using the definition for Λ_i in Equation (14), left multiplying by $(V^{-1})^*$ and right multiplying by V^{-1}, then

$$A_i' (V^{-1})^* D V^{-1} A_i - (V^{-1})^* D V^{-1} < 0, \tag{32}$$

whence Equation (26) follows by taking real parts. □

3.3 Affine perturbation bounds

As in the continuous-time case (Haimovich & Seron, 2013), global ultimate bounds can also be obtained under a simple sufficient condition when the bounding function δ in Equation (16) is of affine form. Theorem 3.4 below derives such bounds and associated invariant sets, and further relates the results with the existence of a quadratic function so that ultimate bounds can be obtained via standard Lyapunov techniques in the case when no delay is present. The proof is given in Section A.3 in the Appendix.

Theorem 3.4: *Consider the system (6) with perturbation bound given by Equations (7) and (8). Let $V \in \mathbb{C}^{n\times n}$ be invertible and consider the matrix Λ as defined in Equation*

(14). *Let $\psi : \mathbb{R}^n_{+0} \to \mathbb{R}^n_{+0}$ be defined by Equation* (15) *and suppose that there exists*

$$\tilde{\delta}(x) \triangleq \bar{F}x + \bar{w} \tag{33}$$

for some $\bar{F} \in \mathbb{R}^{n\times n}_{+0}$ and $\bar{w} \in \mathbb{R}^n_{+0}$, satisfying $\tilde{\delta}(x) \succeq \psi(x)$ for all $x \in \mathbb{R}^n_{+0}$ and such that $\boldsymbol{\rho}(R) < 1$, where

$$R \triangleq \Lambda + \bar{F}. \tag{34}$$

Define

$$\tilde{b} \triangleq (\mathrm{I} - R)^{-1}\bar{w}. \tag{35}$$

Then,

(*a*) *Invariance. If $|V^{-1}x(t)| \preceq \tilde{b}$ for $-\bar{\tau} \le t \le 0$, then $|V^{-1}x(t)| \preceq \tilde{b}$ for all $t \ge -\bar{\tau}$.*
(*b*) *Global ultimate bounds.* $\limsup_{t\to\infty}|V^{-1}x(t)| \preceq \tilde{b}$.
(*c*) *Tighter global ultimate bounds. Suppose that there exists a continuous and CNI $\delta : \mathbb{R}^n_{+0} \to \mathbb{R}^n_{+0}$ satisfying*

$$\psi(x) \preceq \delta(x) \preceq \tilde{\delta}(x), \quad \textit{for all } x \in \mathbb{R}^n_{+0}. \tag{36}$$

Consider the map $T : \mathbb{R}^n_{+0} \to \mathbb{R}^n_{+0}$ given by Equation (17). *Then,* $\limsup_{t\to\infty}|V^{-1}x(t)| \preceq \lim_{k\to\infty} T^k(\tilde{b}) \preceq \tilde{b}$.
(*d*) *Existence of Lyapunov function. There exists D diagonal and positive definite such that*

$$(\Lambda + \bar{F})' D(\Lambda + \bar{F}) - D < 0. \tag{37}$$

(*e*) *Ultimate bounds via standard Lyapunov techniques. If, in addition, $\bar{\tau} = 0$ (no delay), then for each D as in (d) above, the increment ΔL(t, x) of the function $L(x) \triangleq x'Px$ with $P = \mathbb{R}\mathrm{e}\{(V^{-1})^* D V^{-1}\}$ along any trajectory of Equation* (6) *satisfies ΔL(t, x) < 0 for all t and all x such that $\|x\|$ is big enough.*

Theorem 3.4 gives an invariant region and global ultimate bounds for the case when the perturbation bound $\tilde{\delta}$ has affine form (see Equation (33)). The main additional assumption required by this theorem is that the matrix R constructed as the sum of the system matrix Λ and the perturbation bound matrix $\bar{F}$ (see Equation (34) has spectral radius less than 1. The advantages of the affine form of the perturbation bound are analogous to those for the continuous-time case, namely that the search for a vector β satisfying a componentwise inequality is not required.

Since $0 \preceq |V^{-1}A_iV| \preceq \Lambda$ for all $i \in \underline{N}$, and $\bar{F} \succeq 0$, then $\boldsymbol{\rho}(A_i) = \boldsymbol{\rho}(V^{-1}A_iV) \le \boldsymbol{\rho}(\Lambda) \le \boldsymbol{\rho}(\Lambda + \bar{F}) = \boldsymbol{\rho}(R)$ (see, e.g. Theorem 8.1.18 of Horn & Johnson, 1985). Therefore, the condition $\boldsymbol{\rho}(R) < 1$ required by Theorem 3.4 implies that $\boldsymbol{\rho}(\Lambda) < 1$, and the latter condition implies that $\boldsymbol{\rho}(A_i) < 1$ for all $i \in \underline{N}$. Consequently, a necessary condition for the hypotheses of Theorem 3.4 or Corollary 3.2 to hold is that every subsystem matrix A_i be stable. By contrast, note that Theorem 3.1 does not require the matrix Λ to satisfy $\boldsymbol{\rho}(\Lambda) < 1$. However, the existence of a non-zero and non-negative vector β such that $T(\beta) \preceq \beta$ implies that at least one of the eigenvalues of Λ has magnitude not greater than 1. We illustrate some of these facts in Section 4.

Remark 4: Note that in both Theorems 3.1 and 3.4, every invertible matrix $V \in \mathbb{C}^{n\times n}$ for which the hypotheses of the corresponding theorem hold may be used. In particular, in the affine perturbation bound case in Theorem 3.4, every invertible $V \in \mathbb{C}^{n\times n}$ for which $\boldsymbol{\rho}(\Lambda + \bar{F}) < 1$ can be used. However, such a matrix does not always exist. Numerically, we may seek the matrix V by means of the following optimisation problem:

minimise $\boldsymbol{\rho}(\Lambda + \bar{F})$ over $V \in \mathbb{C}^{n\times n}$ invertible.

Note that it is not necessary to find the global optimum of this possibly non-convex optimisation problem; it suffices to find an invertible V for which $\boldsymbol{\rho}(\Lambda + \bar{F}) < 1$. We illustrate this procedure by means of a numerical example in Section 4.

4. Examples

In this section, we illustrate the results of the previous sections by means of numerical examples. The first example, presented in Section 4.1, illustrates the fact that the matrix Λ considered in Theorem 3.1 is not required to be (Schur) stable, in contrast with the analogous results for continuous-time switching systems. The second example, presented in Section 4.2, illustrates the fact that the class of switching systems for which Corollary 3.2 can be applied is strictly contained in the class of switching linear systems that admit a CQLF, although the switching linear system need not be close to simultaneously triangularisable. To complete the picture of the relationship with CQLF, Section 4.3 gives an instance where a CQLF exists for the switching system but the transformation V required by our method cannot be found. Finally, Section 4.4 demonstrates the application of Theorem 3.4 on an example with nonlinear perturbation bounds with affine overbound. The example also shows that it may be possible to obtain tighter bounds by combining bounds computed via CQLF and the proposed componentwise method, in the case when no delay is present.

4.1 Λ not stable

Consider system (6) with $N = 2$, $n = 2$, no perturbation and with

$$A_1 = \begin{bmatrix} 1.19 & -1.09 \\ -0.81 & 0.91 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0.36 & -0.86 \\ -0.64 & 0.14 \end{bmatrix}.$$

Taking V as in Equation (38) and computing Λ as in Equation (14) yield

$$V = \begin{bmatrix} 1 & 1 \\ -0.75 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 2 & 0 \\ 0.0075 & 0.5 \end{bmatrix}. \qquad (38)$$

Note that $\boldsymbol{\rho}(\Lambda) = 2$ and hence Λ is not (Schur) stable. Since there is no perturbation, the map $\psi(x)$ in Equation (15) is identically zero. Taking $\delta \equiv \psi$, the map T in Equation (17) reduces to $T(x) = \Lambda x$. Every non-negative vector β of the form $\beta = [0,\, a]'$ with $a > 0$ satisfies $T(\beta) = \Lambda\beta = [0,\, a/2]' \preceq \beta$ and hence Theorem 3.1 can still be applied, even though Λ is not stable. Note from this theorem that the line segments defined by $|V^{-1}x(t)| \preceq T^{k}(\beta)$, for $k \geq 0$, represent invariant sets inside which the switching system trajectories ultimately converge to zero. Also note that one of the eigenvalues of Λ has magnitude less than 1 (see comment above Remark **4**).

4.2 *Simultaneous triangularisation and CQLF*

Consider again system (6) with $N = 2$, $n = 2$, no perturbation, but now with

$$A_1 = \begin{bmatrix} -0.2 & -0.4 \\ 0.4 & -0.2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -0.2 & -0.4a \\ 0.4/a & -0.2 \end{bmatrix}$$

for some $a > 0$. For every value of a, the eigenvalues of A_2 are $-0.2 \pm 0.4\mathbf{i}$, identical to those of A_1, and hence both A_1 and A_2 are stable. The eigenvectors of A_1 are $[1, \pm\mathbf{i}]'$ and those of A_2 are $[1, \pm a\mathbf{i}]'$. To be simultaneously triangularisable, it is necessary that both A_1 and A_2 have a common eigenvector. Consequently, loosely speaking we may say that this switching system is farther away from simultaneous triangularisation as a is varied farther away from 1. It can be shown via LMIs that for $a > 3 + \sqrt{8}$, the above switching system does not admit a CQLF. For $a = 3 + \sqrt{8} - 10^{-3}$, which corresponds to a switching system with stable subsystems but so far from simultaneous triangularisation that it is at the verge of not admitting a CQLF, searching for an arbitrary V by means of the optimisation proposed in Remark **4**, we are able to obtain the feasible solution,

$$V = \begin{bmatrix} -9.1808 & -10.913 \\ 3.3976 & -5.5452 \end{bmatrix} + \begin{bmatrix} 8.2018 & 13.386 \\ 3.8032 & -4.5207 \end{bmatrix}\mathbf{i},$$

for which the corresponding Λ is stable.

4.3 *CQLF exists but method not applicable*

Consider again system (6) with $N = 3$, $n = 2$, no perturbation, and with

$$A_1 = \begin{bmatrix} 0.954 & 0.121 \\ -0.726 & 0.920 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0.960 & 0.122 \\ -0.633 & 0.936 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} -0.053 & 0.389 \\ -2.161 & 0.019 \end{bmatrix}.$$

A CQLF for this switching linear system can be computed via LMIs. However, searching for V according to Remark **4** does not give a useful solution, even when the optimisation is run over 1000 times from arbitrary initial conditions.

4.4 *Nonlinear perturbation bounds with affine overbound*

Consider a discrete-time switching system of the form (6) with $N = 2$, $n = 3$, $k_1 = 2$, $k_2 = 1$ and

$$A_1 = \begin{bmatrix} -0.46 & 0.75 & -0.67 \\ -0.18 & -0.99 & 0.2 \\ -0.56 & -0.67 & -0.19 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0.84 & 0.14 & 0.65 \\ -0.16 & 1.09 & -0.5 \\ -0.66 & 0.84 & -0.83 \end{bmatrix}$$
$$H_1 = \begin{bmatrix} 0.01 & 0 \\ -0.2 & 0.01 \\ 0 & 0 \end{bmatrix} H_2 = \begin{bmatrix} 0 \\ 0 \\ -0.03 \end{bmatrix}.$$

The perturbation vectors $w_1(t) \in \mathbb{R}^2$ and $w_2(t) \in \mathbb{R}$ are componentwise bounded by $|w_i(t)| \preceq \delta_i(\theta(t))$ with $\theta(t)$ as defined in Equation (8), $\bar{\tau} = 10$, $\delta_1 : \mathbb{R}^3_{+0} \to \mathbb{R}^2_{+0}$ and $\delta_2 : \mathbb{R}^3_{+0} \to \mathbb{R}_{+0}$ given by

$$\delta_1(\theta) = \begin{bmatrix} \begin{cases} 0.01\sin\theta_1 & \text{if } \theta_1 \leq \pi/2 \\ 0.01 & \text{if } \theta_1 > \pi/2 \end{cases} + \begin{cases} 0.02\sin\theta_2 & \text{if } \theta_2 \leq \pi/2 \\ 0.02 & \text{if } \theta_2 > \pi/2 \end{cases} \\ \begin{cases} 0.03\theta_2 e^{-2\theta_2} & \text{if } \theta_2 \leq 1/2 \\ 0.03/(2e) & \text{if } \theta_2 > 1/2 \end{cases} + 0.02\theta_3 + 1 \end{bmatrix}$$
$$\delta_2(\theta) = 0.02\log[(1+\theta_1)^4(1+\theta_3)] + 0.5,$$

where both δ_1 and δ_2 are continuous and CNI. In turn, these bounding functions have affine bounds

$$\delta_1(\theta) \preceq \begin{bmatrix} 0.01\theta_1 + 0.02\theta_2 \\ 0.03\theta_2 + 0.02\theta_3 + 1 \end{bmatrix} = \bar{F}_1\theta + \bar{w}_1,$$
$$\delta_2(\theta) \leq 0.08\theta_1 + 0.02\theta_3 + 0.5 = \bar{F}_2\theta + \bar{w}_2,$$

where

$$\bar{F}_1 = \begin{bmatrix} 0.01 & 0.02 & 0 \\ 0 & 0.03 & 0.02 \end{bmatrix} \qquad \bar{w}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\bar{F}_2 = [\,0.08 \quad 0 \quad 0.02\,] \qquad \bar{w}_2 = 0.5.$$

4.4.1 *Ultimate bounds via componentwise method*

Since the perturbation bounds δ_1 and δ_2 admit affine bounds, as shown above, then the function ψ in Equation (15) can actually be bounded by an affine CNI function $\tilde{\delta}$ for every $V \in \mathbb{C}^{n\times n}$ invertible. To see this, note that

$$\max_{|w_i| \preceq \delta_i(|V|x)} |V^{-1}H_i w_i| \preceq |V^{-1}H_i|\delta_i(|V|x), \tag{39}$$

for $i = 1, 2$ (note that the right-hand side of Equation (39) may not be a tight bound on its left-hand side only when V has complex components). We thus have

$$\psi(x) \preceq \max_{i\in\{1,2\}} [|V^{-1}H_i|(\bar{F}_i|V|x + \bar{w}_i)] \tag{40}$$
$$\preceq \tilde{\delta}(x) \triangleq \bar{F}x + \bar{w}, \quad \text{with} \tag{41}$$
$$\bar{F} \triangleq \max_{i\in\{1,2\}} |V^{-1}H_i|\bar{F}_i|V|, \tag{42}$$
$$\bar{w} \triangleq \max_{i\in\{1,2\}} |V^{-1}H_i|\bar{w}_i. \tag{43}$$

To apply Theorem 3.4, an invertible matrix $V \in \mathbb{C}^{n\times n}$ for which $\boldsymbol{\rho}(\Lambda + \bar{F}) < 1$ should be found. We may seek such a matrix by means of the optimisation problem outlined in Remark **4**. This yields

$$V = \begin{bmatrix} -4.335 & 1.317 & 0.222 \\ 1.773 & 2.369 & -0.020 \\ 5.274 & 1.477 & -0.010 \end{bmatrix} + \begin{bmatrix} -2.221 & 2.546 & -7.226 \\ 0.908 & 4.693 & 0.393 \\ 2.703 & 2.985 & -1.043 \end{bmatrix}\mathbf{i}.$$

Computation of the vector $\tilde{b}$ as in Equations (34) and (35) gives

$$\tilde{b} = \begin{bmatrix} 0.0705 \\ 0.0322 \\ 0.0444 \end{bmatrix} \qquad |V|\tilde{b} = \begin{bmatrix} 0.7562 \\ 0.3269 \\ 0.5709 \end{bmatrix}.$$

An ultimate bound tighter than the above can be computed by the application of Theorem 3.4(c), which yields

$$b = \begin{bmatrix} 0.0652 \\ 0.0309 \\ 0.0404 \end{bmatrix}, \qquad |V|b = \begin{bmatrix} 0.6982 \\ 0.3081 \\ 0.5314 \end{bmatrix}. \tag{44}$$

Note that the bounds obtained are valid for every non-negative value of the maximum delay $\bar{\tau}$, provided the perturbation satisfies Equations (7) and (8).

4.4.2 *Ultimate bound via quadratic Lyapunov function*

If no delay is present, i.e. if $\bar{\tau} = 0$ in Equation (8), we may compute a CQLF according to Theorem 3.4(d). Solving the LMIs, Equation (37) for D yields $D = \mathrm{diag}(17.73, 282.9, 504.4)$, which allows the computation of the CQLF given by

$$P = \mathbb{R}\mathrm{e}\{(V^{-1})^* D V^{-1}\} = \begin{bmatrix} 6.227 & -7.008 & 7.382 \\ -7.008 & 24.41 & -14.25 \\ 7.382 & -14.25 & 11.38 \end{bmatrix}.$$

The function $L(x) = x'Px$ is a CQLF for the switching linear nominal (unperturbed) part of the system. In addition, according to Theorem 3.4(e), the increment of $L(x)$ along any possible trajectory of the (perturbed) system will be negative for every x such that $\|x\|$ is big enough. We have

$$\begin{aligned} \Delta L(t,x) &= x'(A'_{\sigma(t)}PA_{\sigma(t)} - P)x + 2x'A'_{\sigma(t)}PH_{\sigma(t)}w_{\sigma(t)}(t) \\ &\quad + w'_{\sigma(t)}H'_{\sigma(t)}PH_{\sigma(t)}w_{\sigma(t)} \\ &\le \max_{i\in \underline{N}} \Big[x'(A'_iPA_i - P)x \\ &\quad + \max_{|w|\preceq\delta_i(|x|)} (2x'A'_iPH_iw + w'H'_iPH_iw) \Big]. \end{aligned} \tag{45}$$

The bound on $\Delta L(t, x)$ given by Equation (45) is tight, in the sense that for every x, there exists a value of the switching function $\sigma(t)$ and a possible value of the perturbation $w_{\sigma(t)}(t)$ for which $\Delta L(t, x)$ equals the right-hand side of Equation (45). Also, note that for a given x, the maximum over w in Equation (45) can easily be computed by evaluating its argument only on the vertices of the polyhedral region $\{w \in \mathbb{R}^{k_i} : |w| \preceq \delta_i(|x|)\}$.

A sufficient condition to be able to compute a global ultimate bound by means of $L(x)$ is that $\max_{x'Px\ge k} \Delta L(t,x) < 0$ for some $k > 0$. Numerical search for such a k yields $k = 0.3046$, and it can be verified that $\max_{x'Px=0.3045} \Delta L(t,x) > 0$. Consequently, the state trajectory will satisfy $\limsup_{t\to\infty} x(t)'Px(t) \le 0.3046$. From this latter bound, we may compute the componentwise ultimate bounds $\bar{x}_i = \max_{x'Px=0.3046} x_i$ for $i = 1, 2, 3$:

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} 0.6693 \\ 0.3131 \\ 0.7848 \end{bmatrix} \tag{46}$$

Comparing the ultimate bounds (44) and (46) (recall Remark **3**) shows that the bounds for the second and third components of the state obtained via our componentwise method are better than those obtained via the CQLF.

Moreover, we may combine the bounds (44) and (46) in order to obtain a componentwise bound tighter than that can be obtained by either method applied individually:

$$\limsup_{t\to\infty} |x(t)| \preceq \begin{bmatrix} 0.6693 \\ 0.3081 \\ 0.5314 \end{bmatrix}.$$

5. Application to fault tolerant control

In this section, we utilise the ultimate bound and practical stability results of this paper to investigate the fault tolerance properties of the winding machine control system of Noura et al. (2009, Chapter 3), which was introduced in Section 2.1. We thus consider the system of Equation (1), with

$$A_0 = \begin{bmatrix} 0.4126 & 0 & -0.0196 \\ 0.0333 & 0.5207 & -0.0413 \\ -0.0101 & 0 & 0.2571 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} -1.7734 & 0.0696 & 0.0734 \\ 0.0928 & 0.4658 & 0.1051 \\ -0.0424 & -0.0930 & 2.0752 \end{bmatrix}.$$

These values for A_0 and B_0 correspond to the nominal linearised (incremental) model around the operating points $x_0 = [0.6, 0.5, 0.4]'$ an $u_0 = [-0.15, 0.55, 0.15]'$ of the winding machine model, and discretised with a sampling period $t_s = 0.1$s. The state feedback control (2) was computed, as an illustration, via LQR for (A_0, B_0) with identity state and control weightings, yielding

$$K_0 = \begin{bmatrix} 0.1768 & -0.0149 & -0.0097 \\ -0.0303 & -0.2423 & 0.0291 \\ 0.0040 & -0.0152 & -0.0992 \end{bmatrix}.$$

The perturbation vector $w(t)$ is componentwise bounded as in Equation (5), corresponding to drift, bounded noise and uncompensated nonlinearity, as already mentioned in Section 2.1. The values for α, β and ν are 0.1, 0.1 and 0.2, respectively. Note that the perturbation bound (5) admits an affine overbound, since

$$\begin{bmatrix} \alpha|x^1(t)| \\ \nu \\ \beta|\sin x^3(t)| \end{bmatrix} \preceq \underbrace{\begin{bmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta \end{bmatrix}}_{F_0} |x(t)| + \underbrace{\begin{bmatrix} 0 \\ \nu \\ 0 \end{bmatrix}}_{w_0}. \quad (47)$$

Proceeding as in Section 4.4.1, for any $V \in \mathbb{C}^{n\times n}$ invertible, we bound the function ψ in Equation (15) as

$$\psi(x) \preceq \bar{F}x + \bar{w}, \quad \text{with} \quad (48)$$

$$\bar{F} \triangleq \max_{i\in\{1,\ldots,4\}} |V^{-1}B_iK_0|F_0|V|, \quad (49)$$

$$\bar{w} \triangleq \max_{i\in\{1,\ldots,4\}} |V^{-1}B_iK_0|w_0. \quad (50)$$

To apply Theorem 3.4, an invertible matrix $V \in \mathbb{C}^{n\times n}$ for which $\boldsymbol{\rho}(\Lambda + \bar{F}) < 1$ was computed by means of the optimisation problem outlined in Remark **4**. Using such a V in the evaluation of the vector $\tilde{b}$ defined in Equations (34) and (35), yields[1]

$$\tilde{b} = \begin{bmatrix} 0.0054 \\ 0.0318 \\ 0.0063 \end{bmatrix} \qquad |V|\tilde{b} = \begin{bmatrix} 0.0131 \\ 0.0507 \\ 0.0118 \end{bmatrix}. \quad (51)$$

Thus, we conclude that the controller (2) is tolerant to the considered actuator faults in the presence of sensor measurement errors bounded as Equation (5). Moreover, the componentwise ultimate bound vector $|V|\tilde{b}$ can be used as a measure of the 'degree' of fault tolerance, if one compares it with a similar ultimate bound obtained for the system without actuator faults, that is for $B_{\sigma(t)} \equiv B_0$ in Equation (3). Denoting the latter componentwise ultimate bound as $|V_0|\tilde{b}_0$, we obtain, proceeding as above with $B_i = B_0$ in Equations (48)–(50),

$$|V_0|\tilde{b}_0 = \begin{bmatrix} 0.0037 \\ 0.0388 \\ 0.0033 \end{bmatrix}. \quad (52)$$

Note that the ultimate bound (51) is componentwise larger than Equation (52), as expected, but the deterioration in performance is not dramatic. Keeping a fixed gain K_0 for all possible actuator fault modes and ensuring that the closed-loop system remains stable under fault is a *passive* fault tolerant control approach. An *active* fault tolerant control strategy, on the other hand, would reconfigure the controller by employing a suitable gain K_i to match the detected fault mode B_i (assuming a fault detection and isolation mechanism accurately provides this information). We can use the ultimate bound tools to evaluate the performance of such a strategy by computing suitable gains K_i (e.g. via LQR for (A_0, B_i) with identity weightings), replacing K_0 by K_i in Equations (48)–(50) and evaluating the associated componentwise ultimate bound vector, denoted as $|V_R|\tilde{b}_R$. This yields

$$|V_R|\tilde{b}_R = \begin{bmatrix} 0.0093 \\ 0.0500 \\ 0.0092 \end{bmatrix}. \quad (53)$$

Note that the bound (53) resulting from reconfiguring the controller gain is also componentwise larger than the 'healthy' bound (52), but it improves the bound (51) achieved by the passive approach. Other strategies can be analysed and contrasted in a similar way. The method can also be embellished by expanding the optimisation problem outlined in Remark **4** to include the design of the fixed gain or the reconfiguration gains. One could hence use these results to rank different reconfiguration strategies and evaluate whether they are worth implementing and/or to design strategies that may improve the achieved bounds.

6. Conclusions

We have derived novel componentwise bounds and invariant sets for switching discrete-time systems with perturbation bounds that may depend nonlinearly on a delayed state. The method allows every component of the perturbation vector to have a different bound and provides componentwise bounds on the system state. By means of the use of componentwise bounds, the need for bounding the norm of the system state is avoided, thus reducing conservativeness due to different perturbation components having substantially different bounds. Another contribution of the paper was to establish that the class of switching linear systems to which our componentwise bound and invariant set method can be applied is strictly contained in the class of switching linear systems that admit a CQLF. We have shown the usefulness of our method by applying the method to the analysis of fault tolerance of a winding machine control system. Future work may focus on switched systems where either the switching signal or a control input can be designed in order to ensure a given ultimate bound (cf. Kofman et al., 2008) and on the extension and application of the current results to networked control systems (cf. Haimovich, Kofman, & Seron, 2007).

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Note

1. Note that tighter ultimate bounds can be obtained by the application of Theorem 3.4(c), as was illustrated in Section 4.4.1.

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Appendix

A.1 Proofs of properties of CNI functions

A.1.1 Proof of Lemma 2.2

(i) Applying the CNI property to the inequality $f(\beta) \preceq \beta$ and iterating the process, it follows that $f^{k+1}(\beta) \preceq f^k(\beta)$ for all $k \in \mathbb{Z}_+$. Also, since f maps non-negative vectors to non-negative vectors, then $f^k(\beta) \succeq 0$ for all $k \in \mathbb{Z}_+$. It follows that the vectors $f^k(\beta)$

form a componentwise non-increasing sequence which is lower bounded by 0. Hence, each component must converge to some non-negative real number and thus Equation (10) holds.

(ii) Note that $|f_\gamma^k(\beta) - b| \preceq |f_\gamma^k(\beta) - f^k(\beta)| + |f^k(\beta) - b|$. From Equation (10), given $\epsilon \in \mathbb{R}_+^n$, we can select $k = k(\epsilon)$, such that $|f^k(\beta) - b| \prec \epsilon/2$. From the definition of f_γ and the continuity of f, it follows that for the selected value of k, we may select $\gamma = \gamma(\epsilon) \in \mathbb{R}_+^n$ small enough so that $|f_\gamma^k(\beta) - f^k(\beta)| \prec \epsilon/2$. Then, $|f_\gamma^k(\beta) - b| \prec \epsilon$, whence $f_\gamma^k(\beta) \prec b + \epsilon$.

A.1.2 Proof of Lemma 2.3

By assumption we have $R \succeq 0$ and $\boldsymbol{\rho}(R) < 1$. Let R_ϵ be a slight perturbation of R so that $R_\epsilon \succ R$ and $\boldsymbol{\rho}(R_\epsilon) < 1$. Then, $R_\epsilon \succ 0$ and by the Perron–Frobenius theorem (see, e.g. Theorem 8.2.2 of Horn and Johnson (1985)) $\boldsymbol{\rho}(R_\epsilon) > 0$, and there exists $x \succ 0$ such that $R_\epsilon x = \boldsymbol{\rho}(R_\epsilon)x$. It follows that

$$Rx \prec R_\epsilon x = \boldsymbol{\rho}(R_\epsilon)x \prec x. \tag{A1}$$

(i) Immediate from the fact that $R \succeq 0$.

(ii) Immediate from the assumption $\boldsymbol{\rho}(R) < 1$.

(iii) From Equation (A1), $y \triangleq (\mathrm{I} - R)x \succ 0$. Define $z \triangleq r + v$, and let y_i and z_i denote the ith components of y and z, respectively. Select $\alpha > 0$ so that

$$\alpha > \max_{i \in \{1,\dots,n\}} \left\{ \frac{z_i}{y_i} \right\}, \tag{A2}$$

and define $\beta \triangleq \alpha x$. Note that $\beta \succ 0$. Then, $\alpha y = (\mathrm{I} - R)\beta \succ r + v$. Operating on the latter inequality yields $R\beta + r + v = \ell(\beta) + v \prec \beta$, and the result follows.

(iv) Note that Equation (12) with $v \succeq 0$ implies $\ell(\beta) \prec \beta$. We then have $f(\beta) \preceq \ell(\beta) \prec \beta$. Also, by Lemma 2.2(i), then Equation (10) holds. Since both f and ℓ are CNI and $f(x) \preceq \ell(x)$ for all $x \in \mathbb{R}_{+0}^n$, then $f^k(\beta) \preceq \ell^k(\beta) \prec \beta$ for all $k \in \mathbb{Z}_+$, whence applying limits yields $b \preceq \tilde{b} \prec \beta$. Applying the CNI property of f to the latter inequalities, and iterating, yields $b = f^k(b) \preceq f^k(\tilde{b}) \preceq f^k(\beta)$, whence $b \preceq \lim_{k\to\infty} f^k(\tilde{b}) \preceq b$. We have thus established Equation (13).

A.2 Proof of Theorem 3.1

(a) Since δ is CNI by assumption and $\Lambda \succeq 0$, it follows that T in Equation (17) is CNI. Part (a) then follows by applying Lemma 2.2(i) with $f = T$.

(b) Let $x(t) = Vz(t)$ and rewrite Equation (6) as

$$z(t+1) = V^{-1}A_{\sigma(t)}Vz(t) + V^{-1}H_{\sigma(t)}w_{\sigma(t)}(t).$$

Taking componentwise magnitudes in the above equation and operating yields

$$|z(t+1)| \preceq |V^{-1}A_{\sigma(t)}V||z(t)| + |V^{-1}H_{\sigma(t)}w_{\sigma(t)}(t)|. \tag{A3}$$

Note from Equation (14) that, for all t,

$$|V^{-1}A_{\sigma(t)}V| \preceq \Lambda, \tag{A4}$$

and from Equation (7), for all $i \in \underline{N}$ and all t,

$$|V^{-1}H_i w_i(t)| \preceq \max_{w:|w| \preceq \delta_i(\theta(t))} |V^{-1}H_i w|. \tag{A5}$$

The proof proceeds by induction on $t \geq 0$. Note that, for any $k \geq 0$,

$$|V^{-1}x(\tau)| = |z(\tau)| \preceq T^k(\beta), \quad \text{for all } -\bar{\tau} \leq \tau \leq t \tag{A6}$$

is valid at $t = 0$ by assumption. Next, suppose that Equation (A6) holds at some arbitrary $t \geq 0$. Then, $\theta(t)$ in Equation (8) can be bounded using Equation (A6) as

$$\begin{aligned} \theta(t) &= \max_{t-\bar{\tau} \leq \tau \leq t} |VV^{-1}x(\tau)| \\ &\preceq |V| \max_{t-\bar{\tau} \leq \tau \leq t} |V^{-1}x(\tau)| \preceq |V|T^k(\beta). \end{aligned} \tag{A7}$$

Employing Equations (A4)–(A6), and recalling the fact that the δ_i is CNI, it follows that Equation (A3) can be further bounded as

$$|z(t+1)| \preceq \Lambda T^k(\beta) + \max_{i \in \underline{N}} \left[\max_{w:|w| \preceq \delta_i(|V|T^k(\beta))} |V^{-1}H_i w| \right]. \tag{A8}$$

From Equation (15), the second term on the right-hand side of Equation (A8) is equal to $\psi(T^k(\beta))$. Then, using Equations (16) and (17), we have

$$\begin{aligned} |z(t+1)| &\preceq \Lambda T^k(\beta) + \psi(T^k(\beta)) \\ &\preceq \Lambda T^k(\beta) + \delta(T^k(\beta)) = T^{k+1}(\beta). \end{aligned} \tag{A9}$$

Recalling that $z(t) = V^{-1}x(t)$, we have established that $|V^{-1}x(t+1)| \preceq T^{k+1}(\beta) \preceq T^k(\beta)$, where the last inequality follows from part (a). Hence, Equation (A6) holds at $t + 1$, and the proof by induction is thus complete.

(c) By assumption, $|V^{-1}x(t)| \preceq \beta$ for all $-\bar{\tau} \leq t \leq 0$ and by part (b), we know that this bound holds for all $t \geq -\bar{\tau}$. Then, starting from $t = 0$ and recursively repeating the argument that leads to Equation (A9) above, we can show that the vector $T(\beta)$ bounds $|V^{-1}x(t)|$ for $t = 1, \dots, \bar{\tau} + 1$. Thus, using the invariance result of part (b), it follows that

$$|V^{-1}x(t)| \preceq T(\beta), \quad \text{for all } t \geq 1. \tag{A10}$$

In a similar way, we can show that $T^2(\beta)$ bounds $|V^{-1}x(t)|$ for $t = (\bar{\tau} + 1) + 1, \dots, 2(\bar{\tau} + 1) + 1$ and, by invariance, for all $t \geq (\bar{\tau} + 1) + 1$. Thus, for all $k \geq 0$,

$$|V^{-1}x(t)| \preceq T^k(\beta), \quad \text{for all } t \geq (k-1)(\bar{\tau} + 1) + 1.$$

(d) By part (a), we have $\lim_{k\to\infty} T^k(\beta) = b$. The result then follows from part (c) by taking the latter limit into consideration.

A.3 Proof of Theorem 3.4

Consider the map $\tilde{T} : \mathbb{R}_{+0}^n \to \mathbb{R}_{+0}^n$ defined as

$$\tilde{T}(x) = \Lambda x + \tilde{\delta}(x). \tag{A11}$$

Then, $\tilde{T}(x) = Rx + \bar{w}$, with R as in Equation (34). Since $\boldsymbol{\rho}(R) < 1$, then $\lim_{k\to\infty} \tilde{T}^k(x) = \tilde{b}$ for every $x \in \mathbb{R}_{+0}^n$, with $\tilde{b}$ as in Equation (35). Note that $\tilde{T}(\tilde{b}) = \tilde{b} \preceq \tilde{b}$.

(a) The result follows by applying Theorem 3.1(b) with $\beta = \tilde{b}$.

(b) By Equations (14) and (34), we have $R \succeq 0$ and by assumption $\boldsymbol{\rho}(R) < 1$. Let $v \succeq 0$ be such that $|V^{-1}x(t)| \preceq v$ for all $-\bar{\tau} \leq t \leq 0$. For such v Lemma 2.3(iii) shows that we can find $\beta \succ 0$ such that $\tilde{T}(\beta) + v \prec \beta$, which implies $\tilde{T}(\beta) \prec \beta$ and $v \prec \beta$. Then, (b) directly follows from Theorem 3.1(d).

(c) By Equations (36), (A11) and (17), we have $T(x) \preceq \tilde{T}(x)$ for all $x \in \mathbb{R}^n_{+0}$. As in (b) above, take $\beta \in \mathbb{R}^n_{+0}$ such that $T(\beta) \preceq \tilde{T}(\beta) \preceq \beta$ and $|V^{-1}x(t)| \preceq \beta$ for all $-\bar{\tau} \le t \le 0$. Then, by Theorem 3.1, it follows that $|V^{-1}x(t)| \preceq b$ for all $t \ge -\bar{\tau}$, with $b \triangleq \lim_{k\to\infty} T^k(\beta)$. Further, applying Lemma 2.3(iv) yields $b = \lim_{k\to\infty} T^k(\tilde{b}) \preceq \tilde{b}$.

(d) By Equation (14) and since $\bar{F} \succeq 0$, then $R = \Lambda + \bar{F} \succeq 0$. The application of Theorem 3.3(a) with $\bar{\Lambda} = R$ establishes the required result.

(e) For every $i \in \underline{N}$, define $p_i(t) \triangleq V^{-1}H_i w_i(t)$. Let $x = Vz$ and rewrite Equation (6) as

$$z(t+1) = \Lambda_{\sigma(t)} z(t) + p_{\sigma(t)}(t). \tag{A12}$$

Using Equations (7) and (8) with $\bar{\tau} = 0$, it follows that, for all $i \in \underline{N}$,

$$|p_i(t)| \preceq \max_{i\in\underline{N}} \left[\max_{|w_i| \preceq \delta_i(|Vz(t)|)} |V^{-1}H_i w_i| \right] \tag{A13}$$

$$\preceq \psi(|z(t)|) \preceq \tilde{\delta}(|z(t)|) = \bar{F}|z(t)| + \bar{w}, \tag{A14}$$

where the first inequality in Equation (A14) follows from $|Vz| \preceq |V||z|$ and δ_i CNI. Consider the function $L_z(z) = z^* D z$. We have

$$\begin{aligned}\Delta L_z(t, z(t)) &= L_z(z(t+1)) - L_z(z(t)) \\ &= z(t)^*(\Lambda^*_{\sigma(t)} D \Lambda_{\sigma(t)} - D) z(t) \\ &\quad + 2\mathbb{R}\mathrm{e}\{z(t)^* \Lambda^*_{\sigma(t)} D p_{\sigma(t)}(t)\} + p^*_{\sigma(t)} D p_{\sigma(t)}\end{aligned}$$

$$\begin{aligned}&\le |z(t)|'(M'_{\sigma(t)} D M_{\sigma(t)} - D)|z(t)| \\ &\quad + 2|z(t)|' M'_{\sigma(t)} D |p_{\sigma(t)}| + |p_{\sigma(t)}|' D |p_{\sigma(t)}|\end{aligned} \tag{A15}$$

$$\begin{aligned}&\le |z(t)|'(\Lambda' D \Lambda - D)|z(t)| \\ &\quad + 2|z(t)|' \Lambda' D |p_{\sigma(t)}| + |p_{\sigma(t)}|' D |p_{\sigma(t)}|\end{aligned} \tag{A16}$$

$$\begin{aligned}&\le |z(t)|'[(\Lambda + \bar{F})' D (\Lambda + \bar{F}) - D]|z(t)| \\ &\quad + 2|z(t)|'(\Lambda + \bar{F})' D \bar{w} + \bar{w}' D \bar{w},\end{aligned} \tag{A17}$$

where we have used Equation (14) in Equations (A15) and (A16), and Equation (A14) in Equation (A17). By Equations (37) and (A17), it follows that $\Delta L_z(t, z) < 0$, whenever $\|z\|$ is big enough. Next, taking $x \in \mathbb{R}^n$, we have

$$\begin{aligned}&\Delta L_z(t, V^{-1}x(t)) \\ &\quad = x(t)'[A'_{\sigma(t)}(V^{-1})^* D V^{-1} A_{\sigma(t)} - (V^{-1})^* D V^{-1}] x(t) \\ &\qquad + 2\mathbb{R}\mathrm{e}\{x(t)' A'_{\sigma(t)} (V^{-1})^* D p_{\sigma(t)}\} + p^*_{\sigma(t)} D p_{\sigma(t)} \\ &\quad = x(t)'(A'_{\sigma(t)} P A_{\sigma(t)} - P) x(t) + 2x(t)' A'_{\sigma(t)} P H_{\sigma(t)} w_{\sigma(t)}(t) \\ &\qquad + w_{\sigma(t)}(t)' H'_{\sigma(t)} P H_{\sigma(t)} w_{\sigma(t)}(t) = \Delta L(t, x(t)).\end{aligned}$$

Consequently, we conclude that $\Delta L(t, x(t)) < 0$, whenever $\|x\|$ is big enough.