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Directionality Compensation for Linear Multivariable Anti-windup Synthesis.

Ambrose A. Adegbege* and William P. Heath**

Abstract

We develop new synthesis procedures for optimizing anti-windup control applicable to open-loop exponentially stable multivariable plants subject to hard bounds on the inputs. The optimizing anti-windup control falls into a class of compensator commonly termed directionality compensation. The computation of the control involves the on-line solution of a low-order quadratic program in place of simple saturation. We exploit the structure of the quadratic program to incorporate directionality information into the off-line anti-windup synthesis using a decoupled architecture similar to that proposed in the literature for anti-windup schemes with simple saturation. We demonstrate the effectiveness of the design compared to several schemes using a simulated example.

I. INTRODUCTION

Most practical control problems must deal with constraints imposed by equipment limitations such as actuator nonlinearities. One approach that has received much attention in dealing with such problems is the anti-windup technique [2], [3], [4], [5], [6], [7], [8]. It is also a common practice to incorporate an additional artificial non-linearity (directionality compensator) in multivariable anti-windup designs to address the problem of directionality [9], [10], [11], [12], [13], [14]. In general, the design of a directionality compensator is carried out independently of the linear control design and with the assumption that the resulting optimizing structure inherits the

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stability of the unsaturated loop (e.g. [10], [12]). Such directionality compensators often take the form of solutions to some convex optimization problems; these are solved either implicitly ([15]) or explicitly ([10], [11], [12]) on-line during control computation. When the control policy is obtained by an explicit solution of an on-line optimization problem at each time step, the resulting scheme is termed optimizing (e.g. [16], [17]). In this paper, we distinguish such on-line optimization from the off-line synthesis required before implementation. We assume the control action is available instantaneously, as the underlying optimization is low-order, independent of state, piecewise affine in the solution space and can be solved order of magnitude faster than the plant bandwidth [18].

The main contribution of this paper is the synthesis of anti-windup with both stability and performance guarantees for systems incorporating directionality compensation in the form of a quadratic program (QP). In particular, we note that the information from the directional compensator that is resolved on-line can be incorporated into the off-line anti-windup synthesis to guarantee closed-loop stability as well as improved performance.

This paper extends the preliminary results of [1], where development was restricted to the internal model control (IMC) structure of [9], [15] and it is a natural generalization of [4] to cases where the control non-linearity is coupled and satisfy a generalized sector condition. The resulting off-line synthesis is characterized by two matrices: F from the linear anti-windup conditioning and E from the directionality compensation. We design E to minimize some integral squared error performance objective and then synthesize F via a convex search over linear matrix inequalities (LMIs).

Other related works include [17], [19], [20] where sufficient conditions for closed-loop stability were derived in terms of the Karoush-Kuhn-Tucker (KKT) conditions associated with the input non-linearity. These approaches allow only for *a posteriori* stability checks when the anti-windup compensator has already been designed using an existing design technique (e.g. [15]). Others are [21], [22] where non-diagonal stability multipliers are employed in the analysis of systems with decoupled and repeated nonlinearities. Synthesis using the non-diagonal multipliers of ([21], [22]) may be problematic [23, Remark 3]. In [2], an algebraic loop was deliberately introduced into the static anti-windup configuration for improved performance. Under certain conditions ([24]), the algebraic loop leads to a QP whose solution may be considered a directionality compensator. However, the well-posedness and practical implementations of such algebraic loops are nontrivial [25]. The synthesizing LMI of [2] is also not always feasible (see e.g.[3]) and the choice of performance specification do not relate directly to the true goal of anti-windup design which is the swift recovery of linear performance ([4], [26]). Here, we exploit an extra design freedom for incorporating the system's directional characteristics as well as stability and performance requirements for a dynamic anti-windup synthesis which is always feasible. This extra design freedom has previously been exploited in [27] but only to eliminate algebraic loop in the anti-windup construction.

Notations: Let $\mathcal{L}_2^m$ be the Hilbert space of $\mathbb{R}^m$ valued functions f on $[0, \infty)$ such that $\int_0^\infty |f(t)|^2 dt$ is finite. The expression $\langle f, g \rangle$ stands for the inner product of signals $f, g \in \mathcal{L}_2^m$ defined by $\int_0^\infty f(t)^T g(t) dt$. For a general operator $\Pi : \mathcal{L}_2^m \to \mathcal{L}_2^m$, Π^* denotes the adjoint. Let Π be a bounded self-adjoint operator satisfying $\Pi = \Pi^*$; then a bounded operator $\psi : \mathcal{L}_2^m \to \mathcal{L}_2^m$ is said to satisfy the Integral Quadratic Constraint (IQC) defined by Π or simply $\psi \in$ IQC(Π) if the following inequality holds [28]

$$\left\langle \begin{bmatrix} u \\ v \end{bmatrix}, \Pi \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle \geq 0 \text{ for all } v = \psi(u), \; u \in \mathcal{L}_2^m. \tag{1}$$

The signs $\preceq$ and $\succeq$ denote element by element inequalities.

II. Problem Setup

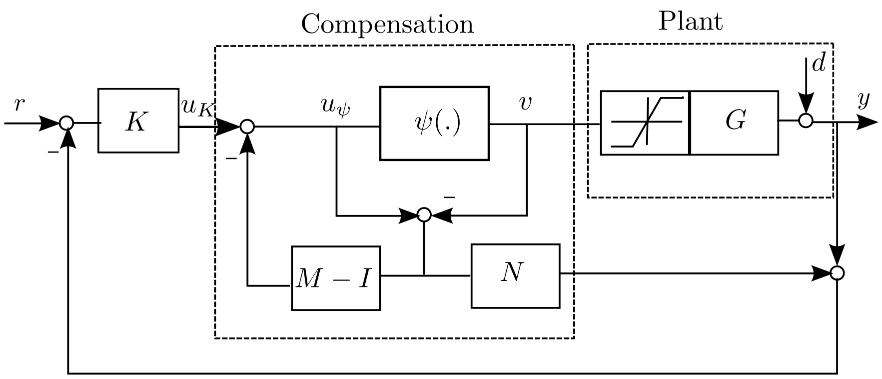

Fig. 1: Configuration for anti-windup and directionality compensation.

We consider the optimizing structure of Fig. 1. The plant is given by $y = Gu + d$ with only bounded u admissible and where G represents the plant dynamics which are assumed to be perfectly known. To avoid confusion, we discriminate between u_K, u_{lin} and u_ψ as follows: u_K

is the output of controller K for the compensated closed-loop, u_{lin} is the output of controller K when the closed-loop is in the linear range i.e. the input constraints are not active and u_ψ is the input to the directionality compensator defined by $\psi(.)$. We denote the state space realizations of G and its right co-prime factorization $G = NM^{-1}$ respectively as

$$G \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \text{ and } \begin{bmatrix} M \\ N \end{bmatrix} \sim \left[\begin{array}{c|c} A+BF & B \\ \hline F & I \\ C+DF & D \end{array}\right] \tag{2}$$

where F is such that $A+BF$ is Hurwitz. The linear controller denoted as K is assumed to have been designed to meet some acceptable linear stability and performance criteria. This implies that the unconstrained closed-loop system

$$\begin{aligned} y_{lin} &= (I+GK)^{-1}GKr + (I+GK)^{-1}d \\ u_{lin} &= K(I+GK)^{-1}(r-d) \end{aligned} \tag{3}$$

is internally stable [29]. Usually, the linear controller K is designed such that the closed loop system (3) is decoupled. The signal y_{lin} is the unconstrained (linear) plant output. The exogenous signals r and d represent the reference and the disturbance signals respectively. The control input bound is modeled using a saturation function block as follows:

$$\hat{u} = \text{sat}(u) = \begin{bmatrix} \text{sat}(u_1) \\ \vdots \\ \text{sat}(u_m) \end{bmatrix} \text{ where } \text{sat}(u_i) = \begin{cases} u_i^{max} & u_i \succ u_i^{max} \\ u_i & u_i^{min} \preceq u_i \preceq u_i^{max} \\ u_i^{min} & u_i \prec u_i^{min} \end{cases} \tag{4}$$

denotes the saturation non-linearity associated with each of the manipulated input $u_i(t)$ for some $u_i^{min} \preceq 0$ and $u_i^{max} \succeq 0$.

Since the multivariable saturation non-linearity (4) acts element by element, the direction of the actual plant input vector $\hat{u}$ necessarily differs from that of the controller output vector u [9], [13]. Such directional change may be unduly amplified especially for plants with strong structural couplings [9]. For such class of systems, a naive application of the classical anti-windup techniques has been shown to result in significant performance deterioration or even instability [30], [9]. This performance loss is generally attributed to the effects of directionality (e.g. [31], [10], [11], [12]). The artificial non-linearity, represented as $\psi(.)$, is introduced in Fig. 1 such that the difference between the unconstrained response y_{lin} and the constrained response y

is minimized in some sense. Let $E \in \mathbb{R}^{m\times m}$ be a non-singular structural matrix that represents the plant's directional characteristics; the directional compensator is such that its optimal solution v minimizes the Euclidean norm of the difference between Ev and Eu_ψ while not violating the constraints imposed by the actuator limits. This constrained weighted least distance problem can be stated as

$$\min_v \frac{1}{2}(v-u_\psi)^T H(v-u_\psi), \text{ subject to } u_i^{min} \preceq v_i \preceq u_i^{max}, i = 1, \cdots, m \tag{5}$$

or in the following standard form

$$\psi(u_\psi) = \arg\min_v \frac{1}{2} v^T H v - v^T H u_\psi, \text{ subject to } Lv \preceq b \text{ with } b \succeq 0 \tag{6}$$

where $H = E^T E = H^T > 0 \in \mathbb{R}^{m\times m}$. The fixed terms $L \in \mathbb{R}^{2m\times m}$ and $b \in \mathbb{R}^{2m}$ in the inequality constraints are obtained from (4) as

$$L = \begin{bmatrix} -I_m, & I_m \end{bmatrix}^T \text{ and } b = \begin{bmatrix} -(u^{min})^T, & (u^{max})^T \end{bmatrix}^T \tag{7}$$

where $u^{min} = \begin{bmatrix} u_1^{min}, & \cdots, & u_m^{min} \end{bmatrix}^T \preceq 0$ and $u^{max} = \begin{bmatrix} u_1^{max}, & \cdots, & u_m^{max} \end{bmatrix}^T \succeq 0$ are the lower and upper bounds on the control inputs respectively. It is important to note that $v = \text{sat}(u_\psi)$ solves (6) when E is diagonal or the identity and that v is admissible for any choice of E (i.e. $v = \text{sat}(v)$). Since v is always admissible, we can safely ignore the saturation block preceding the plant model G in Fig. 1. The objective function of (6) is strictly convex by virtue of H being positive definite. Since the constraints describe a box which contains the origin, the feasible set is not empty [17], the quadratic program always has a unique solution [32].

III. Equivalent Representations

We take advantage of the structure of the non-linearity $\psi(.)$ to obtain equivalent representations of the optimizing anti-windup framework of Fig. 1. These alternative representations are directly linked to the two design variables F and E, and hence provide insights on their design or synthesis. First, we transform $\psi(.)$ into related nonlinearities satisfying some sector conditions.

Lemma 1: Let $v = \psi(u_\psi)$ be the quadratic program defined by (6) and let $w = \xi(u_\psi)$ be the quadratic program

$$\xi(u_\psi) = \arg\min_w \frac{1}{2} w^T H w, \text{ subject to } Lu_\psi - Lw \preceq b, \tag{8}$$

with $H = E^T E = H^T > 0$ and where L and b are defined as in (7). The interconnection of $w = \xi(u_\psi)$ with $v = u_\psi - w$ is equivalent to $v = \psi(u_\psi)$. ■

Lemma 2: Let the quadratic program (6) be set as $v = \psi(u_\psi)$ with $H = E^T E = H^T > 0$ and L and b are defined as in (7). Let $\bar{v} = \phi(\bar{u})$ be the quadratic program

$$\phi(\bar{u}) = \arg\min_{\bar{v}} \frac{1}{2}\bar{v}^T\bar{v} - \bar{v}^T\bar{u}, \text{ subject to } R\bar{v} \preceq b. \tag{9}$$

Suppose $\phi(\bar{u}) = E\psi(u_\psi)$, $\bar{u} = Eu_\psi$ and $R = LE^{-1}$, then by input-output scaling, $\psi(u)$ and $\phi(\bar{u})$ are equivalent. Furthermore, $\phi(\bar{u})$ belongs to the sector$[0\ 1]$. ■

Remark 1: If E is chosen to be diagonal or the identity, $\psi(.)$ and $\xi(.)$ reduce to the decentralized saturation and dead-zone non-linearities respectively (see e.g. [19], [20]). This is the case in most existing anti-windup formulations where E is used to eliminate algebraic loops (e.g. [27], [4]).□

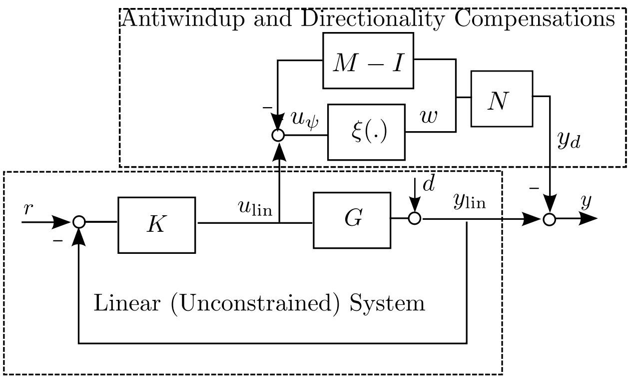

Fig. 2: Equivalent Structure I.

Using the transformation in Lemma 1 and the relationships in (3), the optimizing structure in Fig. 1 can be represented in the decoupled structure of Fig. 2. This decoupled structure is attractive for anti-windup synthesis (e.g. [4], [33]) and allows for easy incorporation of information from the directionality compensator $\psi(.)$ or $\xi(.)$ into the offline synthesis. Alternatively, using the linear transformations in Lemma 2, the optimizing anti-windup structure in Fig. 1 can be re-arranged as shown in Fig. 3 where $\widetilde{M}$ and $\widetilde{N}$ are given by:

$$\begin{bmatrix} \widetilde{M} \\ \widetilde{N} \end{bmatrix} \sim \left[\begin{array}{c|c} A + BF & BE^{-1} \\ \hline F & E^{-1} \\ C + DF & DE^{-1} \end{array}\right]. \tag{10}$$

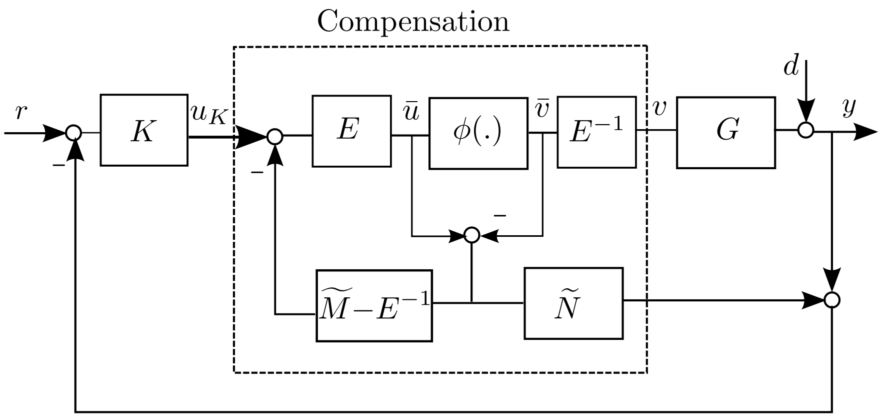

Fig. 3: Equivalent Structure II.

with F and E being design variables and E is invertible. The co-prime factorization (10) is more general than (2) as it offers an extra degree of freedom via the static matrix E which can be exploited in the anti-windup design. The anti-windup design then reduces to finding appropriate parameters F and E which ensure both the stability and the performance of the closed-loop system. This interpretation is similar to the co-prime-factor parametrization in [27], [33], but here the optimizing framework provides some additional insights as to the role and the choice of E. We also advocate that E should not be used as a free design parameter but should be chosen based on some structural characteristics of the plant. From the equivalent structure of Fig. 3, the block E^{-1} may be considered a static pre-compensator for the plant and may be chosen such that the response is decoupled either in the transient period or at steady states (e.g. [12], [17], [34]). Similarly, the block E may be considered a static post-compensator for the controller and may be chosen such that KE is decoupled (e.g. [10]). Existing directionality compensation schemes [10], [12], [17] correspond to particular choices of E. We highlight some specific choices here and refer readers to [35] for detailed derivations.

1. Optimization based Conditioning Technique [10]

$$E = \lim_{s\to\infty} [K(s)]^{-1}. \tag{11}$$

2. Optimal Directionality Compensation [12]

$$E = \lim_{s\to\infty} [\text{diag}(s^{r_i})G(s)]. \tag{12}$$

where $r_i = \min(r_{i1}, r_{i2}, \ldots, r_{im})$ and $r_{i,j}$ is the relative degree of output y_i with respect to manipulated input u_j [12]. In this case, it is usual to call E the characteristic or decoupling matrix of $G(s)$ as it captures the structural couplings between its input and output channels [34].

3. Steady State Directionality Compensation [17], [34]

$$E = \lim_{s \to 0} [G(s)] \, . \tag{13}$$

Thus, it is possible to choose E such that during saturation, the map between the admissible input v and the plant output y is decoupled at certain frequencies. Suppose that E is chosen based on the above structural considerations such that performance deterioration due to the effects of directionality is minimized in some sense. Then, F can be chosen based on convex search such that the anti-windup system is guaranteed stable and that recovery of linear performance is hastened. For the special case where E is diagonal, the directionality compensation (6) reduces to the repeated saturation non-linearity (4) (see Remark 1). In this instance, $v = \text{sat}(u_\psi)$ is the optimal solution to (6) and the inclusion of E in the anti-windup synthesis is of no use (see [27] and also Corollary 2). We discuss in the next section a two-step procedure for the synthesis of F and E and also comment on the procedure for the simultaneous synthesis of F and E.

IV. Anti-windup Compensator Synthesis

We use the machinery of integral quadratic constraints (IQC) to synthesize appropriate anti-windup compensator while incorporating information from the directionality compensator. We only consider static IQCs, since it is our intention to employ such IQCs in the convex synthesis of a suitable anti-windup compensator. Consider the feedback structure of Fig. 4. Standard IQC

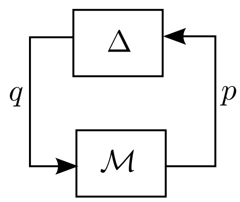

Fig. 4: Interconnections for Synthesis

theory [28] gives sufficient conditions for the stability of such interconnection if for $\Delta \in \text{IQC}(\Pi)$, the following inequality is satisfied

$$\begin{bmatrix} \mathcal{M}(j\omega) \\ I \end{bmatrix}^T \Pi \begin{bmatrix} \mathcal{M}(j\omega) \\ I \end{bmatrix} < 0 \quad \forall\ \omega \in \mathbb{R}. \tag{14}$$

Suppose we define $y_d = y - y_{lin}$, a fictitious block Δ_p satisfying the $\mathcal{L}_2$ gain condition $\|u_{lin}\|^2 - \frac{1}{\gamma^2}\|y_d\|^2 \leq 0$ with $\gamma > 0$ (e.g. [36]) and the nonlinear map $w = \xi(u_\psi)$ satisfying $w^T H(w - u_\psi) \leq 0$ with $w = u_\psi - v$ and $H = H^T > 0$. The closed-loop structure Fig. 2 can be transformed into the feedback interconnection of Fig. 4 with $\Delta = \text{diag}(\xi, \Delta_p)$ and

$$\mathcal{M} = \begin{bmatrix} I - M & I \\ -N & 0 \end{bmatrix} \tag{15}$$

where $p = \begin{bmatrix} u_\psi \\ y_d \end{bmatrix}$ and $q = \begin{bmatrix} w \\ u_{lin} \end{bmatrix}$. The performance requirement that the $\mathcal{L}_2$ gain of the map from u_{lin} to y_d be less than $\gamma > 0$ is equivalent to checking the interconnection of Fig. 4 is stable for all Δ_p which is norm bounded by $1/\gamma$. This is a well-known convention (see [37, section 4.3] and [21, comment after Theorem 2]). Also, such a performance specification has been observed to be central to the anti-windup design problem (e.g. [4], [38]) for the saturated case. Using appropriate IQCs satisfied by the input-output maps of ξ and Δ_p, we state the following result.

Theorem 1: Given a stable linear time invariant plant G with co-prime factorization (2). Let $\mathcal{M}$ be given by (15) and $\Delta = \text{diag}(\xi, \Delta_p)$ be a bounded operator such that ξ and Δ_p satisfy respectively the IQCs conditions

$$\left\langle \begin{bmatrix} u_\psi \\ w \end{bmatrix}, \begin{bmatrix} 0 & H\Lambda \\ \Lambda H & -(H\Lambda + \Lambda H) \end{bmatrix} \begin{bmatrix} u_\psi \\ w \end{bmatrix} \right\rangle \geq 0\ \forall\ w = \xi(u_\psi),\ u_\psi \in \mathcal{L}_2^m \text{ and} \tag{16}$$

$$\left\langle \begin{bmatrix} y_d \\ u_{lin} \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} y_d \\ u_{lin} \end{bmatrix} \right\rangle \geq 0\ \forall\ u_{lin} = \Delta_p(y_d),\ y_d \in \mathcal{L}_2^m. \tag{17}$$

Suppose there exist $P = P^T > 0$, diagonal $\Lambda > 0$ and $\alpha = \gamma^2 > 0$ satisfying matrix inequality (18)

$$\begin{bmatrix} \tilde{A}^T P + P\tilde{A}^T & PB - F^T H\Lambda & 0 \\ B^T P - \Lambda H F & -\Lambda H - H\Lambda & \Lambda H \\ 0 & H\Lambda & -\alpha I \end{bmatrix} + \begin{bmatrix} \tilde{C}^T \\ D^T \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{C}^T \\ D^T \\ 0 \end{bmatrix}^T < 0 \tag{18}$$

with $\tilde{A} = A + BF$, $\tilde{C} = C + DF$. Then, the feedback interconnection of $\mathcal{M}$ and Δ is stable for all $u_\psi, y_d \in \mathcal{L}_2^m$ and the $\mathcal{L}_2$-gain from u_{lin} to y_d is less than γ. ■

Theorem 1 is not suitable for synthesis but allows for an *a posteriori* stability check when both the anti-windup and the directionality compensations have already been designed (i.e F and E are both fixed). The IQC condition (16) is a generalization of the multivariable circle criterion for sector-bounded nonlinearities [39]. In what follows, we modify (18) for convex synthesis. We first consider the two-step design procedure.

A. *Two-step design procedure for F and E*

The following corollary provides an anti-windup synthesis procedure for a given directionality compensation parameter E.

Corollary 1: Consider the optimizing structure of Fig. 2 with a stable linear time invariant plant G described by (2) and non-linearity described by the quadratic program (8). Suppose E is fixed with $H = E^T E$ and $T = H^{-1}$. Further Suppose there exist $Q = Q^T > 0$, $\gamma > 0$, diagonal $U > 0$ and X satisfying the linear matrix inequality (19)

$$\begin{bmatrix} AQ + QA^T + BX + X^T B^T & BUT - X^T & 0 & QC^T + X^T D^T \\ TUB^T - X & -UT - TU & I & TUD^T \\ 0 & I & -\gamma I & 0 \\ CQ + DX & DUT & 0 & -\gamma I \end{bmatrix} < 0 \tag{19}$$

Then, there exists a plant-order anti-windup compensator which renders the interconnection of Fig. 2 stable with an $\mathcal{L}_2$ gain from u_{lin} to y_d less than γ. The design parameter F characterizing the co-prime factorization (2) is recovered using $F = XQ^{-1}$ where X and Q are feasible solutions of LMI (19).■

After E has been chosen based on some structural considerations as highlighted in section III, the design parameter F can be synthesized using (19) which is linear in its variables.

B. *Simultaneous Synthesis of F and E*

Corollary 2: Consider the optimizing structure of Fig. 2 with a stable linear time invariant plant G described by (2) and non-linearity described by the quadratic program (8). Suppose there exist $Q = Q^T > 0$, $\gamma > 0$, W and X satisfying the linear matrix inequality (20)

$$\begin{bmatrix} AQ + QA^T + BX + X^T B^T & BW - X^T & 0 & QC^T + X^T D^T \\ WB^T - X & -W - W^T & I & WD^T \\ 0 & I & -\gamma I & 0 \\ CQ + DX & DW & 0 & -\gamma I \end{bmatrix} < 0 \qquad (20)$$

Then, there exists a plant-order anti-windup compensator which renders the interconnection of Fig. 2 stable with an $\mathcal{L}_2$ gain from u_{lin} to y_d less than γ. The design parameters F and E characterizing the co-prime factorization (10) are recovered using $F = XQ^{-1}$, $E = \sqrt{W^{-1}U}$ where X, Q and W are feasible solutions of LMI (20) and $U > 0$ is arbitrary. ■

Note that the feasibility of LMI (20) guarantees the invertibility of W. From the (2,2) element of LMI (20), we have that $-W - W^T < 0$. It follows that $W > 0$ and hence W^{-1} exists.

Remark 2: Note that matrix W in LMI (20) is not restricted to be diagonal as in other LMI-based anti-windup designs (e.g. [4, LMI (13)]). Although allowing W to assume a full block structure may be considered a greater degree of freedom in the choice of F, the recovered F from LMI (20) tends to place the poles of the anti-windup compensator far into the left half-plane. This feature is characteristic of LMI-based dynamic anti-windup schemes (see [4], [40] for details). □

Remark 3: The main difference between Corollaries 1 and 2 is the explicit inclusion of E via $T = H^{-1}$ in (19). LMI (20) is independent of H. This may suggest that H can be chosen arbitrarily via $H = W^{-1}U$ where U is an arbitrary matrix. It is important to note that a naive choice of U may not preserve the structure of the directionality compensator. An alternative is to restrict W in LMI (20) to be symmetric (i.e $W = W^T$) while U is chosen to be diagonal positive definite. With this, any H recovered from $H = W^{-1}U$ (for $W = W^T$ satisfying (20)) is guaranteed to be symmetric positive definite (i.e. $H = H^T > 0$). It follows that any such H will preserve the structure of the directionality compensator. Similar conditions have earlier been employed to guarantee the well-posedness of algebraic loops (e.g. [41], [25]). □

In existing anti-windup literature (e.g. [2], [3], [4]), U is usually chosen such that $H = I$. In this case, LMI (19) reduces to a special case of (20) where W is diagonal (compare with [4, LMI (13)]). However, when H is chosen as discussed in section III, LMI (19) allows for the incorporation of the plant directional characteristics into the anti-windup optimization.

Remark 4: Since the $\mathcal{L}_2$ performance channels in Corollary (1) are scaled versions of those in Corollary (2), the attained $\mathcal{L}_2$ gains are essentially the same (e.g. see [36, pg 95]).$\square$

Remark 5: Note that there is always a choice of F such that LMIs (19) and (20) are feasible. Choosing $F = 0$ implies $M = I$. In this instance, LMI (19) reduces to a version of the bounded real lemma (e.g. [36]) and the $\mathcal{L}_2$ gain computed when $H = I$ corresponds to the infinity-norm of the stable plant G. This case also recovers the conventional IMC anti-windup structure of [9].$\square$

Since the solution of LMI (20) tends to produce fast dynamics with large closed-loop poles, it is common, for ease of implementation, to constrain the anti-windup poles to regions comparable to the unconstrained closed-loop poles. In the spirit of [42], the anti-windup poles can be constrained to a region formed by the intersection of the negative half s-plane and a disc of radius r. This can easily be achieved by solving in tandem with LMI (20) the following LMI region [42]

$$\begin{bmatrix} -rP & AP + BX \\ PA^T + X^T B^T & -rP \end{bmatrix} < 0 \tag{21}$$

where r is the radius of the disk. Similarly, by incorporating robustness constraints [4] into LMI (20), the anti-windup poles can be restricted to favorable regions for implementation.

V. SIMULATION EXAMPLE

In order to demonstrate the effectiveness of the proposed design procedure, we consider an ill-conditioned example typical of distillation column control [29]. This is a well-studied problem because of the strong directionality and interaction that exist in the plant as well as its high sensitivities to diagonal input nonlinearities and uncertainties. We compare three anti-windup design approaches, namely the optimal directionality compensation scheme [12], the dynamic anti-windup without directionality compensation [4] and the proposed anti-windup with directionality compensation (Corollary 1).

The plant model is given by the transfer function matrix

$$G(s) = \frac{1}{75s + 1} \begin{bmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{bmatrix} \tag{22}$$

with both inputs constrained as $|u_i| \leq 100$, $i = 1, 2$. In the absence of control input saturations, the linear controller is designed to achieve a completely decoupled closed-loop response described by

$$G_F(s) = \frac{1}{1.43s + 1} I.$$

The unity feedback controller which achieves this decoupled response is given by [29]

$$K(s) = \frac{75s + 1}{1.43s} \begin{bmatrix} 45.38 & -35.77 \\ 44.80 & -36.23 \end{bmatrix}. \tag{23}$$

The plant's characteristic matrix is obtained as $\dfrac{1}{75} \begin{bmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{bmatrix}$ with condition number 141.732. For the directionality compensation scheme [12], we chose $H = E^T E$ with E as the plant's non-singular characteristic matrix. Fig. 5 shows the input and output responses of the plant to a set-point changes from $[0 \quad 0]^T$ to $[0.7 \quad 0]^T$ at time $t = 0$ and from $[0.7 \quad 0]^T$ to $[0.7 \quad 0.4]^T$ at time $t = 50$ for the different compensation schemes. Note that the unconstrained case requires a very aggressive control action during transients to achieve the decoupled response. During the transient periods following the two set-point changes, control actions due to the directionality compensator never violate the saturation limits. During the first transient, u_1 stays on the positive saturation limit while u_2 gradually approaches it. This process is reversed during the second transient where u_2 stays on the negative saturation limit while u_1 approaches it. For the saturated case (without anti-windup and directionality compensations), the saturation limits are violated during the two transient periods leading to clipping of both controls at the saturation limits. The effects of clipping are clear on the output response: sluggishness and inverse response. The directionality scheme [12] results in an improved transient response as compared to the saturated (uncompensated) response. Note that this scheme does not offer any stability guarantee. While the problem of directionality is solved by incorporating directionality compensation, the proposed scheme (Corollary 1) recovers linear performance faster and it is closest to the unconstrained response. This superior performance confirms the benefit of combining the optimality of an on-line optimization with the efficiency of convex off-line anti-windup synthesis while guaranteeing closed-loop stability.

Finally, we compare the proposed scheme (Corollary 1) with that of [4, LMI 23]. Following [4], we consider three different performance-robustness weight combinations, namely $w_p =$

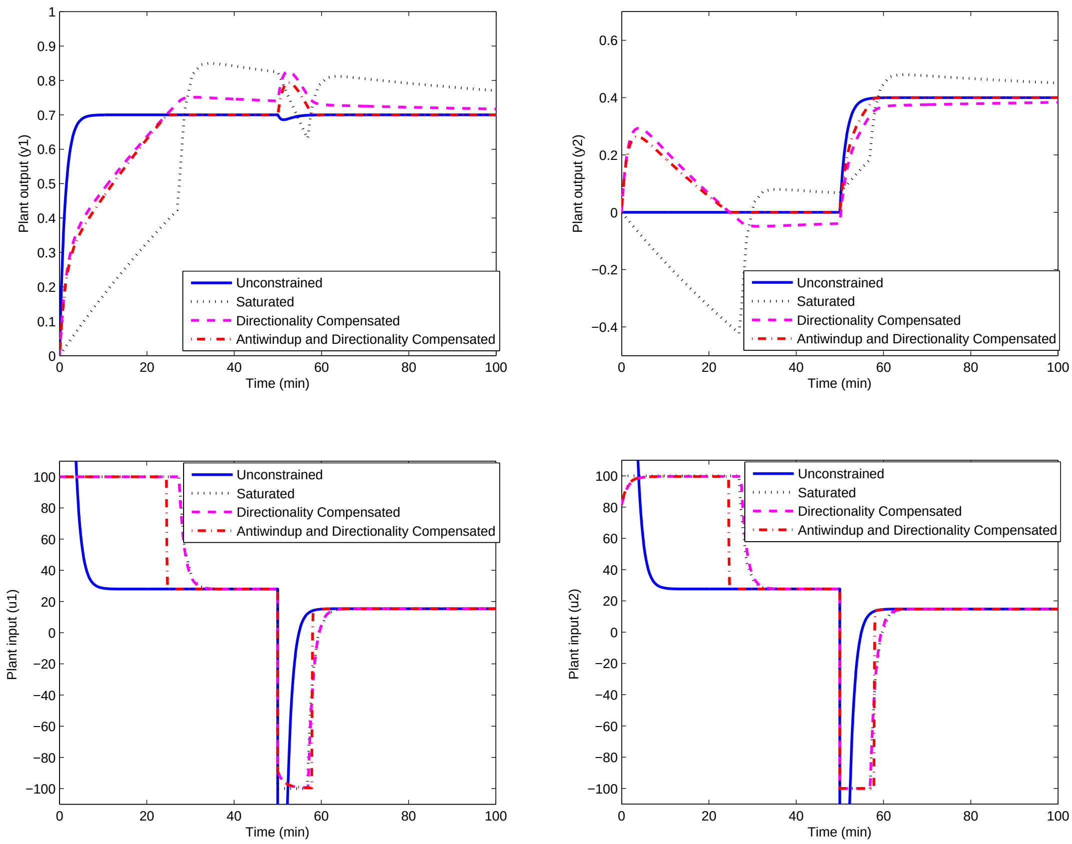

Fig. 5: Responses of the unconstrained (solid), saturated (Dotted), directionality compensated [12] (Dashed) and anti-windup with directionality compensation (Dashdotted) to a step reference. The superior performance of the anti-windup with directionality compensation over [12] can be attributed to its efficiency and swiftness in recovering linear performance after a period of nonlinear operation.

$0.01, W_r = 1$ (case 1), $w_p = 1, W_r = 0.01$ (case 2) and $W_p = 1, W_r = 1$ (case 3). The feedback gain recovered from LMI (19) is

$$F_H = \begin{bmatrix} 162.1456 & -308.6913 \\ 307.7003 & -454.2451 \end{bmatrix}. \tag{24}$$

For the three different cases of [4], we have

$$F_{Case\ 1} = \begin{bmatrix} 0.2623 & -0.2635 \\ -0.2635 & 0.2631 \end{bmatrix} * 10^{-7}, F_{Case\ 2} = \begin{bmatrix} -20.1877 & 20.2183 \\ 20.7024 & -20.7379 \end{bmatrix}$$
$$\text{and } F_{Case\ 3} = \begin{bmatrix} -0.1539 & 0.1541 \\ 0.1541 & -0.1543 \end{bmatrix}. \tag{25}$$

As discussed in Remark 5 (also in [4]), case 1 recovers the internal model control (IMC) anti-windup since $F_{Case\ 1} \approx 0$. It is well-known that the IMC anti-windup is robust to input-multiplicative uncertainties but may result in a very poor performance ([9], [4], [8]). In case 2, performance is emphasized over robustness while case 3 places equal emphasizes on robustness and performance. The input and output responses of the three different cases are compared to those of the proposed scheme in Fig. 6. Note that for all the three cases considered, the dynamic anti-windup [4] seeks to restore the plant inputs to the linear region as quickly as possible and hence the initial sluggish and inverse (on the second channel) responses observed in Fig. 6. As for the proposed scheme, only one of the control inputs is allowed to stay on the constraint during each of the transient periods. This ensured that the plant is driven in the right direction eliminating both the initial sluggish and inverse responses.

For completeness, we mention that Corollary 2 yields similar responses to [4] when the closed-loop poles are subjected to same restrictions. Note that for this case we can choose H as the identity (i.e. $H = I$). Without restrictions, the feedback gain F recovered from LMI (20) is given by

$$F_I = \begin{bmatrix} -1.4552 & 1.4575 \\ 0.2047 & -0.2052 \end{bmatrix} * 10^5. \tag{26}$$

This results in a compensator with very fast poles requiring a very high sampling frequency for implementation. These fast poles can be constrained to a suitable region either by solving LMI (21) (e.g. with $r = 0.05$) in tandem with LMI (20) or by applying the robustness and performance weights of [4].

VI. CONCLUSIONS

We have presented a multivariable optimizing anti-windup design which guarantees closed-loop stability while compensating for the effects of both windup and directionality. Directionality

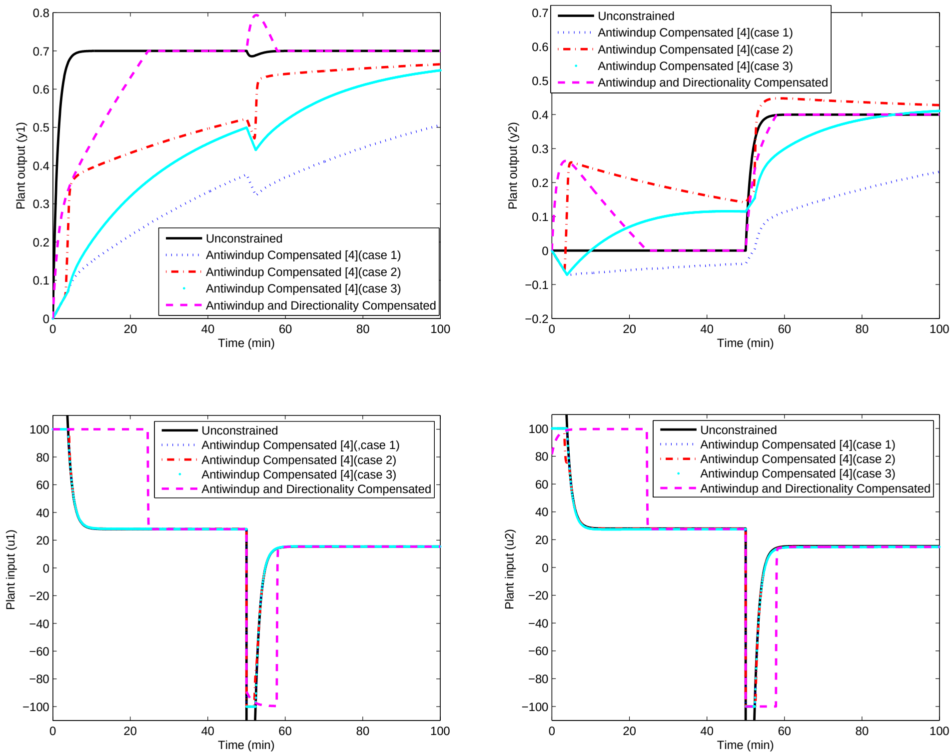

Fig. 6: Responses of anti-windup schemes with (Corollary 1) and without [4] directionality compensation: unconstrained (solid), [4, Case 1] (Dotted), [4, Case 2] (Dashdot), [4, Case 3] (Point) and LMI-based anti-windup with directionality compensation 1 (Dashed) to set-point changes from $[0 \quad 0]^T$ to $[0.7 \quad 0]^T$ at time $t = 0$ and from $[0.7 \quad 0]^T$ to $[0.7 \quad 0.4]^T$ at time $t = 50$. The antiwindup with directionality compensation can be seen to have a significant improvement over [4] in that it combines the optimality of directionality compensation with the efficiency of model recovery anti-windup techniques.

compensation is achieved through an on-line optimization while windup is addressed through an off-line convex dynamic anti-windup synthesis. The resulting synthesis problem is characterized by two gain matrices F and E. In particular, we advocate that E should not be used as a free-design parameter but should be chosen based on the structural characteristics of the plant. Such structural information can then be incorporated into the convex off-line anti-windup synthesis to guarantee both closed-loop stability and performance. The simulated examples demonstrate the benefits that ensue: both from introducing directionality compensation into an anti-windup structure and from applying our proposed design procedures. The results are especially beneficial when the plant is ill-conditioned or has lightly damped modes. We have, however, restricted our discussions to the nominal case where there are no model uncertainties. An area of further work is to incorporate robustness into the optimizing anti-windup design (see [43]).

VII. Appendix

To prove Theorem 1, we need to establish the relations between the nonlinearities ψ and ξ. This follows from the proofs of Lemmas 1 and 2 below.

Proof of Lemma 1: The necessary and sufficient KKT conditions [32] for ξ are given by

$$Hw - L^T\lambda = 0,\ Lu_\psi - Lw - b + s = 0,\ s \succeq 0,\ \lambda \succeq 0,\ \text{and } \lambda^T s = 0. \tag{27}$$

If we substitute $w = u_\psi - v$ into (27), we obtain

$$Hv - Hu_\psi + L^T\lambda = 0,\ Lv - b + s = 0,\ s \succeq 0,\ \lambda \succeq 0,\ \text{and } \lambda^T s = 0. \tag{28}$$

The conditions in (28) are exactly the KKT conditions for ψ. It follows that if w is the unique optimal solution of (8), then the optimal solution of (6) is uniquely determined by $v = u_\psi - w$.■

Proof of Lemma 2: The necessary and sufficient KKT conditions [32] for ϕ are given by

$$\bar{v} - \bar{u} - R^T\lambda = 0,\ R\bar{v} - b + s = 0,\ s \succeq 0,\ \lambda \succeq 0,\ \text{and } \lambda^T s = 0. \tag{29}$$

Equivalence follows by substituting $\bar{v} = Ev$, $\bar{u} = Eu_\psi$ and $L = RE$ into (29) and pre-multiplying the first condition by E^T (since E is invertible) to obtain (28). Pre-multiplying the first KKT condition in (29) by $\bar{v}^T$ and substituting gives

$$\bar{v}^T\bar{v} - \bar{v}^T\bar{u} = -b^T\lambda \leq 0. \tag{30}$$

Hence, we may say $\phi(\bar{u})^T\left[\phi(\bar{u}) - \bar{u}\right] \leq 0$ or analogously $\phi(\bar{u}) \in \text{sector}[0, I]$. ■

Proof of Theorem 1: From (30) and using the relation $w = u_\psi - v$, we have the following generalized sector condition

$$w^T Hw - w^T Hu_\psi \leq 0. \tag{31}$$

For any diagonal $\Lambda > 0$, the inequality $w^T\Lambda H(w - u_\psi) \leq 0$ also holds. In IQC notation, we can write $\xi \in \text{IQC}(\Pi_\xi)$ with $\Pi_\xi = \begin{bmatrix} 0 & H\Lambda \\ \Lambda H & -(H\Lambda + \Lambda H) \end{bmatrix}$. Hence, IQC condition (16) holds for ξ. Also, the $\mathcal{L}_2$ gain performance requirement in terms of the fictitious operator Δ_p can be expressed in the IQC notation as $\Delta_p \in \text{IQC}(\Pi_{\Delta_p})$ where $\Pi_{\Delta_p} = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$. Then $\Delta = \text{diag}(\phi, \Delta_p) \in \text{IQC}(\Pi)$ where Π is the diagonal augmentation of Π_ϕ and Π_{Δ_p} given as

$$\Pi = \begin{bmatrix} 0 & 0 & H\Lambda & 0 \\ 0 & I & 0 & 0 \\ \Lambda H & 0 & -(\Lambda H + H\Lambda) & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} \tag{32}$$

with diagonal $\Lambda > 0$. From (2), the state space realization for $\mathcal{M}$ in (15) is derived as

$$\mathcal{M} \sim \begin{bmatrix} \dot{x} \\ u_\psi \\ y_d \end{bmatrix} = \left[\begin{array}{c|cc} A + BF & B & 0 \\ \hline -F & 0 & I \\ -(C + DF) & -D & 0 \end{array}\right] \begin{bmatrix} x \\ w \\ u_{lin} \end{bmatrix}. \tag{33}$$

Using (32), (33) and the IQC frequency condition (14), the application of KYP lemma [44] leads to the matrix inequality condition

$$\begin{bmatrix} P\tilde{A} + \tilde{A}^T P + \tilde{C}^T \tilde{C} & PB - F^T H\Lambda + \tilde{C}^T D & 0 \\ B^T P + D^T \tilde{C} - \Lambda H F & -\Lambda H - H\Lambda + D^T D & \Lambda H \\ 0 & H\Lambda & -\gamma^2 I \end{bmatrix} < 0 \tag{34}$$

where $\tilde{A} = A + BF$ and $\tilde{C} = C + DF$ and where $P = P^T > 0$. Rearranging (34) gives (18).■

Proof of Corollary 1: Substituting $X = FQ$ into (18) and followed by repeated congruence transformations using diag$(I,\ T^{-1},\ I,\ I)$ and diag$(Q^{-1},\ U^{-1},\ I,\ I)$ gives

$$\begin{bmatrix} \tilde{A}^T Q^{-1} + Q^{-1}\tilde{A} & Q^{-1}B - F^T T^{-1} U^{-1} & 0 & \tilde{C}^T \\ B^T Q^{-1} - U^{-1} T^{-1} F & -U^{-1}T^{-1} - T^{-1}U^{-1} & U^{-1}T^{-1} & D^T \\ 0 & T^{-1}U^{-1} & -\gamma I & 0 \\ \tilde{C} & D & 0 & -\gamma I \end{bmatrix} < 0. \tag{35}$$

Substituting for $Q^{-1} = P$, $U^{-1} = \Lambda$ and $T^{-1} = H$ in (35) followed by the application of Schur's complement gives (18). The result follows by applying Theorem 1. ■

Proof of Corollary 2: Substituting $W = UT$ in LMI (20) gives LMI (19) and the result follows by applying Theorem 1. ■

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