Adaptive second-order sliding mode control with uncertainty compensation

G. Bartolini[a], A. Levant [b], A. Pisano[a*] and E. Usai[a]

[a] *Department of Electrical and Electronic Engineering (DIEE), University of Cagliari, Cagliari (Italy); E-mails: giob@diee.unica.it (G. Bartolini), pisano@diee.unica.it (A. Pisano), eusai@diee.unica.it (E. Usai)*
[b] *Applied Mathematics Dept., Tel-Aviv University, Tel-Aviv, Israel; E-mail: levant@post.tau.ac.il (A. Levant)*

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*Corresponding author. Email: pisano@diee.unica.it

This paper endows the second-order sliding mode control (2-SMC) approach with additional capabilities of learning and control adaptation. We present a 2-SMC scheme that estimates and compensates for the uncertainties affecting the system dynamics. It also adjusts the discontinuous control effort on-line, so that it can be reduced to arbitrarily small values. The proposed scheme is particularly useful when the available information regarding the uncertainties is conservative, and the classical "fixed-gain" SMC would inevitably lead to largely oversized discontinuous control effort. Benefits from the viewpoint of chattering reduction are obtained, as confirmed by computer simulations.

Keywords: Second-order sliding modes, Uncertain systems, Nonlinear systems, Adaptation.

1. Introduction

In this paper we consider the problem of controlling single-input nonlinear uncertain systems, which are minimum phase and of relative degree one with respect to the sliding output, by modulating the discontinuous time derivative of the control input. Sliding Mode Control (SMC) proves to be capable to cope with complex characteristics such as nonlinear uncertain dynamics, model uncertainties and unmodeled perturbations. Since its origins, SMC has evolved into a robust and powerful control design technique for a wide range of applications (Bartolini et al. (2008); Shtessel et al. (2013)). The peculiar aspect of conventional SMC is the discontinuous nature of the underlying control action, providing excellent system performance, which includes insensitivity to matching uncertainties and finite-time convergence.

In spite of the excellent theoretical properties, direct practical application of such discontinuous effort can generate undesirable output chattering. To attenuate this problem, the concept of higher order sliding modes was introduced and, specifically, several second-order sliding mode (Bartolini et al. (1999b); Levant (1993, 2007); Orlov et al. (2003)) and higher order sliding mode (HOSM) (Levant (2003, 2005)) algorithms were presented. Lyapunov-based convergence analysis of the Twisting second-order sliding mode control (2-SMC) algorithm, which plays a major role in the present treatment, were presented in (Orlov (2004); Polyakov (2009))

Since then, the number of publications on HOSM theory and applications has grown exponentially (see e.g. (Pisano et al. (2011)) and the references therein).

The chattering phenomenon is particularly felt when the a-priori known bounds on the uncertainties are very conservative. The overestimation of the uncertainties to cope with, indeed, implies unnecessary large control effort and a corresponding degradation of accuracy in practical implementation.

To counteract the conservatism in the prior evaluation of the system's uncertainties bounds, a number of interesting *adaptive* 2-SMC solutions relying on Lyapunov-based adaptation mechanisms were presented (see, for instance, Gonzalez et al. (2012); Kochalummoottil et al. (2012); Plestan et al. (2010a); Plestan et al. (2010b); Shtessel et al. (2010); Taleb et al. (2013)). On-line inspection of the sliding accuracy was also recently used in Bartolini et al. (2013) to detect 2-SM existence and consequently adapt the parameters of the twisting controller. A different adaptation philosophy, originally introduced in Bartolini et al. (1999a), has led to a readily implementable and straightforward adaptive 2-SMC strategy. Instead of a Lyapunov-based or accuracy-based gain adaptation (as in the aforementioned works), this adaptation mechanism takes advantage of the inherent nature of real (non ideal) sliding modes, existing in actual variable structure systems operating at finite switching frequency. Its adaptation policy depends on counting the zero-crossings of the sliding variable during appropriate receding-horizon adaptation time windows. Then, the occurrence of real sliding mode is verified by checking whether such count is large enough in accordance with a sliding mode existence criterion. The resulting adaptive controller is endowed with the capability of adjusting on-line the discontinuous control gain bidirectionally, maintaining the control magnitude at the minimum admissible level (rather than the worst-case conservative level, as in fixed-gains SMC). This adaptive 2-SMC algorithm has been previously applied in Capisani et al. (2011) and Pisano et al. (2012) to adjust the parameters of the Twisting 2-SMC algorithm Bartolini et al. (1999b) in robotic and automotive applications, respectively. Remarkably, the method was experimentally verified in Capisani et al. (2011) by means of an industrial manipulator. More recently (Evangelista et al. (2014); Pisano et al. (2013)) a similar logic was applied to adjust the parameters of the Suboptimal 2-SMC algorithm (Bartolini et al. (1999b)), in the framework of wind-energy conversion systems (Evangelista et al. (2014)) and by combining it with a switched adaptation algorithm (varying gains depending on the current operating region in the state space), see Pisano et al. (2013).

The contribution of this note consists in identifying a new scheme, embedding the aforementioned adaptation mechanism and combining it with a disturbance identification and compensation method. This modified architecture allows to reduce the discontinuous switching component of the Twisting 2-SMC law to an arbitrarily small value. In order to give some globality properties to the treatment, we refer to some additional input-state stability (ISS) conditions (Sontag et al. (1995)) guaranteeing the boundedness of the system motion when confined to a boundary layer of the sliding manifold. Summarizing, the main contribution of the present work against the related literature, (Bartolini et al. (1999a); Capisani et al. (2011); Pisano et al. (2012)) in particular, is that of integrating the adaptation logic, based on detecting the enforcement of a real sliding mode, with a novel scheme that estimates and compensates for the uncertainties appearing in the system dynamics. Additionally, we allow the uncertain nonlinearities entering the system's dynamics to possess state-dependent upper bounds with arbitrary growth rate, thereby including in the treatment systems possibly featuring the finite escape time phenomenon, a class of systems that was not treated in the above quoted literature.

After the problem formulation, and the list of the assumptions regarding the class of systems considered in this note (Section II), the description of the proposed methodology is split in two steps that are dealt with in Sections III and IV. In Section III it is presented the adaptation algorithm. The idea is to decrease or increase the control input stepwise, at the end of any observation time interval of length T, if the number of "switchings" (i.e. zero crossings) of the sliding output during the previous observation interval is greater or lower, respectively, than a prescribed threshold. Then it is shown that due to this machinery the system trajectories remain confined into a boundary layer of the sliding manifold of size $O(T^2)$. In Section IV we add a bounded component to the original switching control signal. This additional component, that can be seen as an artificial disturbance, is produced by a suitable nonlinear filter driven by the plant control input. The actual control signal, which is the derivative of the plant control input, is therefore made up of two components, one of which is continuous, the artificial disturbance, and the other one being discontinuous with time varying magnitude provided by the adaptation algorithm described in Section III. The overall uncertain system behavior can be described by an homogeneous differential inclusion, and this special structure is exploited to prove that after a finite time process the boundary layer size reduces to $O(T^3)$ and, at the same time, the discontinuous part of the control signal contracts from a finite, possibly large, value to an arbitrarily small infinitesimal $O(T)$ quantity. In Section V several simulation examples are provided to validate the theory.

2. The uncertain control system and relevant assumptions

We consider the nonlinear single-input dynamics

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{a}(\mathbf{x}) + \mathbf{b}(\mathbf{x})[u(t) + d(t)] \\ \mathbf{x}(t_0) &= \mathbf{x}_0 \end{aligned} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $t \in [t_0, \infty)$ is the time variable, $u(t) \in \mathbb{R}$ is the control input, $d(t) \in \mathbb{R}$ is an unknown disturbance and $\mathbf{a}, \mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n$ are smooth uncertain vector fields. Particularly, $\mathbf{b}(\mathbf{x})$ is required to be a complete vector field (Isidori (1995)).

Assumption 1 It can be found an algebraic constraint

$$\sigma(\mathbf{x}) = 0, \qquad \sigma : \mathbb{R}^n \to \mathbb{R} \tag{2}$$

such that:

1a) The "constraint variable" σ has a globally defined relative degree one

1b) The state trajectories solutions of the differential-algebraic equation (DAE) (1)-(2) are bounded and satisfy the design specifications.

The control task is that of finding a *continuous* control action $u(t)$, with discontinuous time derivative, capable of steering the system output σ as close as possible to zero in spite of hard model uncertainty while counteracting the chattering phenomenon.

Under the conditions of the Assumption 1, it is possible (see Isidori (1995)) to define a vector $\mathbf{w} \in \mathbb{R}^{n-1}$ and, correspondingly, a diffeomorphic map

$$\mathbf{x} = \Upsilon(\mathbf{w}, \sigma), \quad \Upsilon : \mathbb{R}^n \to \mathbb{R}^n \tag{3}$$

preserving the origin. The $\mathbf{w}$ dynamics, generally referred to as the "internal dynamics" (Isidori (1995)), can be expressed as

$$\dot{\mathbf{w}} = g(\mathbf{w}, \sigma). \tag{4}$$

Assumption 2. The internal dynamics (4) are input-state stable (ISS) (Sontag et al. (1995))

According to Sontag et al. (1995), Assumption 2 implies that there exist a $\mathcal{K}$-$\mathcal{L}$ function $\alpha : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ and a $\mathcal{K}$ function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for every $\mathbf{w}_0 \in \mathbb{R}^{n-1}$ and every $\sigma \in \mathbb{R}$ the unique maximal solution of the initial value problem (4), $\mathbf{w}(t_0) = \mathbf{w}_0$, has interval of existence $\mathbb{R}^+$ and

$$\|\mathbf{w}\| \le \alpha(\|\mathbf{w}_0\|, t) + \xi(\|\sigma\|_\infty) \qquad \forall t \in [t_0, \infty) \tag{5}$$

Assumption 3. There exist known $\mathcal{K}$-function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|\mathbf{w}\| \le \gamma(\|\mathbf{x}\|) \tag{6}$$

Functions α, ξ and γ will be explicitly used in the parametrization of the proposed control laws and are therefore required to be known.

Consider the first total derivative of $\sigma(\mathbf{x})$ in the transformed $(\mathbf{w}, \sigma)$ coordinates

$$\dot{\sigma}(t) \;=\; \varphi_1(\mathbf{w}, \sigma, t) + K(\mathbf{w}, \sigma)u \tag{7}$$

where

$$K(\mathbf{w}, \sigma) \;=\; \left.\frac{\partial \sigma(\mathbf{x})}{\partial \mathbf{x}} \mathbf{b}(\mathbf{x})\right|_{\mathbf{x}=\Upsilon(\mathbf{w},\sigma)} \tag{8}$$

$$\varphi_1(\mathbf{w}, \sigma, t) \;=\; \left.\frac{\partial \sigma(\mathbf{x})}{\partial \mathbf{x}} \mathbf{a}(\mathbf{x})\right|_{\mathbf{x}=\Upsilon(\mathbf{w},\sigma)} + K(\mathbf{w}, \sigma) d(t) \tag{9}$$

Assumption 4 There exist known positive constants K_m, K_M, K_D such that

$$0 < K_m \;\le\; K(\mathbf{w}, \sigma) \le K_M \tag{10}$$

$$|\dot{K}(\mathbf{w}, \sigma, u, t)| \;\le\; K_D \tag{11}$$

Consider also the second total derivative of $\sigma(\mathbf{x})$, which takes the form

$$\ddot{\sigma}(t) \;=\; \varphi_2(\mathbf{w}, \sigma, t, u) + K(\mathbf{w}, \sigma)\dot{u} \tag{12}$$

where

$$\varphi_2(\mathbf{w},\sigma,t,u) = \varphi_{21}(\mathbf{w},\sigma)+[\varphi_{22}(\mathbf{w},\sigma)+\dot{K}(\mathbf{w},\sigma,u,t)][u+d(t)]+K(\mathbf{w},\sigma)\dot{d}(t) \tag{13}$$

Assumption 5. There exist known positive constants D_1, D_2 such that

$$|d(t)|\le D_1 \qquad |\dot{d}(t)|\le D_2 \tag{14}$$

Define the following functions

$$\phi(\mathbf{w},\sigma,t) = -\frac{\varphi_1(\mathbf{w},\sigma,t)}{K(\mathbf{w},\sigma)} \tag{15}$$

$$\beta(\mathbf{w},\sigma,t,u) = -\frac{\varphi_2(\mathbf{w},\sigma,t,u)}{K(\mathbf{w},\sigma)} \tag{16}$$

After lengthy but straightforward computations it can be shown that $\ddot{\phi}$ and $\dot{\beta}$ can be expressed as follows

$$\begin{array}{c}\ddot{\phi}=\phi_1(\mathbf{w},\sigma,t)+\phi_2(\mathbf{w},\sigma,t)u+\phi_3(\mathbf{w},\sigma,t)u^2+\phi_4(\mathbf{w},\sigma,t)\dot{u}\\ \dot{\beta}=\beta_1(\mathbf{w},\sigma,t)+\beta_2(\mathbf{w},\sigma,t)u+\beta_3(\mathbf{w},\sigma,t)u^2+\beta_4(\mathbf{w},\sigma,t)u^3+\beta_5(\mathbf{w},\sigma,t)\dot{u}\end{array} \tag{17}$$

for some implicitly defined functions ϕ_i $(i=1,2,...,4)$ and β_i $(i=1,2,...,5)$.

Assumption 6. There exist known non-decreasing functions $\Phi_1,\Phi_{21},\Phi_{22},\Theta_1,....,\Theta_4,\Gamma_1,...,\Gamma_4:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ such that for all $t\in[t_0,\infty)$ the following inequalities hold

$$\begin{array}{ll}|\varphi_1(\mathbf{w},\sigma,t)|\le\Phi_1(\|\mathbf{w}\|,|\sigma|) & \\ |\varphi_{2i}(\mathbf{w},\sigma)|\le\Phi_{2i}(\|\mathbf{w}\|,|\sigma|) & i=1,2\\ |\beta_i(\mathbf{w},\sigma,t)|\le\Theta_i(\|\mathbf{w}\|,|\sigma|) & i=1,...,5\\ |\phi_i(\mathbf{w},\sigma,t)|\le\Gamma_i(\|\mathbf{w}\|,|\sigma|) & i=1,...,4\end{array} \tag{18}$$

Note that due to relations (18) the growth rate of the underlying uncertain functions with respect to $\|\mathbf{w}\|$ and $|\sigma|$ is arbitrary. Thus, the open-loop system may exhibit the finite escape-time phenomenon.

3. The properties of a control adaptation mechanism

Let t_0 be the initial time, and let $k>0$ be an arbitrary constant. The initializing control strategy

$$\dot{u}=-U_i(\mathbf{x},u,\sigma)\mathrm{sign}(\sigma) \qquad t_0\le t\le t_c \tag{19}$$

$$U_i(\mathbf{x},u,\sigma)=\frac{1}{K_m}\left[\Phi_{21}(\gamma(\|\mathbf{x}\|),|\sigma|)+[\Phi_{22}(\gamma(\|\mathbf{x}\|),|\sigma|)+K_D](|u|+D_1)+K_MD_2+k\right], \quad k>0 \tag{20}$$

is applied in the initial transient to globally steer in finite time $t_c\ge t_0$ the variable $\dot{\sigma}$ to zero.

At $t=t_c$, a constant Φ is evaluated to subsequently be used for setting the parameters of the adaptive control law to be applied in the successive time interval $t\ge t_C$. The procedure of computing the constant Φ, which is based on certain measurements to be taken on-line at $t=t_c$, is outlined in the following Subsection.

3.1 *On-line computation of the control coefficients*

Define

$$\begin{aligned} \|\mathbf{w}\|^* &= \alpha(\gamma(\|\mathbf{x}(t_c)\|), t_c) + \xi(|\sigma(t_c)| + q), \quad q > 0 \qquad (21)\\ \rho &= 1 + \eta K_M \max\left\{\frac{1}{K_m}, \frac{2}{\alpha K_m - K_M}\right\}, \quad \eta > 1 \qquad (22)\end{aligned}$$

where $q > 0$ and $\eta > 1$ are arbitrary constants. Find the unique positive root Σ_D^* of the equation

$$\Sigma_D^* = \sqrt{2\rho\mathcal{F}(\Sigma_D^*)(|\sigma(t_c)| + q)} \qquad (23)$$

where

$$\begin{aligned} \mathcal{F}(\Sigma_D^*) &= \Phi_{21}(\|\mathbf{w}\|^*, |\sigma(t_c)| + q) + \frac{1}{K_m}\left[\Phi_{22}(\|\mathbf{w}\|^*, |\sigma(t_c)| + q) + K_D\right] \\ &\times \left[\Phi_1(\|\mathbf{w}\|^*, |\sigma(t_c)| + q) + \Sigma_D^* + K_m D_1\right] + K_M D_2 \qquad (24)\end{aligned}$$

Now consider any $\Sigma_D \geq \Sigma_D^*$, and define

$$u^* = \frac{1}{K_m}\left[\Sigma_D + \Phi_1(\|\mathbf{w}\|^*, |\sigma(t_c)| + q)\right] \qquad (25)$$

Finally, compute the constant Φ according to

$$\Phi = \Phi_{21}(\|\mathbf{w}\|^*, |\sigma(t_c)| + q) + \left[\Phi_{22}(\|\mathbf{w}\|^*, |\sigma(t_c)| + q) + K_D\right]\left[u^* + D_1\right] + K_M D_2. \qquad (26)$$

The meaning of constant Φ is that under the action of the adaptive control law applied at $t > t_c$, the following inequality holds

$$|\varphi_2(\mathbf{w}, \sigma, t, u)| \leq \Phi, \quad t \geq t_c \qquad (27)$$

Furthermore, the quantities $\|\mathbf{w}\|$, $|\sigma|$ and $|u|$ will never exceed, at any $t \geq t_c$, the constant upper bounds $\|\mathbf{w}\|^*$, $|\sigma(t_c)| + q$ and u^*. These statements are demonstrated in the proof of Theorem 1.

3.2 *The adaptive algorithm*

In the time interval $t > t_c$, the following adaptive version of the "Twisting" algorithm (see Levant (1993)) is applied

$$\begin{aligned} \dot{u}(t) &= U_{M_j} S(\sigma, \dot{\sigma}), \quad t \in [t_c + (j-1)T, t_c + jT), \quad j = 1, 2, \ldots \qquad (28)\\ S(\sigma, \dot{\sigma}) &= \begin{cases} -\eta\frac{K_M}{K_m} \cdot sign\sigma & \sigma \cdot \dot{\sigma} > 0 \\ -sign\sigma & \sigma \cdot \dot{\sigma} \leq 0 \end{cases} \qquad (29)\end{aligned}$$

The adaptation procedure is defined as follows. Starting from $t = t_c$, consider the sequence of adjacent time intervals

$$t \in \mathcal{T}_j \equiv [t_c + (j-1)T, t_c + jT) \qquad j = 1, 2, \ldots \qquad (30)$$

of width T, and modify the control amplitude U_{M_j} in (28) at the end of the time interval $\mathcal{T}_j$ according to

$$
\begin{aligned}
U_{M_1} &= \eta\Phi \max\{\tfrac{1}{K_m}, \tfrac{2}{\alpha K_m - K_M}\} \\
U_{M_{j+1}} &= \begin{cases} max(U_{M_j} - \Lambda_1 T, 0) & if\ N_{sw,j} \geq N^* \\ min(U_{M_j} + \Lambda_2 T, U_{M_1}) & if\ N_{sw,j} < N^* \end{cases}
\end{aligned} \tag{31}
$$

where $N_{sw,j}$ is the number of sign commutations (zero-crossings) of σ in the interval $\mathcal{T}_j$ and N^* is an integer number to be properly chosen.

Definition 1 *The number $N_{sw,j}$ of sign commutations of $\sigma(t)$ in the time interval $\mathcal{T}_j$ is defined as the cardinality of the set $\mathcal{S}_j = \{t \in \mathcal{T}_j : \sigma(t) = 0,\ \ \dot{\sigma}(t) \neq 0\}$.*

Roughly speaking, we decrement the control magnitude stepwise by $\Lambda_1 T$ while the number of zero-crossings of σ is large enough, otherwise we increment the control magnitude stepwise by $\Lambda_2 T$. Parameter $\Lambda_1 > 0$ is free to be chosen whereas Λ_2 must be computed on line at $t = t_c$, on the basis of available state information, and it must fulfill the following inequality (33)

$$
\Lambda_1 \ > \ 0 \tag{32}
$$

$$
\begin{aligned}
\Lambda_2 \ > \ & \Lambda_1 + 2\Big[\Theta_1(\|\mathbf{w}\|^*, |\sigma(t_c)| + q) + \Theta_2(\|\mathbf{w}\|^*, |\sigma(t_c)| + q)u^* + \Theta_3(\|\mathbf{w}\|^*, |\sigma(t_c)| + q)(u^*)^2 \\
& + \Theta_4(\|\mathbf{w}\|^*, |\sigma(t_c)| + q)(u^*)^3 + \Theta_5(\|\mathbf{w}\|^*, |\sigma(t_c)| + q)(\Phi + K_M \alpha U_M)\Big]
\end{aligned} \tag{33}
$$

Parameter N^* in (31) can be selected by taking advantage of the following Lemma.

Lemma 1 *Consider the second-order sliding variable dynamics (12) and let $N_{sw,j}$ be the number of zero crossings of σ during the time interval $\mathcal{T}_j$ of length T. If condition*

$$
N_{sw,j} \geq N^*, \qquad N^* \geq 2 \tag{34}
$$

is satisfied, and $|\ddot{\sigma}| \leq a_1$ for some $a_1 > 0$, then the inequalities

$$
|\sigma| \leq a_1 T^2 \qquad\qquad |\dot{\sigma}| \leq a_1 T \tag{35}
$$

hold in the whole time interval $\mathcal{T}_j$.

Proof of Lemma 1. If function σ has at least two zero crossings within the interval $\mathcal{T}_j$, then, by virtue of the Rolle theorem, its first derivative $\dot{\sigma}$ has at lest one zero crossing within the same interval. Since $|\ddot{\sigma}| \leq a_1$ by assumption, simple time-integration yields (35). $\square$

Clearly, Lemma 1 offers the rule for choosing the parameter N^* of the adaptation algorithm.

The performance guaranteed by the application of the proposed adaptive control algorithm is established in the following Theorem.

Theorem 1. *Consider system (1) satisfying Assumptions 1-6. Choose the control input as*

$$
\dot{u} = \begin{cases} U_i(\mathbf{x}, u, \sigma) & t_0 \leq t < t_c, \\ U_{M_j} S(\sigma, \dot{\sigma}) & t \in \mathcal{T}_j \ \ (j = 1, 2, \ldots) \end{cases} \tag{36}
$$

where functions $U_i(\mathbf{x}, u, \sigma)$ and $S(\sigma, \dot{\sigma})$ are defined in (20) and (29), the adaptation of U_{M_j} is governed by (31), where Φ is the constant evaluated as explained in the Subsection III.A, $N^* \geq 2$, and Λ_1, Λ_2 are evaluated according to (32)-(33).

If the period T of the adaptation process is such that

$$T < \frac{1}{b_1}\sqrt{|\sigma(t_c)+q|} \qquad b_1 = (\Phi + K_M\alpha U_M)\left[1+\frac{\Phi + K_M\alpha U_M}{2\,(K_M\alpha U_M - \Phi)}\right] \tag{37}$$

then the following inequalities are achieved after a finite-time transient process

$$\begin{aligned} &|\sigma(t)| \le b_1 T^2 \\ &|\dot{\sigma}(t)| \le \sqrt{2b_1(\Phi + K_M\alpha U_M)}T \end{aligned} \tag{38}$$

Proof See the Appendix.

Remark 1. The existence of a diffeomorphic map bringing system (1), along with the sliding output $\sigma = \sigma(\mathbf{x})$ into the normal form (4), (7) is guaranteed by the properties 1a) and 1b) (see, e.g. (khalil , 2002, Th. 13.1)). It appears complicated to devise formal conditions on the vector fields of the original system (1) ensuring that the Assumptions 2, 3 and 6 hold for the transformed system's dynamics. However, the required boundedness restrictions of (5), (6) and (18), appear to be general enough and not particularly restrictive. Systems expressed in the Brunowsky form, for instance, certainly fulfill all the given requirements when a linear sliding manifold is considered (as it is shown in the Simulation section). Furthermore, at the price of certain geometric extra-conditions supporting the diffeomorphic transformation (3), the approach can be extended to the time-varying scenario where the vector fields $a(\mathbf{x})$, $b(\mathbf{x})$ and the sliding manifold $\sigma(\mathbf{x})$ depend explicitly on the time variable (i.e., $a = a(\mathbf{x}, t)$, $b = b(\mathbf{x}, t)$ and $\sigma = \sigma(\mathbf{x}, t)$). This generalization, however, does not affect the main purpose of the paper of presenting a novel combined adaptation/uncertainty compensation policy, and it is skipped for simplicity's sake. □

Theorem 1 does not guarantee the convergence of U_{M_j} towards some attracting set vanishing with T, but just the fact that the proposed adaptation globally steers the systems towards an invariant boundary layer depending on, and vanishing with, the T parameter. In order to reach a more effective result (particularly, convergence of U_{M_j} towards an $O(T)$ vicinity of zero) we need a further modification of the control scheme, that is discussed in the following section.

4. Main result

In this section we present a modified version of the adaptive controller providing some enhanced features. Particularly, we modify the second row of (36) as follows

$$\dot{u} = \begin{cases} U_i(\mathbf{x}, u, \sigma) & t_0 \le t < t_c, \\ U_{M_j} S(\sigma, \dot{\sigma}) + v(t) & t \in \mathcal{T}_j \quad j = 1, 2, \dots \end{cases} \tag{39}$$

where $v(t)$ is the output of the following non linear filter driven by the plant control u

$$\begin{aligned} &z(t_c) = u(t_c) \ ; \ w(t_c) = 0 \ ; \ v(0) = 0 \\ &v_1 = \dot{z} = -\lambda_2 |z-u|^{1/2}\mathrm{sign}(z-u) + w \\ &\dot{w} = -\lambda_1 \mathrm{sign}(z-u) \end{aligned} \tag{40}$$

$$\dot{v} = \begin{cases} -v_1 & if\ |v_1| > \lambda_3 \\ -\lambda_4\ \mathrm{sign}(v - v_1) & if\ |v_1| \le \lambda_3 \end{cases} \tag{41}$$

in which $\lambda_1, \dots, \lambda_4$ are constant tuning parameters. Roughly speaking, we add a continuous component $v(t)$ whose aim is to estimate the uncertain term $\beta(\mathbf{w},\sigma,t,u)$ defined in (16). Indeed, considering the second row of (39) into (12) yields the closed-loop dynamics

$$\ddot{\sigma}(t) = K\left[(v-\beta) + U_{M_j}S(\sigma,\dot{\sigma})\right] \tag{42}$$

so that as the "estimation error" $\varepsilon_\beta = v - \beta$ tends to zero, the discontinuous control magnitude U_{M_j} may tend to zero as well. Note that by construction (and considering the tuning inequalities (48) to subsequently be specified) the following conditions hold

$$|v| \le \lambda_3, \qquad |\dot{v}| \le \lambda_4. \tag{43}$$

The uniformly bounded and Lipschitz signal v can be seen as an additive disturbance matching $\dot{u}$, yielding the "perturbed" dynamics (42) having the uncertain drift term $K(v-\beta) = Kv+\varphi_2$. Since, by (43) and Assumption 4, the term Kv is uniformly bounded and globally Lipschitz, then all the conditions of Theorem 1 are satisfied and accuracy (38), in particular, is thus preserved provided that the procedure of computing the Φ parameter is slightly modified. The on-line computation of the controller parameters, to be made at the time instant $t = t_c$, is outlined and commented in the following subsection.

4.1 *On-line computation of the control coefficients*

In the previous section, the motivation for computing the constant Φ was that of deriving an upper bound to the magnitude of function φ_2 (the drift term of $\ddot{\sigma}$ in eq. (12)), according to (27). Now, the $\ddot{\sigma}$ dynamics (42) has the drift term $K(v-\beta) = Kv + \varphi_2$, which is why we now aim instead to compute a constant Φ such that

$$|K(\mathbf{w},\sigma)v(t) - \varphi_2(\mathbf{w},\sigma,t,u))| \le \Phi, \quad t \ge t_c \tag{44}$$

The underlying procedure is outlined a follows, which straightforwardly derives from that of the Subsection III.A and takes advantage of relation (43) and Assumption 4. Define $\|\mathbf{w}\|^*$ and ρ as in (21), (22), where $q > 0$ and $\eta > 1$ are arbitrary constants. Find the unique positive root Σ_D^* of the equation (23), where

$$\begin{aligned} \mathcal{F}(\Sigma_D^*) \;&=\; K_M\lambda_3 + \Phi_{21}(\|\mathbf{w}\|^*, |\sigma(t_c)|+q) + \frac{1}{K_m}\left[\Phi_{22}(\|\mathbf{w}\|^*, |\sigma(t_c)|+q) + K_D\right] \\ &\times\; \left[\Phi_1(\|\mathbf{w}\|^*, |\sigma(t_c)|+q) + \Sigma_D^* + K_m D_1\right] + K_M D_2 \end{aligned} \tag{45}$$

Now consider any $\Sigma_D \ge \Sigma_D^*$, define u^* as in (25), and compute the constant Φ according to

$$\Phi = K_M\lambda_3 + \Phi_{21}(\|\mathbf{w}\|^*, |\sigma(t_c)|+q) + \left[\Phi_{22}(\|\mathbf{w}\|^*, |\sigma(t_c)|+q) + K_D\right]\left[u^* + D_1\right] + K_M D_2 \tag{46}$$

Additionally, compute the parameter

$$\begin{aligned} \Delta_1 = \;&\Gamma_1(\|\mathbf{w}\|^*, |\sigma(t_c)|+q) + \Gamma_2(\|\mathbf{w}\|^*, |\sigma(t_c)|+q)u^* \\ &+\Gamma_3(\|\mathbf{w}\|^*, |\sigma(t_c)|+q)(u^*)^2 + \Gamma_4(\|\mathbf{w}\|^*, |\sigma(t_c)|+q)(\Phi + K_M\alpha U_M) \end{aligned} \tag{47}$$

which overestimates the magnitude of signal $\ddot{\phi}$ in (17), choose $\lambda_1, \ldots, \lambda_4$ according to

$$\lambda_1 > \Delta_1, \qquad \lambda_2^2 > 2\tfrac{(\lambda_1+\Delta_1)^2}{\lambda_1-\Delta_1}, \qquad \lambda_3 > \tfrac{1}{K_m}\Phi, \qquad \lambda_4 > \max\left\{\Delta_1, \lambda_3\right\}, \tag{48}$$

and, finally, set $\Lambda_1 > 0$ and Λ_2 such that (33) holds.

4.2 *Convergence analysis*

Exploiting the homogeneity property of the closed-loop dynamics we prove (see Theorem 2) that controller (39)-(41) along with the adaptation rule (31) guarantee that the estimation error $\varepsilon_\beta = v - \beta$ and the discontinuous magnitude U_{M_j} reduce to $O(T)$ after a finite transient and, at the same time, that the sliding accuracy improves to $|\sigma| \leq O(T^3)$.

Theorem 2. *Consider system (1) satisfying Assumptions 1-6. Compute the control parameters as specified in the subsection IV.A. Choose the control input as in (39)-(41), (20), (29)-(31).*

Then, if the period of the adaptation process meets the restriction (37), the following inequalities are achieved after a finite-time transient process

$$|\sigma| \leq c_1 T^3 \qquad |\dot{\sigma}| \leq c_2 T^2 \qquad |\ddot{\sigma}| \leq c_3 T \qquad |U_{M_j}| \leq c_4 T \tag{49}$$

with $c_1, ..., c_4$ positive constants independent of T.

Proof. The proof shall exploit some homogeneity properties of the obtained closed-loop system, which can be expressed in a more convenient form as

$$\begin{aligned}
\ddot{\sigma} &= K\left[\delta + U_{M_j} S(\sigma, \dot{\sigma})\right] && (50)\\
\dot{\delta} &= -\lambda_4 \mathrm{sign}\left[\delta - \dot{\omega} - \frac{\dot{K}}{K^2}\dot{\sigma}\right] - \ddot{\phi} + \frac{\ddot{K}K - 2\dot{K}^2}{K^3}\dot{\sigma} + \frac{\dot{K}}{K^2}\ddot{\sigma} && (51)\\
\dot{\omega} &= -\lambda_2 \left|\omega - \frac{1}{K}\dot{\sigma}\right|^{1/2} \mathrm{sign}\left[\omega - \frac{1}{K}\dot{\sigma}\right] + \xi && (52)\\
\dot{\xi} &= -\ddot{\phi} - \lambda_1 \mathrm{sign}\left[\omega - \frac{1}{K}\dot{\sigma}\right] && (53)
\end{aligned}$$

where signals δ, ω and ξ are defined as

$$\delta = \frac{\dot{K}}{K^2}\dot{\sigma} + v - \dot{\phi} \qquad\qquad \omega = z - \phi \qquad\qquad \xi = w - \dot{\phi} \tag{54}$$

Define the vector $\pi = [\sigma, \dot{\sigma}, \delta, \omega, \xi]^T$. Since, according to (43), signal v and its time derivative are bounded, Theorem 1 can be invoked to assess that σ and $\dot{\sigma}$ converge in finite-time towards the invariant bounded domain (38). By virtue of this, and taking into account assumptions (10)-(11), (14), (18), to the uncertain differential equation (50)-(53) can be associated a differential inclusion of the type

$$\dot{\pi} \in F(\pi, U_{M_j}, T) = \begin{bmatrix} \dot{\sigma} \\ [K_m, K_M]\left[\delta + U_{M_j} S(\sigma, \dot{\sigma})\right] \\ -\lambda_4 \mathrm{sign}(\delta - \dot{\omega} + [-\kappa_1, \kappa_1]T) + [-\kappa_2, \kappa_2] \\ -\lambda_2 \left|\omega - \frac{1}{[K_m, K_M]}\dot{\sigma}\right|^{1/2} \mathrm{sign}\left(\omega - \frac{1}{[K_m, K_M]}\dot{\sigma}\right) + \xi \\ -\ddot{\phi} - \lambda_1 \mathrm{sign}(\omega - \frac{1}{[K_m, K_M]}\dot{\sigma}) \end{bmatrix} \tag{55}$$

where κ_1 and κ_2 are positive constants. The differential inclusion (55) is understood here in the Filippov sense (Filippov (1988)), i.e. its right-hand side is enlarged up to the minimal closed convex upper-semicontinuous vector-set function.

Recall that U_{M_j} obeys the discrete dynamics (31). Here T is the system parameter. Fix some value $T = T_0$. Convergence to the invariant bounded domain (38) assures, by (7), that u differs from ϕ by a small "measurement error" related to the actual size of the invariant boundary layer.

According to Levant (1998), conditions (48), where Δ_1 is straightforwardly shown to represent an upperbound to $|\ddot{\phi}|$ at any $t \geq t_c$, guarantee that after a finite transient signals v_1 and v differ from $\dot{\phi}$ by a bounded quantity. Thus, ω and ξ converge to a bounded domain containing the origin, and moreover $\dot{\omega}$ and δ are bounded. As a result, the solutions of (55) enter in finite time the invariant set

$$L : \{|\sigma| \leq d_1, |\dot{\sigma}| \leq d_2, |\delta| \leq d_3, |\omega| \leq d_4, |\xi| \leq d_5\} \triangleq L(d_1, d_2, d_3, d_4, d_5) \tag{56}$$

where $d_i = d_i(T_0)$, $i = 1, 2, ..., 5$, are proper positive constants.

Let us now evaluate the attracting set for the system (31), (55) with the parameter $T = \nu T_0$, where ν is a positive scaling parameter. Define the following linear transformations

$$\begin{array}{l} d_\nu : (\sigma, \dot{\sigma}, \delta, \omega, \xi, U_{M_j}) \longmapsto (\nu^3\sigma, \nu^2\dot{\sigma}, \nu\delta, \nu\omega, \nu\xi, \nu U_{M_j}), \\ G_\nu : (t, \pi, U_{M_j}, T) \longmapsto (\nu t, d_\nu(\pi, U_{M_j}), \nu T). \end{array} \tag{57}$$

Obviously the transformation G_ν yields a one-to-one correspondence between the solutions of the hybrid systems (31), (55) with parameters T and νT. In other words, the system is homogeneous with the homogeneity degree equal to -1.

Taking into account that $\nu = T/T_0$, and applying the transformation G_ν obtain that for the arbitrary adaptation period T the trajectories of the closed loop hybrid system (31), (55) in finite time enter the invariant set $L^* = L\left(\frac{d_1}{T_0^3}T^3, \frac{d_2}{T_0^2}T^2, \frac{d_3}{T_0}T, \frac{d_4}{T_0^2}T^2, \frac{d_5}{T_0}T\right)$ and stay there forever, which provides for the needed asymptotics. Theorem 2 is proven. $\Box$

5. Simulations

Consider the nonlinear system

$$\begin{cases} \dot{x}_i = x_{i+1} & i = 1, 2 \\ \dot{x}_3 = x_1^2 - x_2 x_3 + 5sin(t) + u \end{cases} \tag{58}$$

with the initial conditions $x_1(0) = x_2(0) = x_3(0) = 1.$ and let $\sigma = x_3 + 4x_2 + 4x_1$ be the chosen sliding variable, which clearly possesses a globally defined relative degree one. The trajectories of the DAE (1)-(2) are governed by the reduced-order asymptotically stable linear system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -4x_1 - 4x_2 \end{cases} \tag{59}$$

along with the linear relation $x_3 = -4x_1 - 4x_2$, thereby fulfilling both the requirements 1a) and 1b) of Assumption 1.

In the present scenario, a diffeomorphic map (3) fulfilling all the given assumptions 2-6 takes the simple form of the nonsingular linear mapping

$$\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -4 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \sigma \end{bmatrix}, \tag{60}$$

with the internal dynamics state vector being thus specified to $\mathbf{w} = [x_1, x_2]^T$. The dynamics of the transformed state is straightforwardly derived as follows

$$\begin{cases} \dot{w}_1 = w_2 \\ \dot{w}_2 = -4w_1 - 4w_2 + \sigma \\ \dot{\sigma} = -16w_1 - 12w_2 + 4\sigma + w_1^2 + 4w_2^2 + 4w_1w_2 - w_2\sigma + 5sin(t) + u \end{cases} \tag{61}$$

The internal dynamics, given by the first two lines of (61), are linear and asymptotically stable when $\sigma = 0$, therefore three constants α_1^*, α_2^* and ξ^* can be found such that relation (5) specifies to

$$\|\mathbf{w}(t)\| \leq \alpha_1^* \|\mathbf{w}(0)\| e^{-\alpha_2^* t} + \xi^* \sup_{\tau \in [0,t]} |\sigma(\tau)| \qquad \forall t \in [0, \infty) \tag{62}$$

Also, assumption 3 is straightforwardly satisfied due to (60). Function $K(\mathbf{w}, \sigma)$ takes the constant unit value, hence the constants in Assumption 4 are $K_m = K_M = 1$ and $K_D = 0$. Similarly, due to the chosen matching disturbance $d(t) = 5sin(t)$ the Assumption 5 is in force with the constants $D_1 = D_2 = 5$. The derivation of the expressions in (17), as well as the verification of Assumption 6, are straightforward but yield extremely lengthy developments which are skipped for the brevity sake.

The proposed strategy has been simulated using the parameters $\Lambda_1 = 1$, $\Lambda_2 = 4$, $N^* = 30$, $U_{M1} = 500$, $\lambda_1 = 100$, $\lambda_2 = 25$, $\lambda_3 = 200$, $\lambda_4 = 500$, and two tests using different values of T have been performed: TEST 1 ($T = T_1 = 25ms$) and TEST 2 ($T = T_2 = 2.5ms$).

Fig. 1 shows the time profiles of σ and U_{M_j} corresponding to TEST 1 (note that the U_{M_j} reduction process activates only after a boundary layer of the sliding manifold $\sigma = 0$ has been attained within which the zero crossings of σ take place fast enough).

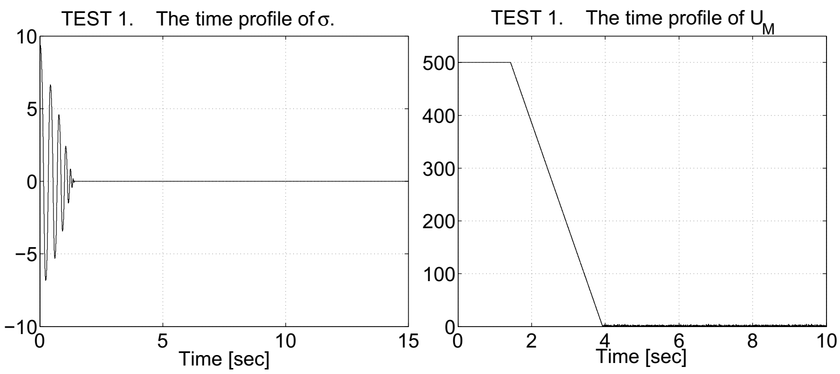

Figure 1. TEST 1 (T = 25 ms). Time evolutions of σ (left) and U_{M_j} (right).

Fig. 2 and 3 are devoted to check the attainment of (49) by comparing the steady-state behavior of σ and U_{M_j} in the two tests. It is expected that in the steady state of TEST 2 the variables σ and U_{M_j} contract by a factor 1000 and 10, respectively, compared to the magnitudes achieved during TEST 1. The comparison between the plots of figures 2 and 3 confirm the expected accuracy thereby supporting the theory.

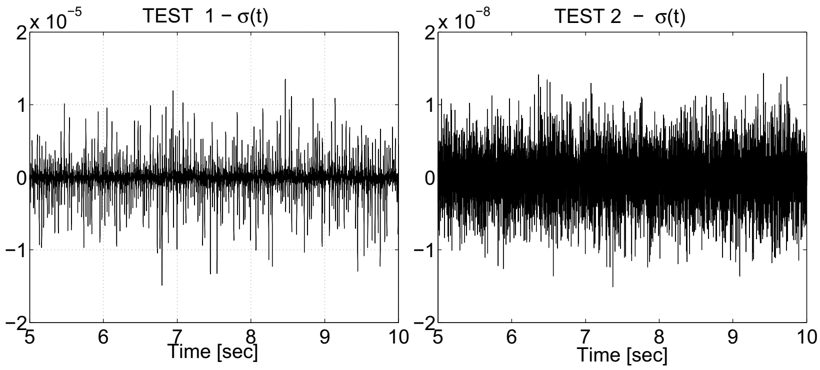

Figure 2. TEST 1 and TEST 2. Zoom on the steady-state behavior of σ

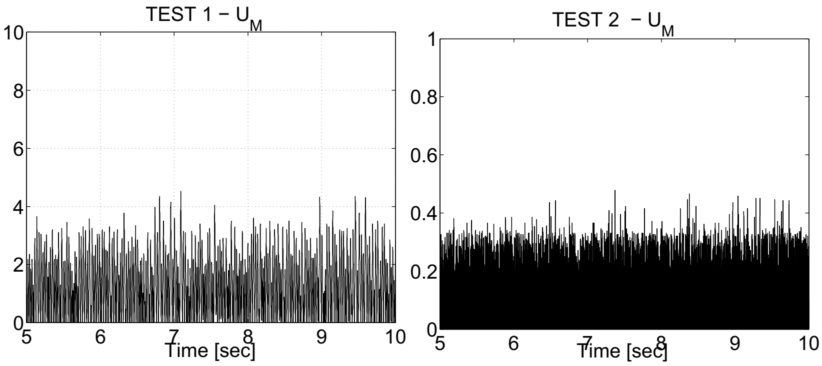

Figure 3. TEST 1 and TEST 2. Zoom on the steady-state behavior of U_{M_j}

	$\sup_{t\in[5,10]} \lvert\sigma\rvert$	$\sup_{t\in[5,10]} \lvert U_{M_j}\rvert$
Algorithm (36)	$3.12 \cdot 10^{-4}$	26.2
Algorithm (39)	$1.45 \cdot 10^{-5}$	4.4

Table 1. Comparative analysis between Algorithms (36) and (39)

Finally, in Table 1, the results of a comparative analysis between the algorithms (36) and (39) is presented. The two algorithms are implemented using the same parameters of TEST 1. The table compares the steady state accuracy of $|\sigma|$ and $|U_{M_j}|$, highlighting that the proposed algorithm (39) provides both higher sliding accuracy and chattering attenuation as compared to the algorithm (36) which does not implement the uncertainty estimation and compensation.

6. Conclusions

An adaptation mechanism, capable of adjusting on-line the discontinuous control magnitude in a second-order sliding mode control system, has been integrated with a method to estimate and compensate for the uncertainty affecting the system's dynamics. The combined scheme is capable of reducing the discontinuous control effort to an arbitrarily small quantity. Next activities will be devoted to analyzing the discrete-time version of the proposed adaptive controller. Validation of the proposed scheme on experimental platforms is another interesting task of future research.

Proof of Theorem 1.

It follows from (10)-(14) and the second row of (18) that in the transient interval $[t_0, t_c]$ the initializing control appearing in the first row of (36) enforces the differential inequality $\ddot{\sigma}\sigma \leq -k^2|\sigma|$, thereby ensuring that for any initial condition $(\sigma(t_0), \dot{\sigma}(t_0))$ the axis $\dot{\sigma} = 0$ will be hit after a finite transient $t_c \geq t_c$.

To analyze the system trajectory in the phase plane $\sigma - \dot{\sigma}$ (see Bartolini et al. (1999b)), let us refer to the Figure 4, which shows the piece-wise parabolic limiting trajectories (continuous lines) and a possible actual trajectory (dashed line) starting from the point $P_1 \equiv (\sigma(t_c), 0) = (\sigma_1, 0)$. From $t = t_c$ the adaptive Twisting Algorithm, appearing in the second row of (36), is applied.

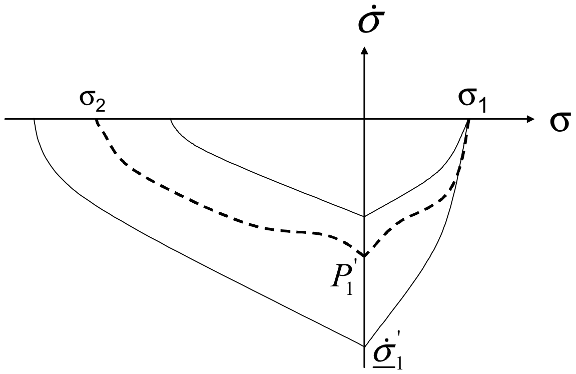

Figure 4. Limit and actual trajectories in the $\sigma - \dot{\sigma}$ plane.

Define the set

$$\Omega = \{(\sigma, \dot{\sigma}) : |\sigma| \leq |\sigma(t_c)| + q, \quad |\dot{\sigma}| \leq \Sigma_D\} \tag{63}$$

where q and Σ_D are the parameters already introduced in the Subsection III.A. Particularly, q is an arbitrary positive parameter, and Σ_D is any value such that $\Sigma_D \geq \Sigma_D^*$, where Σ_D^* is the unique positive root of the equation (23)-(24). We will show that the set Ω is invariant for the closed-loop trajectories from $t = t_c$ on.

Let, without loss of generality, $\sigma(t_c) = \sigma_1 > 0$. Clearly, at $t = t_c$ the $\sigma - \dot{\sigma}$ trajectory belongs to Ω. As long as the $\sigma - \dot{\sigma}$ trajectory does not leave the set, the following estimations are in force. First, by Assumptions 2 and 3 one gets

$$\|\mathbf{w}\| \leq \|\mathbf{w}\|^* \tag{64}$$

where $\|\mathbf{w}\|^*$ was defined in (21). Secondly, by (7) one concludes that

$$u = \frac{1}{K(\mathbf{w}, \sigma)} [\dot{\sigma}(t) - \varphi_1(\mathbf{w}, \sigma, t)] \tag{65}$$

Therefore, by the Assumption 6, coupled to (64), one concludes that into the set Ω the following estimation

$$|u| \leq u^* \tag{66}$$

holds, where u^* is the constant that was defined in (25). As a result, by (13), and by considering relations (64), (66) and the Assumptions 4-6, one concludes that into the set Ω the function

$\varphi_2(\mathbf{w}, \sigma, u, t)$ fulfills the inequality

$$|\varphi_2(\cdot)| \quad \leq \quad \Phi \tag{67}$$

where Φ is the constant that was defined in (26).

From now on, by considering the limit solutions of the differential inclusion $\ddot{\sigma} \in [-\Phi, \Phi] + [K_m \; K_M]\dot{u}$, and referring to their graphical representation of Figure 4, it can be concluded (refer to the similar analysis of Bartolini et al. (1999b)) that $\underline{\dot{\sigma}}_1^{'}$, the lower intersection of the limit trajectories with the vertical axis, satisfies the following relationships

$$|\underline{\dot{\sigma}}_1^{'}| = \sqrt{2|\sigma_1(t_c)|(\Phi + K_M U_{M1})} = \sqrt{2\rho\Phi|\sigma_1(t_c)|} \leq \sqrt{2\rho\Phi(|\sigma_1(t_c)| + q)} \tag{68}$$

where ρ is the constant that was defined in (22). To derive (68), it is considered that the adaptive gain U_{M_j} keeps constant to its initial value U_{M_1} while the system trajectory is approaching the vicinity of the origin of the $\sigma - \dot{\sigma}$ plane. Note that $\underline{\dot{\sigma}}_1^{'}$ overcomes any possible value of $|\dot{\sigma}|$ over the whole considered trajectory between the points P_1 and $P_2 \equiv (\sigma_2, 0)$.

To ensure the invariance of set Ω, it suffices to show that

$$|\underline{\dot{\sigma}}_1^{'}| < \Sigma_D \tag{69}$$

and then notice that the convergence properties of the Twisting algorithm guarantee that $|\sigma_2| < |\sigma_1|$ (see Levant (1993)). The inequality (69), whose left and right hand sides both depend on Σ_D admits a non-empty solution interval $\Sigma_D \in (\Sigma_D^*, \infty)$ since the left hand-side of (69) asymptotically grows like $O(\sqrt{\Sigma_D})$. It is easy to check that the computation of Σ_D^* yields the nonlinear equation (23)-(24) that was previously introduced. Thus, the set Ω turns out to be invariant at any $t \geq t_c$.

From this point on, the convergence properties of the Twisting algorithm yield a sequence of contractive rotations of the trajectories around the origin of the $\sigma - \dot{\sigma}$ plane, thereby leading to the establishment of a practical second-order sliding mode after a finite transient (see Levant (1993)).

As soon as condition (34) is achieved at some $j = M_1$, the stepwise reduction of U_{M_j} is activated.

The dominance over the uncertainties (formalized by condition (29)) will be lost after a finite number of intervals, and at some $j = M_2 > M_1$ the sliding mode existence criterion (34) will be violated. By Lemma 1, at the end of the time interval $\mathcal{T}_{M_2-1}$ the variables σ and $\dot{\sigma}$ are bounded as in (35) with $a_1 = sup|\ddot{\sigma}| = \Phi + K_M \alpha U_{M1}$.

If parameter Λ_2 is such that

$$\Lambda_2 > \Lambda_1 + 2sup|\dot{\beta}| \tag{70}$$

then condition (29) is already restored during the interval $\mathcal{T}_{M_2+1}$, i.e. one interval after the violation of the 2-sliding criterion (34). It can be verified that (33) is equivalent to (70), the term $sup|\dot{\beta}|$ being explicitly evaluated on the basis of (17), (18) and (6).

The maximal deviation undergone by $|\sigma|$ and $|\dot{\sigma}|$ can be evaluated by studying the limit trajectories obtained starting from the initial condition (35). The resulting algebraic computations, skipped for the sake of brevity, yield conditions (38). While U_{M_j} continues to grow, contractive rotations of the system trajectories around the origin of the $\sigma - \dot{\sigma}$ plane preserve the inequalities (38).

To conclude the proof it must be included a constraint on T guaranteeing that the limit set (38) is entirely contained into the invariant rectangle (63). From the inequality

$$b_1 T^2 < |\sigma(t_c)| + q \tag{71}$$

it readily follows condition (37) (the arbitrary positive parameter q allows us to enlarge as desired the admissible range for T). This concludes the proof of Theorem 1. □

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