Detectability, Observability and Lyapunov-Type Theorems of Linear Discrete Time-Varying Stochastic Systems with Multiplicative Noise

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Abstract

The objective of this paper is to study detectability, observability and related Lyapunov-type theorems of linear discrete-time time-varying stochastic systems with multiplicative noise. Some new concepts such as uniform detectability, $\mathcal{K}^\infty$-exact detectability (resp. $\mathcal{K}^{WFT}$-exact detectability, $\mathcal{K}^{FT}$-exact detectability, $\mathcal{K}^N$-exact detectability) and $\mathcal{K}^\infty$-exact observability (resp. $\mathcal{K}^{WFT}$-exact observability, $\mathcal{K}^{FT}$-exact observability, $\mathcal{K}^N$-exact observability) are introduced, respectively, and nice properties associated with uniform detectability, exact detectability and exact observability are also obtained. Moreover, some Lyapunov-type theorems associated with generalized Lyapunov equations and exponential stability in mean square sense are presented under uniform detectability, $\mathcal{K}^N$-exact observability and $\mathcal{K}^N$-exact detectability, respectively.

Key words: Discrete-time time-varying stochastic systems, generalized Lyapunov equations, uniform detectability, exact detectability, exact observability.

1. Introduction

It is well-known that observability and detectability are fundamental concepts in system analysis and synthesis; see, e.g., [1], [6], [8], [14]–[16], [19], [20], [25]–[27], [34], [36]. In the linear system theory, detectability is a weaker concept than observability, since it describes the fact that all unobservable states are asymptotically stable. Over the last two decades, the classical detectability in the linear system theory

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has been extended to stochastic systems in different ways. For example, the definition of stochastic detectability for time-invariant Itô stochastic systems can be found in [7], [8], which is dual to mean square stabilization. In [6], [33], [36], the notions of exact observability and exact detectability were presented for Itô stochastic systems, which led to the stochastic Popov-Belevitch-Hautus (PBH) criteria like those for deterministic systems. Another natural concept of detectability for Itô stochastic systems was given in [6] based on the idea that any non-observed states corresponded to stable models of the system. In [19], the exact detectability in [36] and detectability in [6] were proved to be equivalent, and a unified treatment was proposed for detectability and observability of Itô stochastic systems. Based on the standard notions of detectability and observability for time-varying linear systems [1], [23], studied in [20] were detectability and observability of discrete time-invariant stochastic systems as well as the properties of Lyapunov equations. Recently, the exact detectability and observability were extended to stochastic systems with Markov jumps and multiplicative noise in [5], [22], [27], [37].

As it is well known that the classical Lyapunov theorem is very essential in stability theory, which asserts that if a matrix F is Schur stable, then for any $Q \geq 0$, the classical Lyapunov equation $-P + F^T PF + Q = 0$ admits a unique solution $P \geq 0$; Conversely, if (F, Q) is detectable, $Q \geq 0$, and the Lyapunov equation $-P + F^T PF + Q = 0$ admits a unique solution $P \geq 0$, then F is Schur stable. The classical Lyapunov theorem was generalized to deterministic time-varying systems in [1] and will be extended to stochastic time-varying systems in this paper under any one assumption of uniform detectability, $\mathcal{K}^N$-exact detectability and $\mathcal{K}^N$-exact observability.

Recently it has become known that discrete-time stochastic systems with multiplicative noise are ideal models in the fields of investment portfolio optimization [12], system biology [31] and so on. So the discrete-time stochastic H_2/H_∞ control and filtering design have been extensively studied in recent years; see, e.g., [2], [8], [10], [13], [32], [34] and the references therein. As it is well-known, time-varying systems may be utilized to model more realistic systems and are more challenging in mathematics than time-invariant ones. So far, the majority of the existing results is focused on detectability of time-invariant systems only, except for a few about time-varying systems; see [1], [9]–[11], [14], [17], [26], [29], [30], [35]. Because linear time-invariant systems are not sufficient to describe many practical phenomena, this motivates researchers to study time-varying systems. In the classical work [1], uniform detectability of the deterministic linear discrete-time time-varying (LDTV) system

$$\begin{cases} x_{k+1} = F_k x_k, \ x_0 \in \mathcal{R}^n \\ \quad y_k = H_k x_k, \ k = 0, 1, 2, \cdots \end{cases} \tag{1}$$

was defined and discussed. By the duality of stochastic stabilizability, another definition called “stochastic detectability” was introduced in [8] for LDTV Markov systems, which is not equivalent to uniform detectability in time-varying case.

Mainly motivated by the preceding discussion and the authors’ series works [33]–[36], this paper will study detectability, observability and Lyapunov-type equation related to LDTV stochastic systems with multiplicative noise. Firstly, the classical uniform detectability of [1] for such systems is extended, and some properties on uniform detectability are obtained. By means of our Lemma 2.2 given later, we obtain the observability Gramian matrix $\mathcal{O}_{k+s,k}$ and the state transition matrix $\phi_{l,k}$, which are deterministic matrices and easy to be applied in practice. Specifically, we prove an important theorem that uniform detectability preserves invariance under an output feedback control law, which is expected to be useful in stochastic H_2/H_∞ control. As an application, under the assumption of uniform detectability, Lyapunov-type theorems on stochastic stability are also presented.

Secondly, we extend exact detectability of linear continuous-time stochastic Itô systems [6], [33] to LDTV systems. We introduce four concepts called $\mathcal{K}^N$-exact detectability, $\mathcal{K}^{FT}$-exact detectability, $\mathcal{K}^{WFT}$-exact detectability and $\mathcal{K}^\infty$-exact detectability, and they in turn become weaker in the sense that the former implies the latter in a sequence. Although in linear time-invariant system

$$\begin{cases} x_{k+1} = Fx_k, \ x_0 \in \mathcal{R}^n \\ \quad y_k = Hx_k, \quad k = 0, 1, 2, \cdots \end{cases} \tag{2}$$

these four concepts are equivalent with $N = n-1$, but they are different from the others in the time-varying case, which reveals the essential difference between time-invariant and time-varying systems. It is shown that uniform detectability implies $\mathcal{K}^\infty$-exact detectability (see Lemma 3.1.3), and stochastic detectability [8] implies the above four types of exact detectability (see Proposition 3.1.1 and Remark 3.1.3). It seems that there is no inclusion relation among uniform detectability, $\mathcal{K}^N$-exact detectability, $\mathcal{K}^{FT}$-exact detectability and $\mathcal{K}^{WFT}$-exact detectability, although they can be unified in the linear discrete time-invariant systems [20]. Two important Lyapunov-type theorems under $\mathcal{K}^N$-exact detectability for periodic systems are obtained (see Theorems 3.2.1–3.2.2), which reveal the important relation between the exponential stability and the existence of positive definite solutions of generalized Lyapunov equations (GLEs).

Parallel to various definitions on exact detectability, we also introduce $\mathcal{K}^N$-exact observability, $\mathcal{K}^{FT}$-exact observability, $\mathcal{K}^{WFT}$-exact observability and $\mathcal{K}^\infty$-exact observability, which are respectively stronger than $\mathcal{K}^N$-exact detectability, $\mathcal{K}^{FT}$-exact detectability, $\mathcal{K}^{WFT}$-exact detectability and $\mathcal{K}^\infty$-exact detectability. For the linear time-invariant system (2), $\mathcal{K}^{n-1}$-, $\mathcal{K}^{FT}$-, $\mathcal{K}^{WFT}$- and $\mathcal{K}^\infty$-exact observability are equivalent, but they are different definitions for the linear time-varying system (1). We present a rank criterion for $\mathcal{K}^\infty$- and a criterion for $\mathcal{K}^N$-exact observability based on the Gramian matrix $\mathcal{O}_{k+N,k}$. Finally, under the assumption of $\mathcal{K}^N$-exact observability, a Lyapunov-type theorem is derived from Theorem 3.2.1.

The rest of the paper is organized as follows. In Section 2, we define uniform detectability and discuss its properties. Lyapunov-type theorems are given under uniform detectability. Section 3 introduces some new concepts about exact detectability and exposes nice properties. This section also presents Lyapunov-type stability theorems based on $\mathcal{K}^N$-exact detectability. Moreover, the relation among uniform detectability, exact detectability and stochastic detectability is clarified via some examples. Section 4 introduces various definitions for exact observability, which are stronger than those of Section-3.1. Section 5 provides some comments on this study. Finally, Section 6 concludes the paper with some remarks.

Notation: $\mathcal{R}^n$: the set of all real n-dimensional vectors. $\mathcal{S}_n$: the set of all $n \times n$ symmetric matrices whose entries may be complex numbers. $\mathcal{C}$: the set of all complex numbers. $\mathcal{R}^{m\times n}$: the set of all $m \times n$ real matrices. $\|x\|$: the norm of a vector or matrix. $A > 0$ (resp. $A \geq 0$): A is a real symmetric positive definite (resp. positive semi-definite) matrix. I: the identity matrix. $\sigma(L)$: the spectrum set of the operator or matrix L. A^T: the transpose of matrix A. $\mathcal{N}_{k_0} := \{k_0, k_0+1, k_0+2, \cdots,\}$, especially, $\mathcal{N}_1 = \{1, 2, \cdots,\}$, $\mathcal{N}_0 = \{0, 1, 2, \cdots,\}$. $l^2_{\mathcal{F}_k} := \{x(\omega) : x \text{ is } \mathcal{F}_k - \text{measurable}, E\|x\|^2 < \infty\}$.

2. Uniform Detectability and Related Lyapunov-type Theorems

In this section, we will define one important concept for LDTV stochastic systems, called "uniform detectability". And then, we will obtain Lyapunov-type theorems under uniform detectability, which are extensions of classical Lyapunov theorem.

2.1 Uniform Detectability

Consider the following LDTV stochastic system

$$\begin{cases} x_{k+1} = F_k x_k + G_k x_k w_k, \ x_0 \in \mathcal{R}^n \\ \quad y_k = H_k x_k, \ k = 0, 1, 2, \cdots, \end{cases} \tag{3}$$

where x_k is the n-dimensional state vector, y_k is the m-dimensional measurement output, $\{w_k\}_{k\geq 0}$ represents a one-dimensional independent white noise process defined on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_k, \mathcal{P})$ with $\mathcal{F}_k = \sigma(w(0), \cdots, w(k))$. Assume that $Ew_k = 0$, $E[w_k w_j] = \delta_{kj}$, where δ_{kj} is a Kronecker function defined by $\delta_{kj} = 0$ for $k \neq j$ while $\delta_{kj} = 1$ for $k = j$. x_0 is assumed to be deterministic for simplicity purposes, and F_k, G_k and H_k are time-varying matrices of appropriate dimension. In practice, one is more concerned about the $l^2_{\mathcal{F}_k}$-solution $\{x_k\}_{k\in\mathcal{N}_0}$ of stochastic difference equation

$$x_{k+1} = F_k x_k + G_k x_k w_k,\ x_0 \in \mathcal{R}^n. \tag{4}$$

Definition 2.1. *The stochastic vector-valued sequence $\{\tilde{x}_k\}_{k\in\mathcal{N}_0}$ is called a solution of system (4) if (i) $\tilde{x}_0 = x_0$; (ii) $\tilde{x}_k$ solves (4) for $k = 1, 2, \cdots$; (iii) $\tilde{x}_k \in l^2_{\mathcal{F}_{k-1}}$, where $\mathcal{F}_{-1} = \{\phi, \Omega\}$ is assumed to be a trivial sigma algebra. System (4) is said to have a unique solution if for any two of its solutions $\{\tilde{x}_k\}_{k\in\mathcal{N}_0}$ and $\{\bar{x}_k\}_{k\in\mathcal{N}_0}$, $\mathcal{P}(\tilde{x}_k = \bar{x}_k, k \in \mathcal{N}_0) = 1$.*

Remark 2.1. It can be found that, in most present literature, the condition (iii) in Definition 2.1 is not particularly pointed out when defining solutions of system (4), which is in fact an essential requirement as done in stochastic differential equations [21]. This makes an fundamental difference of (4) from deterministic difference equations, as will be seen in the following examples.

Example 2.1. It is easy to see that the following forward difference equation

$$x_{k+1} = F_k x_k,\ x_0 \in \mathcal{R}^n,\ k = 0, 1, \cdots, N$$

always admits a unique solution on $[0, N+1]$. In addition, if $F_k, k = 0, 1, \cdots, N$ are nonsingular, then the backward difference equation

$$x_{k+1} = F_k x_k,\ x_{N+1} \in \mathcal{R}^n,\ k = 0, 1, \cdots, N$$

also has a unique solution on $[0, N+1]$.

Example 2.2. Obviously, the linear stochastic difference equation (4) always has a unique $l^2_{\mathcal{F}_{k-1}}$-solution x_k on any interval $[0, N+1]$. However, even if $F_k, k = 0, 1, \cdots, N$, are nonsingular, the following stochastic difference equation

$$x_{k+1} = F_k x_k + G_k x_k w_k,\ x_{N+1} \in l^2_{\mathcal{F}_N} \tag{5}$$

with terminal state given does not always admit an $l^2_{\mathcal{F}_{k-1}}$-solution. For example, if we take $F_k = 1$, $G_k = 0$, and the terminal state $x_2 = w_1$ in (5), then $x_1 = w_1 \notin l^2_{\mathcal{F}_0}$, $x_0 = w_1 \notin l^2_{\mathcal{F}_{-1}}$.

Remark 2.2. A class of backward stochastic difference equations arising from the study of discrete stochastic maximum principle can be found in [18].

To define and better understand the uniform detectability for system (3), we first give some lemmas.

Lemma 2.1. *(i) For system (3), $E\|x_l\|^2 = E\|\phi_{l,k}x_k\|^2$ for $l \geq k$, where it is assumed that $\phi_{k,k} = I$, and $\phi_{l,k}$ is given by the following iterative relation*

$$\phi_{l,k} = \begin{bmatrix} \phi_{l,k+1}F_k \\ \phi_{l,k+1}G_k \end{bmatrix}, \; l > k. \tag{6}$$

(ii) $x_k \in l^2_{\mathcal{F}_{k-1}}$ if F_i and G_i are bounded for $0 \leq i \leq k-1$.

Proof: (i) can be shown by induction. For $k = l-1$, we have

$$\begin{aligned} E\|x_l\|^2 &= E[(F_{l-1}x_{l-1} + G_{l-1}x_{l-1}w_{l-1})^T(F_{l-1}x_{l-1} + G_{l-1}x_{l-1}w_{l-1})] \\ &= E[x_{l-1}^T(F_{l-1}^TF_{l-1} + G_{l-1}^TG_{l-1})x_{l-1}] \\ &= E\|\phi_{l,l-1}x_{l-1}\|^2. \end{aligned}$$

Hence, (6) holds for $k = l-1$. Assume that for $k = m < l-1$, $E\|x_l\|^2 = E\|\phi_{l,m}x_m\|^2$. Next, we prove $E\|x_l\|^2 = E\|\phi_{l,m-1}x_{m-1}\|^2$. It can be seen that

$$\begin{aligned} E\|x_l\|^2 &= E[x_m^T\phi_{l,m}^T\phi_{l,m}x_m] \\ &= E[(F_{m-1}x_{m-1} + G_{m-1}x_{m-1}w_{m-1})^T\phi_{l,m}^T\phi_{l,m}(F_{m-1}x_{m-1} + G_{m-1}x_{m-1}w_{m-1})] \\ &= E[x_{m-1}^T(F_{m-1}^T\phi_{l,m}^T\phi_{l,m}F_{m-1} + G_{m-1}^T\phi_{l,m}^T\phi_{l,m}G_{m-1})x_{m-1}] \\ &= E\|\phi_{l,m-1}x_{m-1}\|^2. \end{aligned}$$

This completes the proof of (i). And (ii) is obvious. The proof of this lemma is complete. ∎

Lemma 2.2. *For system (3), there holds $\sum_{i=k}^{l} E\|y_i\|^2 = E\|H_{l,k}x_k\|^2$ for $l \geq k \geq 0$, where*

$$H_{l,k} = \begin{bmatrix} H_k \\ (I_2 \otimes H_{k+1})\phi_{k+1,k} \\ (I_{2^2} \otimes H_{k+2})\phi_{k+2,k} \\ \vdots \\ (I_{2^{l-k}} \otimes H_l)\phi_{l,k} \end{bmatrix} \tag{7}$$

with $H_{k,k} = H_k$ and $\phi_{j,k}(j = k+1, \cdots, l)$ given by (6).

Proof: We prove this lemma by induction. First, by a straight and simple computation, the conclusion holds in the case of $k = l, l-1$. Next, we assume that for $k = m < l-1$, $\sum_{i=m}^{l} E\|y_i\|^2 = E\|H_{l,m}x_m\|^2$

holds, then it only needs to prove $\sum_{i=m-1}^{l} E\|y_i\|^2 = E\|H_{l,m-1}x_{m-1}\|^2$. It can be verified that

$$\begin{aligned}
\sum_{i=m-1}^{l} E\|y_i\|^2 &= \sum_{i=m}^{l} E\|y_i\|^2 + E\|y_{m-1}\|^2 \\
&= E\|H_{l,m}x_m\|^2 + E\|y_{m-1}\|^2 \\
&= E[x_m^T H_{l,m}^T H_{l,m} x_m] + E[x_{m-1}^T H_{m-1}^T H_{m-1} x_{m-1}] \\
&= E[(F_{m-1}x_{m-1} + G_{m-1}x_{m-1}w_{m-1})^T H_{l,m}^T H_{l,m}(F_{m-1}x_{m-1} + G_{m-1}x_{m-1}w_{m-1})] \\
&\quad + E[x_{m-1}^T H_{m-1}^T H_{m-1} x_{m-1}] \\
&= E\left\{ x_{m-1}^T \begin{bmatrix} H_{m-1} \\ H_{l,m}F_{m-1} \\ H_{l,m}G_{m-1} \end{bmatrix}^T \begin{bmatrix} H_{m-1} \\ H_{l,m}F_{m-1} \\ H_{l,m}G_{m-1} \end{bmatrix} x_{m-1} \right\}.
\end{aligned} \tag{8}$$

By (7), it follows that

$$\begin{bmatrix} H_{m-1} \\ H_{l,m}F_{m-1} \\ H_{l,m}G_{m-1} \end{bmatrix} = \begin{bmatrix} H_{m-1} \\ H_m F_{m-1} \\ (I_2 \otimes H_{m+1})\phi_{m+1,m}F_{m-1} \\ \vdots \\ (I_{2^{t-m}} \otimes H_l)\phi_{t,m}F_{m-1} \\ H_m G_{m-1} \\ (I_2 \otimes H_{m+1})\phi_{m+1,m}G_{m-1} \\ \vdots \\ (I_{2^{t-m}} \otimes H_l)\phi_{l,m}G_{m-1} \end{bmatrix}. \tag{9}$$

On the other hand, it can be deduced from (6) and (7) that

$$H_{l,m-1} = \begin{bmatrix} H_{m-1} \\ (I_2 \otimes H_m) \begin{bmatrix} F_{m-1} \\ G_{m-1} \end{bmatrix} \\ (I_{2^2} \otimes H_{m+1}) \begin{bmatrix} \phi_{m+1,m}F_{m-1} \\ \phi_{m+1,m}G_{m-1} \end{bmatrix} \\ \vdots \\ (I_{2^{t-m+1}} \otimes H_l) \begin{bmatrix} \phi_{l,m}F_{m-1} \\ \phi_{l,m}G_{m-1} \end{bmatrix} \end{bmatrix}. \tag{10}$$

Combining (9) and (10) together results in

$$\begin{bmatrix} H_{m-1} \\ H_{l,m}F_{m-1} \\ H_{l,m}G_{m-1} \end{bmatrix}^T \begin{bmatrix} H_{m-1} \\ H_{l,m}F_{m-1} \\ H_{l,m}G_{m-1} \end{bmatrix} = H_{l,m-1}^T H_{l,m-1}. \tag{11}$$

Hence, $\sum_{i=m-1}^{l} E\|y_i\|^2 = E\|H_{l,m-1}x_{m-1}\|^2$. This lemma is shown.

■

Based on Lemmas 2.1–2.2, we are now in a position to define the uniform detectability for system (3).

Definition 2.2. *System (3) or $(F_k, G_k|H_k)$ is said to be uniformly detectable if there exist integers $s, t \geq 0$, and positive constants d, b with $0 \leq d < 1$ and $0 < b < \infty$ such that whenever*

$$E\|x_{k+t}\|^2 = E\|\phi_{k+t,k}x_k\|^2 \geq d^2 E\|x_k\|^2, \tag{12}$$

there holds

$$\sum_{i=k}^{k+s} E\|y_i\|^2 = E\|H_{k+s,k}x_k\|^2 \geq b^2 E\|x_k\|^2, \tag{13}$$

where $k \in \mathcal{N}_0$, and $\phi_{k+t,k}$ and $H_{k+s,k}$ are the same as defined in Lemma 2.2.

Obviously, without loss of generality, in Definition 2.2 we can assume that $t \leq s$. By Lemmas 2.1–2.2, the uniform detectability of $(F_k, G_k|H_k)$ implies, roughly speaking, that the state trajectory decays faster than the output energy does. In what follows, $\mathcal{O}_{k+s,k} := H_{k+s,k}^T H_{k+s,k}$ is called an observability Gramian matrix, and $\phi_{l,k}$ a state transition matrix from x_k to x_l of stochastic system (3). So (13) can be written as $E[x_k^T \mathcal{O}_{k+s,k} x_k] \geq b^2 E\|x_k\|^2$. If $G_k \equiv 0$ for $k \geq 0$, then system (3) reduces to the following deterministic system

$$\begin{cases} x_{k+1} = F_k x_k, \ \ x_0 \in \mathcal{R}^n, \\ \quad y_k = H_k x_k, \end{cases} \tag{14}$$

which was discussed in [1], [23].

Similarly, uniform observability can be defined as follows:

Definition 2.3. *System (3) or $(F_k, G_k|H_k)$ is said to be uniformly observable if there exist an integer $s \geq 0$ and a positive constant $b > 0$ such that*

$$E\|H_{k+s,k}x_k\|^2 \geq b^2 E\|x_k\|^2$$

holds for each initial condition $x_k \in l^2_{\mathcal{F}_{k-1}}$, $k \in \mathcal{N}_0$.

Remark 2.3. Different from the uniform detectability concept, uniform observability needs that any models (unstable and stable) should be reflected by the output. This section concentrates on the uniform detectability of system (3), since it is weaker than uniform observability. Uniform observability is also an important concept, which will be further studied in the future.

Definition 2.4. *System (3) is said to be exponentially stable in mean square (ESMS) if there exist* $\beta \geq 1$ *and* $\lambda \in (0,1)$ *such that for any* $0 \leq k_0 \leq k < +\infty$*, there holds*

$$E\|x_k\|^2 \leq \beta E\|x_{k_0}\|^2 \lambda^{(k-k_0)}. \tag{15}$$

Proposition 2.1. *If system (3) is ESMS, then for any bounded matrix sequence* $\{H_k\}_{k\geq 0}$*, system (3) is uniformly detectable.*

Proof: By Definition 2.4, for any $k, t \geq 0$, we always have

$$E\|x_{k+t}\|^2 = E\|\phi_{k+t,k}x_k\|^2 \leq \beta E\|x_k\|^2\lambda^t, \ \beta > 1, \ 0 < \lambda < 1. \tag{16}$$

By (16), $\beta\lambda^t \to 0$ as $t \to \infty$. Set a large $t_0 > 0$ such that $0 \leq d^2 := \beta\lambda^{t_0} < 1$. Then, for any fixed $t > t_0$, (12) holds only for $x_k = 0$, which makes (13) valid for any $s \geq t > t_0$ and $b > 0$ with an equality. So system (3) is uniformly detectable. ∎

Remark 2.4. For system (14), Definition 2.2 reduces to Definition 2.1 in [1]. It is easy to prove that uniform detectability coincides with classical detectability of the linear time-invariant system (2).

The following lemma will be used throughout this paper.

Lemma 2.3 (see [14])**.** *For a nonnegative real sequence* $\{s_k\}_{k\geq k_0}$*, if there exist constants* $M_0 \geq 1$*,* $\delta_0 \in (0,1)$*, and an integer* $h_0 > 0$ *such that* $s_{k+1} \leq M_0 s_k$ *and* $\min_{k+1\leq i\leq k+h_0} s_i \leq \delta_0 s_k$*, then*

$$s_k \leq [M_0^{h_0}\delta_0{}^{-1}](\delta_0^{h_0})^{k-k_0}s_{k_0}, \ \forall k \geq k_0.$$

The following proposition extends Lemma 2.2 in [1].

Proposition 2.2. *Suppose that* $(F_k, G_k|H_k)$ *is uniformly detectable, and* F_k *and* G_k *are uniformly bounded, i.e.,* $\|F_k\| \leq M, \|G_k\| \leq M, M > 0$*. Then* $\lim_{k\to\infty} E\|y_k\|^2 = 0$ *implies* $\lim_{k\to\infty} E\|x_k\|^2 = 0$.

Proof: If there exists some integer k_0 such that for all $k \geq k_0$, $E\|x_{k+t}\|^2 = E\|\phi_{k+t,k}x_k\|^2 < d^2E\|x_k\|^2$, then $\min_{k+1\leq i\leq k+t} E\|x_i\|^2 < d^2E\|x_k\|^2$. Moreover, $E\|x_{i+1}\|^2 = E\|\phi_{i+1,i}x_i\|^2 = E[x_i^T(F_i^TF_i + G_i^TG_i)x_i] \leq 2M^2E\|x_i\|^2 \leq M_0E\|x_i\|^2$, where $M_0 = \max\{2M^2, 1\} \geq 1$. By Lemma 2.3, not only does

$\lim_{k\to\infty} E\|x_k\|^2 = 0$, but also is system (3) ESMS. Otherwise, there exists a subsequence $\{k_i\}_{i\geq 0}$ such that $E\|\phi_{k_i+t,k_i}x_{k_i}\|^2 \geq d^2E\|x_{k_i}\|^2$. Now, for $k \in (k_i, k_{i+1})$, we write $k = k_i+1+t\alpha+\beta$ with $\beta < t$, then

$$E\|x_{k_i+1+\alpha t}\|^2 \leq d^\alpha E\|x_{k_i+1}\|^2,$$

$$E\|x_{k_i+1+\alpha t+\beta}\|^2 \leq (2M^2)^\beta E\|x_{k_i+1+\alpha t}\|^2,$$

$$E\|x_{k_i+1}\|^2 \leq 2M^2E\|x_{k_i}\|^2.$$

Therefore, we have

$$\begin{aligned} E\|x_k\|^2 = E\|x_{k_i+1+\alpha t+\beta}\|^2 &\leq (2M^2)^\beta d^\alpha E\|x_{k_i+1}\|^2 \\ &\leq (2M^2)^{\beta+1}d^\alpha E\|x_{k_i}\|^2. \end{aligned} \tag{17}$$

Obviously, in order to show $\lim_{k\to\infty} E\|x_k\|^2 = 0$, we only need to show $\lim_{k_i\to\infty} E\|x_{k_i}\|^2 = 0$. If it is not so, then there are a subsequence $\{n_i\}_{i\geq 0}$ of $\{k_i\}_{i\geq 0}$ and $\varsigma > 0$, such that $E\|x_{n_i}\|^2 > \varsigma$, $E\|\phi_{n_i+t,n_i}x_{n_i}\|^2 \geq d^2E\|x_{n_i}\|^2$. By Definition 2.2,

$$\sum_{i=n_i}^{n_i+s} E\|y_i\|^2 = E[x_{n_i}^T\mathcal{O}_{n_i+s,n_i}x_{n_i}] \geq b^2E\|x_{n_i}\|^2 > b^2\varsigma. \tag{18}$$

Taking $n_i \to \infty$ in (18), we have $0 > b^2\varsigma > 0$, which is a contradiction. Hence, the proof is complete. ■

In the remainder of this section, we will prove the output feedback invariance for uniform detectability. Consider the following LDTV stochastic control system

$$\begin{cases} x_{k+1} = (F_kx_k + M_ku_k) + (G_kx_k + N_ku_k)w_k, \\ y_k = H_kx_k, \ \ k = 0, 1, 2, \cdots. \end{cases} \tag{19}$$

Applying an output feedback control law $u_k = K_ky_k$ to (19) yields the following closed-loop system

$$\begin{cases} x_{k+1} = (F_k + M_kK_kH_k)x_k + (G_k + N_kK_kH_k)x_kw_k, \\ y_k = H_kx_k, \ \ k = 0, 1, 2, \cdots. \end{cases} \tag{20}$$

Theorem 2.1. *If $(F_k, G_k|H_k)$ is uniformly detectable, then so is $(F_k + M_kK_kH_k, G_k + N_kK_kH_k|H_k)$.*

Proof: By Lemma 2.2, the observability Gramian for system (20) is $\bar{\mathcal{O}}_{k+s,k} = \bar{H}_{k+s,k}^T\bar{H}_{k+s,k}$, where

$$\bar{H}_{k+s,k} = \begin{bmatrix} H_k \\ (I_2 \otimes H_{k+1})\bar{\phi}_{k+1,k} \\ (I_{2^2} \otimes H_{k+2})\bar{\phi}_{k+2,k} \\ \vdots \\ (I_{2^s} \otimes H_{k+s})\bar{\phi}_{k+s,k} \end{bmatrix}, \ \bar{\phi}_{k+i,k} = \begin{bmatrix} \bar{\phi}_{k+i,k+1}\bar{F}_k \\ \bar{\phi}_{k+i,k+1}\bar{G}_k \end{bmatrix}, \ i = 1, \cdots, s.$$

$$\bar{F}_j = F_j + M_j K_j H_j,\ \bar{G}_j = G_j + N_j K_j H_j,\ j = k, k+1, \cdots, k+s.$$

To prove that $(\bar{F}_k, \bar{G}_k|H_k)$ is uniformly detectable, it suffices to show that there are constants $\bar{b} > 0$, $0 < \bar{d} < 1$, $s, t \geq 0$ such that for $\xi \in l^2_{\mathcal{F}_{k-1}}$, $k \in \mathcal{N}_0$, whenever

$$E[x_k^T \bar{\mathcal{O}}_{k+s,k} x_k] < \bar{b}^2 E\|x_k\|^2, \tag{21}$$

we have

$$E\|\bar{\phi}_{k+t,k} x_k\|^2 < \bar{d}^2 E\|x_k\|^2. \tag{22}$$

It is easy to show

$$\bar{H}_{k+s,k} = Q_{k+s,k} H_{k+s,k},\ Q_{k+s,k} = \begin{bmatrix} I & 0 & \cdots & 0 \\ * & I & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & I \end{bmatrix}.$$

where * represents terms involving H_i, M_i, K_i and N_i, $i = k, k+1, \cdots, k+s$. Hence, for any $x_k \in l^2_{\mathcal{F}_{k-1}}$,

$$\rho E[x_k^T \mathcal{O}_{k+s,k} x_k] \leq E[x_k^T \bar{\mathcal{O}}_{k+s,k} x_k] \leq \varrho E[x_k^T \mathcal{O}_{k+s,k} x_k], \tag{23}$$

where $\rho = \lambda_{\min}(Q^T_{k+s,k} Q_{k+s,k}) > 0$, $\varrho = \lambda_{\max}(Q^T_{k+s,k} Q_{k+s,k}) > 0$. In addition, by observation, for any $l > k \geq 0$,

$$\bar{\phi}_{l,k} = \phi_{l,k} + R_{l,k} H_{l,k},$$

where $R_{l,k}$ is a matrix involving H_i, M_i, K_i and N_i, $i = k, k+1, \cdots, l-1$. If we take $0 < \bar{b} \leq \sqrt{\rho} b$, then it follows from (23) that $E[x_k^T \mathcal{O}_{k+s,k} x_k] < \frac{1}{\rho} E[x_k^T \bar{\mathcal{O}}_{k+s,k} x_k] \leq \frac{\bar{b}^2}{\rho} E\|x_k\|^2 \leq b^2 E\|x_k\|^2$. By the uniform observability of $(F_k, G_k|H_k)$, it follows that

$$\begin{aligned} E\|\bar{\phi}_{k+t,k} x_k\|^2 &= E\|\phi_{k+t,k} x_k + R_{k+t,k} H_{k+t,k} x_k\|^2 \\ &\leq 2E\|\phi_{k+t,k} x_k\|^2 + 2\mu^2 E\|H_{k+t,k} x_k\|^2 \\ &\leq 2d^2 E\|x_k\|^2 + 2\mu^2 E[x_k^T \mathcal{O}_{k+s,k} x_k] \\ &\leq \big(2d^2 + 2\mu^2 \frac{\bar{b}^2}{\rho}\big) E\|x_k\|^2 \\ &= \bar{d} E\|x_k\|^2, \end{aligned} \tag{24}$$

where $\mu = \sup_k \|R_{k+t,k}\|$, $\bar{d} = 2d^2 + 2\mu^2 \frac{\bar{b}^2}{\rho}$. If we take $\bar{b}$ to be sufficiently small, then $\bar{d} < 1$, which yields the uniform detectability of $(\bar{F}_k, \bar{G}_k|H_k)$. Hence, the proof of this theorem is complete. ■

Theorem 2.1 reveals that the output feedback does not change uniform detectability.

Example 2.3. For simplicity, we set $s=1$. Then it can be computed that

$$\bar{H}_{k+1,k}=\begin{bmatrix} H_k \\ (I_2\otimes H_{k+1})\bar{\phi}_{k+1,k}\end{bmatrix}=\begin{bmatrix} H_k \\ H_{k+1}(F_k+M_kK_kH_k) \\ H_{k+1}(G_k+N_kK_kH_k)\end{bmatrix},$$

$$H_{k+1,k}=\begin{bmatrix} H_k \\ (I_2\otimes H_{k+1})\phi_{k+1,k}\end{bmatrix}=\begin{bmatrix} H_k \\ H_{k+1}F_k \\ H_{k+1}G_k\end{bmatrix}.$$

Obviously,

$$Q_{k+1,k}=\begin{bmatrix} I & 0 & 0 \\ H_{k+1}M_kK_k & I & 0 \\ H_{k+1}N_kK_k & 0 & I\end{bmatrix}.$$

Example 2.4. By definition, we have

$$\bar{\phi}_{k+1,k}=\begin{bmatrix} F_k+M_kK_kH_k \\ G_k+N_kK_kH_k\end{bmatrix},\quad \phi_{k+1,k}=\begin{bmatrix} F_k \\ G_k\end{bmatrix}.$$

Hence, $\bar{\phi}_{k+1,k}=\phi_{k+1,k}+R_{k+1,k}H_{k+1,k}$ with $R_{k+1,k}=\begin{bmatrix} M_kK_k & 0 & 0 \\ N_kK_k & 0 & 0\end{bmatrix}$.

2.2 Lyapunov-Type Theorems under Uniform Detectability

In the following, we will further study the following time-varying GLE

$$-P_k+F_k^TP_{k+1}F_k+G_k^TP_{k+1}G_k+H_k^TH_k=0,\ \ k=0,1,2,\cdots, \tag{25}$$

under uniform detectability. The aim of this subsection is to extend the classical Lyapunov theorem to GLE (25). To study (25), we first introduce the following finite time backward difference equation

$$\begin{cases} -P_{k,T}+F_k^TP_{k+1,T}F_k+G_k^TP_{k+1,T}G_k+H_k^TH_k=0, \\ P_{T,T}=0,\ \ k=0,1,\cdots,T-1;\ \ T\in\mathcal{N}_1. \end{cases} \tag{26}$$

Obviously, equation (26) has nonnegative definite solutions $P_{k,T}\geq 0$.

Proposition 2.2.1. *$P_{k,T}$ is monotonically increasing with respect to T, i.e., for any $k_0\leq T_1\leq T_2<+\infty$,*

$$P_{k_0,T_1}\leq P_{k_0,T_2},\ \ k_0\in\{0,1,\cdots,T_1\}.$$

Proof: Obviously, P_{k,T_1} and P_{k,T_2} solve

$$\begin{cases} -P_{k,T_1}+F_k^TP_{k+1,T_1}F_k+G_k^TP_{k+1,T_1}G_k+H_k^TH_k=0, \\ P_{T_1,T_1}=0,\ \ k=0,1,\cdots,T_1-1, \end{cases} \tag{27}$$

and

$$\begin{cases} -P_{k,T_2} + F_k^T P_{k+1,T_2} F_k + G_k^T P_{k+1,T_2} G_k + H_k^T H_k = 0, \\ P_{T_2,T_2} = 0, \ k = 0, 1, \cdots, T_2 - 1, \end{cases} \tag{28}$$

respectively. Consider the following LDTV stochastic system with a deterministic initial state x_{k_0}:

$$\begin{cases} x_{k+1} = F_k x_k + G_k x_k w_k, \\ x_{k_0} \in \mathcal{R}^n, \ k = k_0, k_0 + 1, \cdots. \end{cases} \tag{29}$$

Associated with (29), in view of (27), we have

$$\begin{aligned} \sum_{k=k_0}^{T_1-1} E[x_k^T H_k^T H_k x_k] &= \sum_{k=k_0}^{T_1-1} E[x_k^T H_k^T H_k x_k + x_{k+1}^T P_{k+1,T_1} x_{k+1} - x_k^T P_{k,T_1} x_k] \\ &\quad + x_{k_0}^T P_{k_0,T_1} x_{k_0} - E[x_{T_1}^T P_{T_1,T_1} x_{T_1}] \\ &= \sum_{k=k_0}^{T_1-1} E[x_k^T (-P_{k,T_1} + F_k^T P_{k+1,T_1} F_k + G_k^T P_{k+1,T_1} G_k + H_k^T H_k) x_k] \\ &\quad + x_{k_0}^T P_{k_0,T_1} x_{k_0} \\ &= x_{k_0}^T P_{k_0,T_1} x_{k_0}. \end{aligned} \tag{30}$$

Similarly,

$$\sum_{k=k_0}^{T_2-1} E[x_k^T H_k^T H_k x_k] = x_{k_0}^T P_{k_0,T_2} x_{k_0}. \tag{31}$$

From (30)-(31), it follows that

$$0 \le \sum_{k=k_0}^{T_1-1} E[x_k^T H_k^T H_k x_k] = x_{k_0}^T P_{k_0,T_1} x_{k_0} \le \sum_{k=k_0}^{T_2-1} E[x_k^T H_k^T H_k x_k] = x_{k_0}^T P_{k_0,T_2} x_{k_0}. \tag{32}$$

The above expression holds for any $x_{k_0} \in \mathcal{R}^n$, which yields $P_{k_0,T_1} \le P_{k_0,T_2}$. Thus, the proof is complete. ■

Proposition 2.2.2. *If system (3) is ESMS, and H_k is uniformly bounded (i.e., there exists $M > 0$ such that $\|H_k\| \le M$, $\forall k \in \mathcal{N}_0$), then the solution $P_{k,T}$ of (26) is uniformly bounded for any $T \in \mathcal{N}_1$ and $k \in [0, T]$.*

Proof: By (30), for any deterministic $x_k \in \mathcal{R}^n$, we have

$$\begin{aligned} x_k^T P_{k,T} x_k &= \sum_{i=k}^{T-1} E[x_i^T H_i^T H_i x_i] \le \sum_{i=k}^{\infty} E[x_i^T H_i^T H_i x_i] \\ &\le M^2 \|x_k\|^2 \beta \sum_{i=k}^{\infty} \lambda^{(i-k)} = M^2 \|x_k\|^2 \beta \frac{1}{1-\lambda}, \end{aligned}$$

which leads to that $0 \le P_{k,T} \le \frac{\beta M^2}{1-\lambda} I$ since x_k is arbitrary. Hence, the proof is complete. ■

Combining Proposition 2.2.1 with Proposition 2.2.2 yields that $P_k := \lim_{T\to\infty} P_{k,T}$ exists, which is a solution of (25). Hence, we obtain the following Lyapunov-type theorem.

Theorem 2.2.1 (Lyapunov-Type Theorem)**.** *If system (3) is ESMS and $\{H_k\}_{k\in\mathcal{N}_0}$ is uniformly bounded, then (25) admits a unique nonnegative definite solution $\{P_k\}_{k\in\mathcal{N}_0}$.*

The converse of Theorem 2.2.1 still holds.

Theorem 2.2.2 (Lyapunov-Type Theorem)**.** *Suppose that $(F_k, G_k|H_k)$ is uniformly detectable and F_k and G_k are uniformly bounded with an upper bound $M > 0$. If there is a bounded nonnegative definite symmetric matrix sequence $\{P_k\}_{k\geq 0}$ solving GLE (25), then system (3) is ESMS.*

Proof: For system (3), we take a Lyapunov function as

$$V_k(x) = x^T(P_k + \varepsilon I)x,$$

where $\varepsilon > 0$ is to be determined. For simplicity, in the sequel, we let $V_k := V_k(x_k)$. It is easy to compute

$$\begin{aligned}
EV_k - EV_{k+1} &= E[x_k^T(P_k + \varepsilon I)x_k] - E[x_{k+1}^T(P_{k+1} + \varepsilon I)x_{k+1}] \\
&= E[x_k^T(P_k + \varepsilon I)x_k] - E[(F_k x_k + G_k x_k w_k)^T(P_{k+1} + \varepsilon I)(F_k x_k + G_k x_k w_k)] \\
&= E[x_k^T(P_k - F_k^T P_{k+1} F_k - G_k^T P_{k+1} G_k)x_k] + \varepsilon E[x_k^T(I - F_k^T F_k - G_k^T G_k)x_k] \\
&= E\|y_k\|^2 + \varepsilon E[x_k^T(I - F_k^T F_k - G_k^T G_k)x_k] \\
&= E\|y_k\|^2 + \varepsilon E\|x_k\|^2 - \varepsilon E\|x_{k+1}\|^2. \qquad (33)
\end{aligned}$$

Identity (33) yields

$$\begin{aligned}
EV_k - EV_{k+s+1} &= [EV_k - EV_{k+1}] + [EV_{k+1} - EV_{k+2}] + \cdots + [EV_{k+s} - EV_{k+s+1}] \\
&= \sum_{i=k}^{k+s} E\|y_i\|^2 + \varepsilon E\|x_k\|^2 - \varepsilon E\|x_{k+s+1}\|^2. \qquad (34)
\end{aligned}$$

When $\sum_{i=k}^{k+s} E\|y_i\|^2 \geq b^2 E\|x_k\|^2$, we first note that

$$\begin{aligned}
E\|x_{k+s+1}\|^2 &= E\{x_{k+s}^T(F_{k+s}^T F_{k+s} + G_{k+s}^T G_{k+s})x_{k+s}\} \\
&\leq 2M^2 E\|x_{k+s}\|^2 \leq (2M^2)^2 E\|x_{k+s-1}\|^2 \leq \cdots \\
&\leq (2M^2)^{s+1} E\|x_k\|^2. \qquad (35)
\end{aligned}$$

Then, by (34), we still have

$$\begin{aligned}
EV_k - EV_{k+s+1} &\geq b^2 E\|x_k\|^2 + \varepsilon E\|x_k\|^2 - \varepsilon(2M^2)^{s+1} E\|x_k\|^2 \\
&= [b^2 + \varepsilon - \varepsilon(2M^2)^{s+1}] E\|x_k\|^2. \qquad (36)
\end{aligned}$$

From (36), it readily follows that

$$
\begin{aligned}
EV_{k+s+1} &\leq EV_k - \{b^2 + \varepsilon[1-(2M^2)^{s+1}]\}E\|x_k\|^2 \\
&\leq \left\{1 - \frac{[b^2 + \varepsilon[1-(2M^2)^{s+1}]]}{\lambda_{max}(P_k + \varepsilon I)}\right\} EV_k.
\end{aligned} \tag{37}
$$

Considering that $\{P_k \geq 0\}_{k\in\mathcal{N}_0}$ is uniformly bounded, if ε is taken to be sufficiently small, then there must exist a $\delta \in (0,1)$ such that

$$EV_{k+s+1} \leq \delta EV_k. \tag{38}$$

When $\sum_{i=k}^{k+s} E\|y_i\|^2 \leq b^2 E\|x_k\|^2$, by uniform detectability we have $E\|x_{k+t}\|^2 \leq d^2 E\|x_k\|^2$. From (34), it follows that

$$EV_k - EV_{k+t} \geq \varepsilon E\|x_k\|^2 - \varepsilon d^2 E\|x_k\|^2 = \varepsilon(1-d^2)E\|x_k\|^2. \tag{39}$$

Similarly, we can show that there exists a constant $\delta_1 \in (0,1)$ such that

$$EV_{k+t} \leq \delta_1 EV_k. \tag{40}$$

Set $\delta_0 := \max\{\delta, \delta_1\}$, in view of (38) and (40), we have

$$\min_{k+1\leq i\leq k+s+1} EV_i \leq \delta_0 EV_k,\ \forall k \geq 0. \tag{41}$$

From identity (33), we know

$$EV_{k+1} \leq EV_k + \varepsilon E\|x_{k+1}\|^2 \leq EV_k + \varepsilon EV_{k+1}. \tag{42}$$

Taking $0 < \varepsilon < 1$ in (42), it is easy to derive that there exists a positive constant $M_0 \geq 1$ satisfying

$$EV_{k+1} \leq M_0 EV_k,\ \forall k \geq 0. \tag{43}$$

Applying Lemma 2.3 with $s_k = EV_k$, $h_0 = s+1$, $\beta = [M_0^{h_0} \delta_0{}^{-1}]$, $\lambda = \delta_0^{h_0}$, it follows that

$$EV_k \leq \beta\lambda^{(k-k_0)} EV_{k_0} \leq \lambda_{\max}(P_k + \varepsilon I)\beta\lambda^{(k-k_0)} E\|x_{k_0}\|^2,$$

which implies that system (3) is ESMS due to the fact that $\{P_k\}_{k\geq 0}$ is uniformly bounded. ■

The above theorem directly yields the following result.

Corollary 2.2.1. *Suppose that there exists $\epsilon > 0$ such that $H_k^T H_k > \epsilon I$ for $k \in \mathcal{N}_0$. Additionally, if there is a uniformly bounded symmetric matrix sequence $\{P_k \geq 0\}_{k\geq 0}$ solving GLE (25), then system (3) is ESMS.*

3. Exact Detectability and Related Lyapunov-type Theorems

We recall that for the linear time-invariant system

$$\begin{cases} x_{k+1} = Fx_k + Gx_k w_k, \ x_0 \in \mathcal{R}^n \\ y_k = Hx_k, \ k = 0, 1, 2, \cdots \end{cases} \tag{44}$$

its exact observability was defined in [20], [34], while the same definition for linear continuous-time time-invariant Itô systems was given in [36]. For the LDTV stochastic system (3), the complete observability that is different from the uniform observability [8] was defined in [35]. In this section, we will study exact detectability of the stochastic system (3), from which it can be found that there are some essential differences between the time-varying and time-invariant systems. In addition, Lyapunov-type theorems are also presented.

3.1 Exact Detectability

We first give several definitions.

Definition 3.1.1. *For system (3), $x_{k_0} \in l^2_{\mathcal{F}_{k_0-1}}$ is called a k_0^∞-unobservable state if $y_k \equiv 0$ a.s. for $k \in [k_0, \infty)$, and $x_{k_0} \in l^2_{\mathcal{F}_{k_0-1}}$ is called a $k_0^{s_0}$-unobservable state if $y_k \equiv 0$ a.s. for $k \in [k_0, k_0 + s_0]$.*

Remark 3.1.1. From Definition 3.1.1, we point out the following obvious facts: (i) If x_{k_0} is a k_0^∞-unobservable state, then for any $s_0 \geq 0$, it must be a $k_0^{s_0}$-unobservable state; (ii) If x_{k_0} is a $k_0^{s_1}$-unobservable state, then for any $0 \leq s_0 \leq s_1$, it must be a $k_0^{s_0}$-unobservable state.

Example 3.1.1. In system (3), if we take $H_k \equiv 0$ for $k \geq k_0$, then any state $x_{k_0} \in l^2_{\mathcal{F}_{k_0-1}}$ is a k_0^∞-unobservable state. For any $k_0 \geq 0$, $x_{k_0} = 0$ is a trivial k_0^∞-unobservable state.

Different from the linear time-invariant system (44), even if $x_{k_0} = \zeta$ is a k_0^∞-unobservable state, $x_{k_1} = \zeta$ may not be a $k_1^{s_1}$-unobservable state for any $s_1 \geq 0$, which is seen from the next example.

Example 3.1.2. Consider the deterministic linear time-varying system with $G_k = 0$ and

$$H_k = F_k = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & \text{if } k \text{ is even}, \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, & \text{if } k \text{ is odd}. \end{cases}$$

Obviously, $x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a 0^∞-unobservable state due to $y_k = 0$ for $k \geq 0$, but $x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not a 1^{s_1}-unobservable state for any $s_1 \geq 0$ due to $y_1 = H_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq 0$, let alone 1^∞-unobservable state.

Definition 3.1.2. *System (3) is called k_0^∞-exactly detectable if all k_0^∞-unobservable state ξ is exponentially stable, i.e., there are constants $\beta \geq 1$, $0 < \lambda < 1$ such that*

$$E\|x_k\|^2 \leq \beta E\|\xi\|^2 \lambda^{(k-k_0)}, \quad \forall k \geq k_0. \tag{45}$$

Similarly, system (3) is called $k_0^{s_0}$-exactly detectable if (45) holds for all $k_0^{s_0}$-unobservable state ξ.

Definition 3.1.3. *System (3) (or $(F_k, G_k|H_k)$) is said to be $\mathcal{K}^\infty$-exactly detectable if it is k^∞-exactly detectable for any $k \geq 0$. If there exists a nonnegative integer sequence $\{s_k\}_{k\geq 0}$ with the upper limit $\overline{\lim}_{k\to\infty} s_k = +\infty$ such that system (3) is k^{s_k}-exactly detectable, i.e., for any k^{s_k}-unobservable state ξ_k,*

$$E\|x_t\|^2 \leq \beta E\|\xi_k\|^2 \lambda^{(t-k)}, \ \beta \geq 1, \ 0 < \lambda < 1, \ t \geq k,$$

then system (3) (or $(F_k, G_k|H_k)$) is said to be weakly finite time or $\mathcal{K}^{WFT}$-exactly detectable. If $\overline{\lim}_{k\to\infty} s_k < +\infty$, then system (3) (or $(F_k, G_k|H_k)$) is said to be finite time or $\mathcal{K}^{FT}$-exactly detectable.

A special case of $\mathcal{K}^{FT}$-exact detectability is the so-called $\mathcal{K}^N$-exact detectability, which will be used to study GLEs.

Definition 3.1.4. *If there exists an integer $N \geq 0$ such that for any time $k_0 \in [0, \infty)$, system (3) (or $(F_k, G_k|H_k)$) is k_0^N-exactly detectable, then system (3) (or $(F_k, G_k|H_k)$) is said to be $\mathcal{K}^N$-exactly detectable.*

From Definitions 3.1.3–3.1.4, we have the following inclusion relation

$$\mathcal{K}^N\text{-exact detectability} \implies \mathcal{K}^{FT}\text{-exact detectability}$$
$$\implies \mathcal{K}^{WFT}\text{-exact detectability} \implies \mathcal{K}^\infty\text{-exact detectability.}$$

In this paper, we will mainly use $\mathcal{K}^\infty$- and $\mathcal{K}^N$-exact detectability. Obviously, $\mathcal{K}^N$-exact detectability implies $\mathcal{K}^\infty$-exact detectability, but the converse is not true. We present the following examples to illustrate various relations among several definitions on detectability. For illustration simplicity, we only take the concerned examples to be deterministic.

Example 3.1.3. In system (14), we take $F_k = 1$ for $k \geq 0$, and

$$H_k = \begin{cases} 1, \text{ for } k = n^2, \ n = 1, 2, \cdots, \\ 0, \text{ otherwise.} \end{cases}$$

In this case, system (14) (or $(F_k|H_k)$) is $\mathcal{K}^\infty$-exactly detectable, and the zero vector is the unique k^∞-unobservable state. $(F_k|H_k)$ is also $\mathcal{K}^{WFT}$-exactly detectable, where $s_k = k^2 - k \to \infty$. However, $(F_k|H_k)$ is not $\mathcal{K}^{FT}$-exactly detectable, and, accordingly, is not $\mathcal{K}^N$-exactly detectable for any $N \geq 0$.

Example 3.1.4. In system (14), if we take $F_k = 1$ and $H_k = \frac{1}{k}$ for $k \geq 0$, then $(F_k|H_k)$ is $\mathcal{K}^N$-exactly detectable for any $N \geq 0$, but $(F_k|H_k)$ is not uniformly detectable. This is because for any $t \geq 0$, $0 \leq d < 1$ and $\xi \in \mathcal{R}$, we always have $|\phi_{k+t,k}\xi|^2 = |\xi|^2 \geq d^2|\xi|^2$. But there do not exist $b > 0$ and $s \geq 0$ satisfying (13), because $\xi^T \mathcal{O}_{k+s,k}\xi = |\xi|^2 \sum_{i=k}^{k+s} \frac{1}{i^2}$ while $\lim_{k\to\infty} \sum_{i=k}^{k+s} \frac{1}{i^2} = 0$.

Example 3.1.5. In system (14), if we take $F_k = 1$ for $k \geq 0$, and $H_{2n} = 1$ and $H_{2n+1} = 0$ for $n = 0, 1, 2, \cdots$, then $(F_k|H_k)$ is uniformly detectable and $\mathcal{K}^1$-exactly detectable, but it is not $\mathcal{K}^0$-exactly detectable.

The following lemma is obvious.

Lemma 3.1.1. *At any time k_0, $x_{k_0} = 0$ is not only a k_0^∞- but also a $k_0^{s_0}$-unobservable state for any $s_0 \geq 0$.*

By Lemma 3.1.1, if we let $\Theta_{k_0}^\infty$ denote the set of all the k_0^∞-unobservable states of system (3) at time k_0, then $\Theta_{k_0}^\infty$ is not empty. Furthermore, it is easy to show that $\Theta_{k_0}^\infty$ is a linear vector space.

Lemma 3.1.2. *For $k_0 \in \mathcal{N}_0$, if there does not exist a nonzero $\zeta \in \mathcal{R}^n$ such that $H_{k_0}\zeta = 0$, $(I_{2^{l-k_0}} \otimes H_l)\phi_{l,k_0}\zeta = 0$, $l = k_0+1, k_0+2, \cdots$, then $y_k \equiv 0$ a.s. with $k \geq k_0$ implies $x_{k_0} = 0$ a.s..*

Proof: From $y_{k_0} \equiv 0$ a.s., it follows that

$$E[x_{k_0}^T H_{k_0}^T H_{k_0} x_{k_0}] = 0. \tag{46}$$

From $y_l \equiv 0$ a.s., $l = k_0 + 1, \cdots$, it follows from Lemma 2.2 that

$$E\big[x_{k_0}^T \phi_{l,k_0}^T (I_{2^{l-k_0}} \otimes H_l^T)(I_{2^{l-k_0}} \otimes H_l)\phi_{l,k_0} x_{k_0}\big] = 0. \tag{47}$$

Let $R_{k_0} = E[x_{k_0} x_{k_0}^T]$, rank$R_{k_0} = r$. When $r = 0$, this implies $x_{k_0} = 0$ a.s., and this lemma is shown. For $1 \leq r \leq n$, by the result of [24], there are real nonzero vectors $z_1, z_2, \cdots, z_r$ such that $R_{k_0} = \sum_{i=1}^r z_i z_i^T$.

By (46), we have

$$\begin{aligned} E[x_{k_0}^T H_{k_0}^T H_{k_0} x_{k_0}] &= \text{trace} E[H_{k_0}^T H_{k_0} x_{k_0} x_{k_0}^T] \\ &= \text{trace}\{H_{k_0}^T H_{k_0} E[x_{k_0} x_{k_0}^T]\} \\ &= \text{trace}\{H_{k_0}^T H_{k_0} \sum_{i=1}^{r} z_i z_i^T\} \\ &= \sum_{i=1}^{r} [z_i^T H_{k_0}^T H_{k_0} z_i] = 0, \end{aligned} \tag{48}$$

which gives $H_{k_0} z_i = 0$ for $i = 1, 2, \cdots, r$. Similarly, (47) yields

$$(I_{2^{l-k_0}} \otimes H_l)\phi_{l,k_0} z_i = 0,\ \ i = 1, 2, \cdots, r.$$

According to the given assumptions, we must have $z_i = 0$, $i = 1, 2, \cdots, r$, which again implies $x_{k_0} = 0$ a.s.. ∎

By Lemma 3.1.2, it is known that under the conditions of Lemma 3.1.2, $x_{k_0} = 0$ is the unique k_0^∞-unobservable state, i.e., $\Theta_{k_0}^\infty = \{0\}$.

Lemma 3.1.3. *Uniform detectability implies $\mathcal{K}^\infty$-exact detectability.*

Proof: For any ${k_0}^\infty$-unobservable state $x_{k_0} = \xi$, by Definition 2.2 and Definition 3.1.3, we must have $E\|\phi_{k+t,k} x_k\|^2 < d^2 E\|x_k\|^2$ or $x_k \equiv 0$ for $k \geq k_0$; otherwise, it will lead to a contradiction since

$$0 = \sum_{i=k}^{k+s} E\|y_i\|^2 \geq b\|x_k\|^2 > 0.$$

Under any case, the following system

$$\begin{cases} x_{k+1} = F_k x_k + G_k x_k w_k, \\ x_{k_0} = \xi \in \Theta_{k_0}^\infty, \\ y_k = H_k x_k,\ \ k = 0, 1, 2, \cdots \end{cases} \tag{49}$$

is ESMS, so $(F_k, G_k|H_k)$ is exactly detectable. ∎

Remark 3.1.2. When system (3) reduces to the deterministic time-invariant system (2), the uniform detectability, $\mathcal{K}^{n-1}$-exact detectability and $\mathcal{K}^\infty$-exact detectability coincide with the classical detectability of linear systems [16]. Examples 3.1.4–3.1.5 show that there is no inclusion relation between uniform detectability and $\mathcal{K}^N$-exact detectability for some $N > 0$. We conjecture that if $(F_k, G_k|H_k)$ is uniformly detectable, then there is a sufficiently large $N > 0$ such that $(F_k, G_k|H_k)$ is $\mathcal{K}^N$-exactly detectable.

Corresponding to Theorem 2.1, we also have the following theorem for exact detectability, but its proof is very simple.

Theorem 3.1.1. *If $(F_k, G_k|H_k)$ is $\mathcal{K}^\infty$-exactly detectable, then so is $(F_k + M_k K_k H_k, G_k + N_k K_k H_k|H_k)$ for any output feedback $u_k = K_k y_k$.*

Proof: We prove this theorem by contradiction. Assume that $(F_k + M_k K_k H_k, G_k + N_k K_k H_k|H_k)$ is not $\mathcal{K}^\infty$-exactly detectable. By Definition 3.1.3, for system (20), although the measurement equation becomes $y_k = H_k x_k \equiv 0$ for $k \in \mathcal{N}_0$, the state equation

$$x_{k+1} = (F_k + M_k K_k H_k)x_k + (G_k + N_k K_k H_k)x_k w_k \tag{50}$$

is not ESMS. In view of $y_k = H_k x_k \equiv 0$, (50) is equivalent to

$$x_{k+1} = F_k x_k + G_k x_k w_k. \tag{51}$$

Hence, under the condition of $y_k = H_k x_k \equiv 0$ for $k = 0, 1, 2, \cdots$, if (50) is not ESMS, then so is (51), which contradicts the $\mathcal{K}^\infty$-exact detectability of $(F_k, G_k|H_k)$. ■

It should be pointed out that Theorem 3.1.1 does not hold for $\mathcal{K}^N$-exact detectability. That is, even if $(F_k, G_k|H_k)$ is $\mathcal{K}^N$-exactly detectable for $N \geq 0$, $(F_k + M_k K_k H_k, G_k + N_k K_k H_k|H_k)$ may not be so, and such a counterexample can be easily constructed.

Proposition 3.1.1. *If there exists a matrix sequence $\{K_k, k = 0, 1, \cdots,\}$ such that*

$$x_{k+1} = (F_k + K_k H_k)x_k + G_k x_k w_k \tag{52}$$

is ESMS, then $(F_k, G_k|H_k)$ is $\mathcal{K}^\infty$-exactly detectable.

Proof: Because (52) is ESMS, by Proposition 2.1 and Lemma 3.1.3, $(F_k + K_k H_k, G_k|H_k)$ is $\mathcal{K}^\infty$-exactly detectable. By Theorem 3.1.1, for any matrix sequence $\{L_k, k = 0, 1, \cdots,\}$, $(F_k + K_k H_k + L_k H_k, G_k|H_k)$ is also $\mathcal{K}^\infty$-exactly detectable. Taking $L_k = -K_k$, we obtain that $(F_k, G_k|H_k)$ is $\mathcal{K}^\infty$-exactly detectable. Thus, this proposition is shown. ■

Remark 3.1.3. In some previous references such as [8], [29], if system (52) is ESMS for some matrix sequence $\{K_k\}_{k\in\mathcal{N}_0}$, then $(F_k, G_k|H_k)$ is called stochastically detectable or detectable in conditional mean [29]. Proposition 3.1.1 tells us that stochastic detectability implies $\mathcal{K}^\infty$-exact detectability, but the converse is not true. Such a counterexample can be easily constructed; see the following Example 3.1.6. The $\mathcal{K}^\infty$-exact detectability implies that any k_0^∞-unobservable initial state ξ leads to an exponentially stable trajectory for any $k_0 \geq 0$. However, in the time-invariant system (44), the stochastic detectability of (44) (or $(F, G|H)$ for short) is equivalent to that there is a constant output feedback gain matrix K, rather than necessarily a time-varying feedback gain matrix sequence $\{K_k\}_{k\in\mathcal{N}_0}$, such that

$$x_{k+1} = (F + KH)x_k + Gx_k w_k \tag{53}$$

is ESMS; see [8].

Example 3.1.6. Let $G_k = 3$ for $k \geq 0$, and

$$F_k = H_k = \begin{cases} 1, & \text{for } k = 3n, \ n = 1, 2, \cdots, \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 2.1, for any output feedback $u_k = K_k y_k$, we have $Ex_k^2 = 3^{(k-k_0)} Ex_{k_0}^2$ for $k > k_0$, where x_k is the closed-loop trajectory of

$$x_{k+1} = (F_k + K_k H_k)x_k + 3x_k w_k,$$

which is not ESMS. So $(F_k, G_k|H_k)$ is not stochastically detectable. However, $(F_k, G_k|H_k)$ is not only $\mathcal{K}^\infty$- but also $\mathcal{K}^3$-exactly detectable, and 0 is the unique k^3-unobservable state.

Remark 3.1.4. According to the linear system theory, for the deterministic linear time-invariant system (2), the $\mathcal{K}^\infty$- and $\mathcal{K}^{n-1}$-exact detectability are equivalent. By the $\mathcal{H}$-representation theory [35], for (44), the $\mathcal{K}^\infty$- and $\mathcal{K}^{[\frac{n(n+1)}{2}-1]}$-exact detectability are also equivalent. So, in what follows, system (44) (or $(F, G|H)$) is simply called exactly detectable.

Remark 3.1.5. In Example 3.1.3, $(F_k|H_k)$ is stochastically detectable, but it is not $\mathcal{K}^N$-exactly detectable for any $N \geq 0$. In Example 3.1.6, $(F_k|H_k)$ is not stochastically detectable, but it is $\mathcal{K}^N$-exactly detectable for $N \geq 3$. Hence, it seems that there is no inclusion relation between stochastic detectability and $\mathcal{K}^N$-exact detectability.

3.2 Lyapunov-Type Theorems under Exact Detectability

At present, we do not know whether Theorem 2.2.2 holds under exact detectability, but we are able to prove a similar result to Theorem 2.2.2 for a periodic system, namely, in (3), $F_{k+\tau} = F_k$, $G_{k+\tau} = G_k$, $H_{k+\tau} = H_k$. Periodic systems are a class of very important time-varying systems, which have been studied by many researchers; see [3], [8], [10].

Theorem 3.2.1 (Lyapunov-Type Theorem)**.** *Assume that system (3) is a periodic system with the period $\tau > 0$. If system (3) is $\mathcal{K}^N$-exactly detectable for any fixed $N \geq 0$ and $\{P_k > 0\}_{k \geq 0}$ is a positive definite matrix sequence which solves GLE (25), then the periodic system (3) is ESMS.*

Proof: By periodicity, $P_k = P_{k+\tau}$. Select an integer $\bar{\kappa} > 0$ satisfying $\bar{\kappa}\tau - 1 \geq N$. For $\kappa \geq \bar{\kappa}$, we introduce the following backward difference equation

$$\begin{cases} -P_0^{\kappa\tau-1}(k) + F_k^T P_0^{\kappa\tau-1}(k+1)F_k + G_k^T P_0^{\kappa\tau-1}(k+1)G_k + H_k^T H_k = 0, \\ P_0^{\kappa\tau-1}(\kappa\tau) = 0, \ k = 0, 1, \cdots, \kappa\tau - 1. \end{cases} \tag{54}$$

Set $V_k = x_k^T P_k x_k$, then associated with (54), we have

$$
\begin{aligned}
EV_0 - EV_{\kappa\tau} &= x_0^T P_0 x_0 - E[x_{\kappa\tau}^T P_{\kappa\tau} x_{\kappa\tau}] = x_0^T P_0 x_0 - E[x_{\kappa\tau}^T P_0 x_{\kappa\tau}] \\
&= \sum_{i=0}^{\kappa\tau-1} E\|y_i\|^2 = x_0^T P_0^{\kappa\tau-1}(0) x_0, \qquad (55)
\end{aligned}
$$

where the last equality is derived by using the completing squares technique. We assert that $P_0^{\kappa\tau-1}(0) > 0$. Otherwise, there exists a nonzero x_0 satisfying $x_0^T P_0^{\kappa\tau-1}(0) x_0 = 0$ due to $P_0^{\kappa\tau-1}(0) \geq 0$. As so, by $\mathcal{K}^N$-exact detectability, (55) leads to

$$
\begin{aligned}
0 = \sum_{i=0}^{\kappa\tau-1} E\|y_i\|^2 &\geq \lambda_{\min}(P_0)\|x_0\|^2 - \lambda_{\max}(P_0)\beta\lambda^{\kappa\tau}\|x_0\|^2 \\
&= (\lambda_{\min}(P_0) - \lambda_{\max}(P_0)\beta\lambda^{\kappa\tau})\|x_0\|^2, \qquad (56)
\end{aligned}
$$

where $\beta > 1$ and $0 < \lambda < 1$ are defined in (15). If κ is taken sufficiently large such that $\kappa \geq \kappa_0 > 0$ with $\kappa_0 > 0$ being a minimal integer satisfying $\lambda_{\min}(P_0) - \lambda_{\max}(P_0)\beta\lambda^{\kappa_0\tau} > 0$, then (56) yields $x_0 = 0$, which contradicts $x_0 \neq 0$.

If we let $P_{(n-1)\kappa\tau}^{n\kappa\tau-1}((n-1)\kappa\tau + k)$ denote the solution of

$$
\begin{cases}
-P_{(n-1)\kappa\tau}^{n\kappa\tau-1}((n-1)\kappa\tau + k) + F_{(n-1)\kappa\tau+k}^T P_{(n-1)\kappa\tau}^{n\kappa\tau-1}((n-1)\kappa\tau + k + 1) F_{(n-1)\kappa\tau+k} \\
\qquad + G_{(n-1)\kappa\tau+k}^T P_{(n-1)\kappa\tau}^{n\kappa\tau-1}((n-1)\kappa\tau + k + 1) G_{(n-1)\kappa\tau+k} + H_{(n-1)\kappa\tau+k}^T H_{(n-1)\kappa\tau+k} = 0, \\
P_{(n-1)\kappa\tau}^{n\kappa\tau-1}(n\kappa\tau) = 0,\ k = 0, 1, \cdots, \kappa\tau - 1;\ n = 1, 2, \cdots,
\end{cases}
$$

then by periodicity, $P_0^{\kappa\tau-1}(0) = P_{(n-1)\kappa\tau}^{n\kappa\tau-1}((n-1)\kappa\tau) > 0$, and

$$
\begin{aligned}
EV_{(n-1)\kappa\tau} - EV_{n\kappa\tau} = \sum_{i=(n-1)\kappa\tau}^{n\kappa\tau-1} E\|y_i\|^2 &= E[x_{(n-1)\kappa\tau}^T P_{(n-1)\kappa\tau}^{n\kappa\tau-1}((n-1)\kappa\tau) x_{(n-1)\kappa\tau}] \\
&= E[x_{(n-1)\kappa\tau}^T P_0^{\kappa\tau-1}(0) x_{(n-1)\kappa\tau}] \geq \varrho_0 \|x_{(n-1)\kappa\tau}\|^2,
\end{aligned}
$$

where $\varrho_0 = \lambda_{\min}(P_0^{\kappa\tau-1}) > 0$. Generally, for $0 \leq s \leq \kappa\tau - 1$, we define $P_{(n-1)\kappa\tau+s}^{n\kappa\tau+s-1}((n-1)\kappa\tau + s + k)$ as the solution to

$$
\begin{cases}
-P_{(n-1)\kappa\tau+s}^{n\kappa\tau+s-1}((n-1)\kappa\tau + s + k) + F_{(n-1)\kappa\tau+s+k}^T P_{(n-1)\kappa\tau+s}^{n\kappa\tau+s-1}((n-1)\kappa\tau + s + k + 1) F_{(n-1)\kappa\tau+s+k} \\
\quad + G_{(n-1)\kappa\tau+s+k}^T P_{(n-1)\kappa\tau+s}^{n\kappa\tau+s-1}((n-1)\kappa\tau + s + k + 1) G_{(n-1)\kappa\tau+s+k} + H_{(n-1)\kappa\tau+s+k}^T H_{(n-1)\kappa\tau+s+k} = 0, \\
P_{(n-1)\kappa\tau+s}^{n\kappa\tau+s-1}(n\kappa\tau + s) = 0,\ k = 0, 1, \cdots, \kappa\tau - 1;\ n = 1, 2, \cdots.
\end{cases}
$$

It can be shown that $P_{(n-1)\kappa\tau+s}^{n\kappa\tau+s-1}((n-1)\kappa\tau + s) = P_s^{\kappa\tau+s-1}(s) > 0$ and

$$
\sum_{i=(n-1)\kappa\tau+s}^{n\kappa\tau+s-1} E\|y_i\|^2 = E[x_{(n-1)\kappa\tau+s}^T P_s^{\kappa\tau+s-1}(s) x_{(n-1)\kappa\tau+s}],
$$

provided that we take $\kappa \geq \max_{0\leq s\leq \kappa\tau-1}\kappa_s$, where $\kappa_s > 0$ is the minimal integer satisfying $\lambda_{\min}(P_s) - \lambda_{\max}(P_s)\beta\lambda^{\kappa_s\tau} > 0$.

Summarizing the above discussions, for any $k \geq 0$ and $\hat{\kappa} > \max\{\bar{\kappa}, \max_{0\leq s\leq \kappa\tau-1}\kappa_s\}$, we have

$$EV_k - EV_{k+\hat{\kappa}\tau} = \sum_{i=k}^{k+\hat{\kappa}\tau-1} E\|y_i\|^2 \geq \rho E\|x_k\|^2,$$

where $\rho = \min_{0\leq s\leq \hat{\kappa}\tau-1}\rho_s > 0$ with $\rho_s = \lambda_{\min}[P_s^{\hat{\kappa}\tau+s-1}(s)]$. The rest is similar to the proof of Theorem 2.2.2 and thus is omitted. ■

In Theorem 3.2.1, if $\{P_k > 0\}_{k\geq 0}$ is weaken as $\{P_k \geq 0\}_{k\geq 0}$, then we have

Theorem 3.2.2 (Lyapunov-Type Theorem). *Assume that system (3) is a periodic system with the period $\tau > 0$. If (i) system (3) is $\mathcal{K}^N$-exactly detectable for any fixed $N \geq 0$; (ii) $\{P_k \geq 0\}_{k\geq 0}$ is a positive semi-definite matrix sequence which solves GLE (25); (iii) Ker(P_0) = Ker$(P_1) = \cdots =$ Ker$(P_{\tau-1})$, then the periodic system (3) is ESMS.*

Proof: From GLE (25), it is easy to show (e.g., see Theorem 3.2 in [34]) that Ker$(P_k) \subset$ Ker(H_k), F_kKer$(P_k) \subset$ Ker(P_{k+1}), G_kKer$(P_k) \subset$ Ker(P_{k+1}). In addition, in view of Ker$(P_0) = \cdots =$ Ker$(P_{\tau-1})$ and $P_{\tau+k} = P_k$, there is a common orthogonal matrix S such that for any $k \geq 0$, there hold

$$S^TP_kS = \begin{bmatrix} 0 & 0 \\ 0 & P_k^{22} \end{bmatrix},\ P_k^{22} \geq 0,\ S^TH_k^TH_kS = \begin{bmatrix} 0 & 0 \\ 0 & (H_k^{22})^TH_k^{22} \end{bmatrix},$$

$$S^TF_kS = \begin{bmatrix} F_k^{11} & F_k^{12} \\ 0 & F_k^{22} \end{bmatrix},\quad S^TG_kS = \begin{bmatrix} G_k^{11} & G_k^{12} \\ 0 & G_k^{22} \end{bmatrix}.$$

Pre- and post-multiplying S^T and S on both sides of GLE (25) gives rise to

$$-S^TP_kS + S^TF_k^TS\cdot S^TP_{k+1}S\cdot S^TF_kS + S^TG_k^TS\cdot S^TP_{k+1}S\cdot S^TG_kS + S^TH_k^TS\cdot S^TH_kS = 0,$$

which is equivalent to

$$-P_k^{22} + (F_k^{22})^TP_{k+1}^{22}F_k^{22} + (G_k^{22})^TP_{k+1}^{22}G_k^{22} + (H_k^{22})^TH_k^{22} = 0. \tag{57}$$

Set $\eta_k = \begin{bmatrix} \eta_{1,k} \\ \eta_{2,k} \end{bmatrix} = S^Tx_k = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}^T x_k$, then it follows that

$$\begin{cases} \eta_{1,k+1} = F_k^{11}\eta_{1,k} + G_k^{11}\eta_{1,k}w_k + F_k^{12}\eta_{2,k} + G_k^{12}\eta_{2,k}w_k, \\ \eta_{2,k+1} = F_k^{22}\eta_{2,k} + G_k^{22}\eta_{2,k}w_k, \\ \quad y_k = H_kS\eta_k. \end{cases} \tag{58}$$

It can be easily seen that $y_k = H_k S\eta_k \equiv 0, a.s.$ iff $H_k^{22}\eta_{2,k} \equiv 0, a.s.$, for which a sufficient condition is $\eta_{2,k} = 0$. By $\mathcal{K}^N$-exact detectability, $\eta_{1,k+1} = F_k^{11}\eta_{1,k} + G_k^{11}\eta_{1,k}w_k$ is ESMS. To show that $\eta_{2,k+1} = F_k^{22}\eta_{2,k} + G_k^{22}\eta_{2,k}w_k$ is ESMS, we consider the following reduced-order state-measurement equation

$$\begin{cases} \eta_{2,k+1} = F_k^{22}\eta_{2,k} + G_k^{22}\eta_{2,k}w_k, \\ \bar{y}_k = H_k^{22}\eta_{2,k}. \end{cases} \tag{59}$$

Obviously, system (59) is still a periodic system and has the same period $\tau > 0$ as (3).

In the following, we show that (59) is also $\mathcal{K}^N$-exactly detectable. Because system (3) is $\mathcal{K}^N$-exactly detectable, for any $k \geq 0$, $y_i \equiv 0$ a.s. for $i = k, \cdots, k+N$, implies that there are constants $\beta_0 > 1$ and $0 < \lambda_0 < 1$ such that

$$E\|x_t\|^2 = E\|\phi_{k+t,k}x_k\|^2 \leq \beta_0 E\|x_k\|^2 \lambda_0^{(t-k)}, \ t \geq k \tag{60}$$

for any k^N-unobservable state x_k. Take $x_k = S\eta_k = S\begin{bmatrix} 0 \\ \eta_{2,k} \end{bmatrix}$, with $\eta_{2,k}$ being a k^N-unobservable state of (59), i.e., $\bar{y}_i = H_k^{22}\eta_{2,i} = 0$ for $i = k, \cdots, k+N$. Then $y_i = H_i S\begin{bmatrix} 0 \\ \eta_{2,k} \end{bmatrix} = 0$ for $i = k, \cdots, k+N$. Hence, (60) holds. Substituting $x_k = S\begin{bmatrix} 0 \\ \eta_{2,k} \end{bmatrix}$ into (60) yields

$$E\|\eta_{2,t}\|^2 \leq \beta_0 E\|\eta_{2,k}\|^2 \lambda_0^{(t-k)}, \ t \geq k. \tag{61}$$

So (59) is $\mathcal{K}^N$-exactly detectable.

Associated with (59), the GLE (57) admits a positive definite solution sequence $\{P_k > 0\}_{k \geq 0}$. Applying Theorem 3.2.1, the subsystem (59) is ESMS. Since $\eta_{1,k+1} = F_k^{11}\eta_{1,k} + G_k^{11}\eta_{1,k}w_k$ has been shown to be ESMS, there are constants $\beta_1 > 1$ and $0 < \lambda_1 < 1$ such that

$$E\|\eta_{1,t}\|^2 \leq \beta_1 E\|\eta_{1,k}\|^2 \lambda_1^{(t-k)}, \ t \geq k. \tag{62}$$

Set $\beta := \max\{\beta_0, \beta_1\}$, $\lambda = \max\{\lambda_0, \lambda_1\}$, then the composite system (58) is ESMS with

$$E\|\eta_t\|^2 = E\|\eta_{1,t}\|^2 + E\|\eta_{2,t}\|^2 \leq \beta E\|\eta_k\|^2 \lambda^{(t-k)}, \ t \geq k,$$

which deduces that the periodic system (3) is ESMS because (3) and (58) are equivalent. ■

Finally, we consider the linear time-invariant stochastic system (44) and present a Lyapunov-type theorem as a complementary result of Theorem 19 [20]. Associated with (44), we introduce the linear symmetric operator $\mathcal{L}_{F,G}$, called the generalized Lyapunov operator (GLO), as follows:

$$\mathcal{L}_{F,G}Z = FZF^T + GZG^T, \quad Z \in \mathcal{S}_n.$$

Moreover, for system (44), the GLE (25) becomes

$$-P+F^TPF+G^TPG+H^TH=0. \tag{63}$$

Theorem 3.2.3. *Suppose that $\sigma(\mathcal{L}_{F,G})\subset\bar{\odot}:=\{\lambda:|\lambda|\leq 1\}$ and $(F,G|H)$ is exactly detectable. If P is a real symmetric solution of (63), then $P\geq 0$ and (F,G) is stable, i.e., the state trajectory of (44) is asymptotically mean square stable.*

In order to prove Theorem 3.2.3, we need to cite the well-known Krein-Rutman Theorem as follows:

Lemma 3.2.1 (see [28]). *Let $\beta:=\max_{\lambda_i\in\sigma(\mathcal{L}_{F,G})}|\lambda_i|$ be the spectral radius of $\mathcal{L}_{F,G}$. Then there exists a nonzero $X\geq 0$ such that $\mathcal{L}_{F,G}X=\beta X$.*

Proof of Theorem 3.2.3: Because $\sigma(\mathcal{L}_{F,G})\subset\bar{\odot}$, the spectral radius $\beta\leq 1$. If $\beta<1$, then this means that (F,G) is stable by [20, Lemma 3], which yields $P\geq 0$ according to [20, Lemma 17]. If $\beta=1$, then by Lemma 3.2.1, there exists a nonzero $X\geq 0$, such that $\mathcal{L}_{F,G}X=X$. Therefore, we have

$$\begin{aligned}0\geq\langle -H^TH,X\rangle&=\langle -P+\mathcal{L}^*_{F,G}(P),X\rangle=\langle -P,X\rangle+\langle P,\mathcal{L}_{F,G}(X)\rangle\\&=\langle -P,X\rangle+\langle P,X\rangle=\langle 0,X\rangle=0,\end{aligned} \tag{64}$$

where $\langle A,B\rangle:=\mathrm{trace}(A^TB)$, $\mathcal{L}^*_{F,G}$ is the adjoint operator of $\mathcal{L}_{F,G}$, and $\mathcal{L}^*_{F,G}(P)=F^TPF+G^TPG$. From (64) it follows that $\mathrm{trace}(H^THX)=0$, which implies $HX=0$ due to $X\geq 0$. However, according to [20, Theorem 8-(4)], $\mathcal{L}_{F,G}X=X$ together with $HX=0$, contradicts the exact detectability of $(F,G|H)$. Hence, we must have $0\leq\beta<1$, and this theorem is verified. ■

Remark 3.2.1. Following the line of Theorem 3.2.3, Conjecture 3.1 in [33] can also be verified.

4. Exact Observability

This section introduce another definition called "exact observability" for system (3), which is stronger than exact detectability and also coincides with the classical observability when system (3) reduces to the deterministic linear time-invariant system (2).

We first give the following definitions:

Definition 4.1. *System (3) is called k_0^∞-exactly observable if $x_{k_0}=0$ is the unique k_0^∞-unobservable state. Similarly, system (3) is called $k_0^{s_0}$-exactly observable if $x_{k_0}=0$ is the unique $k_0^{s_0}$-unobservable state.*

Definition 4.2. *System (3) (or $(F_k,G_k|H_k)$) is said to be $\mathcal{K}^\infty$-exactly observable if for any $k\in[0,\infty)$, system (3) is k^∞-exactly observable. If for any time $k\in[0,\infty)$, there exists a nonnegative integer $N\geq 0$*

such that system (3) is k^N-exactly observable, then system (3) (or $(F_k, G_k|H_k)$) is said to be $\mathcal{K}^N$-exactly observable. Similarly, the $\mathcal{K}^{WFT}$- and $\mathcal{K}^{FT}$-exact observability can be defined.

Combining Lemmas 3.1.1–3.1.2 together, a sufficient condition for the exact observability is presented as follows.

Theorem 4.1. *If rank$H_{\infty,k} = n$ for any $k \geq 0$, then $(F_k, G_k|H_k)$ is $\mathcal{K}^\infty$-exactly observable. In particular, if rank$H_{k+s_0,k} = n$ for some fixed integer $s_0 \geq 0$ and any $k \geq 0$, then system (3) is not only $\mathcal{K}^\infty$- but also $\mathcal{K}^{s_0}$-exactly observable. Here $H_{l,k}$ is defined in Lemma 2.2, and*

$$H_{\infty,k} = \begin{bmatrix} H_k \\ (I_2 \otimes H_{k+1})\phi_{k+1,k} \\ (I_{2^2} \otimes H_{k+2})\phi_{k+2,k} \\ \vdots \\ (I_{2^{l-k}} \otimes H_l)\phi_{l,k} \\ \vdots \end{bmatrix}.$$

The next corollary follows immediately from Theorem 4.1.

Corollary 4.1. *If H_k is nonsingular for $k \geq 0$, then system (3) is $\mathcal{K}^0$-exactly observable.*

By Definitions 4.1–4.2, k_0^∞ (resp. $k_0^{s_0}$)-exact observability is stronger than k_0^∞ (resp. $k_0^{s_0}$)-exact detectability. Likewise, $\mathcal{K}^\infty$ (resp. $\mathcal{K}^{WFT}$, $\mathcal{K}^{FT}$, $\mathcal{K}^N$)-exact observability is stronger than $\mathcal{K}^\infty$ (resp. $\mathcal{K}^{WFT}$, $\mathcal{K}^{FT}$, $\mathcal{K}^N$)-exact detectability. A necessary and sufficient condition for the $\mathcal{K}^N$-exact observability was presented in [35] based on the $\mathcal{H}$-representation theory developed therein. Below, we give another equivalent theorem based on Lemma 2.2.

Theorem 4.2. *(i) System (3) is $\mathcal{K}^N$-exactly observable iff for any $k \in \mathcal{N}_0$, the Gramian $\mathcal{O}_{k+N,k}$ is a positive definite matrix. (ii) If system (3) is $\mathcal{K}^{WFT}$-exactly observable and $\{P_k \geq 0\}_{k\geq 0}$ solves the GLE (25), then $P_k > 0$ for any $k \geq 0$.*

Proof: We note that $y_i \equiv 0$ a.s. for $i = k, k+1, \cdots, k+N$, is equivalent to $\sum_{i=k}^{k+N} E\|y_i\|^2 = 0$. By Lemma 2.2, $\sum_{i=k}^{k+N} E\|y_i\|^2 = E[x_k^T \mathcal{O}_{k+N,k} x_k] = 0$. So system (3) is exactly observable iff $\sum_{i=k}^{k+N} E\|y_i\|^2 = E[x_k^T \mathcal{O}_{k+N,k} x_k] = 0$ implies $x_k = 0$ a.s., which is equivalent to $\mathcal{O}_{k+N,k} > 0$ due to $\mathcal{O}_{k+N,k} \geq 0$. Hence, (i) is proved.

Now we prove (ii) by contradiction. If some P_{k_0} is not strictly positive definite, then there exists a nonzero $x_{k_0} \in l^2_{\mathcal{F}_{k_0-1}}$ such that $E[x_{k_0}^T P_{k_0} x_{k_0}] = 0$. By the $\mathcal{K}^{WFT}$-exact observability of $(F_k, G_k|H_k)$, there is $s_0 \geq 0$ such that system (3) is $k_0^{s_0}$-exactly observable. Since the following identity

$$E[x_k^T P_k x_k] - E[x_{s+1}^T P_{s+1} x_{s+1}] \;=\; \sum_{i=k}^{s} E\|y_i\|^2 \tag{65}$$

holds for any $s \geq k \geq 0$, it follows that

$$0 \leq \sum_{i=k_0}^{k_0+s_0} E\|y_i\|^2 = -E[x_{k_0+s_0+1}^T P_{k_0+s_0+1} x_{k_0+s_0+1}] \leq 0$$

and accordingly $y_i \equiv 0$ a.s. for $i \in [k_0, k_0+s_0]$. By the $k_0^{s_0}$-exact observability, $x_{k_0} = 0$, which contradicts $x_{k_0} \neq 0$. Hence, (ii) is proved. ■

Remark 4.1. Theorem 4.2-(i) shows that the $\mathcal{K}^N$-exact observability is weaker than the uniform observability given in [8], where it was proved that system (3) is uniformly observable iff there are $N \geq 0$ and $\gamma > 0$ such that $\mathcal{O}_{k+N,k} \geq \gamma I$ for any $k \in \mathcal{N}_0$.

Remark 4.2. There is no inclusion relation between uniform detectability and exact observability. For example, in Example 3.1.4, $(F_k|H_k)$ is $\mathcal{K}^N$-exactly observable, but it is not uniformly detectable. On the other hand, in Example 3.1.5, $(F_k|H_k)$ is uniformly detectable, but it is not $\mathcal{K}^0$-exactly observable.

Similar to exact detectability, we also have the following inclusion relation for exact observability:

$$\begin{aligned} &\mathcal{K}^N\text{-exact observability} \implies \mathcal{K}^{FT}\text{-exact observability} \\ \implies &\mathcal{K}^{WFT}\text{-exact observability} \implies \mathcal{K}^{\infty}\text{-exact observability}. \end{aligned}$$

The following Lyapunov-type theorem can be viewed as a corollary of Theorem 3.2.1.

Theorem 4.3 (Lyapunov-Type Theorem)**.** *Assume that system (3) is a periodic system with the period $\tau > 0$. If system (3) is $\mathcal{K}^N$-exactly observable for any $N \geq 0$ and $\{P_k \geq 0\}_{k \geq 0}$ solves GLE (25), then the periodic system (3) is ESMS.*

Proof: By Theorem 4.2-(ii), $P_k > 0$ for $k \geq 0$. Because $\mathcal{K}^N$-exact observability must be $\mathcal{K}^N$-exact detectability, this theorem is an immediate corollary of Theorem 3.2.1. ■

5. Additional Comments

At the end of this paper, we give the following comments:

(i) In [22], [27], [37], exact observability and detectability of linear stochastic time-invariant systems with Markov jump were studied. How to extend various definitions of this paper to linear time-varying Markov jump systems is an interesting research topic that merits further study.

(ii) Following the line of [35] that transforms the system (3) into a deterministic time-varying system, it is easy to give some testing criteria for uniform detectability and observability of (3) by means of the existing results on deterministic time-varying systems [23]. In addition, applying the infinite-dimensional operator theory, the spectral criterion for stability of system (3) is also a valuable research issue.

(iii) In view of Remarks 3.1.3–3.1.4, we know that, for linear time-invariant system (44), stochastic detectability implies exact detectability. In [37], it was shown that exact detectability is equivalent to the so-called “W-detectability” (see [37, Definition 3]). A new definition called “weak detectability” was introduced in [30], where a counter-example (see Example 15 in [30]) shows that W-detectability does not imply weak detectability. In particular, it was proved in [30] that weak detectability can be derived from stochastic detectability. It is easy to prove that weak detectability implies exact detectability. In summary, we have the following inclusion relation:

$$\text{stochastic detectability} \Rightarrow \text{weak detectability} \Rightarrow \text{exact detectability} \Leftrightarrow \text{W-detectability}.$$

As stated in [30], the converse implication that whether W-detectability or exact detectability implies weak detectability is an open question.

(iv) Lemmas 2.1–2.2 are important, which will have potential applications in mean stability analysis and system synthesis.

(v) This paper reveals some essential differences between linear time-varying and time-invariant systems. For example, for linear time-invariant system (44), exact detectability and exact observability can be uniquely defined, but they exhibit diversity for LDTV system (3). Moreover, many equivalent relations in linear time-invariant system (44) such as

$$\text{uniform detectability} \Leftrightarrow \text{exact detectability}, \qquad \text{uniform observability} \Leftrightarrow \text{exact observability}$$

do not hold for LDTV system (3).

6. Conclusion

This paper has introduced the new concepts on detectability and observability for LDTV stochastic systems with multiplicative noise. Uniform detectability defined in this paper can be viewed as an extended version of that in [1]. Various definitions on exact detectability and observability are extensions of those in [6], [20], [33], [34], [36] to LDTV stochastic systems. Different from time-invariant systems, defining exact detectability and exact observability for the time-varying stochastic system (3) is much more complicated.

We have also obtained some Lyapunov-type theorems under uniform detectability of LDTV systems, $\mathcal{K}^N$-exact detectability and $\mathcal{K}^N$-exact observability of linear discrete periodic systems. We believe that all these new concepts that have been introduced herein will play important roles in control and filtering design of LDTV systems.

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