Optimal control of discrete-time linear fractional order systems with multiplicative noise

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Abstract

A finite horizon linear quadratic(LQ) optimal control problem is studied for a class of discrete-time linear fractional systems (LFSs) affected by multiplicative, independent random perturbations. Based on the dynamic programming technique, two methods are proposed for solving this problem. The first one seems to be new and uses a linear, expanded-state model of the LFS. The LQ optimal control problem reduces to a similar one for stochastic linear systems and the solution is obtained by solving Riccati equations. The second method appeals to the Principle of Optimality and provides an algorithm for the computation of the optimal control and cost by using directly the fractional system. As expected, in both cases the optimal control is a linear function in the state and can be computed by a computer program. Two numerical examples proves the effectiveness of each method.

1 Introduction

Fractional calculus(FC) began to engage mathematicians' interest in the 17th century as evidenced by a letter of Leibniz to L'Hôspital, dated 30^{th} September 1695, which talks about the possibility of non-integer order differentiation. Later on, famous mathematicians as Fourier, Euler and Laplace contributed to the foundation of this new branch of mathematics with various concepts and results. Nowadays, the most popular definitions of the non-integer order integral or derivative are the Riemann-Liouville, Caputo and Grunwald-Letnikov definitions. For a historical survey and the current state of the art, the reader is referred to [14], [21], [11],[15], [6] and the references therein.

FC finds use in different fields of science and engineering including the electrochemistry, electromagnetism, biophysics, quantum mechanics, radiation physics, statistics or control theory (see [15], [11], [14]). Such an example comes from the field of autonomous guided vehicles, which lateral control seems to be

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improved by using fractional adaptation schemes [17]. Also, partial differential equations of fractional order were applied to model the wave propagation in viscoelastic media or the dissipation in seismology or in metallurgy [13].

The optimal control theory was intensively developed during the last century for deterministic systems defined by integer-order derivatives, in both continuous- and discrete- time cases [4]. Since many real-world phenomena are affected by random factors that exercised a decisive influence on the processes behavior, stochastic optimal control theory had a similar evolution in the recent decades (see [5], [8], [7], [19] and the references therein).

However, only a few papers address optimal control problems for fractional systems (see e.g. [18], [1], [2], [12], [10], [9]) and fewer consider stochastic fractional systems [16], [3].

In this paper we formulate a finite-horizon LQ optimal control problem for stochastic discrete-time LFSs defined by the Grunwald-Letnikov fractional derivative. As far as we know, this subject seems to be new.

We use the classical dynamic programming technique to derive two methods for solving the proposed optimal control problem. Obviously these methods apply to deterministic discrete-time LFSs.

The first one is new and uses an equivalent linear *expanded-state* model of the stochastic LFS. As the name says, the state of this model is expanded and formed by the actual state and all the past states of the LFS [15]. The quadratic cost functional is rewritten accordingly and the original optimal control problem reduces to a LQ optimal control problem for linear stochastic systems. Since the control weight matrix of the new optimization problem is not positive, we modify it with a parameter $\varepsilon > 0$ for achieving the positivity condition. This perturbation is chosen such that the optimal value of the new performance index (denoted $I_{x_0,N,\varepsilon}(U)$) is independent of ε and coincides with the optimal value of the original cost functional ($I_{x_0,N}(u)$) (see Proposition 2). The optimal value of the performance index $I_{x_0,N,\varepsilon}(U)$ is a quadratic form in the initial expanded-state and can be computed by solving a classical matrix Riccati equation. The optimal feedback law U is linear in state, and involves the solution of the same Riccati equation. The optimal control sequence u of the original optimal control problem is computable from U.

The second method is a stochastic version of the new algorithm proposed in [9] for deterministic LFSs. It uses the *Optimality Principle* for computing recursively (and starting with the terminal time) the optimal control sequence $u_{N-1},..,u_0$ and the optimal cost.

The main difference between the two methods is that the dynamic programing approach is applied in the first case to a linear stochastic system, while, in the second case, the same technique is applied directly to a stochastic LFS.

To compare the two methods, a numerical example is solved by using two computer algorithms written for this purpose. As expected, the mathematical results are the same, but the run-time of the program that implements the algorithm provided by the first method seems to be shorter.

The paper is organized as follows. In Section 2, we shortly review necessary notions from FC and we state the optimal control problem $\mathcal{O}$. In Section 3 we reformulate the problem $\mathcal{O}$ by using the equivalent linear *expanded-state* model of the stochastic LFS and a parametrized cost functional $I_{x_0,N,\varepsilon}(U)$. As mentioned above, the optimal control and cost can be computed with the solution of an associated Riccati equation. Finally, a numerical example illustrates the effectiveness of this first method.

In Section 4 we present the first two steps of the recursive algorithm (called Algorithm A) which starts with the terminal time and provides the optimal control sequence and cost that solves problem $\mathcal{O}$. The general step of the Algorithm A is described in the Appendix. The numerical example presented at the end in Section 4 is solved by using Algorithm A. It proves the applicability of the second method. Some conclusions are drawn in the last section.

2 Notations and statement of the problem

As usual, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^d, d \in \mathbb{N}^* = \mathbb{N} - \{0\}$ is the real Hilbert space of real d-dimensional vectors and $\mathbb{R}^{d\times n}$, $n \in \mathbb{N}^*$ is the linear space of $d \times n$ real matrices. We also denote by $\left(\mathbb{R}^d\right)^n$ the Hilbert space of all n dimensional vectors from $\mathbb{R}^d$. Obviously it is isomorphic with $\mathbb{R}^{d\times n}$. In this paper we do not distinguish between a linear operator on $\mathbb{R}^{d\times n}$ (or $\mathbb{R}^d$) and the associated matrix. Also we shall write $\langle .,.\rangle$ for the inner product and $\|.\|$ for norms of elements and operators. For any linear operator T acting on finite dimensional real spaces, we denote by T^* the adjoint operator of T. We say that $T : \mathbb{R}^n \to \mathbb{R}^n$ is nonnegative (we write $T \geq 0$) if $T = T^*$ and $\langle Tx, x\rangle \geq 0$, for all $x \in \mathbb{R}^n$; T is positive (we write $T > 0$) if $T \geq 0$ and there is $\delta > 0$ such that $\langle Tx, x\rangle \geq \delta \|x\|^2$, for all $x \in \mathbb{R}^n$. The identity operator on $\mathbb{R}^n$ will be denoted by $I_{\mathbb{R}^d}$.

Let $\alpha \in (0, 2)$ and $h > 0$ be fixed. We recall that for all $j \in \mathbb{N}$, $\left(\begin{array}{c}\alpha\\ j\end{array}\right)$ denotes the generalized binomial coefficient defined by

$$\left(\begin{array}{c}\alpha\\ j\end{array}\right) = \left\{\begin{array}{c} 1, j = 0 \\ \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha+1-j)}{j!}, j \in \mathbb{N}^* \end{array}\right. .$$

Then, for any sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^d, d \in \mathbb{N}$

$$\Delta^{[\alpha]}x_{k+1} = \frac{1}{h^\alpha}\sum_{j=0}^{k+1}(-1)^j \left(\begin{array}{c}\alpha\\ j\end{array}\right) x_{k+1-j}, h > 0$$

is the discrete fractional-order operator that arises in the Grünwald-Letnikov definition of the fractional order derivatives (see for e.g. [15]).

Let $\{\xi_k\}_{k\in\mathbb{N}}$ be a sequence of real-valued, mutually independent random variables on the probability space $(\Omega, \mathcal{G}, P)$ that satisfies the condition $E\left[\xi_k\right] = 0, E\left[\xi_k^2\right] = 1, k \in \mathbb{N}$. (Here $E[\xi]$ is the mean (expectation) of ξ_k.) The $\sigma-$algebra generated by $\{\xi_i, 0 \leq i \leq n-1\}, n \in \mathbb{N}^*$ will be denoted by $\mathcal{G}_n$. We consider the stochastic discrete-time fractional system with control

$$\begin{aligned} \Delta^{[\alpha]} x_{k+1} &= \mathbb{A} x_k + \xi_k \mathbb{B} x_k + \mathbb{D} u_k + \xi_k \mathbb{F} u_k, k \in \mathbb{N} \\ x_0 &= x \in \mathbb{R}^d, \end{aligned} \tag{1}$$

where $\mathbb{A}$, $\mathbb{B} \in \mathbb{R}^{d\times d}$, $\mathbb{D}$, $\mathbb{F} \in \mathbb{R}^{d\times m}, m \in \mathbb{N}$ and the control $u = \{u_k\}_{k\in\mathbb{N}}$ belongs to a class of admissible controls $\mathcal{U}^a$ formed by all sequences u which elements u_k are $\mathcal{G}_k$-measurable, $\mathbb{R}^m$-valued random variables satisfying $E\left[\|u_k\|^2\right] < \infty$ for all $k \in \mathbb{N}$.

A finite segment of an admissible control sequence u is of the form u_k, u_{k+1}, ... , u_N. In the sequel we shall denote by $\mathcal{U}^a_{k,N-1}$ the set of segments $u_k, u_{k+1}, ...,$ u_{N-1} of admissible controls $u \in \mathcal{U}^a$.

Multiplying (1) by h^α and denoting $A_0 = h^\alpha \mathbb{A} + \alpha I_{\mathbb{R}^d}$, $T = h^\alpha \mathbb{T}$, for any $T = B$, D, F, $\mathbb{T} = \mathbb{B}$, $\mathbb{D}$, $\mathbb{F}$, $c_j := (-1)^j \begin{pmatrix} \alpha \\ j+1 \end{pmatrix}$ and $A_j = c_j I_{\mathbb{R}^d}$, system (1) can be equivalently rewritten as

$$x_{k+1} = \sum_{j=0}^{k} A_j x_{k-j} + \xi_k B x_k + D u_k + \xi_k F u_k, \tag{2}$$

$$x_0 = x \in \mathbb{R}^d. \tag{3}$$

Let $x_0 \in \mathbb{R}^d$ and $N \in \mathbb{N}$ be fixed, $C \in \mathbb{R}^{p\times d}, S \in \mathbb{R}^{d\times d}, S \geq 0$ and $K \in \mathbb{R}^{m\times m}, K > 0$.

Our *optimal control problem* $\mathcal{O}$ is to minimize the cost functional

$$I_{x_0,N}(u) = \tag{4}$$

$$\sum_{n=0}^{N-1} E\left[\left(\|Cx_n\|^2 + < Ku_n, u_n >\right)\right] + E < Sx_N, x_N >$$

subject to (2)-(3), over the class $\mathcal{U}^a_{0,N-1}$ of segments of admissible controls.

3 An equivalent optimal control problem for a non-fractional linear system

In this section we first present an equivalent linear *expanded-state* model (see (6)-(7)) of the stochastic LFS. Then we show that optimal control problem $\mathcal{O}$ is "equivalent" with a LQ optimal control problem associated with (6)-(7). The word "equivalent" means here that the two optimal control problems have the same optimal costs and an optimal control sequence (OCS) of the one can be obtained from an OCS of the other. Since the solution of the new optimal control problem can be obtained by solving a backward discrete-time Riccati equation, we get a solution of $\mathcal{O}$(see Theorem 3).

3.1 A linear expanded-state model

Let $\mathcal{A},\mathcal{B}:\left(\mathbb{R}^d\right)^N\to\left(\mathbb{R}^d\right)^N$ be the linear operators defined by the matrices

$$\mathcal{A}=\begin{pmatrix} A_0 & c_1 I_{\mathbb{R}^d} & ... & c_{N-1}I_{\mathbb{R}^d}\\ I_{\mathbb{R}^d} & 0 & . & 0\\ . & I_{\mathbb{R}^d} & . & .\\ . & . & I_{\mathbb{R}^d} & 0\end{pmatrix},\mathcal{B}=\begin{pmatrix} B & 0 & . & . & .\\ 0 & 0 & . & . & .\\ . & . & . & . & .\\ . & . & . & . & .\end{pmatrix}. \tag{5}$$

Also let $\mathcal{D}_k,\mathcal{F}_k:(\mathbb{R}^m)^N\to(\mathbb{R}^d)^N, k=0,..,N-1$ be given by

$$\mathcal{D}_k\left(v_0,v_1,...,v_{N-1}\right)=\left(Dv_k,0,...,0\right)\in(\mathbb{R}^d)^N$$

and

$$\mathcal{F}_k\left(v_0,v_1,...,v_{N-1}\right)=\left(Fv_k,0,...,0\right)\in(\mathbb{R}^d)^N,$$

for all $(v_0,v_1,...,v_{N-1})\in(\mathbb{R}^m)^N$.

Similarly, for all $k=0,..,N-1$, we define $\mathcal{K}_k:(\mathbb{R}^m)^N\to(\mathbb{R}^m)^N$,$\mathcal{C}:(\mathbb{R}^d)^N\to(\mathbb{R}^p)^N$

$$\begin{aligned}\mathcal{K}_k\left(v_0,v_1,...,v_{N-1}\right) &= (0,...,Kv_k,0,...,0)\in\mathbb{R}^{m\times N}\\ \mathcal{C}\left(v_0,v_1,...,v_N-1\right) &= \left(Cv_0,0,...,0\right)\end{aligned}$$

and $\mathcal{S}:(\mathbb{R}^d)^N\to(\mathbb{R}^d)^N$

$$\mathcal{S}\left(v_0,...,v_{N-1}\right)=\left(Sv_0,0,...,0\right).$$

Obviously, $\mathcal{K}_k,\mathcal{S}\geq 0$. Let $x_0,x_1,...,x_k,...$ be a solution of (2). For any $k<N$,$X_k^T=\left(x_k,x_{k-1},...,x_0,0,..,\underset{N}{0}\right)\in\left(\mathbb{R}^d\right)^N$ is a solution of the discrete-time system with independent random perturbations

$$\begin{aligned}X_{k+1} &= \mathcal{A}X_k+\xi_k\mathcal{B}X_k+\mathcal{D}_kU_k+\xi_k\mathcal{F}_kU_k, &(6)\\ X_0 &= \left(x_0,0,...,\underset{N}{0}\right), &(7)\end{aligned}$$

where the control $U=\{U_k\}_{k\in\mathbb{N}}\subset(\mathbb{R}^m)^N$ belongs to the set $\mathbb{U}^a$ of admissible controls sequences $\{U_k\}_{k\in\mathbb{N}}$ having the property that U_k are $(\mathbb{R}^m)^N$-valued, $\mathcal{G}_k$-measurable random variables satisfying $E\left[\|U_k\|^2\right]<\infty$ for all $k\in\mathbb{N}$. The system (6)-(7) is a classical linear discrete-time control system with independent random perturbations. We know (see, e.g [8]) that for all $k\in\mathbb{N}^*$, X_k is $\mathcal{G}_k$-measurable and the pair X_k,ξ_n is independent for all $n\geq k>0$.

Computing X_N from (6)-(7),we note that $X_N=(x_N,x_{N-1},...,x_1)$ and x_N, the n-th solution of (2)-(3), is the first component of X_N. Then $\langle\mathcal{S}X_N,X_N\rangle=\langle Sx_n,x_n\rangle$. Also, for all $k<N$ we have

$$\begin{aligned}\mathcal{C}X_k &= \mathcal{C}\left(x_k,x_{k-1},...,x_0,0,..0\right)=\left(Cx_k,0,...,0\right),\\ \mathcal{K}_kU_k &= \mathcal{K}_k\left(\overline{u}_0,\overline{u}_1,...,u_k..,\overline{u}_{N-1}\right)=\left(0,0,...,\underset{k+1}{Ku_k},...,0\right).\end{aligned}$$

Then, the cost functional (4) can be equivalently rewritten as

$$I_{x_0,N}(U) = E\left[\sum_{k=0}^{N-1} \langle \mathcal{C}^*\mathcal{C}X_k, X_k\rangle + \langle \mathcal{S}X_N, X_N\rangle + \langle \mathcal{K}_k U_k, U_k\rangle\right]. \tag{8}$$

Substituting X_N given by (6)-(7) in (8), we get

$$\begin{aligned} I_{x_0,N}(U) \quad = \quad & \sum_{n=0}^{N-2} E[\langle \mathcal{C}^*\mathcal{C}X_k, X_k\rangle + \langle \mathcal{K}_k U_k, U_k\rangle] \\ & + E[\langle (\mathcal{C}^*\mathcal{C} + \mathcal{A}^*\mathcal{S}\mathcal{A} + \mathcal{B}^*\mathcal{S}\mathcal{B})\, X_{N-1}, X_{N-1}\rangle \\ & + 2\left\langle \left(\mathcal{D}^*_{N-1}\mathcal{S}\mathcal{A} + \mathcal{F}^*_{N-1}\mathcal{S}\mathcal{B}\right) X_{N-1}, U_{N-1}\right\rangle + \\ & \left\langle \left(\mathcal{K}_{N-1} + \mathcal{D}^*_{N-1}\mathcal{S}\mathcal{D}_{N-1} + \mathcal{F}^*_{N-1}\mathcal{S}\mathcal{F}_{N-1}\right) U_{N-1}, U_{N-1}\right\rangle]. \end{aligned} \tag{9}$$

To obtain the last equality we have applied the property of X_{N-1} and U_{N-1} of being independent of ξ_{N-1}. Thus for any appropriate deterministic linear operators V and T, we have

$$\begin{gathered} E\left[\left\langle VX_{N-1} + \xi_{N-1}TU_{N-1}, VX_{N-1} + \xi_{N-1}TU_{N-1}\right\rangle\right] = \\ E\left[\langle VX_{N-1}, VX_{N-1}\rangle\right] + 2E\left[\xi_{N-1}\right] E\left[\langle VX_{N-1}, TU_{N-1}\rangle\right] + \\ E\left[\xi^2_{N-1}\right] E\left[\langle TU_{N-1}, TU_{N-1}\rangle\right] \\ = E\left[\langle VX_{N-1}, VX_{N-1}\rangle\right] + E\left[\langle TU_{N-1}, TU_{N-1}\rangle\right] \end{gathered}$$

and (9) follows.

Now let $\mathbb{U}^a_{0,N-1}$ be the class of all finite segments $U_0, ..., U_{N-1}$ of sequences $U \in \mathbb{U}^a$. It is not difficult to see that the optimal control problem $\mathcal{O}$ is equivalent with the minimizing optimal control problem $\mathcal{O}_1$ defined by system (6)-(7), $I_{x_0,N}(U)$ and $\mathbb{U}^a_{0,N-1}$. Indeed, for any $u \in \mathcal{U}^a_{0,N-1}$, the segment $U = \{U_k = \left(0, .., \underset{k+1}{u_k}, ...0\right), k = 0, .., N-1\}$ belongs to $\mathbb{U}^a_{0,N-1}$ and $I_{x_0,N}(U) = I_{x_0,N}(u)$. Conversely, given $U \in \mathbb{U}^a_{0,N-1}$, we define $u = \{u_k = U_{kk}, k = 0, .., N-1\}$. Thus, $u \in \mathcal{U}^a_{0,N-1}$ and $I_{x_0,N}(U) = I_{x_0,N}(u)$. Now it is clear that $\widetilde{U}$ is optimal for $I_{x_0,N}(U)$ if and only if $\widetilde{u} = \{\widetilde{u}_k = \widetilde{U}_{kk}, k = 0, .., N-1\}$ is optimal for $I_{x_0,N}(u)$ and $I_{x_0,N}(\widetilde{U}) = I_{x_0,N}(\widetilde{u})$.

The problem $\mathcal{O}_1$ is a linear quadratic optimal control problem for stochastic systems. However $\mathcal{K}_k$ does not satisfy the condition $\mathcal{K}_k > 0$, $k = 0, .., N-1$ and we cannot solve $\mathcal{O}_1$ by a direct application of the known results from the optimal control theory of stochastic discrete-time systems (see [4], [8]).

Therefore, we replace the optimal cost $I_{x_0,N}(U)$ from $\mathcal{O}_1$ with the optimal cost

$$\begin{aligned} I_{x_0,N,\varepsilon}(U) \quad = \quad & \sum_{k=0}^{N-2} E\left[\langle \mathcal{C}^*\mathcal{C}X_k, X_k\rangle\right] + E\left[\langle (\mathcal{K}_k + \varepsilon\mathcal{I}_k)\, U_k, U_k\rangle\right] \\ & + E[\langle (\mathcal{C}^*\mathcal{C} + \mathcal{A}^*\mathcal{S}\mathcal{A} + \mathcal{B}^*\mathcal{S}\mathcal{B})\, X_{N-1}, X_{N-1}\rangle \\ & + 2\left\langle \left(\mathcal{D}^*_{N-1}\mathcal{S}\mathcal{A} + \mathcal{F}^*_{N-1}\mathcal{S}\mathcal{B}\right) X_{N-1}, U_{N-1}\right\rangle \\ & + \left\langle \mathbb{K}^{\varepsilon}_{N-1} U_{N-1}, U_{N-1}\right\rangle] \end{aligned} \tag{10}$$

where $\varepsilon > 0$ is fixed,

$$\mathbb{K}^{\varepsilon}_{N-1} = \mathcal{K}_{N-1} + \varepsilon \mathcal{I}_{N-1} + \mathcal{D}^*_{N-1} \mathcal{S} \mathcal{D}_{N-1} + \mathcal{F}^*_{N-1} \mathcal{S} \mathcal{F}_{N-1} \tag{11}$$

and $\mathcal{I}_k (v_0, v_1, ..., v_{N-1}) = \left(v_0, v_1, ..., \underset{k+1}{0}, ..., v_{N-1} \right), k = 0, .., N-1$. We obtain a new optimal control problem $\mathcal{O}_\varepsilon$.

The hypothesis $K > 0$, implies that $\mathcal{K}_k + \varepsilon \mathcal{I}_k > 0$, for all $k = 0, .., N-1$. Thus we can apply the classical results based on the Principle of Optimality stating that the optimal cost is a quadratic form in the state, with the weighting matrix computable via a recursion that involves the solution of a backward discrete-time Riccati equation.

3.2 Backward discrete-time Riccati equation of control

We associate with $\mathcal{O}_\varepsilon$ the backward discrete-time Riccati equation

$$\begin{gathered} R^{\varepsilon}_n = \mathcal{A}^* R^{\varepsilon}_{n+1} \mathcal{A} + \mathcal{B} R^{\varepsilon}_{n+1} \mathcal{B} + \mathcal{C}^* \mathcal{C} - \left(\mathcal{D}^*_n R^{\varepsilon}_{n+1} \mathcal{A} + \mathcal{F}^*_n R^{\varepsilon}_{n+1} \mathcal{B} \right)^* \cdot \\ \left(\mathcal{K}_n + \varepsilon \mathcal{I}_n + \mathcal{D}^*_n R^{\varepsilon}_{n+1} \mathcal{D}_n + \mathcal{F}^*_n R^{\varepsilon}_{n+1} \mathcal{F}_n \right)^{-1} \left(\mathcal{D}^*_n R^{\varepsilon}_{n+1} \mathcal{A} + \mathcal{F}^*_n R^{\varepsilon}_{n+1} \mathcal{B} \right), \\ \text{for } n < N-1 \text{ and} \end{gathered} \tag{12}$$

$$\begin{gathered} R^{\varepsilon}_{N-1} = \mathcal{C}^* \mathcal{C} + \mathcal{A}^* \mathcal{S} \mathcal{A} + \mathcal{B}^* \mathcal{S} \mathcal{B} - \\ \left(\mathcal{D}^*_{N-1} \mathcal{S} \mathcal{A} + \mathcal{F}^*_{N-1} \mathcal{S} \mathcal{B} \right)^* \cdot \left(\mathbb{K}^{\varepsilon}_{N-1} \right)^{-1} \left(\mathcal{D}^*_{N-1} \mathcal{S} \mathcal{A} + \mathcal{F}^*_{N-1} \mathcal{S} \mathcal{B} \right). \end{gathered} \tag{13}$$

Setting

$$W_{N-1} = - \left(\mathbb{K}^{\varepsilon}_{N-1} \right)^{-1} \left(\mathcal{D}^*_{N-1} \mathcal{S} \mathcal{A} + \mathcal{F}^*_{N-1} \mathcal{S} \mathcal{B} \right), \tag{14}$$

we observe that

$$\begin{gathered} R^{\varepsilon}_{N-1} = \mathcal{C}^* \mathcal{C} + \mathcal{A}^* \mathcal{S} \mathcal{A} + \mathcal{B}^* \mathcal{S} \mathcal{B} + \left(\mathcal{D}^*_{N-1} \mathcal{S} \mathcal{A} + \mathcal{F}^*_{N-1} \mathcal{S} \mathcal{B} \right)^* W_{N-1} + \\ W^*_{N-1} \left(\mathcal{D}^*_{N-1} \mathcal{S} \mathcal{A} + \mathcal{F}^*_{N-1} \mathcal{S} \mathcal{B} \right) + \\ W^*_{N-1} \left(\mathcal{K}_{N-1} + \varepsilon \mathcal{I}_{N-1} + \mathcal{D}^*_{N-1} \mathcal{S} \mathcal{D}_{N-1} + \mathcal{F}^*_{N-1} \mathcal{S} \mathcal{F}_{N-1} \right) W_{N-1} \\ = \mathcal{C}^* \mathcal{C} + \left(\mathcal{A} + \mathcal{D}_{N-1} W_{N-1} \right)^* \mathcal{S} \left(\mathcal{A} + \mathcal{D}_{N-1} W_{N-1} \right) + \\ \left(\mathcal{B} + \mathcal{F}_{N-1} W_{N-1} \right)^* \mathcal{S} \left(\mathcal{B} + \mathcal{F}_{N-1} W_{N-1} \right) + W^*_{N-1} \left(\mathcal{K}_{N-1} + \varepsilon \mathcal{I}_{N-1} \right) W_{N-1}. \end{gathered}$$

Now it is clear that $R^{\varepsilon}_{N-1} \geq 0$. Denoting

$$\begin{gathered} W_n = -(\mathcal{K}_n + \mathcal{D}^*_n R^{\varepsilon}_{n+1} \mathcal{D}_n + \mathcal{F}^*_n R^{\varepsilon}_{n+1} \mathcal{F}_n + \varepsilon I_n)^{-1} \cdot \\ \left(\mathcal{D}^*_n R^{\varepsilon}_{n+1} \mathcal{A} + \mathcal{F}^*_n R^{\varepsilon}_{n+1} \mathcal{B} \right), n = 0, .., N-2, \end{gathered} \tag{15}$$

and applying formula (4.8) from [20], we obtain

$$\begin{gathered} R^{\varepsilon}_n = \left(\mathcal{A} + \mathcal{D}_n W_n \right)^* R^{\varepsilon}_{n+1} \left(\mathcal{A} + \mathcal{D}_n W_n \right) + \left(B + \mathcal{F}_n W_n \right)^* R^{\varepsilon}_{n+1} \left(B + \mathcal{F}_n W_n \right) \\ + \mathcal{C}^* \mathcal{C} + W^*_n \left(\mathcal{K}_n + \varepsilon I_n \right) W_n, n = 0, .., N-2. \end{gathered}$$

Using the induction, we deduce that Riccati equation (12) has a unique non-negative solution $R^{\varepsilon}_n, n = 0, .., N-1$.

Lemma 1 *The cost functional (10) can be equivalently rewritten as*

$$I_{x_0,N,\varepsilon}(U) = E\left[\langle R_0^\varepsilon X_0, X_0\rangle\right] + \tag{16}$$
$$\left\langle \mathbb{K}_{N-1}^\varepsilon \left(W_{N-1}X_{N-1} - U_{N-1}\right), \left(W_{N-1}X_{N-1} - U_{N-1}\right)\right\rangle + \tag{17}$$
$$\sum_{k=0}^{N-2} E\left[\left\|\left(\mathcal{K}_n + \mathcal{D}_n^* R_{n+1}^\varepsilon \mathcal{D}_n + \mathcal{F}_n^* R_{n+1}^\varepsilon \mathcal{F}_n + \varepsilon \mathcal{I}_n\right)^{1/2}\left(W_n X_n - U_n\right)\right\|^2\right],$$

where R_n^ε is the unique solution of (12)-(13).

Proof. Let X_{n+1} be defined by (6). We have

$$\begin{aligned} E\left[\left\langle R_{n+1}^\varepsilon X_{n+1}, X_{n+1}\right\rangle\right] &= E[\left\langle\left(\mathcal{A}^* R_{n+1}^\varepsilon \mathcal{A} + \mathcal{B} R_{n+1}^\varepsilon \mathcal{B}\right) X_n, X_n\right\rangle + \\ &\quad 2\left\langle\left(\mathcal{D}_n^* R_{n+1}^\varepsilon \mathcal{A} + \mathcal{F}_n^* R_{n+1}^\varepsilon \mathcal{B}\right) X_n, U_n\right\rangle \\ &\quad + \left\langle\left(\mathcal{D}_n^* R_{n+1}^\varepsilon \mathcal{D}_n + \mathcal{F}_n^* R_{n+1}^\varepsilon \mathcal{F}_n\right) U_n, U_n\right\rangle] \end{aligned}$$

and, taking into account (12) and (15), we obtain

$$\begin{aligned} E\left[\left\langle R_{n+1}^\varepsilon X_{n+1}, X_{n+1}\right\rangle\right] &= E[\langle R_n^\varepsilon X_n, X_n\rangle - \langle \mathcal{C}^*\mathcal{C} X_n, X_n\rangle - \\ &\quad \langle(\mathcal{K}_n + \varepsilon\mathcal{I}_n)U_n, U_n\rangle] - \\ &\quad + E[\left\langle(\mathcal{K}_n + \mathcal{D}_n^* R_{n+1}^\varepsilon \mathcal{D}_n + \mathcal{F}_n^* R_{n+1}^\varepsilon \mathcal{F}_n + \varepsilon\mathcal{I}_n)W_n X_n, W_n X_n\right\rangle \\ &\quad - 2E\left[\left\langle(\mathcal{K}_n + \mathcal{D}_n^* R_{n+1}^\varepsilon \mathcal{D}_n + \mathcal{F}_n^* R_{n+1}^\varepsilon \mathcal{F}_n + \varepsilon\mathcal{I}_n)W_n X_n, U_n\right\rangle\right] \\ &\quad + E\left[\left\langle(\mathcal{K}_n + \mathcal{D}_n^* R_{n+1}^\varepsilon \mathcal{D}_n + \mathcal{F}_n^* R_{n+1}^\varepsilon \mathcal{F}_n + \varepsilon\mathcal{I}_n)U_n, U_n\right\rangle\right] \\ &= E\left[\langle R_n^\varepsilon X_n, X_n\rangle - \langle \mathcal{C}^*\mathcal{C} X_n, X_n\rangle - \langle(\mathcal{K}_n + \varepsilon\mathcal{I}_n)U_n, U_n\rangle\right] \\ &\quad + E\left\langle\left(\mathcal{K}_n + \mathcal{D}_n^* R_{n+1}^\varepsilon \mathcal{D}_n + \mathcal{F}_n^* R_{n+1}^\varepsilon \mathcal{F}_n + \varepsilon\mathcal{I}_n\right)\left(W_n X_n - U_n\right), \left(W_n X_n - U_n\right)\right\rangle. \end{aligned}$$

for all $n = 0, .., N-2$. Summing for $n = 0$ to $N-2$ the last equality, we obtain

$$\begin{aligned} E\left[\left\langle R_{N-1}^\varepsilon X_{N-1}, X_{N-1}\right\rangle\right] &= E\left[\langle R_0^\varepsilon X_0, X_0\rangle\right] - \\ &\quad \sum_{k=0}^{N-2} E\left[\langle \mathcal{C}^*\mathcal{C} X_k, X_k\rangle\right] + E\left[\langle\left(\mathcal{K}_k + \varepsilon\mathcal{I}_k\right) U_k, U_k\rangle\right] \\ &\quad + \sum_{k=0}^{N-2} E\left[\left\|\left(\mathcal{K}_n + \mathcal{D}_n^* R_{n+1}^\varepsilon \mathcal{D}_n + \mathcal{F}_n^* R_{n+1}^\varepsilon \mathcal{F}_n + \varepsilon\mathcal{I}_n\right)^{1/2}\left(W_n X_n - U_n\right)\right\|^2\right]. \end{aligned} \tag{18}$$

On the other hand, from (13), (11) and (14), we know that

$$\begin{aligned} E\left[\left\langle R_{N-1}^\varepsilon X_{N-1}, X_{N-1}\right\rangle\right] &= E[\langle\left(\mathcal{C}^*\mathcal{C} + \mathcal{A}^*\mathcal{S}\mathcal{A} + \mathcal{B}^*\mathcal{S}\mathcal{B}\right) X_{N-1}, X_{N-1}\rangle \\ &\quad - \left\langle \mathbb{K}_{N-1}^\varepsilon W_{N-1} X_{N-1}, W_{N-1} X_{N-1}\right\rangle] \\ &= E[\langle\left(\mathcal{C}^*\mathcal{C} + \mathcal{A}^*\mathcal{S}\mathcal{A} + \mathcal{B}^*\mathcal{S}\mathcal{B}\right) X_{N-1}, X_{N-1}\rangle \\ &\quad - \left\langle \mathbb{K}_{N-1}^\varepsilon \left(W_{N-1}X_{N-1} - U_{N-1}\right), \left(W_{N-1}X_{N-1} - U_{N-1}\right)\right\rangle \\ &\quad + 2\left\langle\left(\mathcal{D}_{N-1}^*\mathcal{S}\mathcal{A} + \mathcal{F}_{N-1}^*\mathcal{S}\mathcal{B}\right) X_{N-1}, U_{N-1}\right\rangle + \left\langle \mathbb{K}_{N-1}^\varepsilon U_{N-1}, U_{N-1}\right\rangle]. \end{aligned}$$

Replacing the above formula in (18), we obtain (16) and the conclusion follows. ■

3.3 Main results

In this section we shall prove that problem $\mathcal{O}$ has a solution derived from the solution of problem $\mathcal{O}_\varepsilon$.

Proposition 2 *For all $\varepsilon > 0$*

$$\min_{U\in\mathbb{U}^a_{0,N-1}} I_{x_0,N,\varepsilon}(U) = \min_{u\in\mathcal{U}^a_{0,N-1}} I_{x_0,N}(u).$$

Proof. Let $u = \{u_0, .., u_{N-1}\} \in \mathcal{U}^a_{0,N-1}$. If $U = \{U_0, .., U_{N-1}\}$, $U_k = \left(0, .., \underset{k}{u_k}, ...0\right)$, then $U \in \mathbb{U}^a_{0,N-1}$ and $I_{x_0,N,\varepsilon}(U) = I_{x_0,N}(u)$. Thus

$$\min_{u\in\mathcal{U}^a_{0,N-1}} I_{x_0,N}(u) \geq \min_{U\in\mathbb{U}} I_{x_0,N,\varepsilon}(U). \tag{19}$$

On the other hand if $U \in \mathbb{U}^a_{0,N-1}$ and $u = \{u_0, .., u_{N-1}\}$ is defined by $u_k = U_{kk}, k = 0, 1, .., N-1$, then $u \in \mathcal{U}^a_{0,N-1}$ and $I_{x_0,N}(u) = I_{x_0,N}(U) \leq I_{x_0,N,\varepsilon}(U)$. Replacing U in the above inequality by $\widetilde{U}^\varepsilon$ the optimal control which minimizes $I_{x_0,N,\varepsilon}(U)$ (we know that it exists), we see that $I_{x_0,N}(\widetilde{u}) \leq I_{x_0,N,\varepsilon}(\widetilde{U}^\varepsilon) = \min_{U\in\mathbb{U}^a_{0,N-1}} I_{x_0,N,\varepsilon}(U)$, where $\widetilde{u} = \{\widetilde{u}_0, .., \widetilde{u}_{N-1}\}$ and $\widetilde{u}_k = \widetilde{U}_{kk}$. Therefore $\min_{u\in\mathcal{U}^a_{0,N-1}} I_{x_0,N}(u) \leq \min_{U\in\mathbb{U}^a_{0,N-1}} I_{x_0,N,\varepsilon}(U)$. In view of (19) we get the conclusion. ■

The next theorem is a direct consequence of Lemma 1 and of the above proposition.

Theorem 3 *Let $\{R^\varepsilon_n\}_{n=0,..,N-1}$ be the unique solution of the Riccati equation (12)-(13) and let $W_n, n = 0, .., N-1$ be defined by (15), (14). The control sequence $\widetilde{U} = \{\widetilde{U}_0 = W_0X_0, ..., \widetilde{U}_n = W_nX_n, ..., \widetilde{U}_{N-1} = W_{N-1}X_{N-1}\}$ minimizes the cost functional $I_{x_0,N,\varepsilon}(U)$ and $\min_{U\in\mathbb{U}^a_{0,N-1}} I_{x_0,N,\varepsilon}(U) = E\left[\langle R^\varepsilon_0 X_0, X_0\rangle\right]$.*

Moreover, $\{R^\varepsilon_n\}_{n=0,..,N-1}$ does not depend on ε and the control $\widetilde{u} = \{\widetilde{u}_0, ..., \widetilde{u}_{N-1}\}$, defined by $\widetilde{u}_k = \widetilde{U}_{kk}, k = 0, 1, .., N-1$ is also optimal for $I_{x_0,N}(u)$. The optimal cost is

$$\min_{u\in\mathcal{U}^a_{0,N-1}} I_{x_0,N}(u) = E\left[\langle R^\varepsilon_0 X_0, X_0\rangle\right].$$

Proof. The proof is a simple exercise for the reader. ■

The following numerical example illustrates the applicability of the theory.

Example 4 *Let $\alpha = \frac{1}{2}, h = 1, d = 2, m = 1, x_0 = \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix}$ and $\mathbb{A} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $\mathbb{B} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \mathbb{D} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbb{F} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 2 & -1 \end{pmatrix}, K = 1, S = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Then $A_0 = \begin{pmatrix} 3/2 & 0 \\ 1 & 1/2 \end{pmatrix}$ and $T = \mathbb{T}$ for $\mathbb{T} = \mathbb{B}, \mathbb{D}, \mathbb{F}$. We consider the optimal control problem $\mathcal{O}$ for $N = 4$. Using a computer program,* *we compute in four simple steps the solution R_0^ε of the Riccati equation (12)-(13). The first five lines and columns of the matrix that defines the operator R_0^ε are the following*

$$\begin{pmatrix} 266.5781 & 33.3776 & 14.1358 & 16.6723 & 7.5490 & . \\ 33.3776 & 149.0853 & -9.7899 & 8.6506 & -4.8235 & . \\ 14.1358 & -9.7899 & 1.7822 & 0.4176 & 0.9168 & . \\ 16.6723 & 8.6506 & 0.4176 & 1.3910 & 0.2413 & . \\ 7.5490 & -4.8235 & 0.9168 & 0.2413 & 0.4730 & . \\ . & . & . & . & . & . \end{pmatrix}$$

Using the intermediate values R_n^ε, (15) and Theorem 3, we obtain the following optimal control sequence

$$\begin{aligned} \widetilde{u}_3 &= -0.3333x_{31} - 0.6000x_{32} - 0.0167x_{21} + 0.0167x_{22} - \\ &\quad 0.0083x_{11} + 0.0083x_{12} - 0.0052x_{01} + 0.0052x_{00} && (20) \\ \widetilde{u}_2 &= -0.4788x_{21} - 0.6894x_{22} - 0.0234x_{11} + 0.0109x_{12} - \\ &\quad 0.0121x_{01} + 0.0054x_{02} && (21) \\ \widetilde{u}_1 &= -0.4563x_{11} - 0.7023x_{12} - 0.0253x_{01} + 0.0107x_{02} && (22) \\ \widetilde{u}_0 &= -0.4582x_{01} - 0.7484x_{02}, && (23) \end{aligned}$$

where $x_n = \begin{pmatrix} x_{n1} \\ x_{n2} \end{pmatrix}$, $n = 1, 2, 3$ is the state vector of the fractional system. The optimal cost is

$$\min_{u \in \mathcal{U}^a_{0,N-1}} I_{x_0,N}(u) = x_0^T \begin{pmatrix} 266.5781 & 33.3776 \\ 33.3776 & 149.0853 \end{pmatrix} x_0 = 28.086.$$

4 A dynamic programming approach for the fractional system

In this section we apply the Principle of Optimality to derive a direct algorithm for solving the optimal control problem $\mathcal{O}$. As in [9], the optimal control is a state feedback law, computable via a recursion commencing at the terminal time and evolving backwards. The obtained result is a stochastic counterpart of the one provided in [9] for deterministic fractional systems.

Algorithm A

Consider the optimal control problem $\mathcal{O}$. An optimal control process P_0 is defined by the control policy $u = \{u_k\}_{k \in \{0,..,N-1\}}$ and the corresponding trajectory $x = \{x_k\}_{k \in \{0,..,N\}}$. Let

$$\begin{aligned} P_m \quad : \quad & x_m, x_{m+1} .., x_{N-1}, x_N \\ & u_m, .., u_{N-2}, u_{N-1}. \end{aligned}$$

be a final segment of P_0 starting at a time $t = m$, when system (2) is in the state x_m obtained from the initial state x_0 with the optimal control sequence $u_0, .., u_{m-1}$. The performance functional on this final segment is

$$I_{m,x_0,..,x_m}(u) = \tag{24}$$
$$\sum_{n=m}^{N-1} E\left[\left(\|Cx_n\|^2 + < Ku_n, u_n >\right)\right] + E < Sx_N, x_N > .$$

The Principle of Optimality says that any final segment P_m of P_0 must be optimal for $I_{m,x_0,..,x_m}$.

Thus for $m = N-1$, the process

$$\begin{aligned} P_{N-1} \quad : \quad & x_{N-1}, x_N \\ & u_{N-1} \end{aligned}$$

should be optimal for the cost

$$\begin{aligned} I_{N-1,x_0,..,x_{N-1}}(u) \quad = \quad & E\left[\left(\|Cx_{N-1}\|^2 + \langle Ku_{N-1}, u_{N-1}\rangle\right)\right] + \\ & E\left[\langle Sx_N, x_N\rangle\right]. \end{aligned} \tag{25}$$

This condition and the following computations leads to a formula for the optimal control u_{N-1}. Writing (2) for $k = N-1$, we obtain x_N. Substituting x_N in (25), we get

$$I_{N-1,x_0,..,x_{N-1}}(u) = \tag{26}$$
$$= E < S(\sum_{j=0}^{N-1} A_j x_{N-1-j} + \xi_{N-1} B x_{N-1} + Du_{N-1} + \xi_{N-1} F u_{N-1}),$$
$$\sum_{j=0}^{N-1} A_j x_{N-1-j} + \xi_{N-1} B x_{N-1} + Du_{N-1} + \xi_{N-1} F u_{N-1} > +$$
$$+ E\left[\left(\|Cx_{N-1}\|^2 + < Ku_{N-1}, u_{N-1} >\right)\right]$$

Since x_n, u_n are $\mathcal{G}_n$-measurable and ξ_p-independent for all $p \geq n, n, p \in \mathbb{N}$, we have

$$\begin{aligned} E\left[\langle Tx_i, \xi_n S v_n, \rangle\right] \quad &= \quad E\left[\xi_n\right] E\left[\langle Tx_i, Sv_n, \rangle\right] = 0, \\ E\left[\langle \xi_n T v_n, \xi_n S v_n, \rangle\right] \quad &= \quad E\left[\xi_n^2\right] E\left[\langle Tv_n, Sv_n, \rangle\right] = E\left[\langle Tv_n, Sv_n, \rangle\right] \end{aligned}$$

for all $i \leq n$, $v = u, x$ and S, T matrices of appropriate dimensions. Therefore,

$$\begin{gathered}
I_{N-1,x_0,..,x_{N-1}}(u) = E\left\langle S\sum_{j=0}^{N-1} A_j x_{N-1-j}, \sum_{j=0}^{N-1} A_j x_{N-1-j}\right\rangle + \\
E\left\langle SBx_{N-1}, Bx_{N-1}\right\rangle + 2E\left\langle D^*S\sum_{j=0}^{N-1} A_j x_{N-1-j}, u_{N-1}\right\rangle + \\
E\left\langle F^*SFu_{N-1}, u_{N-1}\right\rangle + 2E\left\langle F^*SBx_{N-1}, u_{N-1}\right\rangle \\
+E\left\langle D^*SDu_{N-1}, u_{N-1}\right\rangle + E\left[\left(\left\|Cx_{N-1}\right\|^2 + \left\langle Ku_{N-1}, u_{N-1}\right\rangle\right)\right] \\
= E\left[\left\|\sqrt{S}\sum_{j=0}^{N-1} A_j x_{N-1-j}\right\|^2\right] + E\left[\left\|\sqrt{S}Bx_{N-1}\right\|^2\right] + E\left[\left\|Cx_{N-1}\right\|^2\right] \\
+2E\left\langle D^*S\sum_{j=1}^{N-1} A_j x_{N-1-j} + \left(F^*SB + D^*SA_0\right)x_{N-1}, u_{N-1}\right\rangle \\
+E\left\langle \left(F^*SF + D^*SD + K\right)u_{N-1}, u_{N-1}\right\rangle.
\end{gathered}$$

Setting

$$\begin{gathered}
v_{N-1}\left(x_0, .., x_{N-1}\right) = D^*S\sum_{j=1}^{N-1} A_j x_{N-1-j} + \left(F^*SB + D^*SA_0\right)x_{N-1}, \\
w_{N-1}\left(x_0, .., x_{N-1}\right) = E\left[\left\|\sqrt{S}\sum_{j=0}^{N-1} A_j x_{N-1-j}\right\|^2\right] \\
+E\left[\left\|\sqrt{S}Bx_{N-1}\right\|^2\right] + E\left[\left\|Cx_{N-1}\right\|^2\right] \\
J_{N-1} = F^*SF + D^*SD + K > 0
\end{gathered} \tag{27}$$

and using a squares completion technique, we see that

$$\begin{gathered}
I_{N-1,x_0,..,x_{N-1}}(u) = w_{N-1}\left(x_0, .., x_{N-1}\right) + \\
2E\left\langle v_{N-1}\left(x_0, .., x_{N-1}\right), u_{N-1}\right\rangle + E\left\langle J_{N-1}u_{N-1}, u_{N-1}\right\rangle.
\end{gathered}$$

The cost functional $I_{N-1,x_0,..,x_{N-1}}(u)$ can be equivalently rewritten as

$$\begin{aligned}
I_{N-1,x_0,..,x_{N-1}}(u) \quad &= \quad E\left\langle J_{N-1}\left(u_{N-1} + J_{N-1}^{-1}v_{N-1}\right), \left(u_{N-1} + J_{N-1}^{-1}v_{N-1}\right)\right\rangle \\
&\quad\quad +w_{N-1} - E\left\langle J_{N-1}^{-1}v_{N-1}, v_{N-1}\right\rangle.
\end{aligned}$$

As a function of u_{N-1}, $I_{N-1,x_0,..,x_{N-1}}(u)$ is optimal for

$$u_{N-1}^* = -J_{N-1}^{-1}v_{N-1} \tag{28}$$

and its optimal value is

$$\min_{u_{N-1}\in\mathcal{U}^a_{N-1,N-1}} I_{N-1,x_0,..,x_{N-1}}(u) = w_{N-1} - E\left\langle J^{-1}_{N-1}v_{N-1}, v_{N-1}\right\rangle. \tag{29}$$

In view of (27),

$$u^*_{N-1}\left(x_0,..,x_{N-1}\right) = \sum_{j=0}^{N-1} W_{j,N-1}x_{N-1-j}, \tag{30}$$

where

$$\begin{aligned} W_{j,N-1} &= -\left(F^*SF + D^*SD + K\right)^{-1} D^*SA_j,\ j\in\{1,..,N-1\} \\ W_{0,N-1} &= -\left(F^*SF + D^*SD + K\right)^{-1}\left(F^*SB + D^*SA_0\right). \end{aligned} \tag{31}$$

From the above proof we deduce that $u^*_{N-1}\left(x_0,..,x_{N-1}\right) = \sum_{j=0}^{N-1} W_{j,N-1}x_{N-1-j}$ is optimal for $I_{N-1}(u)$ for any trajectory $(x_0,..,x_{N-1})$. Substituting (28) to (26) we obtain

$$\min_{u_{N-1}\in\mathcal{U}^a_{N-1,N-1}} I_{N-1,x_0,..,x_{N-1}}(u) = \mathcal{O}(x_0,..,x_{N-1}) =$$

$$E < S(\sum_{j=0}^{N-1} A_j x_{N-1-j} + \xi_{N-1}Bx_{N-1} + \left(D+\xi_{N-1}F\right)\sum_{j=0}^{N-1} W_{j,N-1}x_{N-1-j}),$$

$$\sum_{j=0}^{N-1} A_j x_{N-1-j} + \xi_{N-1}Bx_{N-1} + \left(D+\xi_{N-1}F\right)\sum_{j=0}^{N-1} W_{j,N-1}x_{N-1-j}) > +$$

$$+E\left[\|Cx_{N-1}\|^2\right] + E[< K\sum_{j=0}^{N-1} W_{j,N-1}x_{N-1-j}, \sum_{j=0}^{N-1} W_{j,N-1}x_{N-1-j} >]$$

$$= E < S(\sum_{j=0}^{N-1} V^{S,1}_{N-1,j}x_{N-1-j} + \xi_{N-1}\sum_{j=0}^{N-1} V^{S,2}_{N-1,j}x_{N-1-j}),$$

$$\sum_{j=0}^{N-1} V^{S,1}_{N-1,j}x_{N-1-j} + \xi_{N-1}\sum_{j=0}^{N-1} V^{S,2}_{N-1,j}x_{N-1-j} > +$$

$$+E\left[\|Cx_{N-1}\|^2\right] + E[< K\sum_{j=0}^{N-1} W_{j,N-1}x_{N-1-j}, \sum_{j=0}^{N-1} W_{j,N-1}x_{N-1-j} >]$$

Denoting

$$\begin{aligned} V^{S,2}_{N-1,0} &= B + FW_{0,N-1}, V^{S,2}_{N-1,j} = FW_{j,N-1}, j\neq 0, \\ V^{S,1}_{N-1,j} &= A_j + DW_{j,N-1}, V^{K,1}_{N-1,j} = W_{j,N-1}, j\in\{0,..,N-1\} \end{aligned} \tag{32}$$

we obtain the optimal value of the cost:

$$\mathcal{O}(x_0,..,x_{N-1}) = E\left[\left\|\sqrt{S}\sum_{j=0}^{N-1} V^{S,1}_{N-1,j}x_{N-1-j}\right\|^2\right] +$$

$$E\left[\left\|\sqrt{S}\sum_{j=0}^{N-1} V^{S,2}_{N-1,j}x_{N-1-j}\right\|^2\right] +$$

$$E\left[\|Cx_{N-1}\|^2\right] + E\left[\left\|\sqrt{K}\sum_{j=0}^{N-1} V^{K,1}_{N-1,j}x_{N-1-j}\right\|\right]$$

Now we assume that

$$\begin{aligned} P_{N-2} \quad : \quad & x_{N-2}, x_{N-1}, x_N \\ & u_{N-2}, u_{N-1} \end{aligned}$$

is a final segment of the process P_0.

Then P_{N-2} should be optimal for $I_{N-2,x_0,..,x_{N-2}}(u)$. Since

$$\min_{u_{N-2},u_{N-1}\in\mathcal{U}^a_{N-2,N-1}} I_{N-2,x_0,..,x_{N-2}}(u) =$$

$$\min_{u_{N-2},u_{N-1}\in\mathcal{U}^a_{N-2,N-1}} \{I_{N-1,x_0,..,x_{N-1}}(u) + E\left[\|Cx_{N-2}\|^2\right] +$$

$$E\left[\langle Ku_{N-2}, u_{N-2}\rangle\right]\} =$$

$$\min_{u_{N-2}\in\mathcal{U}^a_{N-2,N-2}} \{ \min_{u_{N-1}\in\mathcal{U}^a_{N-1,N-1}} I_{N-1,x_0,..,x_{N-1}}(u) + E\left[\|Cx_{N-2}\|^2\right] +$$

$$+E\left[\langle Ku_{N-2}, u_{N-2}\rangle\right]\} =$$

$$\min_{u_{N-2}\in\mathcal{U}^a_{N-2,N-2}} \{\mathcal{O}(x_0,..,x_{N-1}) + E\left[\|Cx_{N-2}\|^2 + < Ku_{N-2}, u_{N-2} >\right]\},$$

it follows that u_{N-1} is given by (30) and u_{N-2} should be computed. Substituting x_{N-1} given by (2) in $\mathcal{O}(x_0,..x_{N-2},x_{N-1})$, we see that $\mathcal{O}(x_0,..x_{N-2},x_{N-1}) = \phi(x_0,..x_{N-2},u_{N-2})$ and u_{N-2} solves the optimal control problem

$$\min_{u_{N-2},u_{N-1}\in\mathcal{U}^a_{N-2,N-1}} I_{N-2,x_0,..,x_{N-2}}(u) =$$

$$= \min_{u_{N-2}\in\mathcal{U}^a_{N-2,N-2}} \{\phi(x_0,..x_{N-2},u_{N-2}) + E\left[\left(\|Cx_{N-2}\|^2 + < Ku_{N-2}, u_{N-2} >\right)\right]\}.$$

Using again the squares completion technique, we can prove that the optimal control u_{N-2} is a linear function of $x_0,..x_{N-2}$ and $\min_{u_{N-2},u_{N-1}\in\mathcal{U}^a_{N-2,N-1}} I_{N-2}(u)$ is a function of the trajectory $x_0,..,x_{N-2}$. Repeating the above arguments, we find u_{N-3}, u_{N-4} and so on. The general step of the above algorithm is described in detail in the Appendix. At the step q we find the optimal control u_{N-q} as a linear function of $x_0, .., x_{N-q}$ of the form $\sum_{j=0}^{N-q} W_{j,N-q} x_{N-q-j}$ where the coefficients $W_{j,N-q}$ are given by a set of recurrent formulas (see (36), (38), (39),(40) in the Appendix). At a first sight this algorithm is more complicated than the one described in Section 3.

Example 5 *Consider the optimal control problem $\mathcal{O}$ under the hypotheses of Example 4. Implementing in MATLAB the Algorithm A, we obtain the following results. Since* $W_{0,3} = \begin{pmatrix} -0.3333 & -0.600 \end{pmatrix}$, $W_{1,3} = \begin{pmatrix} -0.0167 & 0.0167 \end{pmatrix}$, $W_{2,3} = \begin{pmatrix} -0.0083 & 0.0083 \end{pmatrix}$ *and* $W_{3,3} = \begin{pmatrix} -0.0052 & 0.0052 \end{pmatrix}$ *we deduce by (30) that the optimal control u_3 have the same formula as the one obtained in Example 4. Further, we compute* $V_{3,j}^{S,1}, V_{3,j}^{S,2}, V_{3,j}^{K,1}$, $j = 0, 1, 2, 3$. *Writing (35) for* $q = 2$ *we get* $J_2 = 67.3667$. *Also the coefficients of x_0, x_1 and x_2 from (36) are* $\begin{pmatrix} 0.8151 & -0.3630 \end{pmatrix}$, $\begin{pmatrix} 1.5750 & -0.7333 \end{pmatrix}$ *and* $\begin{pmatrix} 32.2583 & 46.4417 \end{pmatrix}$, *respectively. Since $W_{j,N-q}$ is obtained by multiplying the coefficient of x_{N-q-j} from v_{N-q} with $-J_{N-q}^{-1}$, we get* $W_{0,2} = -J_2^{-1} \begin{pmatrix} 32.2583 & 46.4417 \end{pmatrix} = \begin{pmatrix} -0.478\,85 & -0.689\,39 \end{pmatrix}$, $W_{1,2} = \begin{pmatrix} -0.0233 & 0.0108 \end{pmatrix}$ *and* $W_{2,2} = \begin{pmatrix} -0.0120 & 0.0053 \end{pmatrix}$. *Thus*

$$\begin{aligned} u_2 &= \begin{pmatrix} -0.478\,85 & -0.689\,39 \end{pmatrix} x_2 + \begin{pmatrix} -0.0233 & 0.0108 \end{pmatrix} x_1 + \\ &\quad \begin{pmatrix} -0.0120 & 0.0053 \end{pmatrix} x_0 \end{aligned}$$

and the formula of $\widetilde{u}_2$ obtained in Example 4 is recovered. At the next step are computed the coefficients $V_{3,j}^{S,l}, l = 1, .., 4, V_{3,j}^{S,l}, l = 1, 2, 3, V_{3,j}^{K,1}, l = 1, 2,\ j = 0, 1, 2$. *With (35) written for $q = 1$ we obtain $J_1 = 196.1711$. The coefficients of x_0 and x_1 from (36) are* $\begin{pmatrix} -0,0253 & 0.0107 \end{pmatrix}$, $\begin{pmatrix} 4.8670 & -2.2318 \end{pmatrix}$ *and*

$$u_1 = \begin{pmatrix} -0.456\,34 & -0.70234 \end{pmatrix} x_1 + \begin{pmatrix} -0,0253 & 0.0107 \end{pmatrix} x_0.$$

Continuing the procedure, we obtain $u_0 = \begin{pmatrix} -0.4589 & -0.7481 \end{pmatrix}$ *and* $\mathcal{O}(x_0) = x_0^T \begin{pmatrix} 266.8471 & 32.9452 \\ 32.9452 & 149.4033 \end{pmatrix} x_0 = 28.074.$

5 Conclusions

This paper provides two methods of solving the LQ optimal control problem $\mathcal{O}$. Both of them are based on the dynamic programming approach. The first one seems to be new and easier. It consists in a reformulation of the problem for an associated linear non-fractional system (6)-(7), defined on spaces of higher dimensions. The second one uses the Principle of Optimality to derive a dynamic programming algorithm for the optimal control of the LFS. This algorithm is a stochastic counterpart of the one obtained in [9] for deterministic LFSs; it keeps the dimensions of the state space of system (2)-(3), but it is more laborious. The computer program implementing it is not such simple and fast as the one that implements the first method. A future analysis of these algorithms from the computer science point of view will highlight the real advantages and disadvantages of each method.

6 Appendix

The general step of the algorithm A

Our problem is to find the final segment

$$P_{N-q} \quad : \quad x_{N-q}, x_{N-q+1}.., x_{N-1}, x_N$$
$$u_{N-q}, .., u_{N-2}, u_{N-1}.$$

of P_0 which minimizes $I_{N-q,,x_0,..,x_{N-q}}(u)$. Assume that the optimal controls $u_{N-1}, ..., u_{N-q+1}$,$q \geq 2$ were determined and the optimal cost

$$I_{N-q+1,,x_0,..,x_{N-q+1}}(u)$$

has the form

$$\mathcal{O}(x_0, ..x_{N-q}, x_{N-q+1}) := \min_{u_{N-q+1},...,u_{N-1} \in \mathcal{U}^a_{N-q+1,N-1}} I_{N-q+1,x_0,..,x_{N-q+1}}(u) =$$
$$\sum_{l=1}^{2^{q-1}} E \left\| \sqrt{S} \sum_{j=0}^{N-q+1} V^{S,l}_{N-q+1,j} x_{N-q+1-j} \right\|^2 +$$
$$\sum_{l=1}^{2^{q-1}-1} E[\left\| \sqrt{K} \sum_{j=0}^{N-q+1} V^{K,l}_{N-q+1,j} x_{N-q+1-j} \right\|^2$$
$$\sum_{l=1}^{2^{q-1}-2} E[\left\| C \sum_{j=0}^{N-q+1} V^{C,l}_{N-q+1,j} x_{N-q+1-j} \right\|^2 + E\left[\|Cx_{N-q+1}\|^2 \right]$$
$$= \sum_{l=1}^{2^{q-1}} \sigma^{S,l}_{N-q+1} + \sum_{l=1}^{2^{q-1}-1} \sigma^{K,l}_{N-q+1} + \sum_{l=1}^{2^{q-1}-2} \sigma^{C,l}_{N-q+1} + E\left[\|Cx_{N-q+1}\|^2 \right] \tag{33}$$

where $V^{S,l}_{N-q+1,j}, V^{K,l}_{N-q+1,j}$ and $V^{C,l}_{N-q+1,j}$ are matrices of appropriate dimensions depending on the coefficients of the optimal control problem. We shall compute the optimal control u_{N-q} and we shall prove that $\mathcal{O}(x_0, ..x_{N-q})$ is given by a formula of the form (33) where q is replaced by $q-1$.

We know that

$$I_{N-q,,x_0,..,x_{N-q}}(u) = I_{N-q+1,x_0,..,x_{N-q+1}}(u) + E\left[\left(\|Cx_{N-q}\|^2 + < Ku_{N-q}, u_{N-q} > \right) \right].$$

Then

$$\min_{u_{N-q},...,u_{N-1} \in \mathcal{U}^a_{N-q,N-1}} I_{N-q,x_0,..,x_{N-q+1}}(u) =$$
$$\min_{u_{N-q} \in \mathcal{U}^a_{N-q,N-q}} \{ \min_{u_{N-q+1},...,u_{N-1} \in \mathcal{U}^a_{N-q+1,N-1}} I_{x,N-q+1}(u) + E\left[\left(\|Cx_{N-q}\|^2 + \langle Ku_{N-q}, u_{N-q} \rangle \right) \right] \}$$
$$:= \min_{u_{N-q} \in \mathcal{U}^a_{N-q,N-q}} f\left(x_0, .., x_{N-q}, x_{N-q+1}, u_{N-q} \right).$$

Substituting

$$x_{N-q+1}=\sum_{j=0}^{N-q}A_jx_{N-q-j}+\xi_{N-q}Bx_{N-q}+Du_{N-q}+\xi_{N-q}Fu_{N-q} \tag{34}$$

in $\sigma_{N-q+1}^{S,l}$ (see (33)) we obtain $\sigma_{N-q+1}^{S,l}$ as a function of the known $x_j, j=\overline{0,N-q}$ and the unknown u_{N-q}. We have

$$\sigma_{N-q+1}^{S,l}=$$

$$E\left\|\sqrt{S}(\sum_{j=1}^{N-q+1}V_{N-q+1,j}^{S,l}x_{N-q+1-j}+V_{N-q+1,0}^{S,l}\sum_{j=0}^{N-q}A_jx_{N-q-j}+V_{N-q+1,0}^{S,l}Du_{N-q})\right\|^2$$

$$+E\left\|\sqrt{S}V_{N-q+1,0}^{S,l}(Bx_{N-q}+Fu_{N-q})\right\|^2$$

$$=E\left\|\sqrt{S}\sum_{j=0}^{N-q}\left(V_{N-q+1,j+1}^{S,l}+V_{N-q+1,0}^{S,l}A_j\right)x_{N-q-j}\right\|^2+E\left\|\sqrt{S}V_{N-q+1,0}^{S,l}Bx_{N-q}\right\|^2$$

$$+2E<S\sum_{j=0}^{N-q}\left(V_{N-q+1,j+1}^{S,l}+V_{N-q+1,0}^{S,l}A_j\right)x_{N-q-j},V_{N-q+1,0}^{S,l}Du_{N-q}>$$

$$+2E\left\langle\left(V_{N-q+1,0}^{S,l}\right)^*SV_{N-q+1,0}^{S,l}Bx_{N-q},Fu_{N-q})\right\rangle$$

$$+E\left\|\sqrt{S}V_{N-q+1,0}^{S,l}Du_{N-q}\right\|^2+E\left\|\sqrt{S}V_{N-q+1,0}^{S,l}Fu_{N-q}\right\|^2$$

A similar computation leads to a formula for σ_{N-q+1}^{K} and σ_{N-q+1}^{C} obtained from the one above by replacing S by K. Also

$$E\left[\left\|Cx_{N-q+1}\right\|^2\right]=E\left\|C\sum_{j=0}^{N-q}A_jx_{N-q-j}\right\|^2+E\left\|CBx_{N-q}\right\|^2+$$

$$2E<D^*C^*C\sum_{j=0}^{N-1}A_jx_{N-q-j},u_{N-q}>+2E\left\langle F^*C^*CBx_{N-q},u_{N-q}\right\rangle$$

$$+E\left\|CFu_{N-q}\right\|^2+E\left\|CDu_{N-q}\right\|^2.$$

Therefore, substituting (34) in $f\left(x_0,..,x_{N-q},x_{N-q+1},u_{N-q}\right)$ we see that $f\left(x_0,..,x_{N-q},x_{N-q+1},u_{N-q}\right)$ becomes a function $g(x_0,..,x_{N-q},u_{N-q})$ of the form

$$w_{N-q}\left(x_0,..,x_{N-q}\right)+2E\left\langle v_{N-q}\left(x_0,..,x_{N-q}\right),u_{N-q}\right\rangle+E\left\langle J_{N-q}u_{N-q},u_{N-q}\right\rangle,$$

where

$$J_{N-q}=K+\sum_{l=1}^{2^{q-1}}\left[\left(V_{N-q+1,0}^{S,l}D\right)^{*}SV_{N-q+1,0}^{S,l}D+\left(V_{N-q+1,0}^{S,l}F\right)^{*}SV_{N-q+1,0}^{S,l}F\right] \tag{35}$$

$$+\sum_{l=1}^{2^{q-1}-1}\left[\left(V_{N-q+1,0}^{K,l}D\right)^{*}KV_{N-q+1,0}^{K,l}D+\left(V_{N-q+1,0}^{K,l}F\right)^{*}KV_{N-q+1,0}^{K,l}F\right]$$

$$+\sum_{l=1}^{2^{q-1}-2}\left[\left(V_{N-q+1,0}^{C,l}D\right)^{*}C^{*}CV_{N-q+1,0}^{C,l}D+\left(V_{N-q+1,0}^{C,l}F\right)^{*}C^{*}CV_{N-q+1,0}^{C,l}F\right]$$

$$+F^{*}C^{*}CF+D^{*}C^{*}CD>0$$

and

$$v_{N-q}\left(x_0,..,x_{N-q}\right)= \tag{36}$$

$$\sum_{l=1}^{2^{q-1}}\{\left(V_{N-q+1,0}^{S,l}D\right)^{*}S\sum_{j=0}^{N-q}\left(V_{N-q+1,j+1}^{S,l}+V_{N-q+1,0}^{S,l}A_j\right)x_{N-q-j}+$$

$$\left(V_{N-q+1,0}^{S,l}F\right)^{*}SV_{N-q+1,0}^{S,l}Bx_{N-q}\}+$$

$$\sum_{l=1}^{2^{q-1}-1}\{\left(V_{N-q+1,0}^{K,l}D\right)^{*}K\sum_{j=0}^{N-q}\left(V_{N-q+1,j+1}^{K,l}+V_{N-q+1,0}^{K,l}A_j\right)x_{N-q-j}+$$

$$\left(V_{N-q+1,0}^{K,l}F\right)^{*}KV_{N-q+1,0}^{K,l}Bx_{N-q}\}+$$

$$\sum_{l=1}^{2^{q-1}-2}\{\left(V_{N-q+1,0}^{C,l}D\right)^{*}C^{*}C\sum_{j=0}^{N-q}\left(V_{N-q+1,j+1}^{C,l}+V_{N-q+1,0}^{C,l}A_j\right)x_{N-q-j}+$$

$$\left(V_{N-q+1,0}^{C,l}F\right)^{*}C^{*}CV_{N-q+1,0}^{C,l}Bx_{N-q}\}+$$

$$D^{*}C^{*}C\sum_{j=0}^{N-q}A_jx_{N-q-j}+F^{*}C^{*}CBx_{N-q}.$$

Reasoning as in the case $q=1$, we get the optimal control

$$\begin{aligned} u_{N-q}^{*}\left(x_0,..,x_{N-q}\right) &= -J_{N-q}^{-1}v_{N-q} \\ &= \sum_{j=0}^{N-q}W_{j,N-q}x_{N-q-j}. \end{aligned} \tag{37}$$

Taking into account (36), we see that for all $j\in\{0,1,...,N-q\}$, $W_{j,N-q}$ is obtained by multiplying the coefficient of x_{N-q-j}, from v_{N-q}, with $-J_{N-q}^{-1}$.

Replacing (37) in $\sigma^{S,l}_{N-q+1}$, we observe that, for all $l \in \{1,..,2^{q-1}\}$,

$$\sigma^{S,l}_{N-q+1} =$$

$$E\left\|S\sum_{j=0}^{N-q}\left(V^{S,l}_{N-q+1,j+1}+V^{S,l}_{N-q+1,0}A_j\right)x_{N-q-j}+V^{S,l}_{N-q+1,0}Du_{N-q}\right\|^2+$$

$$+E\left\|\sqrt{S}V^{S,l}_{N-q+1,0}(Bx_{N-q}+Fu_{N-q})\right\|^2$$

$$=E\left\|\sqrt{S}\sum_{j=0}^{N-q}V^{S,l}_{N-q,j}x_{N-q-j}\right\|^2+E\left\|\sqrt{S}\sum_{j=0}^{N-q}V^{S,l+2^{q-1}}_{N-q,j}x_{N-q-j}\right\|^2$$

and

$$\sum_{l=1}^{2^{q-1}}\sigma^{S,l}_{N-q+1}=\sum_{l=1}^{2^q}E\left\|\sqrt{S}\sum_{j=0}^{N-q}V^{S,l}_{N-q,j}x_{N-q-j}\right\|^2,$$

where

$$\begin{aligned} V^{S,l}_{N-q,j} &= V^{S,l}_{N-q+1,j+1}+V^{S,l}_{N-q+1,0}\left(A_j+DW_{j,N-q}\right), \\ V^{S,l+2^{q-1}}_{N-q,0} &= V^{S,l}_{N-q+1,0}\left(B+FW_{0,N-q}\right), \\ V^{S,l+2^{q-1}}_{N-q,j} &= V^{S,l}_{N-q+1,0}FW_{j,N-q},\, j\neq 0. \end{aligned} \tag{38}$$

Arguing as above and using (29), we obtain

$$\sum_{l=1}^{2^{q-1}-1}E[\left\|\sqrt{K}\sum_{j=0}^{N-q+1}V^{K,l}_{N-q+1,j}x_{N-q+1-j}\right\|^2+\langle Ku_{N-q},u_{N-q}\rangle=$$

$$=\sum_{l=1}^{2^q-1}E\left\|\sqrt{K}\sum_{j=0}^{N-q}V^{K,l}_{N-q,j}x_{N-q-j}\right\|$$

where for all $l\in\{1,..,2^{q-1}-1\}, j\leq N-q$

$$\begin{aligned} V^{K,l}_{N-q,j} &= V^{K,l}_{N-q+1,j+1}+V^{K,l}_{N-q+1,0}\left(A_j+DW_{j,N-q}\right), \\ V^{K,l+2^{q-1}-1}_{N-q,0} &= V^{K,l}_{N-q+1,0}\left(B+FW_{0,N-q}\right), \\ V^{K,l+2^{q-1}-1}_{N-q,j} &= V^{K,l}_{N-q+1,0}FW_{j,N-q}, V^{K,2^q-1}_{N-q,j}=W_{j,N-q},\, j\neq 0. \end{aligned} \tag{39}$$

Similarly,

$$\sum_{l=1}^{2^{q-1}-2}E[\left\|C\sum_{j=0}^{N-q+1}V^{K,l}_{N-q+1,j}x_{N-q+1-j}\right\|^2+E\left[\|Cx_{N-q+1}\|^2\right]^2=$$

$$\sum_{l=1}^{2^q-2}E\left\|C\sum_{j=0}^{N-q}V^{C,l}_{N-q,j}x_{N-q-j}\right\|$$

where $V^{C,l}_{N-q,j}, 1 \leq l \leq 2^{q-1}-2, j \leq N-q$ are given by

$$
\begin{aligned}
V^{C,l}_{N-q,j} &= V^{C,l}_{N-q+1,j+1} + V^{C,l}_{N-q+1,0}\left(A_j + DW_{j,N-q}\right), \qquad (40)\\
V^{C,l+2^{q-1}-2}_{N-q,0} &= V^{C,l}_{N-q+1,0}\left(B + FW_{0,N-q}\right),\\
V^{C,l+2^{q-1}-2}_{N-q,j} &= V^{C,l}_{N-q+1,0} FW_{j,N-q}, V^{C,2^q-3}_{N-q,j} = A_j + DW_{j,N-q}, j \neq 0\\
V^{C,2^q-2}_{N-q,0} &= B + FW_{0,N-q}, V^{C,2^q-2}_{N-q,j} = FW_{j,N-q}, j \neq 0.
\end{aligned}
$$

Now it is clear that a formula for $\mathcal{O}(x_0, .., x_{N-q})$ can be obtained by replacing q with $q+1$ in (33) and using the coefficients (31), (32), (38), (39) and (40).

The optimal cost $I_{x_0,N}(u)$ is given by $\mathcal{O}(x_0)$, i.e. by formula (33) written for $q = N+1$.

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