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Prediction-based control for LTI systems with uncertain time-varying delays and partial state knowledge

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The stability of a prediction-based controller for LTI systems is studied in presence of time-varying input and output delays. The uncertain delay case is treated as well as the partial state knowledge case. The reduction method is used in order to prove the convergence of the closed-loop system including the state-observer, the predictor and the plant. Explicit conditions that guarantee the closed-loop stability are given thanks to a Lyapunov-Razumikhin analysis. The results are illustrated thanks to simulations.

Keywords: Prediction-based control, uncertain delay, time-varying delay, reduction method, Lyapunov-Razumikhin functional, output feedback

1. Introduction

Time-delay systems (TDS) is a very wide area of research in the control community since delays are almost always present in real systems. The survey papers Richard (2003) and Gu and Niculescu (2003) give an overview about the topic. TDS can be classified according to the variables affected by the delays: input, state, output or any combination of them. As soon as a plant is remotely controlled, input and output delays occur because of communication time-lag. This article deals with the combination of both input and output delays.

1.1 *Input delay systems*

A vast literature is available for the control of input-delay systems. The works on this topic can be divided into two main classes according to the controller design: memory free (or memoryless) and memory controllers. The interest of memory free controllers is that they are easier to implement in practice. The truncated predictor feedback is an example of such a technique: see Zhou, Lin, and Duan (2010) for constant delay and Yoon and Lin (2013), Zhou, Lin, and Duan (2012), Zhou and Lin (2014) for time-varying delays. Sliding mode and adaptive control have also been used in Richard, Gouaisbaut, and Perruquetti (2001) and Choi and Lim (2006) respectively. The drawback of memoryless control is that it is usually not possible to compensate for an arbitrarily long delay except for some particular classes of systems as in Mazenc, Mondié, and Niculescu (2003), Mazenc, Mondié, and Francisco (2004), Lin and Haijun (2007).

The other class of controller is the memory controllers. For systems with a single delay, a mem-

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ory controller is often a prediction-based controller. Note that a predictive controller is always a memory controller but a memory controller is not always a predictive controller. The Smith predictor presented in Smith (1957) is the most famous memory controller. This method is based on a frequency approach to control stable systems with a constant and known delay in the input. The works of Artstein (1982), Kwon and Pearson (1980) and Olbrot (1978) have extended the Smith predictor to state-space representation and unstable systems. In the last decade, the predictive methods have been widely studied and numerous extensions have been proposed. This interest has been supported by the use of new tools for the stability analysis: the backstepping transformation associated to a systematic Lyapunov-Krasovskii functional design has been presented in Krstic (2008) for constant delays and in Krstic (2010) for time-varying delays. For a complete overview of the backstepping method, the reader can refer to Krstic and Bekiaris-Liberis (2010) and Krstic (2009). A different Lyapunov-Krasovskii analysis has been used in Mazenc, Niculescu, and Krstic (2012). A Lyapunov-Krasovskii functional construction for multiple input delays is proposed in Li, Zhou, and Lam (2014). The backstepping transformation has been extensively used in order to design adaptive predictive control when the delay is unknown as in Bresch-Pietri and Krstic (2010) and also in case of parameter uncertainties in Bresch-Pietri and Krstic (2010). These results have been extended to systems with state delay in Bekiaris-Liberis and Krstic (2010). A further extension of this adaptive method in presence of perturbation and for the output feedback case is given in Bresch-Pietri, Chauvin, and Petit (2012). The trajectory tracking in presence of parameter uncertainties and unknown input delay is achieved in Zhu, Su, and Krstic (2015). A new prediction more robust to external perturbation has been presented in Léchappé, Moulay, Plestan, Glumineau, and Chriette (2015). A different memory method using dynamical systems to compute an approximate prediction has been presented in Besançon, Georges, and Benayache (2007) and Najafi, Hosseinnia, Sheikholeslam, and Karimadini (2013) for systems with constant input delay.

The inconvenient of above methods (except adaptive methods) is that they require the exact knowledge of the delay in order to compute the control law. In Michiels and Niculescu (2003), the delay sensitivity of the Smith Predictor is investigated. The maximal delay mismatch that preserves stability is characterized. More recently, this problem has also been studied for state-space systems. In Krstic (2008) a robustness analysis to delay mismatch is performed thanks to the backstepping techniques and a Lyapunov-Krasovskii analysis as well as in Bresch-Pietri and Krstic (2010). It has been shown in Bekiaris-Liberis and Krstic (2013) that the global exponential stability of the nonlinear predictor feedback is preserved when the delay mismatch and its rate are small enough. In Karafyllis and Krstic (2013a), a small-gain analysis allows to compute an upper bound of the delay uncertainty. In Li, Zhou, and Lin (2014), less conservatives bounds are obtained using a delay partition technique and a stability analysis from neutral system theory. All the previous results consider a constant delay or a constant delay estimation or both in the robustness analysis. In practice, the delay and its estimation are usually time-varying that is why, in the present work, the robustness analysis will be carried out with both a time-varying delay and a time-varying estimation.

1.2 *Output delay systems*

Some works have treated the observation problem in presence of output delays: see for example Cacace, Germani, and Manes (2010) for systems with a time-varying delay, Germani, Manes, and Pepe (2002), Ahmed-Ali, Karafyllis, and Lamnabhi-Lagarrigue (2013) for a sub-observer method, Ghanes, de León Morales, and Barbot (2013) for systems with an unknown delay. Above articles do not tackle the control issue of output delay systems. In Cacace, Germani, and Manes (2014a) and Cacace, Germani, and Manes (2014b), the estimated state value fed a memory free controller to control the system with full and partial state knowledge respectively.

1.3 *Input and output delays*

A generalization of the reduction method proposed by Artstein (1982) is presented in Jankovic (2010) when input and output delays (and state) are present in the loop. Predictive techniques are used in Karafyllis and Krstic (2012) and Karafyllis and Krstic (2013b) to control delayed nonlinear systems with Zero-order hold input and sampled measurements. The dynamic predictor method has been extended to nonlinear systems with sample measurement and output feedback in Karafyllis, Krstic, Ahmed-Ali, and Lamnabhi-Lagarrigue (2014). A prediction method for systems with multiple state delays is presented in Yoon and Lin (2015). All the previous works deal with constant delays. In Zhou, Li, and Lin (2013), the truncated predictor method is extended to input and output time-varying delays. A chain of predictors is designed in Cacace, Conte, Germani, and Pepe (2016). The drawback of these two works is that the delay has to be known in advance to compute the controllers. In Selivanov, Fridman, and Fradkov (2015), an adaptive memory free controller is proposed and it does not need the delay value. However, it cannot compensate for arbitrarily large delays.

1.4 *Contributions*

The contributions of this work are stated below:

- The results from Bekiaris-Liberis and Krstic (2013) and Bresch-Pietri and Krstic (2010) are extended to both time-varying and uncertain input and output delay systems. The controller only requires an estimation of the round-trip delay (input delay plus output delay).
- A prediction-based controller is designed using only partial state knowledge. Since both input and output delays are time varying, the closed-loop stability analysis is not straightforward and is a major contribution of this work.
- The reduction method is combined with a Lyapunov-Razumikhin analysis to study the closed-loop stability. As far as the authors' knowledge, this analysis has not been used in this context and allows to obtain simpler stability conditions than with the Lyapunov-Krasovskii analysis.

Most of the works that deal with input and output delays assume that the delays are known and as far as the authors know, no work is available combining input and output uncertain time-varying delays and output predictive feedback.

1.5 *Organization*

The paper is organized as follows. Section 2 gives the problem statement. The stability of the observer, the prediction-based controller and the plant is studied in Section 3. Simulations support previous theoretical results in Section 4. Finally, some perspectives are given in Section 5.

2. Problem statement

The systems considered in this work have the following form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t - h_I(t)) \\ y(t) = Cx(t - h_O(t)) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ and $h_I(t)$ and $h_O(t)$ are *uncertain* time-varying delays. The matrices A, B and C are known and of appropriate dimensions. Throughout this paper, the following assumption holds.

Assumption 1: *The pair (A, B) is controllable so there exists a matrix K such that $A + BK$ is Hurwitz which ensures the existence of a unique symmetric positive matrix P, solution of the Lyapunov equation*

$$(A + BK)^T P + P(A + BK) = -c_u I_n \tag{2}$$

The time-varying delays satisfy the following assumption.

Assumption 2: *The time-varying delays $h_I(t)$ and $h_O(t)$ are uncertain but bounded, i.e. there exist $h_{min}, h_{max} > 0$ such that*

$$h_{min} \leq h_I(t) \leq h_{max} \tag{3}$$

and

$$h_{min} \leq h_O(t) \leq h_{max}. \tag{4}$$

In addition, the delays are differentiable and the time-derivatives are bounded, i.e. there exist $\delta_I, \delta_O > 0$ such that

$$|\dot{h}_I(t)| \leq \delta_I \quad \textit{and} \quad |\dot{h}_O(t)| \leq \delta_O. \tag{5}$$

Remark 2.1: *Assumption 2 is standard since, in practice, communication delays are bounded and cannot have arbitrary fast variations.*

Since only a part of the state is known, an observer is designed in the next sections to estimate the whole state of system (1). Consequently, the following assumption is made.

Assumption 3: *The pair (C, A) is observable so there exists a matrix L such that $A + LC$ is Hurwitz and this ensures the existence of a unique symmetric positive matrix Q, solution of the Lyapunov equation*

$$(A + LC)^T Q + Q(A + LC) = -c_l I_n. \tag{6}$$

In practice, it is difficult to know exactly the delays $h_I(t)$ and $h_O(t)$. That is why, in this sub section, one supposes that only an approximation $\hat{h}(t)$ of the **round-trip delay** (input and output delays) is available. It is assumed that this estimation complies with the assumption below.

Assumption 4: *Denoting by $h(t) = h_I(t) + h_O(t)$ the round trip delay, its estimation $\hat{h}(t)$ is bounded, i.e.*

$$2h_{min} \leq \hat{h}(t) \leq 2h_{max} \tag{7}$$

with h_{min} and h_{max} defined in Assumption 2. In addition, the estimation $\hat{h}$ is differentiable and its time-derivative is bounded, i.e. there exists $\hat{\delta} > 0$ such that

$$|\dot{\hat{h}}(t)| \leq \hat{\delta}. \tag{8}$$

Remark 2.2: *The delay estimations can be available by a direct computation: in the framework of NCS, the information is sent through the network thanks to time-stamped packets. Then, it is possible to compute the value of the delay by comparing the time-stamps as in Hetel, Daafouz, Richard, and Jungers (2011). However, if the clocks in the network are not perfectly synchronized,* *the delay is not exact: this is why it is useful to consider an approximate value. Another way to have a delay approximation is to use a delay estimator: see O'Dwyer (2000) for references on constant delay estimation and Léchappé, De León, Moulay, Plestan, and Glumineau (2015), Léchappé et al. (2016) for some references on the estimation of time-varying delays. An example of delay estimator will be given in Section 4.*

3. Main results

Since the delays are uncertain, a state observer based on the round-trip delay estimation $\hat{h}(t)$ is used to estimate the state:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t - \hat{h}(t)) + L[C\hat{x}(t) - y(t)] \tag{9}$$

Then, an approximate prediction[1] is computed thanks to the estimated state $\hat{x}$ and the uncertain delay $\hat{h}$ as follows

$$z(t) = e^{A\hat{h}(t)}\hat{x}(t) + \int_{t-\hat{h}(t)}^{t} e^{A(t-s)}Bu(s)ds. \tag{10}$$

This approximate prediction z can be used to control the system by defining the following prediction-based controller

$$u(t) = Kz(t). \tag{11}$$

The analysis of the closed-loop convergence of system (1) with controller (11) is studied below.

Theorem 1: *Consider system (1), where h_I and h_O are uncertain and comply with Assumption 2 and $\hat{h}$ complies with Assumption 4. Suppose that system (1) is controlled by the feedback* (11) *where z and $\hat{x}$ are given in* (9) *and* (10) *respectively, and define*

$$\Upsilon(t) \quad = \quad ||x(t)||^2 + ||e(t)||^2 + \sup_{s\in[t-2h_{max},t]} ||u(s)||^2 + \sup_{s\in[t-2h_{max},t]} ||\dot{u}(s)||^2 \tag{12}$$

where $e(t) = \hat{x}(t) - x(t - h_O(t))$. Then, there exist $\delta_O^*, \hat{\delta}^*, D^*, \varepsilon, \varsigma > 0$ such that, provided that

$$\delta_O < \delta_O^*, \quad \hat{\delta} < \hat{\delta}^* \tag{13}$$

and

$$D < D^* \tag{14}$$

where $D = \max_{t\geq 0} |h_O(t) + h_I(t - h_O(t)) - \hat{h}(t)|$, one has

$$\Upsilon(t) \leq \varsigma\Upsilon(0)e^{-\varepsilon t}, \quad \forall t \geq 0 \tag{15}$$

and therefore $\lim_{t\to+\infty} ||x(t)|| = 0$ and $\lim_{t\to+\infty} ||e(t)|| = 0$.

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[1]This prediction has been presented in Bresch-Pietri and Krstic (2010) for an uncertain input delay. In Bresch-Pietri and Krstic (2010), the analysis is carried out only for a constant uncertain delay and when the full state is known.

Proof. By using Leibniz's formula for integral differentiation given in Flanders (1973), the derivative of z defined by (10) is

$$\dot{z}(t) \;=\; A\dot{\hat{h}}e^{A\hat{h}}\hat{x}(t) + e^{A\hat{h}}\dot{\hat{x}}(t) + Bu(t) - (1-\dot{\hat{h}})e^{A\hat{h}}Bu(t-\hat{h}) + A\int_{t-\hat{h}}^{t} e^{A(t-s)}Bu(s)ds. \qquad (16)$$

From (10), one has

$$e^{A\hat{h}(t)}\hat{x}(t) = z(t) - \int_{t-\hat{h}}^{t} e^{A(t-s)}Bu(s)ds \qquad (17)$$

Substituting $\dot{\hat{x}}(t)$ by expression (9) and $e^{A\hat{h}(t)}\hat{x}$ by (17), one obtains that the prediction z is solution of the following equation

$$\begin{aligned}\dot{z}(t) \;=\;& A\dot{\hat{h}}z(t) - A\dot{\hat{h}}\int_{t-\hat{h}}^{t} e^{A(t-s)}Bu(s)ds + Bu(t) + e^{A\hat{h}}A\hat{x}(t) + e^{A\hat{h}}Bu(t-\hat{h}) + e^{A\hat{h}}LCe(t)\\ &-(1-\dot{\hat{h}})e^{A\hat{h}}Bu(t-\hat{h}) + A\int_{t-\hat{h}}^{t} e^{A(t-s)}Bu(s)ds.\end{aligned} \qquad (18)$$

Noting that $e^{A\hat{h}}A = Ae^{A\hat{h}}$, we recognize

$$Az(t) = Ae^{A\hat{h}}\hat{x}(t) + A\int_{t-\hat{h}}^{t} e^{A(t-s)}Bu(s)ds \qquad (19)$$

and using (11), one gets

$$\dot{z}(t) \;=\; (A+BK)z(t) + \dot{\hat{h}}Az(t) + \dot{\hat{h}}e^{A\hat{h}}Bu(t-\hat{h}) - \dot{\hat{h}}A\int_{t-\hat{h}}^{t} e^{A(t-s)}Bu(s)ds + e^{A\hat{h}}LCe(t). \qquad (20)$$

The dynamics of the observation error is

$$\begin{aligned}\dot{e}(t) \;=\;& (A+LC)e(t) - \dot{h}_O Ae(t) + \dot{h}_O Ae^{-A\hat{h}}z(t) - \dot{h}_O Ae^{-A\hat{h}}\int_{t-\hat{h}}^{t} e^{A(t-s)}Bu(s)ds\\ &+\dot{h}_O Bu(\phi(t)) + B\int_{\phi(t)}^{t-\hat{h}} \dot{u}(s)ds\end{aligned} \qquad (21)$$

where $\phi(t) = t - h_O(t) - h_I(t-h_O(t))$. We define the following Lyapunov-Razumikhin candidate function

$$V(z(t), e(t)) = V_1(z(t)) + \alpha V_2(e(t)) \qquad (22)$$

with $\alpha > 0$ and

$$V_1(z(t)) = z^T(t)Pz(t), \qquad (23)$$
$$V_2(e(t)) = e^T(t)Qe(t), \qquad (24)$$

Note that P and Q are defined in (2) and (6). The function $V_1(z(t))$ verifies the inequalities

$$\lambda_{min}(P)||z(t)||^2 \leq V_1(z(t)) \leq \lambda_{max}(P)||z(t)||^2 \tag{25}$$

where $\lambda_{min}(.)$ (respectively $\lambda_{max}(.)$) denotes the minimum (respectively maximum) eigenvalue of a matrix. Similarly the function $V_2(e(t))$ verifies the inequalities

$$\lambda_{min}(Q)||e(t)||^2 \leq V_2(e(t)) \leq \lambda_{max}(Q)||e(t)||^2. \tag{26}$$

Taking the time derivative of V_1 along the trajectories of system (20) leads to the following inequality:

$$\dot{V}_1(z(t)) \leq -\Big[c_u - \hat{\delta}c_1\Big]||z(t)||^2 + \hat{\delta}c_2||z(t)||\ ||u(t-\hat{h})|| + \hat{\delta}c_3||z(t)||\ ||v(t)|| + c_4||z(t)||\ ||e(t)|| \tag{27}$$

with $c_1 = 2\ ||PA||$, $c_2 = 2e^{2||A||h_{max}}||B||\ ||P||$, $c_3 = 2e^{2||A||h_{max}}||PA||\ ||B||$, $c_4 = 2e^{2||A||h_{max}}||P||\ ||LC||$ and

$$||v(t)|| = \int_{t-\hat{h}}^{t} ||u(s)||ds. \tag{28}$$

The time derivative of V_2 along the trajectories of system (21) verifies the following inequality

$$\begin{aligned}\dot{V}_2(e(t)) \leq\ & -\Big[c_l - \delta_O c_5\Big]||e(t)||^2 + \delta_O c_6||e(t)||\ ||z(t)|| + \delta_O c_7||e(t)||\ ||v(t)|| + \delta_O c_8||e(t)||\ ||u(\phi(t))|| \\ & + c_8||e(t)||\ ||w(t)||.\end{aligned} \tag{29}$$

with $c_5 = 2\ ||QA||$, $c_6 = 2||QA||e^{2||A||h_{max}}$, $c_7 = 2||QA||\ ||B||e^{2||A||h_{max}}$, $c_8 = 2||QB||$ and

$$||w(t)|| = \int_{min(\phi(t),t-\hat{h}(t))}^{max(\phi(t),t-\hat{h}(t))} ||\dot{u}(s)||ds. \tag{30}$$

The following Razumikhin condition is assumed: for a given $\kappa > 1$, the inequality

$$V(z(t-s), e(t-s)) \leq \kappa V(z(t), e(t)) \qquad \forall s \in [0, 2h_{max}] \tag{31}$$

holds. Then it follows that

$$||z(t-s)|| \leq c_9\Big(||z(t)|| + \sqrt{\alpha}||e(t)||\Big) \qquad \forall s \in [0, 2h_{max}] \tag{32}$$

with $c_9 = \sqrt{\kappa \max(\lambda_{max}(P), \lambda_{max}(Q))/\lambda_{min}(P)}$. Using (32), we get

$$\hat{\delta}c_2||z(t)||\ ||u(t-\hat{h})|| \leq \hat{\delta}c_{10}||z(t)||\Big(||z(t)|| + \sqrt{\alpha}||e(t)||\Big) \tag{33}$$

with $c_{10} = c_2 c_9||K||$ and

$$\hat{\delta}c_3||z(t)||\ ||v(t)|| \leq \hat{\delta}c_{11}||z(t)||\Big(||z(t)|| + \sqrt{\alpha}||e(t)||\Big) \tag{34}$$

with $c_{11} = c_3 c_9 ||K|| h_{max}$. Similarly for the term in $\dot{V}_2$, one has

$$\delta_O c_7 ||e(t)||\ ||v(t)|| \leq \delta_O c_{12} ||e(t)|| \Big(||z(t)|| + \sqrt{\alpha} ||e(t)|| \Big) \tag{35}$$

with $c_{12} = c_7 c_9 ||K|| h_{max}$ and

$$\delta_O c_8 ||e(t)||\ ||u(\phi(t))|| \leq \delta_O c_{13} ||e(t)|| \Big(||z(t)|| + \sqrt{\alpha} ||e(t)|| \Big) \tag{36}$$

with $c_{13} = c_7 c_9 ||K||$. From (20), one can derive the following maximization

$$||\dot{z}(t)|| \leq c_{14} ||z(t)|| + c_{15}(||z(t)|| + \sqrt{\alpha}||e(t)||) + h_{max} c_{15}(||z(t)|| + \sqrt{\alpha}||e(t)||) + c_{16}||e(t)|| \tag{37}$$

with $c_{14} = ||A + BK|| + \hat{\delta}||A||$, $c_{15} = e^{2||A||h_{max}} ||BK|| c_9$ and $c_{16} = e^{2||A||h_{max}} ||LC||$ so

$$||\dot{z}(t)|| \leq c_{17} ||z(t)|| + (c_{18}\sqrt{\alpha} + c_{16}) ||e(t)|| \tag{38}$$

with $c_{17} = c_{14} + c_{15}(1 + h_{max})$ and $c_{18} = c_{15}(1 + h_{max})$. Then, using (38), one obtains

$$c_8 ||e(t)||\ ||w(t)|| \leq c_{19} D ||e(t)||\ ||z(t)|| + c_{20} D (c_{18}\sqrt{\alpha} + c_{16}) ||e(t)||^2 \tag{39}$$

with $c_{19} = c_8 c_{17} ||K||$ and $c_{20} = c_8 ||K||$ and

$$D = \max_{t \geq 0} |h_O(t) + h_I(t - h_O(t)) - \hat{h}(t)|. \tag{40}$$

Substituting inequalities (33)-(36) and (39) into (27) and (29), one gets

$$\begin{aligned} \dot{V}(z(t), e(t)) \leq & -\Big[c_u - \hat{\delta} c_{21}\Big] ||z(t)||^2 - \Big[\hat{\delta} c_{22}(\alpha) + c_4 + \alpha \delta_O c_{23} + c_{19} \alpha D\Big] ||e(t)||\ ||z(t)|| \\ & -\alpha \Big[c_l - \delta_O c_{24}(\alpha) - c_{25}(\alpha) D\Big] ||e(t)||^2 \end{aligned} \tag{41}$$

with $c_{21} = c_1 + c_{10} + c_{11}$, $c_{22}(\alpha) = (c_{10} + c_{11})\sqrt{\alpha}$, $c_{23} = c_6 + c_{12} + c_{13}$ and $c_{24}(\alpha) = c_5 + (c_{12} + c_{13})\sqrt{\alpha}$ and $c_{25}(\alpha) = c_{20}(c_{18}\sqrt{\alpha} + c_{16})$. Using the Young's inequality given in Mitrinović, Pečarić, and Fink (2013) and the completing the square method from Narasimhan (2009) to get rid of the crossed terms, one has

$$\begin{aligned} \dot{V}(z(t), e(t)) \leq & -\Big[c_u - \hat{\delta}(c_{21} + c_{22}(\alpha)/2) - \alpha \delta_O c_{23}/2 - c_{19} \alpha D/2 - \tfrac{c_4^2}{2\alpha c_l}\Big] ||z(t)||^2 \\ & -\alpha \Big[c_l/2 - \hat{\delta} c_{22}(\alpha)/2\alpha - \delta_O (c_{24}(\alpha) + c_{23}/2) - D(c_{25}(\alpha) + c_{19}/2)\Big] ||e(t)||^2 \end{aligned} \tag{42}$$

Choosing α sufficiently large too minimize the term $\frac{c_4^2}{2\alpha c_l}$, then taking $\hat{\delta}$, δ_O and D sufficiently guarantee the existence of c_{26}, $c_{27} > 0$ such that

$$\dot{V}(z(t), e(t)) \leq -c_{26} ||z(t)||^2 - c_{27} \alpha ||e(t)||^2. \tag{43}$$

As a consequence using (25) and (26), one can deduce that

$$\dot{V}(z(t), e(t)) \leq -\varepsilon V(z(t), e(t)) \tag{44}$$

with $\varepsilon = \min\left(\frac{c_{26}}{\lambda_{max}(P)}, \frac{c_{27}}{\lambda_{max}(Q)}\right)$. Using the Razumikhin theorem reminded in Appendix A, we deduce that the equilibrium points $(z,e) = (0,0)$ of (20)-(21) is asymptotically stable. Inequality (15) can be deduced using Lemma 1 in Appendix B. □

Remark 3.1: *Some numerical estimations of δ_O^*, $\hat{\delta}^*$ and D^* can be computed. Note that these values will be smaller than the real values because the stability analysis is conservative as shown in the simulation section.*

Remark 3.2: *In the case of a Lyapunov-Krasoskii stability analysis, the function $V(z(t), e(t)) = z^T(t)Pz(t) + e^T(t)Qe(t)$ would have been replaced by a functional of the form*

$$V'(e(t), z(t), u_t, \dot{u}_t) = z^T(t)Pz(t) + \alpha e^T(t)Qe(t) + \int_{t-2h_{max}}^{t} (2h_{max} + s - t)(\beta||\dot{u}(s)||^2 + \gamma||u(s)||^2)ds$$

with $\alpha, \beta, \gamma > 0$ and $u_t(\theta) = u(t+\theta)$, $\dot{u}_t(\theta) = \dot{u}(t+\theta)$ for $\theta \in [-2h_{max}, 0]$. The integral term in $||\dot{u}(s)||^2$ is needed to deal with the term in $\dot{u}$ in equation (21) and the integral term in $||u(s)||^2$ is required to maximize the terms in u in equations (20) and (21). As a result, computations in the Lyapunov-Krasovskii analysis involve more terms and result in intricate stability conditions.

Condition (14) shows that the estimated round trip delay $\hat{h}$ has to be close to the delay $h_O(t) + h_I(t - h_O(t))$ to guarantee the closed-loop stability. Note that if the input delay is slow varying or if the output delay is small then, in this case, $h_I(t - h_O(t)) \approx h_I(t)$ and $\hat{h}(t)$ is close to the round trip delay $h_O(t) + h_I(t)$. Note also that even if there is no condition on $\dot{h}_I$ in Theorem 1, the input delay dynamics is indirectly constrained by conditions $\hat{\delta} < \hat{\delta}^*$ and $D < D^*$. Indeed, $\hat{\delta} < \hat{\delta}^*$ means that $\hat{h}$ should be sufficiently slow varying and $D < D^*$ means that $\hat{h}$ has to be close to the real delay. Consequently, if $\hat{\delta} < \hat{\delta}^*$ and $D < D^*$ are verified it means that $\dot{h}_I$ is small.

Remark that if the delays h_I and h_O are known, they can be used to compute the prediction (10). In this case, defining $\hat{h}(t) = h_I(t) + h_O(t)$, the following corollary of Theorem 1 can be obtained.

Corollary 1: *Consider system (1), where h_I and h_O are known and comply with Assumption 2. Suppose that system (1) is controlled by the feedback (11) where z and $\hat{x}$ are given in* (9) *and* (10) *respectively with $\hat{h}(t) = h_I(t) + h_O(t)$, and define*

$$\Upsilon(t) \quad = \quad ||x(t)||^2 + ||e(t)||^2 + \sup_{s\in[t-2h_{max},t]} ||u(s)||^2 + \sup_{s\in[t-2h_{max},t]} ||\dot{u}(s)||^2 \tag{45}$$

where $e(t) = \hat{x}(t) - x(t)$. Then, there exist $\delta_I^*, \delta_O^*, \varepsilon, \varsigma > 0$ such that, provided that

$$\delta_I < \delta_I^* \quad \text{and} \quad \delta_O < \delta_O^*, \tag{46}$$

one has

$$\Upsilon(t) \leq \varsigma \Upsilon(0) e^{-\varepsilon t}, \quad \forall t \geq 0 \tag{47}$$

and therefore $\lim_{t\to+\infty} ||x(t)|| = 0$.

Proof. The proof is deduced from the proof of Theorem 1. Noting that if the delays are known then $D = \max_{t\geq 0} |h_O(t) + h_I(t - h_O(t)) - \hat{h}(t)|$ with $\hat{h}(t) = h_I(t) + h_O(t)$ becomes $D = \max_{t\geq 0} |h_I(t - h_O(t)) - h_I(t)|$. Remark that from the Taylor-Lagrange formula in Kline (1998), there exists $\xi \in [t - h_O(t), t]$

such that $h_I(t - h_O(t)) = h_I(t) - h_O(t)\dot{h}_I(\xi)$ then one deduces that $D \leq h_{max}\delta_I$. Choosing δ_I sufficiently small guarantees that (14) is verified. □

For the known delay case, condition (46) means that the closed-loop system is stable if the input and the output time-varying delays are sufficiently slow varying. In the next section, the qualitative behavior described in Section 3 is illustrated by simulation.

4. Simulation

Consider the second order system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t - h_I(t)) \\ y(t) = Cx(t - h_O(t)) \end{cases} \tag{48}$$

with $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [1, 0]$ and $x(0) = [2, 0]^T$. The state feedback reads as

$$u(t) = Kz(t) \tag{49}$$

with $K = [0.85, -1.8]$ and z defined by (10). The gain L of the observer (9) is equal to $L = [-4, -5]^T$. Different values of h_I, h_O and $\hat{h}$ are tested in order to illustrate the stability conditions of Theorem 1 and Corollary 1. These values are chosen such that Assumptions 2 and 4 are satisfied with $h_{min} = 0.2$ and $h_{max} = 2$. Note that the different values of the delay $h_i + h_O$ have the same order of magnitude as the time constant of the open loop system (0.5 s). In addition, system (48) is open loop unstable so memoryless (non predictive) controllers would not be able to achieve a good level of performance for an arbitrarily large delay because they often require to have a small gain controller as in Choi and Lim (2006).

Remark 4.1: *The computation of z requires an integration. For open-loop stable system the integral term can be computed without discretizing the integral as in Watanabe and Ito (1981). However, for open-loop unstable systems, the integral has to be discretized in a finite number of points. This step has to be done very carefully since it can destabilize the system as pointed out in Van Assche, Dambrine, Lafay, and Richard (1999). Safe implementations of the prediction are given in Mondié and Michiels (2003) and Zhong (2006). In this article, a time-domain approximation with sample-and-hold is used and guarantees the accuracy of the prediction if the sample time is sufficiently small Zhong (2006).*

4.1 *Known delay*

For the first simulation, the delays are defined as follows

$$h_I(t) = \; 0.5 + 0.25\sin(1.5t) \quad \text{for } t \leq 25$$

and

$$h_O(t) = \begin{cases} 0.5 + 0.25\sin(1.5t) & \text{for } t \leq 25 \\ 0.5 + 0.25\sin(0.3t) & \text{for } t > 25. \end{cases}$$

These delays are supposed to be known at time t so $\hat{h}(t) = h_I(t) + h_O(t)$ is used to compute prediction (10). As displayed on Figure 1-Top, between $t = 0$ and $t = 25$ s, the delays are fast and the system is not stabilized by the predictive feedback. However, when the output delay h_O becomes slower, after $t = 25$ s, the system is stabilized by the prediction-based feedback (49). This is in accordance with the existence of an upper-bound δ_O^* for the time-derivative of $\dot{h}_O$ in Corollary 1. Note that the actual upper bound for $|\dot{h}_O|$ is around 10^{-2} for this example since the system is stable for $|\dot{h}_O| < 0.05$ and unstable for $|\dot{h}_O| < 0.3$.

For the second simulation, the delays are chosen as follows

$$h_I(t) = \begin{cases} 0.5 + 0.25\sin(1.2t) & \text{for } t \leq 25 \\ 0.5 + 0.25\sin(0.2t) & \text{for } t > 25 \end{cases}$$

and

$$h_O(t) = \quad 0.5 + 0.25\sin(1.2t) \quad \text{for } t \leq 25$$

In this configuration, the delay h_I is slowed down after $t = 25$ s but the output delay is kept constant. It is clear that the system is unstable first and then becomes stable when the input delay slows down (Figure 2).

This is in accordance with the existence of an upper-bound δ_I^* for the time-derivative of $\dot{h}_I$ in Corollary 1. Note that the actual upper bound for $|\dot{h}_I|$ is also around 10^{-2}.

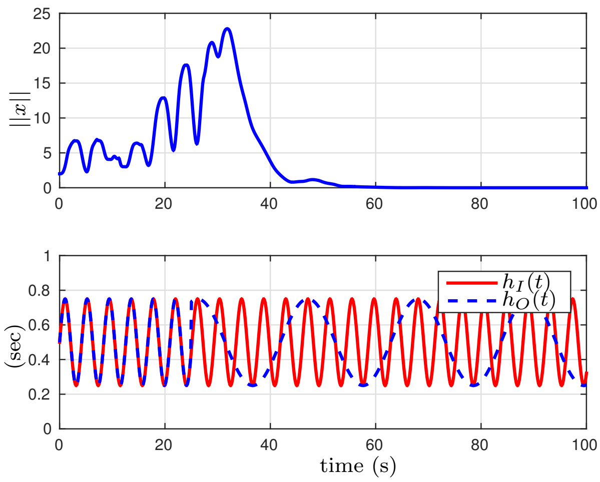

Figure 1. Top: Norm of the state – Bottom: Known Delays $h_I(t)$ and $h_o(t)$ used to compute the prediction

4.2 *Uncertain delay*

In this subsection the delays are defined as follows

$$h_I(t) = h_O(t) = \quad 0.5 + 0.25\sin(2t) \quad \text{for } t \geq 0$$

Since the delays are unknown in this part, the delay estimator presented in Léchappé et al. (2016) is used to obtain an approximation of the delay. The dynamics of the delay estimator is given by

$$\dot{\hat{h}}(t) = \rho_h[u(t - \hat{h}(t)) - u(t - h(t))]\dot{u}(t - \hat{h}(t)). \tag{50}$$

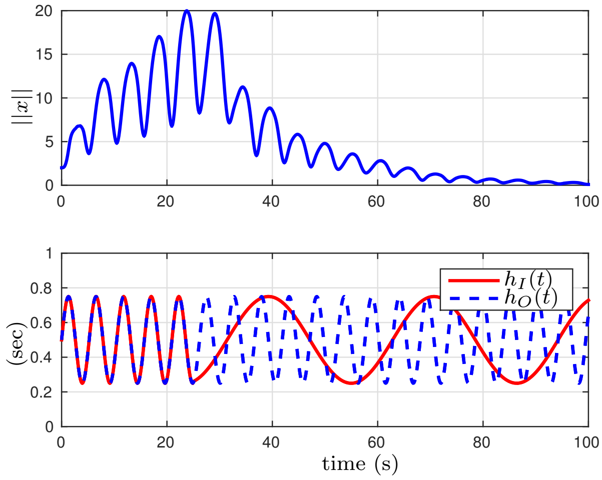

Figure 2. Top: Norm of the state – Bottom: Exact delay $h(t)$ with different dynamics.

where $\rho_h = 0.05$ and $h(t) = h_I(t) + h_O(t)$. Then this estimation is used to compute the estimation $\hat{x}$ in (9) and the prediction (10). The result of this simulation is displayed on Figure 3. Three phases can be observed on this figure.

- From $t = 0$ to $t = 20$ s, the delay estimation error is large so the system diverges.
- Between $t = 20$ s and $t = 40$ s, the delay estimation error is reduced and the system converges to the origin. From Figure 4, one can see that, when $h_O(t) + h_I(t - h_O(t)) - \hat{h}(t)$ becomes sufficiently small, the system becomes stable.
- From $t = 40$ s to $t = 100$ s, the system is stabilized to zero and the input is constant so its time derivative tends to zero. As a consequence, the delay estimator cannot track the delay variations anymore because the delay estimation dynamics $\dot{\hat{h}}$ depends on $\dot{u}$ in (50).

This simulation confirms that the delay estimation error should be sufficiently small (existence of D^* in Theorem 1) to guarantee the stability of the closed-loop system. From Figure 4, one can see that the order of magnitude of the actual value of D^* is 10^{-1}. As mentioned in Remark 3.1, estimations of the theoretical bounds δ_O^*, $\hat{\delta}^*$ and D^* defined in (13)-(14) can be computed thanks to equation (42). For this simulation example, the order of magnitude of the theoretical bounds is 10^{-9} that is much smaller than the actual bounds $\delta_I^* \approx \delta_O^* \approx 10^{-2}$ and $D^* \approx 10^{-1}$.

5. Conclusion

In this article, a stability analysis of a prediction-based controller is proposed when both the input and the output are affected by uncertain time-varying delays. The partial state knowledge case is also treated by designing a state-observer that reconstruct the delayed state. The reduction method and a Lyapunov-Razumikhin analysis are used to prove the closed-loop stability. The results are illustrated by simulation. The extension to uncertain linear systems and nonlinear systems is considered for future developments.

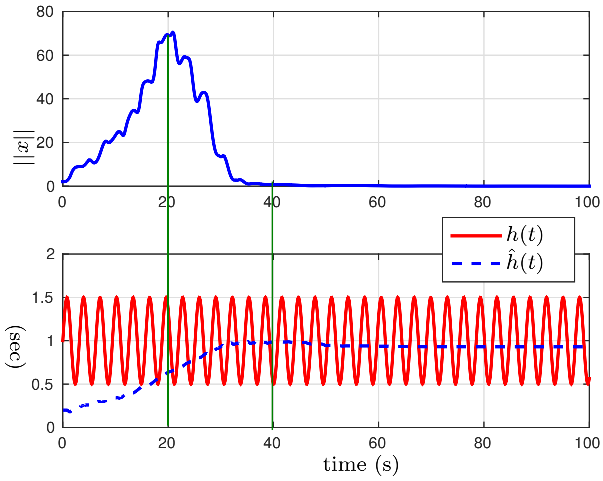

Figure 3. Top: Norm of the state – Bottom: Round trip delay $h(t) = h_I(t) + h_O(t)$ and its estimation $\hat{h}(t)$ from delay estimator in Léchappé et al. (2016).

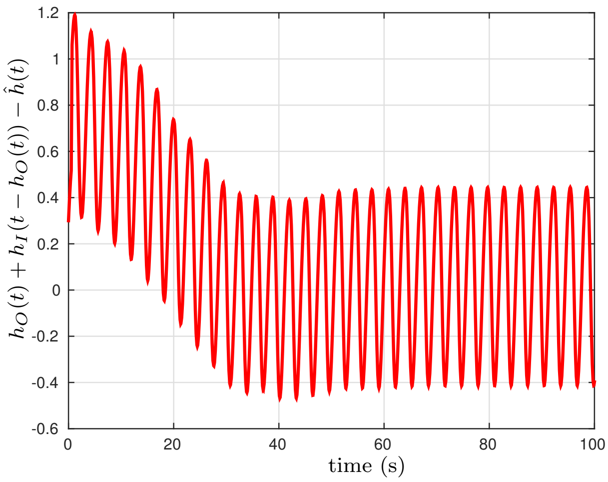

Figure 4. Delay estimation error

Appendix A.

Lyapunov-Razumikhin theorem [Hale and Verduyn Lunel (1993)] is reminded here. Consider the system

$$\dot{x}(t) = f(t, x_t) \tag{A1}$$

where $f : \mathbb{R} \times C[-h, 0] \to \mathbb{R}^n$ is continuous in both arguments and is locally Lipschitz in the second argument and $x_t(\theta) = x(t + \theta)$ for $\theta \in [-h, 0]$. It is also assumed that $f(t, 0) = 0$. Consider also the differentiable function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$ and define the derivative of V along the solution of (A1) as

$$\dot{V}(t, x(t)) = \frac{d}{dt} V(t, x(t)) = \frac{\partial V(t, x(t))}{\partial t} + \frac{\partial V(t, x(t))}{\partial x} f(t, x_t).$$

Theorem 2 (Lyapunov-Razumikhin theorem)**:** *Suppose $f : \mathbb{R} \times C[-h, 0] \to \mathbb{R}^n$ maps $\mathbb{R} \times$ (bounded set of $C[-h, 0]$) into bounded sets of $\mathbb{R}^n$ and that u, v, w: $\mathbb{R}^+ \to \mathbb{R}^+$ are continuous non decreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$ and $u(0) = v(0) = 0$, v is strictly increasing. The trivial solution of* (A1) *is uniformly stable if there exists a differentiable function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$, which is positive-definite*

$$u(|x|) \leq V(t, x) \leq v(|x|) \tag{A2}$$

such that the derivative of V along the solution of (A1) *satisfies*

$$\dot{V}(t, x(t)) \leq -w(|x(t)|) \quad \text{if} \quad V(t+\theta, x(t+\theta)) \leq V(t, x(t)) \quad \forall\theta \in [-h, 0]. \tag{A3}$$

If, in addition, $w(s) > 0$ for $s > 0$, and there exists a continuous non decreasing function $\kappa(s) > s$ for $s > 0$ such that condition (A3) *is strengthened to*

$$\dot{V}(t, x(t)) \leq -w(|x(t)|) \quad \text{if} \quad V(t+\theta, x(t+\theta)) \leq \kappa(V(t, x(t))) \quad \forall\theta \in [-h, 0] \tag{A4}$$

then the trivial solution of (A1) *is uniformly asymptotically stable. If in addition, $\lim_{s \to +\infty} u(s) = +\infty$, then it is globally uniformly asymptotically stable.*

Appendix B.

Lemma 1: *Consider the Lyapunov-Razumikhin function V defined in (22). The time-derivative of V along the trajectories of* (20)-(21) *verifies* (44) *then inequality* (15) *is verified.*

Proof. It follows from (44) that

$$V(t) \leq V(0)e^{-\varepsilon t}. \tag{B1}$$

Furthermore, since $||z(t)||^2 \leq \frac{1}{\lambda_{min}(P)} z^T(t)Pz(t)$ with $\lambda_{min}(P)$ denoting the smallest eigenvalue of P then $||z(t)||^2 \leq \frac{1}{\lambda_{min}(P)} V(t)$ so one gets

$$||z(t)||^2 \leq \frac{1}{\lambda_{min}(P)} V(0)e^{-\varepsilon t}. \tag{B2}$$

From the definition of $V(t)$, one has

$$V(0) \;\; \leq \;\; \lambda_{max}(P)||z(0)||^2 + \alpha\lambda_{max}(Q)||e(0)||^2. \tag{B3}$$

In addition, from (10), Hölder's inequality, Jensen's inequality, the following maximization can be deduced

$$||z(0)||^2 \leq c_1'(||x(0)||^2 + ||e(0)||^2) + c_2' \sup_{s\in[-2h_{max},0]} ||u(s)||^2 \tag{B4}$$

with $c_1' = 2e^{4||A||h_{max}}$ and $c_2' = 2e^{4||A||h_{max}}||B||^2 h_{max}^2$. From (B3) and (B4), it can be deduced that there exists $c_3' > 0$ such that

$$V(0) \leq c_3' \Upsilon(0). \tag{B5}$$

Then, from (B2) and (B5), one deduces

$$||z(t)||^2 \leq c_4' \Upsilon(0) e^{-\varepsilon t} \tag{B6}$$

for all $t \geq 0$ and with $c_4' = \frac{c_3}{\lambda_{min}(P)}$. Since $u(t) = Kz(t)$ for all $t \geq 0$, one has

$$\sup_{s\in[t-2h_{max},t]} ||u(s)||^2 \leq \left\{ \begin{array}{ll} \sup\limits_{s\in[-2h_{max},0]} ||u(s)||^2 + ||K||^2 \sup\limits_{s\in[0,t]} ||z(s)||^2 & \text{if } t < 2h_{max} \\ ||K||^2 \sup\limits_{s\in[t-2h_{max},t]} ||z(s)||^2 & \text{if } t \geq 2h_{max} \end{array} \right. \tag{B7}$$

so

$$\sup_{s\in[t-2h_{max},t]} ||u(s)||^2 \leq \left\{ \begin{array}{ll} \Upsilon(0) + ||K||^2 c_4' \Upsilon(0) & \text{if } t < 2h_{max} \\ c_5' \Upsilon(0) e^{-\varepsilon t} & \text{if } t \geq 2h_{max} \end{array} \right. \tag{B8}$$

with $c_5' = (||K||^2 c_4' + 1) e^{\varepsilon h_{max}}$. Noting that $c_5' \Upsilon(0) e^{-\varepsilon t} \geq \Upsilon(0) + ||K||^2 c_4' \Upsilon(0)$ for all $t \in [0, 2h_{max}]$, one can state that

$$\sup_{s\in[t-2h_{max},t]} ||u(s)||^2 \leq c_5' \Upsilon(0) e^{-\varepsilon t} \tag{B9}$$

for all $t \geq 0$. Similarly, using (20), one gets

$$\sup_{s\in[t-2h_{max},t]} ||\dot{u}(s)||^2 \leq c_6' \Upsilon(0) e^{-\varepsilon t} \tag{B10}$$

with $c_6' > 0$. Moreover, rearranging (10) gives

$$\hat{x}(t) = e^{-A\hat{h}} z(t) - \int_{t-\hat{h}}^{t} e^{A(t-\hat{h}-s)} B u(s) ds \tag{B11}$$

so by the same steps as in (B4), one gets

$$||\hat{x}(t)||^2 \leq c_1 ||z(t)||^2 + c_2 \sup_{s\in[t-2h_{max},t]} ||u(s)||^2 \tag{B12}$$

for all $t \geq 0$. Differentiating (B11) with respect to time, we can show that there exist $c_7', c_8', c_9' > 0$ such that

$$||\dot{\hat{x}}(t)||^2 \quad \leq \quad c_7' ||z(t)||^2 + c_8' \sup_{s\in[t-2h_{max},t]} ||u(s)||^2 + c_9' \sup_{s\in[t-2h_{max},t]} ||\dot{u}(s)||^2 \tag{B13}$$

Then, using (9), (B12) and (B14) one derives that there exist $c_{10}', c_{11}', c_{12}' > 0$ such that

$$||e(t)||^2 \quad \leq \quad c_{10}' ||z(t)||^2 + c_{11}' \sup_{s\in[t-2h_{max},t]} ||u(s)||^2 + c_{12}' \sup_{s\in[t-2h_{max},t]} ||\dot{u}(s)||^2 \tag{B14}$$

From (B14) and since

$$x(t) = e^{-A\hat{h}}z(t) - \int_{t-\hat{h}}^{t} e^{A(t-\hat{h}-s)}Bu(s)ds - e(t) \tag{B15}$$

then one can deduce the existence of $c'_{13}, c'_{14}, c'_{14} > 0$ such that

$$||x(t)||^2 \leq c'_{13}||z(t)||^2 + c'_{14} \sup_{s\in[t-2h_{max},t]} ||u(s)||^2 + c'_{15} \sup_{s\in[t-2h_{max},t]} ||\dot{u}(s)||^2 \tag{B16}$$

Finally, from (B6), (B9), (B10), (B14) and (B16), one can guarantee that there exists $\varsigma > 0$ such that

$$\Upsilon(t) \leq \varsigma\Upsilon(0)e^{-\varepsilon t} \tag{B17}$$

for all $t \geq 0$. This ends the proof. □

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