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Reducing Energy Consumption in Serial Production Lines with Bernoulli Reliability Machines

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Abstract

In this paper, an integrated model to minimize energy consumption while maintaining desired productivity in Bernoulli serial lines is introduced. Exact analysis of optimal allocation of production capacity is carried out for small systems, such as three- and four-machine lines with small buffers. For medium size systems (e.g., three- and four-machine lines with larger buffers, or five-machine lines with small buffers), an aggregation procedure is introduced to evaluate line production rate, and then use it to search optimal allocation of machine efficiency to minimize energy usage. Insights and allocation principles are obtained through the analyses. Finally, for larger systems, a heuristic algorithm is proposed and validated through extensive numerical experiments.

Keywords

Serial line; Bernoulli reliability model; energy efficient manufacturing; key performance indicator; probabilistic models; stochastic models; production modelling; reliability engineering

1. INTRODUCTION

Manufacturing systems need to be green. As outlined in Li *et al.* (2013), green manufacturing includes both manufacturing of green technology products, such as solar panels, battery cells, and other renewable energy products, and improving manufacturing process and control to address key performance indicators (KPIs) of energy and environmental concerns in existing systems (Joung *et al.* (2013)), such as pollution and emission reduction, energy consumption reduction, recycling and remanufacturing. Green manufacturing contributes to providing cleaner energy, reducing waste, emissions and greenhouse gasses, saving natural resources and energy, etc., which have substantial benefits to the society and economy.

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In many manufacturing processes, such as refinery, casting, painting, heating, and photovoltaic process, extensive energy consumption is needed. For example, petroleum refining industry and the chemical industry are the first and second largest users of energy in industry. In steel industry, which consumes about 6% of all energies, about 30% of the energy consumption is used in heating. In automotive industry, more than 60% of the energy consumption is in painting booths and ovens. Therefore, studying energy efficient and environment friendly (EEEF or E^3F) manufacturing systems has received growing interests in recent years. Significant research attention is paid to improving energy efficiency in manufacturing process, particularly, those with energy intensive operations. However, in many studies, the productivity concerns are not addressed.

In another direction, substantial amount of research efforts have been devoted to manufacturing systems. Most of them address issues related to throughput, quality, cost, lead time, and demand satisfaction, etc. But energy consumption is typically not considered in such works. In other words, the productivity improvement and energy reduction are studied separately instead of taking their interdependency into analysis. However, these two areas are tightly coupled. The energy reduction efforts is typically carried out with the loss in production performance. To our best knowledge, only limited studies have addressed the tradeoffs between productivity and energy. Therefore, there is a strong need to develop an integrated model to optimize energy consumption and productivity simultaneously.

The main contribution of this paper is in presenting such an integrated model. Using which, the energy consumption as a sustainability KPI in manufacturing systems (NIST (2009)) is minimized while still maintaining the desired production rate. Specifically, we consider serial production lines with multiple Bernoulli machines and finite buffers. Analytical formulas and procedures are developed for small and medium size systems and a heuristic algorithm is introduced for larger systems.

The remainder of this paper is organized as follows. Section 2 reviews the related literature. System description and problem formulation are presented in Section 3. Section 4 focuses on three- and four-machine lines with small buffers, and Sections 5 and 6 analyze medium and large size systems, respectively. Conclusions are given in Section 7. Proofs and additional examples are provided in the Appendices.

2. LITERATURE REVIEW

Manufacturing systems have received substantial research attention during the last five decades. Numerous studies have been focusing on productivity analysis, quality improvement, product scheduling, cost and lead time reduction, and customer demand satisfaction (see, for example, monographs by Viswanadham and Narahari (1992); Buzacott and Shanthikumar (1993); Papadopoulos *et al.* (1993); Tempelmeier and Kuhn (1993); Gershwin (1994); Li and Meerkov (2009), reviews by Dallery and Gershwin (1992); Papadopoulos and Heavey (1996); Li *et al.* (2009)). Among all these studies, throughput analysis to maintain and improve system productivity has been one of the center topics. Due to the complexity in manufacturing systems, direct analysis may not be possible. Thus, various aggregation and decomposition methods have been proposed to evaluate system performance, see representative papers by Gershwin (1987); Dallery *et al.* (1988); Jacobs and Meerkov (1995); Chiang *et al.* (2001); Li (2005); Colledani *et al.* (2008); Zhao *et al.* (2016).

Due to the importance of energy and environmental concerns, sustainable manufacturing has become more and more critical (Li *et al.* (2013)). Substantial amount of efforts have been devoted to improving energy intensive manufacturing processes. For example, as metal casting is an energy and materials intensive manufacturing process, Ross (2006) creates a material and energy flow model of the typical iron casting facility to analyze the effect of melting technologies on energy, materials and pollution. It is shown that the impact on energy and the associated carbon dioxide emissions change widely with different melting technologies. Mattis *et al.* (1996) investigate the energy expenditure in the forming process of designing injection-molded thermoplastic parts. The influence of component and mold design decisions and process parameters on the process energy efficiency is analyzed through a 3D solid modeling framework that integrates environment, filling and post-filling behavior, and an energy-based process model. Calvanese (2013) intends to minimize energy consumption of machine tools (CNC milling centre) during operations. The energy consumptions in terms of power adsorption are analyzed in a closed analytical form and then numerically optimized to identify the cutting conditions that satisfy the minimum energy criteria.

As paint shop has been the largest energy unit in automotive assembly plants, Kolta (1999) presents various aspects of volatile-organic-compound (VOC) emission control in automotive painting operations, including sources of VOC emissions, monitoring, equipment, processes, and materials, etc. Guerrero *et al.* (2011) study capacity design of repair facility in paint shops to reduce the energy consumption due to excessive repaints. Wang *et al.* (2011) introduce efficient scheduling algorithms for paint operations to reduce energy consumption without investigating on new equipments.

In addition, Sekulic (2009) uses exergy (the maximum amount of available work) as a metric to measure energy availability, and studies insights of sustainability and energy-related metric based on entropy balancing of a non-energy system. A manufacturing process of controlled atmosphere brazing of aluminum is used as an example, which involves mass production of compact heat exchangers for automotive, aerospace, and process industries. Based on the first and second laws of thermodynamics, Gyftopoulos and Beretta (2005) model manufacturing as a sequence of open thermodynamic processes. Aligning in this direction, Gutowski *et al.* (2009) use a thermodynamic framework to characterize the material and energy resources used in manufacturing processes. A total of 20 processes are analyzed and the relevances of thermodynamics (including exergy analysis) for all processes are illustrated. It is shown that the long-lasting focus in manufacturing on product quality not necessarily leads to energy/material conversion efficiency.

However, in most of these studies, issues of production performance are seldom investigated. As both productivity and energy have significant importance, an integrated study to maintain desired production performance while minimizing energy consumption has become a new trend in production systems research. Giret *et al.* (2015) present a state-of-the-art review on sustainable manufacturing operations scheduling, and discuss the relevant challenges, issues and urgent problems. Jaehn (2016) introduces terms and definitions of sustainable operations by focusing on the interactions between economic, social and ecological aspects, and organizes them into various areas arising from a typical enterprise structure.

Along this direction, Chen *et al.* (2011) introduce effective scheduling and control policies of machine startup and shutdown to achieve energy consumption reduction in production systems. Transient analysis of Bernoulli serial lines with time-dependent machine efficiencies is carried out. Tradeoff between productivity and energy efficiency is studied using a constrained optimization approach. Mashaei and Lennartson (2013) derive a control policy to turn off the idle machines and reduce their level of energy consumption in a closed-loop pallet system. The operation of the machines and the motion of the pallets are coordinated to gain the minimal energy consumption as well as to maintain the desired throughput. A mixed integer nonlinear minimization algorithm is developed and a heuristic approach is introduced to reduce time complexity. Xu and Cao (2014) consider improving energy efficiency through effective scheduling of maintenance operations using a discrete-time, discrete-state homogeneous Markov chain model of machine tool deterioration and renewal reward algorithm of average energy efficiency and average productivity. Frigerio and Matta (2015) propose a framework to integrate different control policies for machine switching off in order to minimize the expected energy during the machine idle periods. The energy consumed at each machine state is modeled explicitly and a comparison of the most common practices in manufacturing is reported. In addition, Fernandez *et al.* (2011) present a "just-for-peak" inventory management policy to reduce electricity consumption without sacrificing system throughput during peak periods for manufacturing firms. The electricity cost is considered through balancing with inventory holding and backorder costs in stochastic make-to-stock manufacturing lines by Papier (2016).

In spite of these efforts, an integrated study on productivity and energy efficiency is still in an infant phase. More in-depth research is desirable. To improve energy efficiency and to maintain productivity performance simultaneously, Su *et al.* (2016) present an integrated model for two-machine Bernoulli lines. Analytical formulas to achieve optimal solutions to assign production capacity are derived. However, the study for longer lines is still unavailable. The goal of this paper is to contribute to this end.

3. PROBLEM FORMULATION

Consider a serial production line making one product type with M machines and $M - 1$ buffers separating each pair of consecutive machines (see Figure 1 where the circles represent the machines and the rectangles are the buffers). The following assumptions address the machines, the buffers, the energy consumption, and their mutual interactions.

1. All machines have identical cycle time. During each cycle, machine m_i, $i = 1, \ldots, M$, has probability p_i to be up and $1-p_i$ to be down. When a machine is up, it is capable of processing a part, while when it is down, no production takes place. The status of each machine is determined at the beginning of each cycle, independent of the status in the previous cycle and buffer contents.

2. Buffer b_i, $i = 1, \ldots, M-1$, has a finite capacity $0 \leq N_i < \infty$. The buffer status is determined at the end of each cycle.

3. If buffer b_i, $i = 1, \ldots, M-1$, is empty at the beginning of a cycle, then machine m_{i+1} is starved in this time slot if it is up. Machine m_1 is never starved.

4. If buffer b_i, $i = 1, \ldots, M-1$, is full and machine m_{i+1} does not take a part from b_i at the beginning of a time slot, then machine m_i is blocked during this cycle if it is up. Machine m_M is never blocked.

5. The energy consumption of machine m_i, $i = 1, \ldots, M$, includes the average electrical power to start up and maintain the machine and environment, and the power used to process the parts, $k_i p_i$, which is proportional to its production capacity p_i and k_i is a constant, referred to as energy consumption coefficient.

6. The desired production rate the system needs to maintain is denoted as PR_d.

Remark 3.1

Bernoulli machine reliability model is assumed in this work. Such a model is typically suitable for assembly type systems where machine downtime is comparable to its cycle time. For other systems, a transformation can be introduced that

$$p_i = \frac{c_i}{c_{max}} \cdot \frac{\mu_i}{\lambda_i + \mu_i},$$

where c_i is speed (or capacity) of machine m_i, and $c_{max} = \max_i c_i$. The shortest processing time ($1/c_{max}$) is selected as the unit for cycle time. In addition, λ_i and μ_i are the failure and repair rates of m_i, respectively. Thus, parameter p_i represents the probability to produce a part during cycle time $1/c_{max}$, and it can also be viewed as the relative efficiency or capacity of the machine, or the proportion of time the machine is working during a cycle. The Bernoulli model has been used in many manufacturing system studies successfully (see monograph by Li and Meerkov (2009) and papers by Lim *et al.* (1990); Kuo *et al.* (1996); Li and Meerkov (2001); Arinez *et al.* (2010); Wang and Li (2010); Zhao and Li (2014)). In future work, other machine reliability models will be studied.

Remark 3.2

In this study, we assume electrical power being the practical form of energy considered. Such energy can be divided into two categories. One is the energy to start up and keep the machine at an “on” or “ready” status as well as maintain the necessary environment (such as temperature, humidity, lights), which often require a fixed power. Another is the energy to operate the machine to produce a part, which typically consumes power proportional to the machine’s processing rate (e.g., parameter p_i). In this study, we focus on the second one. Thus, the energy consumption for machine m_i during a cycle is

$$E_i = k_i p_i, \quad i = 1, \ldots, M. \qquad (1)$$

Similar formulation is also given in Calvanese (2013), Gutowski *et al.* (2009) and Su *et al.* (2016).

Then the total energy consumed in a multi-machine line is

$$E = \sum_{i=1}^{M} k_i p_i . \quad (2)$$

The objective of this study is to seek the optimal allocation of machine capacity (or efficiency), denoted as p_i^*'s, $i = 1, \ldots, M$, to minimize energy consumption E (or equivalently, $\sum_{i=1}^{M} k_i p_i$) while ensuring the desired production rate PR_d. Define PR as the production rate of the serial line. Then the problem to be addressed can be formulated as:

$$\min \ E = \sum_{i=1}^{M} k_i p_i, \quad (3)$$

$$\text{s.t. } PR \geq PR_d . \quad (4)$$

Remark 3.3

In general, the failure rates (λ_i in Remark 3.1) may not be fully controllable but predictable or observable. The repair rates (μ_i in Remark 3.1) are relatively easier to adjust and the speed can be controlled. For example, many machines have some freedom to adjust the processing time ($1/c_i$), such as in machining and painting. In this case, it is still possible to control machine parameters to reach the desired p_i. This has been achieved in many case studies (e.g., see Li and Meerkov (2009)).

To solve this problem, we first study three- and four-machine production lines with very small buffers to derive exact solutions. Then, for medium size systems (typical three- or four-machine lines with relatively large buffers), an aggregation procedure is introduced for production rate evaluation and then used for optimal solution search. Finally, for larger systems with five or more machines, a heuristic algorithm is developed.

4. EXACT ANALYSIS FOR SMALL SYSTEMS

For small systems, it is possible to develop Markov chain models to derive exact results. In an appropriately defined state space, transition probabilities can be defined, and steady state probabilities can be calculated by solving balance equations. Then the line production rate can be evaluated. By searching the possible scenarios, the optimal solution to minimize energy consumption while maintaining the desired production rate can be obtained.

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In this section, first, we analyze balanced lines with identical N_i's, $i = 1, \ldots, M-1$, and k_i's, $i = 1, \ldots, M$. Next, lines with unequal N_i's but equal k_i's are studied. Finally, systems with nonequal k_i's are investigated.

4.1 Lines with Identical Buffers and Energy Coefficients

First, consider the simplest three-machine line L_1, where energy coefficients and buffer capacities are assumed to be one.

$$L_1: \quad N_1 = N_2 = 1, \quad k_1:k_2:k_3 = 1:1:1\,.$$

According to Li and Meerkov (2009), the states of a Bernoulli production line can be represented by the occupancy of the buffers. Thus, the state space of L_1 can be described by $S = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$. Let $P_{j,i}$ be the transition probability from state j to state i, and P_i be the steady state probability. Then the corresponding transition probability matrix and balance equations can be obtained. Solving them we obtain:

Proposition 4.1—Under assumptions 1)–6), the steady state probabilities of Line L_1 are

$$P_{(0,0)} = \frac{(1-p_1)^2 p_2 p_3^2}{\Lambda}, \quad P_{(1,0)} = \frac{p_1 p_3 (p_3 + p_1(1-p_2-p_3))}{\Lambda},$$

$$P_{(0,1)} = \frac{(1-p_1)p_1 p_2 p_3}{\Lambda}, \quad P_{(1,1)} = \frac{p_1^2 p_2}{\Lambda}, \quad (5)$$

where

$$\Lambda = p_1^2 p_2 (1-p_3)^2 + p_1^2 p_3 (1-p_3) + p_2 p_3^2 + p_1 p_3 (p_2 + p_3 - 2p_2 p_3)\,. \quad (6)$$

Then the line production rate can be calculated as

$$PR_{L1} = (P_{(0,1)} + P_{(1,1)})p_3 = \frac{(1-p_1)p_1 p_2 p_3^2 + p_1^2 p_2 p_3}{\Lambda}\,. \quad (7)$$

Proof: See Appendix A.

Although a closed-form expression of line production rate can be obtained, finding a closed-form solution of p_i^*'s, $i = 1, 2, 3$ for optimization problem (3) and (4) is impossible. Therefore, *Mathematica* is used to search the optimal p_i^*'s satisfying the desired production rate. The results are illustrated in Figure 2(a).

Similar analysis is applied to another three-machine line L_2 with the only difference that all buffer capacities are increased by 1, and the optimal allocations of p_i's are shown in Figure 2(b).

$$L_2: \quad N_1 = N_2 = 2, \quad k_1 : k_2 : k_3 = 1 : 1 : 1 .$$

For both lines, p_1^* and p_3^* are always equal and smaller than p_2^*, indicating that there exists an inverted bowl pattern for the optimal allocation of production capacity. In addition, the difference between p_1^* (or p_3^*) and p_2^* is typically small, less than 0.05, and is also distributed as an inverted bowl shape with respect to PR_d. Such small difference between p_1^* and p_2^* suggests that by evenly distributing p_i along the line, the result could be close to the optimal solution. Since the inverted bowl shape allocation delivers the maximal production rate and the performance difference between inverted bowl and evenly distributed allocations is very small (Li and Meerkov (2009)), such a result matches with intuition. To minimize energy consumption, the maximal production rate should be equal to the desired one. Thus, the inverted bowl or well balanced allocation should be selected.

Next, a four-machine line (L_3) is investigated with all buffer capacities and energy coefficients are one.

$$L_3: \quad N_1 = N_2 = N_3 = 1, \quad k_1 : k_2 : k_3 : k_4 = 1 : 1 : 1 : 1 .$$

Figure 3 illustrates the optimal allocations of p_i^*'s. As we can see, the same properties are observed. In other words, $p_2^* = p_3^* > p_1^* = p_4^*$ is the optimal allocation, which again shows a flat inverted bowl pattern with very small difference between p_1^* and p_2^*.

4.2 Lines with Nonidentical Buffers and Identical Energy Coefficients

To investigate how buffer capacity affects optimal allocation of production capacity, two three-machine lines, L_4 and L_5, are considered. Both lines have identical energy coefficients (all equal to one) but nonidentical buffers.

$$L_4: \quad N_1 = 1, N_2 = 2, \quad k_1 : k_2 : k_3 = 1 : 1 : 1,$$

$$L_5: \quad N_1 = 2, N_2 = 1, \quad k_1 : k_2 : k_3 = 1 : 1 : 1 .$$

Using the similar analysis approach, the optimal allocations of p_i^*'s are illustrated in Figure 4. Again, larger production capacity is allocated to the middle machine. In addition, an increase in one buffer size leads to a decrease in optimal production capacity of neighbouring machines. Such a decrease is more dominating when the machine is at the

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beginning or the end of the line. For example, an increase in N_1 results in smaller p_1^* and p_2^*, but p_1^* has a much severe decrease. This is due to the fact the middle machine has more weight in determining the production rate than the first and last machines.

4.3 Lines with Identical Buffers and Nonidentical Energy Coefficients

First we consider two three-machine lines (L_6 and L_7) with equal buffer capacities and nonequal but symmetric energy coefficients. In both lines, the middle machines' energy consumption coefficients are larger than those of other machines.

$$L_6: \quad N_1 = N_2 = 1, \quad k_1 : k_2 : k_3 = 1:2:1,$$

$$L_7: \quad N_1 = N_2 = 2, \quad k_1 : k_2 : k_3 = 1:2:1.$$

Using a similar method we can derive the optimal allocation of p_i's numerically. The results are presented in Figure 5 for Lines L_6 and L_7, respectively. Examining the results, it is observed that p_1^* and p_3^* are still equal but greater than p_2^*, which represents a bowl shape. This is due to m_2's higher coefficient in energy consumption. Note that p_1^* and p_3^* will be capped by 1 when PR_d is large. Thus, by considering k_ip_i, we still obtain inverted bowls. Also note that now $p_1^* - p_2^*$ shows an inverted bowl pattern.

Next we study the impact of energy coefficients on optimal allocation of production capacity. In Lines L_8 and L_9, all buffers have capacity one. The energy coefficients are neither equal nor symmetric. Solving the optimization problem, the optimal allocations of p_i's are illustrated in Figure 6.

$$L_8: \quad N_1 = N_2 = 1, \quad k_1 : k_2 : k_3 = 1:2:3,$$

$$L_9: \quad N_1 = N_2 = 1, \quad k_1 : k_2 : k_3 = 1:3:2.$$

From these figures, it is clear that the largest energy coefficient is corresponding to the smallest production capacity (p_3^* in L_8 and p_2^* in L_9). For the other two machines, if the difference between k_i's is large (as in L_8 and L_9), higher production capacity is assigned to the machine with a smaller k_i. In other words, larger energy coefficient leads to smaller production capacity, and the middle machine has higher priority in production capacity allocation.

Finally, when each buffer capacity is increased to two in Lines L_{10} and L_{11}, similar property is observed (see Figure 7).

$$L_{10}: \quad N_1 = N_2 = 2, \quad k_1 : k_2 : k_3 = 1 : 2 : 3,$$

$$L_{11}: \quad N_1 = N_2 = 2, \quad k_1 : k_2 : k_3 = 1 : 3 : 2 .$$

In summary, the following conclusions can be drawn from the above results:

- For lines with identical buffer capacity and energy coefficients, the optimal allocation of production capacity has an inverted bowl shape with very small differences between machines. Evenly distributing production capacity will lead to a close to optimal result.
- For lines with nonidentical buffer capacity but identical energy coefficients, larger buffers lead to lower production capacity in neighboring machines, in an descending order from the center of the line to the two ends.
- For lines with identical buffers but nonidentical energy coefficients, larger coefficients are associated with lower production capacity.
- In general, the middle machines receive more priority in production capacity allocation comparing with the machines at the beginning or end of the line.

5. AGGREGATION APPROACH FOR MEDIUM SIZE SYSTEMS

The exact analysis is only available for small systems, i.e., three-machine lines with small buffers (capacities equal to one or two), or four-machine line with the smallest buffer capacity. For medium size systems, deriving the exact solution of steady state probabilities is not feasible. Therefore, an approximation method is used for performance evaluation. Specifically, the aggregation procedure introduced in Chapter 4 of Li and Meerkov (2009) is used to evaluate line production rate so that the optimization problem defined in (3) and (4) can be solved using *Mathematica*. For the sake of self-contain of this paper, the aggregation procedure is described in Appendix B. Below we will first study medium size lines with identical buffers and energy coefficients, then extend to systems with nonidentical buffers or coefficients.

5.1 Lines with Identical Buffers and Energy Coefficients

First we consider three-machine lines with equal buffer capacity ranging from one to ten, and the same energy coefficients, i.e., $k_i = 1$, $i = 1, 2, 3$. The desired production rate PR_d is set to be 0.75 (and in all subsequent studies).

As one can see from Table 1, larger buffer size always implies smaller production capacity allocation for each machine, resulting in lower energy consumption. The rationale behind this is that larger buffers lead to higher production rate. Thus, to only maintain the desired production rate, machine reliability can be reduced so that the required energy consumption will be smaller. In addition, Table 1 suggests that the production capacity is almost evenly distributed, which is slightly different with the results obtained for small buffer systems in the previous section (such as the inverted bowl shape). However, the difference is small. In the first two rows of Table 1, comparison with the optimal results obtained through exact analysis is provided. Clearly, the differences are smaller than 0.8%, which indicates that the results obtained using aggregation procedure provide close to optimal solutions of allocating p_i^*'s. Such differences are in the same order as throughput difference between an inverted bowl allocation and a well balanced distribution (see Li and Meerkov (2009)). Note that for other rows and in subsequent tables, exact analysis cannot generate optimal results due to high dimension of transition matrices, thus an "N/A" note is presented.

For four-machine lines with identical buffers (capacities one to three) and energy coefficients, similar results can be observed (as shown in Table 2). Again the first row in Table 2 shows the comparison with the optimal solution, and the difference is only 0.7%.

Similar results are obtained for five-machine lines with small buffer capacities (one or two, see Table 3). Note that in this case, exact analysis is unavailable.

Therefore, we conclude that for lines with identical buffers and energy coefficients, evenly distributing production capacity is a practical way to minimize energy usage and maintain the desired productivity level.

5.2 Lines with Nonidentical Buffers and Equal Energy Coefficients

To study the impact of buffer capacity, we consider three- and four-machine lines with different buffer capacities but equal energy coefficients, and results are presented in Tables 4 and 5, respectively.

Again as one can see, in Table 4, an increase in N_1 leads to decreases in p_1^*, and sometimes p_2^* as well. Similarly, larger N_2 also indicates smaller p_2^* and p_3^*. Energy is reduced in both cases. In Table 5, an N_3's increase implies decreases in p_1^* and p_2^*. Thus, the conclusions are consistent, i.e., an increase in N_i leads to decreases in p_i^* and possible p_{i+1}^* as well as the total energy consumed. Comparing to exact analysis, the results from aggregation-based method only have small differences (within 1.0%, see the first two rows in Table 4). The properties obtained from exact analysis still apply.

For five-machine lines with small buffers, as before, similar properties are observed (Table 6).

5.3 Lines with Identical Buffers and Unbalanced Energy Coefficients

To investigate the impact of energy consumption coefficients, lines with nonidentical energy coefficients are considered. In Table 7, three-machine lines with identical buffers ($N_1 = N_2 = 2$) but different energy coefficients are studied. Four different patterns of k_i's are considered: ramp, slope, bowl-type, and inverted bowl-type. As one can observe, an increasing (or decreasing) pattern of en- ergy coefficients results in a decreasing (respectively, increasing) distribution of optimal production capacity. In bowl and inverted bowl shapes of k_i's allocation, p_i^*'s follow an inverted bowl and a regular bowl pattern, respectively. In addition, comparing with the exact solutions, the allocations using the aggregation procedure lead to close to optimal results, with differences less than 1.5%.

When four-machine lines are considered, similar properties still follow, as illustrated in Table 8. This is also true for the five-machine case (see Table 9). In addition, note that the significantly higher energy consumptions in the first two rows of Tables 8 and 9 than those in other lines are due to much larger energy coefficients. Thus, we still claim that in lines with identical buffers, the production capacity needs to be allocated negatively to the energy coefficients distribution.

The above results indicate that, for medium size systems, the method based on aggregation procedure can be applied to search the optimal production capacity allocations for energy consumption minimization while still ensuring the desired line throughput. When systems become more complex, the search space for optimal solution becomes too large so that computation intensity becomes an issue. Thus, a heuristic method will be pursued.

6. HEURISTIC METHOD FOR LARGE SYSTEMS

Although the aggregation approach can be used to study medium size systems, computation intensity still limits its application to larger systems. Therefore, in this section, a heuristic method is proposed for systems with more than five machines. First, a heuristic algorithm is introduced. Then, the performance of the algorithm is studied by comparing with the results from aggregation approach in medium size systems, and with simulation results in larger systems.

6.1 Algorithm Description

The idea of the heuristic algorithm is to forwardly group the energy consumption coefficients of two machines into one, and continue until the last one, then backwardly derive production capacity allocation for every two machines until the first one. Such a process is repeated until convergence and the energy efficiency can be improved while still maintaining the desired production rate.

From Su *et al.* (2016), for two-machine lines, the energy consumption is minimized when they are balanced. Such a property is practically observed for small and medium systems introduced in previous sections. Since the two-machine line is used in each iteration, a basic principle of the algorithm is to balance energy consumption among all the machines. Intuitively, when n machines are grouped into one machine, denoted as m_n^h (where superscript ‘h’ indicates heuristic algorithm), the energy consumption coefficient, k_n^h, can be selected as the sum of the coefficients of all machines from m_1 to m_n, i.e., $\sum_{i=1}^{n} k_i$. However, such a sum will lead to a larger coefficient. In particular, when this machine is further grouped with the next machine (i.e., machine m_{n+1} with coefficient k_{n+1}), the unbalance between coefficients k_n^h and k_{n+1} will lead to unbalanced production capacity. In other words, p_{n+1} needs to be quite large (or almost 1) in order to make $k_{n+1}p_{n+1}$ comparable to $k_n^h p_n^h$, where p_n^h is the production rate of machine m_n^h. Such unbalance will lead to higher energy consumption. Thus, to reduce the unbalance, k_n^h should be smaller than $\sum_{i=1}^{n} k_i$, i.e., a discount factor is needed.

To introduce a discount factor for $\sum_{i=1}^{n} k_i$, we intend to evenly distribute the coefficients. To do this, a proportional ratio, $(\sum_{i=1}^{n} k_i p_i)/(\sum_{i=1}^{n+1} k_i p_i)$, is introduced (assume we know p_i for now). Moreover, since the unbalanced k_i's will lead to more unbalance, the proportion ratio should be further discounted based on the severity of differences in k_i's. Thus, an index β_n is introduced to the power of the discount factor. When k_i's are similar, the effect should be ignored, i.e., $\beta_n = n - 1$. When k_i's are substantially different, the distribution range ($\max_x k_x - \min_x k_x$) is taken into account. Thus, introduce

$$\beta_n = \begin{cases} 1, & \text{if } \max_{i=1}^{n} k_i - \min_{i=1}^{n} k_i = n, \\ n-1, & \text{if } \max_{i=1}^{n} k_i - \min_{i=1}^{n} k_i = 0, \\ \lfloor n - \max_{i=1}^{n} k_i + \min_{i=1}^{n} k_i \rfloor, & \text{otherwise.} \end{cases} \quad (8)$$

Then, the energy coefficients for machine m_n, $n = 2, \ldots, M - 1$, can be calculated forwardly as

$$k_n^h = \left(\frac{\sum_{i=1}^{n} k_i p_i}{\sum_{i=1}^{n+1} k_i p_i} \right)^{\beta_n} \cdot \sum_{i=1}^{n} k_i, \quad n = 2, \ldots, M-1 .$$

Using these k_i^h's, a backward allocation of new p_i's can be carried out. First, consider machines m_{M-1}^h and m_M. Using k_{M-1}^h and k_M, and the desired production rate PR_d, allocation of p_{M-1}^h and p_M can be carried out by using the two-machine analysis method introduced in Su *et al.* (2016). In addition, let p_{M-1}^h be the desired production rate of the first ($M - 1$)-machine line, denoted as $PR_{d,M-1}$. Next, using k_{M-2}^h and k_{M-1}, we can obtain p_{M-2}^h and p_{M-1} to satisfy $PR_{d,M-1}$. Again let $PR_{d,M-2}$ equal to p_{M-2}^h. Repeating this process for k_{M-3}^h and k_{M-2}, we obtain p_{M-2} and $p_{M-3}^h = PR_{d,M-3}$. Continue allocating until all p_i's, $i = 1, \ldots, M - 3$, are calculated.

In each step, the analysis of a two-machine line with parameters k_1, k_2, N_1 and PR_d, can be represented by operators $\Psi_1(\cdot)$ and $\Psi_2(\cdot)$, i.e.,

$$p_i^* = \Psi_i(k_1, k_2, PR_d, N_1), \quad i = 1, 2,$$

where p_1^*, p_2^* are the resulting optimal allocation, and operator $\Psi(\cdot)$ solves the following optimization problem (see Su *et al.* (2016) for solutions):

$$\min \; k_1 p_1 + k_2 p_2 \quad (9)$$

$$\text{s.t.} \quad PR_d = p_2[1 - Q(p_1, p_2, N_1)], \quad (10)$$

$$0 < p_i < 1, \quad i = 1, 2,$$

where

$$Q(x, y, N) = \begin{cases} \dfrac{(1-x)(1-\alpha(x, y))}{1 - \frac{x}{y}\alpha^N(x, y)}, & \text{if } \; x \neq y, \\ \dfrac{1-x}{N+1-y}, & \text{if } \; x = y, \end{cases} \quad (11)$$

$$\alpha(x, y) = \frac{x(1-y)}{y(1-x)}. \quad (12)$$

6.2 Recursive Procedure

Since the above algorithm relies on values of p_i that are unknown, we introduce iterations to continuously update p_i during each iteration. Formally, let $k_i^h(j)$ and $p_i^h(j)$ denote the energy consumption coefficient and the production capacity allocation for machine m_i^h during iteration *j*, respectively. Also denote $p_i(j)$ as the production capacity for machine m_i, and $PR_{d,i}(j)$ be the desired production rate for machines m_1 to m_i, during iteration *j*. Then, the recursive procedure can be formally expressed as:

Procedure 6.1—Step 1: Initialization.

$$k_n^h(0) = \left(\frac{\sum_{i=1}^{n} k_i}{\sum_{i=1}^{n+1} k_i}\right)^{\beta_n} \cdot \sum_{i=1}^{n} k_i, \quad n = 2, \ldots, M-1,$$

$$p_i(0) = 1, \quad i = 1, \ldots, M,$$

$$k_1^h(j) = k_1, \quad \forall j, \quad (13)$$

$$PR_{d,M}(j) = PR_d, \quad \forall j .$$

Let iteration number $j = 1$.

Step 2: Forward calculation of energy consumption coefficients.

$$k_n^h(j) = \left(\frac{\sum_{i=1}^{n} k_i p_i(j-1)}{\sum_{i=1}^{n+1} k_i p_i(j-1)}\right)^{\beta_n} \cdot \sum_{i=1}^{n} k_i, \quad n = 2, \ldots, M-1 . \quad (14)$$

Step 3: Backward allocation of production capacity. For $i = M, \ldots, 2$,

$$p_{i-1}^h(j) = \Psi_1(k_{i-1}^h(j-1), k_i, PR_{d,i}(j), N_{i-1}),$$

$$p_i(j) = \Psi_2(k_{i-1}^h(j-1), k_i, PR_{d,i}(j), N_{i-1}), \quad (15)$$

$$PR_{d,i-1}(j) = p_{i-1}^h(j) .$$

Step 4: Check iteration stop condition. For $j \leq 100$, if

$$\max_i | p_i(j) - p_i(j-1) | < \Delta_1,$$

where Δ_1 is typically set to 10^{-7}, then go to Step 5, otherwise, let $j = j + 1$, and go back to Step 2. For $j > 100$, let

$$k_n^h(j+2) = \frac{k_n^h(j+1) + k_n^h(j)}{2} . \quad (16)$$

Then go back to Step 2.

Step 5: Check desired production rate. If

$$PR - PR_d < \Delta_2,$$

where Δ_2 is typically selected as 10^{-5}, then stop the procedure and assign $p_i(j)$ to p_i^*, $i = 2, \ldots, M$, and $p_1^h(j)$ to p_1^*. Otherwise

$$p_i(j) = p_i(j) - 0.001, \quad i = 2, \ldots, M,$$

$$p_1^h(j) = p_1^h(j) - 0.001,$$

and repeat Step 5.

6.3 Convergence

To study the convergence of Procedure 6.1, we first investigate the bounds between iterations. For convenience, define

$$\delta_n(j) = |k_n^h(j) - k_n^h(j-1)|, \quad j = 1, 2, \ldots. \quad (17)$$

In addition, let

$$B_1(n) = \left(\frac{1}{1 + \frac{1}{PR_d}\frac{k_{n+1}}{\sum_{i=1}^n k_i}}\right)^{\beta_n} \cdot \sum_{i=1}^n k_i,$$

$$B_2(n) = \left(\frac{1}{1 + PR_d\frac{k_{n+1}}{\sum_{i=1}^n k_i}}\right)^{\beta_n} \cdot \sum_{i=1}^n k_i. \quad (18)$$

Then we obtain bounds for k_n^h during each iteration.

Proposition 6.1—Under assumptions 1)–6), both $k_n^h(j)$ and $\delta_n(j)$ in Procedure 6.1 are bounded, *i.e.*,

$$k_n^h(j) \in \begin{cases} (B_1(n), B_2(n)), & if\ \beta_n > 0, \\ (B_2(n), B_1(n)), & if\ \beta_n < 0, \end{cases}$$

$$\delta_n(j) \in [0, |B_1(n) - B_2(n)|). \quad (19)$$

Proof: See Appendix A.

Moreover, for three-machine lines, $\delta_n(j)$ converges to δ_n when j is large. Then δ_n is not only bounded, but also unique.

Proposition 6.2—Under assumptions 1)–6) with $M = 3$, if $0 \ll PR_d < 1$, then δ_n in Procedure 6.1 is unique.

Proof: See Appendix A.

The results of Proposition 6.2 indicate that $k_n^h(j)$ is either oscillating with a small bound or converging. Similar observations are discovered for longer lines. To verify the convergence of the procedure, thousands of experiments are carried out. In all cases, the convergence of the procedure is observed. Thus we formulate it as a numerical observation.

Numerical Observation 1: Under assumptions 1)–6), Procedure 6.1 converges, *i.e.*, the following limits exist

$$\lim_{j \to \infty} p_i(j) := p_i^*, \quad i = 2, \ldots, M, \qquad (20)$$

$$\lim_{j \to \infty} p_1^h(j) := p_1^*. \qquad (21)$$

Remark 6.1—Among over 10,000 experiments with randomly selected parameters, except two or three, Procedure 6.1 converges within 10 iterations. For the two or three exceptions, the difference between $k_i^h(j)$ and $k_i^h(j+1)$ is around 0.001 after 100 iterations. Then the adjustment (16) in Step 4 makes the procedure converges immediately.

6.4 Accuracy

To investigate the accuracy of Procedure 6.1, numerical experiments are carried out to compare the results from Procedure 6.1 with the "optimal" solutions. For three- to five-machine lines, the "optimal" solutions are obtained using the aggregation approach introduced in Section 5. The relative differences are defined as

$$\varepsilon = \frac{\left(\sum k_i p_i^*\right)^h - \left(\sum k_i p_i^*\right)^a}{\left(\sum k_i p_i^*\right)^a} \cdot 100\%,$$

where superscripts "h" and "a" denote heuristic method and aggregation approach, respectively.

As shown in Tables C.1 and C.2 in Appendix C for three- and five-machine lines, in most cases the differences are within 0.1% and the largest ones are still less than 1%. In addition, the iteration steps are also included in these tables, as well as the computation time using the two methods. As one can see, when the system is small, aggregation approach is faster; while when buffers become large, heuristic method is more efficient. Again note that two examples in Table C.2 with large buffers do not have aggregation results due to computation intensity, thus are marked with "N/A".

For six- to ten-machine systems, Monte Carlo simulations are introduced to generate numerous feasible solutions. 5000 lines are randomly generated using the following parameter ranges:

$$M \in [6, 10],$$

$$PR_d \in [0.6, 0.95],$$

$$N_i \in [1, 10],$$

$$k_i \in [1, 10] .$$

Tables C.3 and C.4 in Appendix C illustrate the examples of such experiments for seven- and ten-machine lines. Thousands of experiments are carried out by randomly generating p_i's until the desired production rate is satisfied. Then the one with the smallest $\Sigma_i k_i p_i$ is considered as the "optimal" one. By comparing the "optimal" solution with the one calculated by heuristic algorithm, we observe that only in 205 out of 5,000 examples (i.e., 4.10%) the "optimal" solution leads to smaller amount of consumed energy comparing with the results from heuristic method. Among the 205 cases, 70% of them have gaps within 1%, and 90% within 2%, while the maximum gap is 5.73%.

Therefore, we claim that Procedure 6.1 provides an optimal or a close to optimal solution of allocating production capacity to minimize energy consumptions.

6.5 System Property

The properties obtained through exact analysis and aggregation approach for small and medium size systems, respectively, are still valid for larger systems. For systems with identical buffers and energy coefficients, production capacity of each machine is almost evenly distributed and larger buffer size always implies smaller amount of energy consumption. An example of eight-machine line is illustrated in Table 10(a).

When lines have nonidentical buffers but equal energy coefficients, an increase in N_i will lead to a decrease in p_{i+1} (rather than in p_i) because the forward aggregation of k_i's and backward allocation of p_i's are used in the heuristic algorithm (see Table 10(b)). The total energy usage is reduced when buffer size increases.

For lines with identical buffers and unequal energy coefficients, two examples are shown in Table 10(c). As one can see, the allocation of productivity capacity has a negative pattern to that of energy coefficients. An increase in k_i's leads to a decrease in p_i and an increase in other p_j's ($j \neq i$) as well as the total energy consumed. Therefore, we conclude that the properties observed in previous sections still hold.

7. CONCLUSIONS

In this paper, an integrated model of energy consumption and productivity in Bernoulli serial lines is introduced. Using such a model, we seek to minimize energy consumption while still keeping the desired productivity. For production lines with three and four machines and small buffers, exact investigation is carried out. For medium size systems, an aggregation-based method is used to evaluate productivity. We observe that the inverted bowl allocation of production capacity leads to minimal energy consumption in case of identical buffer and energy coefficients, while evenly distributed allocation renders close to optimal solution. When lines have nonidentical buffers, larger buffer size leads to smaller production capacity for neighboring machines. If energy coefficients are different, production capacity allocation follows a negative pattern to the distribution of energy coefficients. For larger systems, a heuristic algorithm is presented to allocate production capacity. The algorithm forwardly groups the energy coefficients and then backwardly allocates production capacity. The convergence of the procedure is justified numerically. It is shown that the algorithm leads to an optimal or a close to optimal solution. The system properties obtained in small and medium systems are still suitable for large systems.

The results of this work provide production engineers a quantitative tool to effectively operate the system to reduce energy consumption and meet production target. The future work can be directed to the following areas:

- Extending the study to other machine reliability models, such as geometric, exponential, or general reliability machines.
- Generalizing to flexible manufacturing systems that can produce multiple product types.
- Investigating the continuous improvement strategies, such as identifying the energy bottleneck machines, i.e., the machines whose reduction in energy coefficient will lead to the largest reduction in total energy consumption, and the energy bottleneck buffers, whose increase will lead to the largest reduction in system energy.
- Applying the results on the factory floor to design and control the production line for energy efficiency.

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APPENDIX A

PROOFS

Proof of Proposition 4.1

The probability transition matrix can be represented as:

$$P = \begin{bmatrix} 1 - p_1 & p_1 & 0 & 0 \\ 0 & 1 - p_2 & (1 - p_1)p_2 & p_1 p_2 \\ (1 - p_1)p_3 & p_1 p_3 & (1 - p_1)(1 - p_3) & p_1(1 - p_3) \\ 0 & (1 - p_2)p_3 & (1 - p_1)p_2 p_3 & 1 - p_3 + p_1 p_2 p_3 \end{bmatrix}.$$

The balance equations are:

$$P_{(0,0)} = (1 - p_1)P_{(0,0)} + (1 - p_1)p_3 P_{(0,1)},$$

$$P_{(1,0)} = p_1 P_{(0,0)} + (1 - p_2)P_{(1,0)} + p_1 p_3 P_{(0,1)} + (1 - p_2)p_3 P_{(1,1)},$$

$$P_{(0,1)} = (1 - p_1)p_2 P_{(1,0)} + (1 - p_1)(1 - p_3)P_{(0,1)} + (1 - p_1)p_2 p_3 P_{(1,1)},$$

$$P_{(1,1)} = p_1 p_2 P_{(1,0)} + p_1(1 - p_3)P_{(0,1)} + (1 - p_3 + p_1 p_2 p_3)P_{(1,1)},$$

and

$$P_{(0,0)} + P_{(1,0)} + P_{(0,1)} + P_{(1,1)} = 1 .$$

Solving these equations, the analytical expressions of $P_{(0,0)}$, $P_{(1,0)}$, $P_{(0,1)}$ and $P_{(1,1)}$ in (5) and (6) can be derived. Then the line production rate can be calculated.

$$PR_{L1} = (P_{(0,1)} + P_{(1,1)})p_3 = \frac{(1 - p_1)p_1 p_2 p_3^2 + p_1^2 p_2 p_3}{\Lambda} .$$

Proof of Proposition 6.1

From equation (14), for $n = 2, \ldots, M - 1$,

$$k_n^h(j) = \left(\frac{\sum_{i=1}^{n} k_i p_i(j-1)}{\sum_{i=1}^{n+1} k_i p_i(j-1)}\right)^{\beta_n} \cdot \sum_{i=1}^{n} k_i = \left(\frac{\sum_{i=1}^{n} k_i p_i(j-1)}{\sum_{i=1}^{n} k_i p_i(j-1) + k_{n+1} p_{n+1}(j-1)}\right)^{\beta_n} \cdot \sum_{i=1}^{n} k_i$$

$$= \left(\frac{1}{1 + \frac{k_{n+1} p_{n+1}(j-1)}{\sum_{i=1}^{n} k_i p_i(j-1)}}\right)^{\beta_n} \cdot \sum_{i=1}^{n} k_i .$$

From Li and Meerkov (2009), $p_i(j) \in (PR_d, 1)$, we have

$$\frac{k_{n+1} p_{n+1}(j-1)}{\sum_{i=1}^{n} k_i p_i(j-1)} > \frac{k_{n+1} PR_d}{\sum_{i=1}^{n} k_i p_i(j-1)} > PR_d \cdot \frac{k_{n+1}^h(j-1)}{\sum_{i=1}^{n} k_i^h(j-1)},$$

$$\frac{k_{n+1} p_{n+1}(j-1)}{\sum_{i=1}^{n} k_i p_i(j-1)} < \frac{k_{n+1}}{\sum_{i=1}^{n} k_i p_i(j-1)} < \frac{k_{n+1}}{PR_d \cdot \sum_{i=1}^{n} k_i} .$$

When $\beta_n > 0$, it follows that

$$\left(\frac{1}{1 + PR_d \cdot \frac{k_{n+1}}{\sum_{i=1}^{n} k_i}}\right)^{\beta_n} \cdot \sum_{i=1}^{n} k_i > \left(\frac{1}{1 + \frac{k_{n+1}}{PR_d \cdot \sum_{i=1}^{n} k_i}}\right)^{\beta_n} \cdot \sum_{i=1}^{n} k_i,$$

i.e.,

$$B_2(n) > B_1(n), \quad k_n^h(j) \in (B_1(n), B_2(n)) .$$

When $\beta_n < 0$, we obtain

$$B_2(n) < B_1(n), \quad k_n^h(j) \in (B_2(n), B_1(n)) .$$

For $\delta_n(j)$, from definition (17) and bounds of $k_n^h(j)$ and $k_n^h(j-1)$, it follows straightforwardlythat

$$\delta_n(j) \in [0, |B_1(n) - B_2(n)|) .$$

Proof of Proposition 6.2

First assume $\beta_2 > 0$. If $k_2^a(j+1) = k_2^a(j)$, clearly we have $\delta_2(j+1) = 0$ and when $j \to \infty$, $\delta_2 = 0$. If $k_2^a(j+1) < k_2^a(j)$, then from Su *et al.* (2016), we have

$$p_2^h(j+1) > p_2^h(j), \quad p_3(j+1) < p_3(j),$$

which implies that

$$p_1(j+1) > p_1(j), \quad p_2(j+1) > p_2(j).$$

It follows that

$$k_2^a(j+2) = \left(\frac{1}{1+\frac{k_2 p_3(j+1)}{k_1 p_1(j+1) + k_2 p_2(j+1)}}\right)^{\beta_2} \sum_{i=1}^{2} k_i > k_2^a(j+1) = \left(\frac{1}{1+\frac{k_2 p_3(j)}{k_1 p_1(j) + k_2 p_2(j)}}\right)^{\beta_2} \sum_{i=1}^{2} k_i.$$

To compare $k_2^a(j+2)$ and $k_2^a(j)$, the following scenarios are considered:

- If $k_2^a(j+2) = k_2^a(j)$, then

$$\delta_2(j+1) = k_2^a(j+2) - k_2^a(j+1) = k_2^a(j) - k_2^a(j+1) = constant.$$

  When $j \to \infty$, δ_2 is a constant.

- If $k_2^a(j+2) < k_2^a(j)$, then using the similar arguments from $k_2^a(j+1) < k_2^a(j)$, we obtain

$$p_1(j+2) > p_1(j), \quad p_2(j+2) > p_2(j), \quad p_3(j+2) < p_3(j),$$

$$k_2^a(j+3) = \left(\frac{1}{1+\frac{k_2 p_3(j+2)}{k_1 p_1(j+2) + k_2 p_2(j+2)}}\right)^{\beta_2} \sum_{i=1}^{2} k_i > k_2^a(j+1)$$
$$= \left(\frac{1}{1+\frac{k_2 p_3(j)}{k_1 p_1(j) + k_2 p_2(j)}}\right)^{\beta_2} \sum_{i=1}^{2} k_i.$$

  Continue this process we obtain

$$k_2^a(j+2l+1) < k_2^a(j+2l+3) < \ldots < k_2^a(j+2(l+1)) < k_2^a(j+2l), \quad l = 0, 1, 2, \ldots .$$

Thus, when $j \rightarrow \infty$, namely, $k_2^a(j)$ converges and $\delta_2 = 0$.

- If $k_2^a(j+2) > k_2^a(j)$, similarly,

$$p_1(j+2) < p_1(j), \quad p_2(j+2) < p_2(j), \quad p_3(j+2) > p_3(j),$$

$$k_2^a(j+3) = \left(\frac{1}{1+\frac{k_2p_3(j+2)}{k_1p_1(j+2)+k_2p_2(j+2)}}\right)^{\beta_2} \sum_{i=1}^{2} k_i < k_2^a(j+1)$$
$$= \left(\frac{1}{1+\frac{k_2p_3(j)}{k_1p_1(j)+k_2p_2(j)}}\right)^{\beta_2} \sum_{i=1}^{2} k_i .$$

Iterate this process we obtain

$$k_2^a(j+2l) < k_2^a(j+2(l+1)), \quad k_2^a(j+2l+1) > k_2^a(j+2l+3), \quad l = 0, 1, 2, \ldots .$$

From Proposition 6.1, we have

$$\lim_{l \to \infty} k_2^a(j+2l) = \left(\frac{1}{1+PR_d\frac{k_3}{k_1+k_2}}\right)^{\beta_2} (k_1+k_2),$$

$$\lim_{l \to \infty} k_2^a(j+2l+1) = \left(\frac{1}{1+\frac{1}{PR_d}\frac{k_3}{k_1+k_2}}\right)^{\beta_2} (k_1+k_2).$$

According to Su *et al.* (2016), when l is large enough, we obtain

$$k_2^a(j+2l+1) \ll k_3 \ll k_2^a(j+2l),$$

which leads to

$$\left(\frac{1}{1+\frac{1}{PR_d}\frac{k_3}{k_1+k_2}}\right)^{\beta_2} \ll \left(\frac{1}{1+PR_d\frac{k_3}{k_1+k_2}}\right)^{\beta_2}.$$

This results in $PR_d \ll 1$, which contradicts to the condition of this proposition.

If $k_2^a(j+1) > k_2^a(j)$, the proof is similar. Analogously, the case of $\beta_2 < 0$ can be proved.

Appendix B

Aggregation Procedure

Let superscripts ‘f’ and ‘b’ denote the forward and backward aggregations for machine parameters, respectively. For an M-machine $M-1$-buffer serial production line with parameters p_i, $i = 1, \ldots, M$, and N_i, $i = 1, \ldots, M-1$, we have:

Procedure B.1

$$p_i^b(n+1) = p_i[1 - Q(p_{i+1}^b(n+1), p_i^f(n), N_i)], i = 1, \ldots, M-1,$$

$$p_i^f(n+1) = p_i[1 - Q(p_{i-1}^f(n+1), p_i^b(n+1), N_{i-1})], i = 2, \ldots, M, n = 0, 1, 2, \ldots,$$

with initial conditions

$$p_i^f(0) = p_i, \quad i = 1, \ldots, M,$$

and boundary conditions

$$p_1^f(n) = p_1, \quad n = 0, 1, 2, \ldots,$$

$$p_M^b(n) = p_M, \quad n = 0, 1, 2, \ldots,$$

and n is iteration number, $Q(\cdot)$ is defined in (11).

It is shown in Li and Meerkov (2009) that the procedure is convergent with a unique solution, i.e.,

$$\lim_{n \to \infty} p_i^f(n) = p_i^f, \quad \lim_{n \to \infty} p_i^b(n) = p_i^b, \quad i = 1, \ldots, M.$$

The line production rate can be expressed as

$$PR = p_M^f = p_1^b.$$

APPENDIX C

EXAMPLES

Table C.1

Allocation of p_i's using heuristic method: $M = 3$

N_i	k_i	p_i^*	step	time^h	time^a	$\left(\sum_i k_i p_i^*\right)^h$	ε
1,1	1,1,1	0.888, 0.888, 0.919	7	4.19	0.91	2.695	0.03%
2,2	1,1,1	0.850, 0.850, 0.835	5	2.89	2.08	2.535	−0.09%
3,3	1,1,1	0.826, 0.826, 0.809	5	3.28	3.91	2.460	−0.06%
5,5	1,1,1	0.801, 0.801, 0.787	5	8.86	11.77	2.390	0.06%
10,10	1,1,1	0.778, 0.778, 0.770	4	7.52	411.19	2.327	0.05%
2,4	1,1,1	0.839, 0.839, 0.798	5	3.41	3.44	2.475	−0.37%
4,2	1,1,1	0.825, 0.825, 0.832	5	5.73	3.09	2.481	0.27%
2,2	1,1.5,2	0.909, 0.866, 0.793	6	3.67	2.55	3.794	0.05%
2,2	2,1.5,1	0.810, 0.850, 0.894	5	4.73	2.58	3.788	−0.04%
2,2	1,2,1	0.889, 0.799, 0.881	5	5.77	2.41	3.367	0.01%
2,2	2,1,2	0.830, 0.907, 0.800	6	4.08	2.17	4.167	−0.02%

Table C.2

Allocation of p_i's using heuristic method: $M = 5$

N_i	k_i	p_i^*	step	time^h	time^a	$\left(\sum_i k_i p_i^*\right)^h$	ε
1,1,1,1	1,1,1,1,1	0.894, 0.894, 0.928, 0.973, 0.998	6	7.25	70.00	4.687	0.63%
2,2,2,2	1,1,1,1,1	0.876, 0.876, 0.866, 0.862, 0.860	5	6.33	1142.73	4.339	0.04%
3,3,3,3	1,1,1,1,1	0.851, 0.851, 0.837, 0.828, 0.822	5	9.42	265705	4.189	−1.85%
5,5,5,5	1,1,1,1,1	0.822, 0.822, 0.809, 0.800, 0.793	5	11.57	N/A	4.045	N/A
10,10,10,10	1,1,1,1,1	0.792, 0.792, 0.784, 0.777, 0.772	4	11.31	N/A	3.917	N/A

N_i	k_i	p_i^*	step	time^h	time^a	$(\sum_i k_i p_i^*)^h$	ε
2,3,2,2	1,1,1,1,1	0.870, 0.870, 0.845, 0.862, 0.859	4	6.16	2552.97	4.305	0.12%
2,2,3,2	1,1,1,1,1	0.872, 0.872, 0.862, 0.832, 0.859	5	7.17	2043.17	4.298	−0.09%
2,4,4,2	1,1,1,1,1	0.860, 0.860, 0.826, 0.817, 0.859	5	7.92	29019	4.222	−1.68%
4,2,2,4	1,1,1,1,1	0.854, 0.854, 0.862, 0.857, 0.803	5	12.77	24381	4.229	−1.74%
4,2,4,2	1,1,1,1,1	0.850, 0.850, 0.859, 0.816, 0.858	5	12.50	34302	4.234	−1.28%
2,2,2,2	1,1.5,2,2.5,3	0.958, 0.935, 0.892, 0.840, 0.824	6	9.22	1060.75	8.715	0.60%
2,2,2,2	3,2.5,2,1.5,1	0.829, 0.853, 0.874, 0.907, 0.948	5	7.25	1031.45	8.774	0.18%
2,2,2,2	1,2,1,2,1	0.918, 0.848, 0.915, 0.807, 0.907	4	6.77	1071.02	6.049	0.15%
2,2,2,2	1,2,2,2,1	0.936, 0.873, 0.848, 0.836, 0.927	5	7.06	1045.19	6.978	0.23%
2,2,2,2	2,1,1,1,2	0.836, 0.905, 0.903, 0.904, 0.790	6	9.47	1062.20	5.962	0.10%

Table C.3

Allocation of p_i's using heuristic method: $M = 7$

N_i	k_i	p_i^*	step	time^h	$(\sum_i k_i p_i^*)^h$
1,1,1,1,1,1	1,1,1,1,1,1,1	0.894, 0.894, 0.928, 0.973, 0.998, 0.998, 0.998	6	10.91	6.683
2,2,2,2,2,2	1,1,1,1,1,1,1	0.884, 0.884, 0.877, 0.875, 0.875, 0.875, 0.876	5	14.95	6.145
3,3,3,3,3,3	1,1,1,1,1,1,1	0.862, 0.862, 0.851, 0.844, 0.838, 0.834, 0.831	4	12.25	5.922
5,5,5,5,5,5	1,1,1,1,1,1,1	0.834, 0.834, 0.823, 0.814, 0.808, 0.802, 0.798	4	18.28	5.712
10,10,10, 10,10,10	1,1,1,1, 1,1,1	0.801, 0.801, 0.793, 0.787 0.783, 0.778, 0.775	4	15.17	5.519
2,2,2,3,2,2	1,1,1,1,1,1,1	0.883, 0.883, 0.875, 0.873, 0.841, 0.877, 0.875	5	16.53	6.105
2,3,2,2,2,2	1,1,1,1,1,1,1	0.878, 0.878, 0.857, 0.874, 0.873, 0.874, 0.875	4	12.05	6.109
2,2,2,2,3,2	1,1,1,1,1,1,1	0.883, 0.883, 0.876, 0.874, 0.873, 0.836, 0.875	4	10.25	6.101
2,2,4,4,2,2	1,1,1,1,1,1,1	0.877, 0.877, 0.869, 0.831, 0.825, 0.875, 0.876	5	16.73	6.029
4,4,2,2,4,4	1,1,1,1,1,1,1	0.856, 0.856, 0.846, 0.872, 0.871, 0.814, 0.810	5	17.64	5.924
4,2,4,2,4,2	1,1,1,1,1,1,1	0.861, 0.861, 0.870, 0.832, 0.872, 0.816, 0.875	5	19.80	5.988

N_i	k_i	p_i^*	step	time^h	$(\sum_i k_i p_i^*)^h$
2,2,2 2,2,2	1,1.5,2,2.5 3,3.5,4	0.966, 0.952, 0.925, 0.887 0.879, 0.847, 0.841	6	16.03	15.426
2,2,2 2,2,2	4,3.5,3,2.5 2,1.5,1	0.840, 0.857, 0.868, 0.890 0.925, 0.945, 0.964	4	10.44	15.422
2,2,2,2,2,2	1,2,1,2,1,2,1	0.925, 0.868, 0.925, 0.835, 0.921, 0.821, 0.917	5	13.53	8.736
2,2,2,2,2,2	1,1,2,2,2,1,1	0.950, 0.950, 0.873, 0.849, 0.837, 0.930, 0.931	4	9.98	8.878
2,2,2,2,2,2	2,2,1,1,1,2,2	0.860, 0.860, 0.92, 0.921, 0.922, 0.834, 0.829	6	14.75	9.528

Table C.4

Allocation of p_i's using heuristic method: $M = 10$

N_i	k_i	p_i^*	step	time^h	$(\sum_i k_i p_i^*)^h$
1,1,1,1,1 1,1,1,1	1,1,1,1,1 1,1,1,1,1	0.894, 0.894, 0.928, 0.973, 0.998 0.998, 0.998, 0.998, 0.998, 0.998	5	11.66	9.677
2,2,2,2,2 2,2,2,2	1,1,1,1,1 1,1,1,1,1	0.886, 0.886, 0.881, 0.880, 0.881 0.882, 0.884, 0.886, 0.888, 0.889	4	16.44	8.842
3,3,3,3,3 3,3,3,3	1,1,1,1,1 1,1,1,1,1	0.868, 0.868, 0.859, 0.853, 0.849 0.846, 0.844, 0.842, 0.841, 0.839	4	17.78	8.512
5,5,5,5,5, 5,5,5,5	1,1,1,1,1 1,1,1,1,1	0.844, 0.844, 0.834, 0.827, 0.821 0.816, 0.812, 0.809, 0.806, 0.803	4	20.09	8.216
10,10,10,10,10 10,10,10,10	1,1,1,1,1 1,1,1,1,1	0.809, 0.809, 0.802, 0.796, 0.792 0.788, 0.784, 0.781, 0.779, 0.776	4	19.91	7.917
2,2,2,2,3 2,2,2,2	1,1,1,1,1 1,1,1,1,1	0.885, 0.885, 0.880, 0.879, 0.880 0.847, 0.884, 0.886, 0.887, 0.889	5	20.13	8.803
2,3,2,2,2 2,2,2,2	1,1,1,1,1 1,1,1,1,1	0.882, 0.882, 0.864, 0.880, 0.880 0.882, 0.884, 0.886, 0.887, 0.889	5	22.48	8.816
2,2,2,2,2 2,2,3,2	1,1,1,1,1 1,1,1,1,1	0.886, 0.886, 0.881, 0.880, 0.881 0.882, 0.884, 0.886, 0.842, 0.889	3	11.20	8.796
2,2,2,4,4 4,2,2,2	1,1,1,1,1 1,1,1,1,1	0.883, 0.883, 0.877, 0.876, 0.833 0.829, 0.825, 0.887, 0.888, 0.890	5	24.11	8.672
4,4,4,2,2 2,4,4,4	1,1,1,1,1 1,1,1,1,1	0.859, 0.859, 0.850, 0.844, 0.880 0.882, 0.883, 0.819, 0.817, 0.815	5	26.28	8.508

N_i	k_i	p_i^*	step	time^h	$(\sum_i k_i p_i^*)^h$
2,4,2,4,2 4,2,4,2	1,1,1,1,1 1,1,1,1,1	0.877, 0.877, 0.852, 0.877, 0.834 0.882, 0.824, 0.886, 0.818, 0.890	5	23.08	8.617
2,2,2,2,2 2,2,2,2	1,1.5,2,2.5,3 3.5,4,4.5,5,5.5	0.964, 0.956, 0.938, 0.912, 0.907 0.886, 0.886, 0.867, 0.867, 0.850	5	16.61	28.833
2,2,2,2,2 2,2,2,2	5.5,5,4.5,4,3.5 3,2.5,2,1.5,1	0.849, 0.861, 0.866, 0.879, 0.910 0.924, 0.942, 0.952, 0.962, 0.970	4	13.95	29.014
2,2,2,2,2 2,2,2,2	1,2,1,2,1 2,1,2,1,2	0.925, 0.880, 0.925, 0.854, 0.923 0.845, 0.923, 0.842, 0.922, 0.841	5	17.23	13.141
2,2,2,2,2 2,2,2,2	2,2,2,1,1 2,2,2,1,1	0.870, 0.870, 0.863, 0.924, 0.925 0.926, 0.927, 0.852, 0.852, 0.852	5	17.84	14.016
2,2,2,2,2 2,2,2,2	1,1,1,2,2 2,2,1,1,1	0.954, 0.954, 0.954, 0.883, 0.864 0.856, 0.852, 0.938, 0.939, 0.940	5	16.41	12.588
2,2,2,2,2 2,2,2,2	1,1,1,1,1 5,1,1,1,1	0.983, 0.983, 0.983, 0.984, 0.985 0.755, 0.959, 0.960, 0.960, 0.960	4	17.83	12.531

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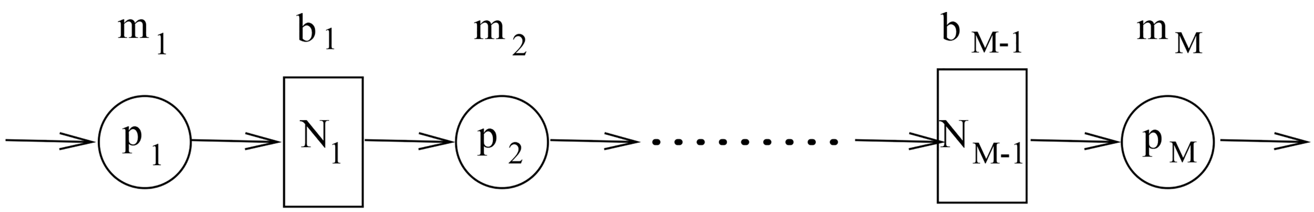
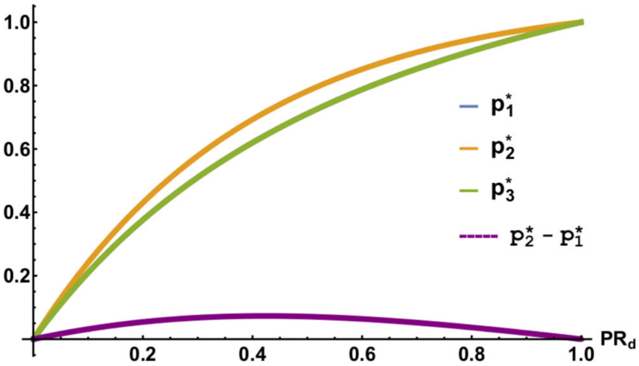


Figure 1.
Bernoulli serial line



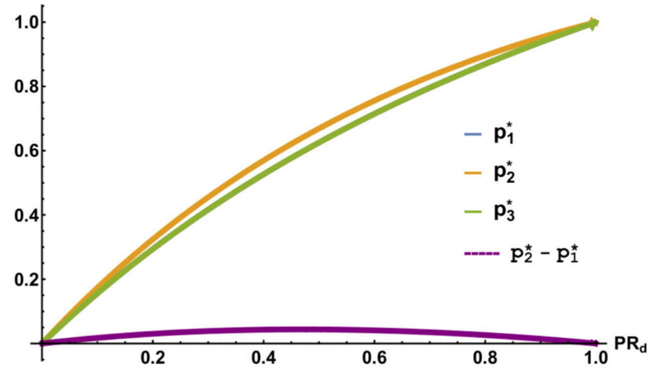


Figure 2.
Optimal p_i^*'s in L_1 and L_2

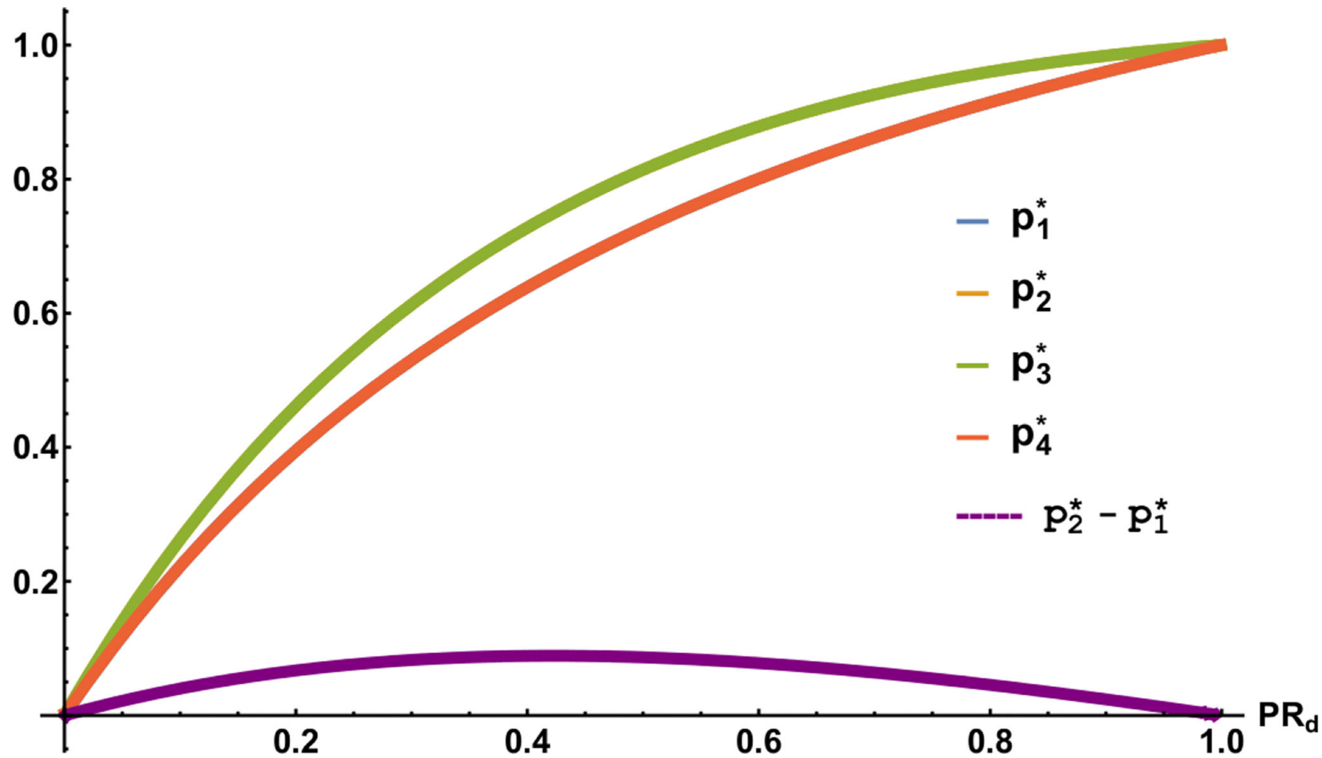


Figure 3.
Optimal p_i^*'s in L_3

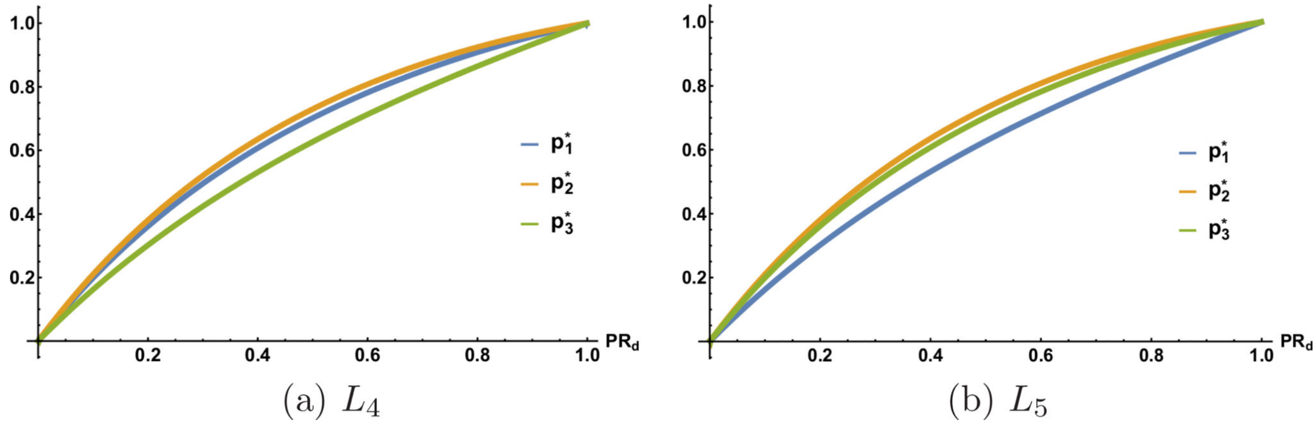


Figure 4.

Optimal p_i^*'s in L_4 and L_5

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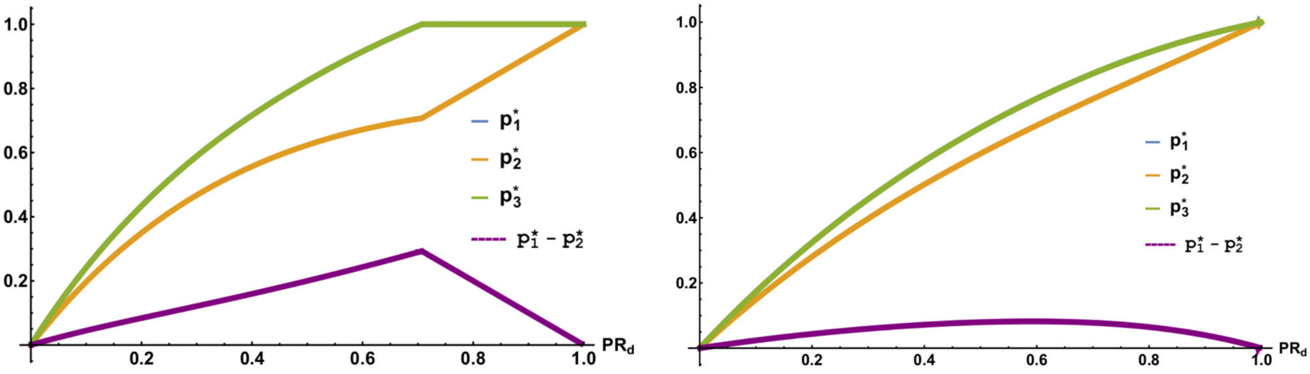

(a) L_6 (b) L_7

Figure 5.

Optimal p_i^*'s in L_6 and L_7

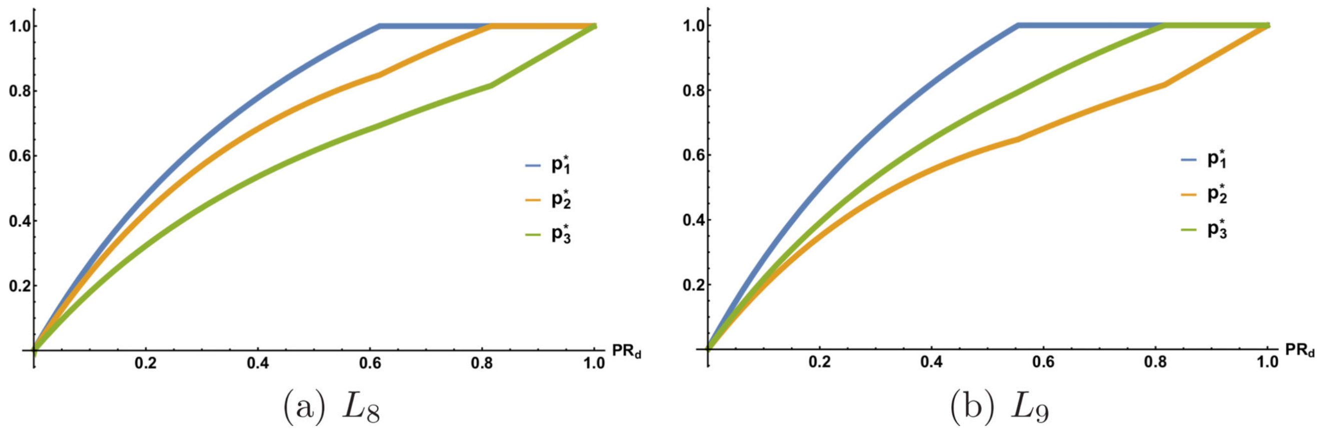

Figure 6.

Optimal p_i^*'s in L_8 and L_9

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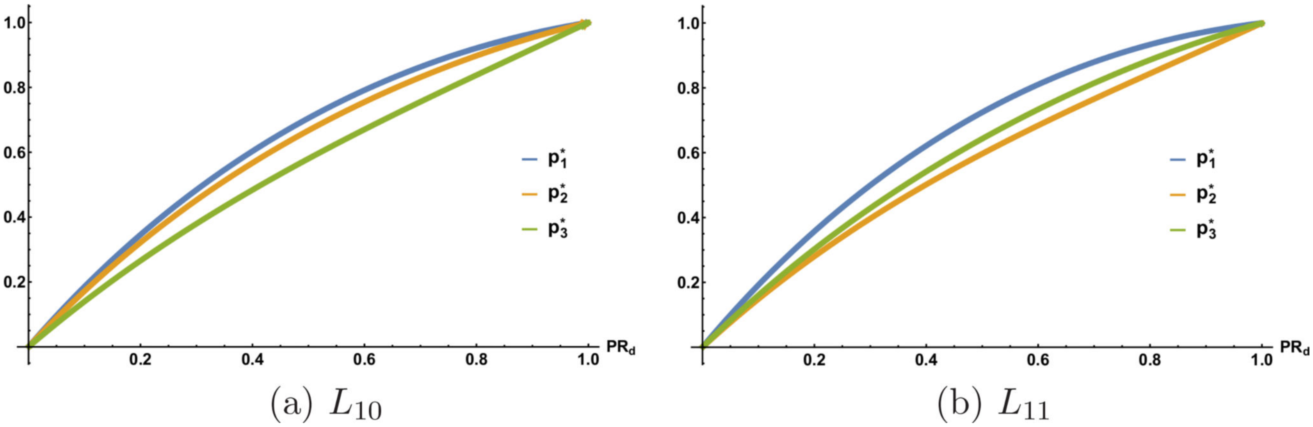

Figure 7.

Optimal p_i^*s allocation in L_{10} and L_{11}

Table 1

Three-machine lines with identical buffers and energy coefficients ($k_i = 1$, $i = 1, 2, 3$)

N_1, N_2	p_1^*, p_2^*, p_3^*	$\sum_i k_i p_i^*$	Comparison
1, 1	0.900, 0.900, 0.900	2.700	0.4%
2, 2	0.852, 0.850, 0.850	2.552	0.8%
3, 3	0.827, 0.825, 0.825	2.476	N/A
4, 4	0.811, 0.810, 0.810	2.431	N/A
5, 5	0.801, 0.800, 0.800	2.400	N/A
6, 6	0.794, 0.793, 0.793	2.379	N/A
7, 7	0.788, 0.787, 0.787	2.362	N/A
8, 8	0.784, 0.783, 0.783	2.350	N/A
9, 9	0.780, 0.780, 0.780	2.340	N/A
10, 10	0.778, 0.777, 0.777	2.332	N/A

Table 2

Four-machine lines with identical buffers and energy coefficients ($k_i = 1$, $i = 1, \ldots, 4$)

N_1, N_2, N_3	$p_1^*, p_2^*, p_3^*, p_4^*$	$\sum_i k_i p_i^*$	Comparison
1, 1, 1	0.923, 0.923, 0.923, 0.923	3.692	0.7%
2, 2, 2	0.873, 0.871, 0.896, 0.896	3.483	N/A
3, 3, 3	0.843, 0.842, 0.840, 0.840	3.367	N/A

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Table 3

Five-machine lines with identical buffers and energy coefficients ($k_i = 1$, $i = 1, \ldots, 5$)

N_1, N_2, N_3, N_4	$p_1^*, p_2^*, p_3^*, p_4^*, p_5^*$	$\sum_i k_i p_i^*$
1,1,1,1	0.938, 0.938, 0.938, 0.938, 0.938	4.689
2,2,2,2	0.887, 0.886, 0.884, 0.883, 0.883	4.423

Table 4

Three-machine lines with nonidentical buffers and equal energy coefficients

N_1, N_2	p_1^*, p_2^*, p_3^*	$\sum_i k_i p_i^*$	**Comparison**
1, 2	0.964, 0.835, 0.835	2.634	1.0%
2, 1	0.834, 0.891, 0.891	2.617	0.3%
2, 3	0.859, 0.830, 0.830	2.521	N/A
3, 2	0.822, 0.843, 0.843	2.509	N/A
2, 4	0.863, 0.820, 0.820	2.502	N/A
4, 2	0.806, 0.839, 0.839	2.484	N/A

Table 5

Four-machine lines with nonidentical buffers and equal energy coefficients

N_1, N_2, N_3	$p_1^*, p_2^*, p_3^*, p_4^*$	$\sum_i k_i p_i^*$
2, 1, 1	0.840, 0.923, 0.923, 0.923	3.608
1, 2, 1	0.994, 0.834, 0.892, 0.892	3.617
1, 1, 2	0.974, 0.974, 0.843, 0.843	3.633
2, 2, 3	0.878, 0.876, 0.850, 0.850	3.454
3, 2, 2	0.836, 0.868, 0.866, 0.866	3.434
2, 3, 2	0.880, 0.842, 0.861, 0.861	3.444
3, 3, 4	0.845, 0.844, 0.830, 0.830	3.350
3, 4, 3	0.846, 0.825, 0.836, 0.836	3.343
4, 3, 3	0.822, 0.840, 0.838, 0.838	3.339

Table 6

Five-machine lines with nonidentical buffers and equal energy coefficients

N_1, N_2, N_3, N_4	$p_1^*, p_2^*, p_3^*, p_4^*, p_5^*$	$\sum_i k_i p_i^*$
2, 3, 2, 2	0.893, 0 .850, 0.880, 0.879, 0.879	4.380
2, 2, 3, 2	0.892, 0 .891, 0.855, 0.874, 0.874	4.302

Table 7

Three-machine lines with identical buffers ($N_1 = N_2 = 2$) but different energy coefficients

$k_1 : k_2 : k_3$	p_1^*, p_2^*, p_3^*	$\sum_i k_i p_i^*$	Comparison
1 : 1.2 : 1.5	0.878, 0.855, 0.824	3.141	0.8%
1 : 2 : 3	0.926, 0.856, 0.800	5.040	0.5%
3 : 2 : 1	0.800, 0.856, 0.928	5.041	0.5%
2 : 1 : 2	0.830, 0.910, 0.828	4.227	1.4%
1 : 2 : 1	0.889, 0.797, 0.888	3.371	0.3%

Table 8

Four-machine lines with identical buffers ($N_i = 2$, $i = 1, 2, 3$) but different energy coefficients

$k_1 : k_2 : k_3 : k_4$	$p_1^*, p_2^*, p_3^*, p_4^*$	$\sum_i k_i p_i^*$
1 : 2 : 3 : 4	0.951, 0.902, 0.856, 0.816	8.587
4 : 3 : 2 : 1	0.818, 0.855, 0.900, 0.955	8.594
1 : 2 : 1 : 2	0.916, 0.839, 0.916, 0.837	5.184
2 : 1 : 1 : 2	0.840, 0.916, 0.916, 0.836	5.184
1 : 2 : 2 : 1	0.916, 0.839, 0.837, 0.916	5.184

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Table 9

Five-machine lines with identical buffers ($N_i = 2$, $i = 1, 2, 3, 4$) but different energy coefficients

$k_1 : k_2 : k_3 : k_4 : k_5$	$p_1^*, p_2^*, p_3^*, p_4^*, p_5^*$	$\sum_i k_i p_i^*$
1 : 1.5 : 2 : 2.5 : 3	0.945, 0.917, 0.890, 0.863, 0.839	8.774
3 : 2.5 : 2 : 1.5 : 1	0.842, 0.864, 0.888, 0.916, 0.947	8.782
1 : 2 : 1 : 2 : 1	0.921, 0.847, 0.921, 0.844, 0.920	6.145
1 : 2 : 2 : 2 : 1	0.931, 0.864, 0.862, 0.860, 0.931	7.034
2 : 1 : 1 : 1 : 2	0.848, 0.921, 0.921, 0.920, 0.843	6.146

Table 10

Eight-machine lines

(a) Identical buffers and energy coefficients ($k_i = 1$, $i = 1, \ldots, 8$)

N_i	p_1^*	$\sum_i k_i p_i^*$
2,2,2,2,2,2,2	0.885, 0.885, 0.879, 0.877, 0.877, 0.878, 0.880, 0.881	7.043

(b) Nonidentical buffer size and identical energy coefficients ($k_i = 1$, $i = 1, \ldots, 8$)

N_i	p_1^*	$\sum_i k_i p_i^*$
2,2,2,3,2,2,2	0.884, 0.884, 0.877, 0.876, 0.845, 0.878, 0.879, 0.881	7.004

(c) Identical buffer size ($N_i = 2$, $i = 1, \ldots, 7$) and nonidentical energy coefficients

k_i	p_1^*	$\sum_i k_i p_i^*$
1,1.5,2,2.5,3,3.5,4,4.5	0.966, 0.955, 0.931, 0.899, 0.893, 0.866, 0.862, 0.838	19.440
1,2,1,2,1,2,1,2	0.926, 0.876, 0.926, 0.847, 0.923, 0.836, 0.922, 0.831	10.475