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This paper mainly studies nonlinear feedback control applied to the nonlinear vehicle dynamics with varying velocity. The main objective of this study is the stabilisation of longitudinal, lateral and yaw angular vehicle velocities. To this end, a nonlinear vehicle model is developed which takes both the lateral and longitudinal vehicle dynamics into account. Based on this model, a method to build a nonlinear state feedback control is first designed by which the complexity of system structure can be simplified. The obtained system is then synthesised by the combined Lyapunov–LaSalle method. The simulation results show that the proposed control can improve stability and comfort of vehicle driving. Moreover, this paper presents a lemma which ensures the trajectory tracking and path-following problem for vehicle. It can also be exploited simultaneously to solve both the tracking and path-following control problems of the vehicle ride and driving stability. We also show how the results of the lemma can be applied to solve the path-following problem, in which the vehicle converges and follows a designed path. The effectiveness of the proposed lemma for trajectory tracking is clearly demonstrated by simulation results.

Keywords: modeling; nonlinear feedback; nonlinear control; trajectory tracking; stabilisation; nonlinearities; simulation

1. Introduction

The vehicle model exhibits strongly nonlinear characteristic and has various devices and complex properties which make their control difficult and interesting problem. In the literature, there is a range of nonlinear models of road vehicles used to describe longitudinal, lateral and yaw motions (Abbassi, Aït-Amirat, & Outbib, 2007a; Andrea & Chou, 2005; Cao, Rakheja, & Su, 2008; Jia, 2000; Sayers, 1990). The majority of the models are characterised by a nonlinear aspect which cannot be tackled by using classical linear approaches (King, Chapman, & Ilic, 1994; Mielczarsky & Zajaczkowski, 1994; Vandergrift, Lewis, & Zhu, 1994; Ackermann, Guldner, Sienel, Steinhauser, & Utkin, 1995). More precisely, deficiencies associated with the linear approaches appears when one moves away from the set point. However the applicability of linear controller design techniques for transient stability enhancement is severely restricted. In high performance applications where a wide range of operating conditions are encountered such as vehicle dynamics, linear control design based on local approximations may be inadequate, and in the worst case fail (Sarkis Bedrossian, 1984).

Thus, a number of authors, Bingzhao, Hong, Yunfeng, and Kazushi (2011), Shraim, Ouladsine, and Fridman (2007) and De Luca, Oriolo, and Samson (1998), Beji and Bestaoui (2005) were interested in the problem of the vehicle control by the nonlinear approach to enhance stability and directional performance of road vehicles. However, the problem of vehicle control in a large operating range, hence by considering nonlinear aspect, is still an open question.

In this work, we are interested in the problem of feedback stabilisation of vehicle velocities and trajectory tracking. A design methodology, for feedback stabilisation of vehicle speed velocities, which fully incorporates all the inherent nonlinearities of the system without any kind of linearisation is presented. Based on this methodology, a lemma on stabilisation of nonlinear systems by adding an integrator, is proposed to achieve the trajectory tracking of vehicle.

Note that the control of nonlinear systems has been the subject of several studies in the literature (Abbassi et al. 2011; Guoguang, Zhaoxia, Rahmani, & Yongguang, 2013; Zhaoxia, Guoguang, Rahmani, & Yongguang, 2013). Thus, a great number of results have been proposed. These results are stated as necessary conditions, see Brockett, Millmann, and Sussmann (1983) and Tsinias (1989), sufficient condition ones (Arstein, 1983) or are established for some classes of nonlinear systems, for instance, to the dissipative systems, see Jurjevic and Quinn (1978) and Outbib and Vivalda (1999). However, it should be noted that these results cannot be applied directly to the considered models in this work.

The paper is organised as follows. In Section 2, a nonlinear vehicle model suitable for the present study is developed. In Section 3, some variable transformations are

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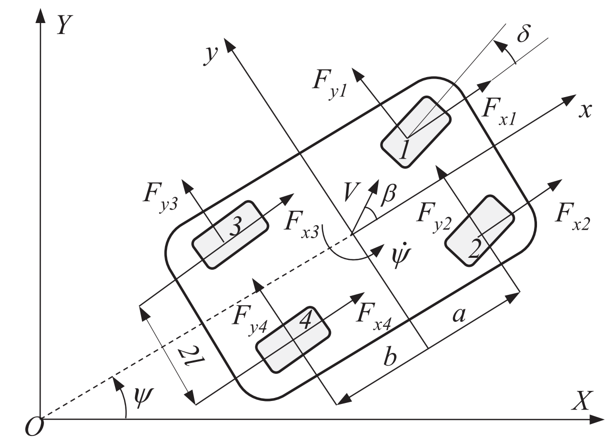


Figure 1. Forces acting on a tyre in x and y directions.

proposed to simplify the structure of the nonlinear model which provides preliminaries for the following sections. Then, a feedback nonlinear control for the stabilisation of the velocities is proposed in Section 4. Simulation results showing different velocities of the controlled vehicle are presented in Section 5. In Section 6, we propose a lemma which solves the trajectory tracking of road vehicle. To evaluate the performance of the proposed lemma, simulation results are further presented in the same section. Finally, the paper concludes in Section 7.

2. Nonlinear vehicle model

A number of advanced nonlinear models for the lateral and yaw motions have been developed to enhance directional performance and stability of road vehicle. In the present study, we consider the model proposed in Abbassi et al. (2007a) and (2007b). The vehicle dynamics are developed in two coordinate directions x, y and one rotation around the z axis. The vehicle model shown in Figure 1 is considered as a front wheel driving and steering, i.e., $\delta_f = \delta$ and the rear steering angle $\delta_r = 0$. Assuming small motions, the equations of motion are obtained by using Lagrangian formalism, and summarised as

$$\begin{cases} M\dot{V}_x = MV_\psi V_y + (F_{x1} + F_{x2})\cos\delta + F_{x3} + F_{x4} \\ \qquad\quad - (F_{y1} + F_{y2})\sin\delta - F_{ax} \\ M\dot{V}_y = -MV_\psi V_x + (F_{x1} + F_{x2})\sin\delta \\ \qquad\quad + (F_{y1} + F_{y2})\cos\delta + F_{y3} + F_{y4} - F_{ay} \\ I_z\dot{V}_\psi = a((F_{x1} + F_{x2})\sin\delta + (F_{y1} + F_{y2})\cos\delta) \\ \qquad\quad - b(F_{y3} + F_{y4}) + l(F_{x4} - F_{x3}) \\ \qquad\quad + l((F_{x2} - F_{x1})\cos\delta + (F_{y1} - F_{y2})\sin\delta) \end{cases} \tag{1}$$

where V_x and V_y are the longitudinal and lateral velocity, respectively. ψ is the yaw angle and $V_\psi = \dot{\psi}$ is the yaw angular velocity. g is the acceleration due to gravity and M is the mass of the vehicle. h is the centre of gravity (c.g.) height of the vehicle from the ground. a and b are longitudinal distances between the vehicle c.g. and the front and rear axles, respectively. I_z is the yaw mass moment of inertia of the vehicle. l is half-track width of the front and rear vehicle ends. The equivalent aerodynamic drag force in the x and y directions can be expressed as

$$F_{ax} = k_{ax}V_x^2, \qquad F_{ay} = k_{ay}V_y^2;$$

with $k_{ax} = \frac{1}{2}\rho C_a S_f V_x^2$ and $k_{ay} = \frac{1}{2}\rho C_a S_l V_y^2$ are the aerodynamic drag. ρ is the mass density of air and C_a is the aerodynamic drag coefficient. S_f and S_l are, respectively, the frontal and lateral area of the vehicle.

The nonlinear normal forces F_{zi}, $i = 1, \ldots, 4$, shown in Figure 2(b), act from the road on the inner tyre depend on the load transfer and are defined as follows:

$$\begin{aligned} F_{zi} &= \frac{b}{2(a+b)}\left[Mg + \frac{Mh(\dot{V}_x - V_yV_\psi)}{b}\right. \\ &\qquad \left. + (-1)^{i+1}\frac{Mh(\dot{V}_y + V_xV_\psi)}{l}\right], \quad i = 1, 2 \\ F_{zi} &= \frac{a}{2(a+b)}\left[Mg - \frac{Mh(\dot{V}_x - V_yV_\psi)}{a}\right. \\ &\qquad \left. + (-1)^{i+1}\frac{Mh(\dot{V}_y + V_xV_\psi)}{l}\right], \quad i = 3, 4 \end{aligned} \tag{2}$$

The rolling resistance R_{ri}, $(i = 1, \ldots, 4)$ shown in Figure 2(b) is modelled as being proportional to the normal load force on each set of tyres. For practical purposes, it is usually expressed as

$$R_{ri} = fF_{zi} \tag{3}$$

where f is the rolling friction coefficient. The tyre side slip angle α_i, $(i = f, r)$ is defined as the angle between the longitudinal direction of the tyre x_w and the orientation of the velocity vector V_w at the centre of the wheel, as shown in Figure 2(a),

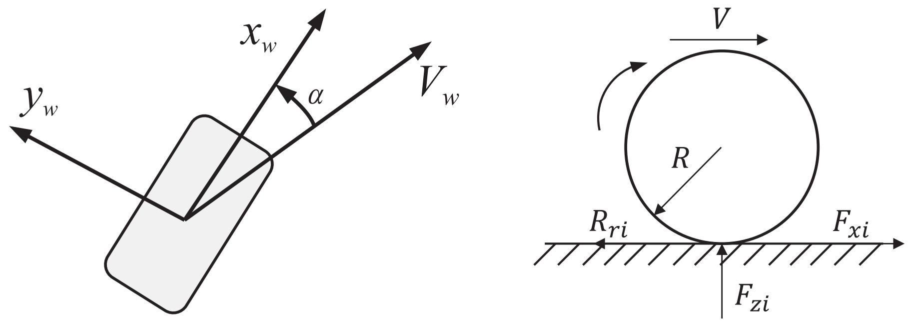
$$\alpha_f = \delta_f - \beta_f, \qquad \alpha_r = \delta_r - \beta_r \tag{4}$$

The front and rear chassis side slip angles β_i,$(i = f, r)$ are expressed as

$$\beta_f = \frac{V_y + aV_\psi}{V_x}, \qquad \beta_r = \frac{V_y - bV_\psi}{V_x} \tag{5}$$

The vehicle is considered as a front wheels driving and steering, i.e., $\delta_f = \delta$ and the rear steering angle $\delta_r = 0$. By substituting Equation (5) in Equation (4), we have

$$\alpha_f = \delta - \frac{V_y + aV_\psi}{V_x}, \qquad \alpha_r = -\frac{V_y - bV_\psi}{V_x} \tag{6}$$



(a) (b)

Figure 2. (a) Tyre side slip angle α and (b) wheel dynamic model.

Experimental results show that the lateral tyre force is proportional to the tyre side slip angle. For the front and rear tyres, they can be written as

$$\begin{aligned} F_{yi} &= C_f \alpha_f, \quad i = 1, 2 \\ F_{yi} &= C_r \alpha_r, \quad i = 3, 4 \end{aligned} \tag{7}$$

From Equation (4), the expressions of the lateral tyres forces in Equation (7) become

$$\begin{aligned} F_{yi} &= C_f \left(\delta - \frac{V_y + aV_\psi}{V_x} \right), \quad i = 1, 2 \\ F_{yi} &= -C_r \left(\frac{V_y - bV_\psi}{V_x} \right), \quad i = 3, 4 \end{aligned} \tag{8}$$

The vehicle is supposed to be a front wheels driving and steering. However, the longitudinal forces F_{xi} are distributed on the front and rear wheels with a distribution coefficient $f_r \in [0, 1]$ (see Alloum, Charara, & Rombaut, 1995) such that

$$\begin{aligned} F_{xi} &= \frac{(F_a - f_r F_b)}{2} - R_{ri}, \quad i = 1, 2 \\ F_{xi} &= \frac{(f_r - 1)\, F_b}{2} - R_{ri}, \quad i = 3, 4 \end{aligned} \tag{9}$$

where F_a, F_b and R_{ri} are the acceleration, braking and rolling resistance forces, respectively.

For a small steering angle δ and by substituting Equations (8) and (9) in the system (1), we have the following equations of the nonlinear vehicle model:

$$\begin{cases} M\dot{V}_x = F_a - F_b + MV_y V_\psi - Mfg + C_f \delta \dfrac{V_y}{V_x} \\ \qquad + aC_f \delta \dfrac{V_\psi}{V_x} - k_{ax} V_x^2 \\ M\dot{V}_y = (F_a - f_r F_b)\,\delta - MV_x V_\psi + C_f \delta \\ \qquad - (C_f + C_r) \dfrac{V_y}{V_x} - (aC_f - bC_r) \dfrac{V_\psi}{V_x} - k_{ay} V_y^2 \\ I_z \dot{V}_\psi = a(F_a - f_r F_b)\delta - MfhV_x V_\psi + aC_f \delta \\ \qquad - (aC_f - bC_r) \dfrac{V_y}{V_x} - \left(a^2 C_f + b^2 C_r\right) \dfrac{V_\psi}{V_x} \end{cases} \tag{10}$$

In a realistic situation, the driver does not act on the brake and accelerate at the same time. For this reason, we can use only one longitudinal force F_l for acceleration/braking. Then, the equations of motion (10) become

$$\begin{cases} \dot{V}_x = \dfrac{1}{M} \Bigg(F_l + c_1 + MV_y V_\psi + C_f \delta \dfrac{V_y}{V_x} \\ \qquad + aC_f \delta \dfrac{V_\psi}{V_x} - k_{ax} V_x^2 \Bigg) \\ \dot{V}_y = \dfrac{1}{M} \Bigg(F_l \delta - MV_x V_\psi + C_f \delta + c_2 \dfrac{V_y}{V_x} \\ \qquad + c_3 \dfrac{V_\psi}{V_x} - k_{ay} V_y^2 \Bigg) \\ \dot{V}_\psi = \dfrac{1}{I_z} \left(aF_l \delta + c_4 V_x V_\psi + aC_f \delta + c_5 \dfrac{V_y}{V_x} + c_6 \dfrac{V_\psi}{V_x} \right) \end{cases} \tag{11}$$

where F_l and δ denote the control inputs of the nonlinear vehicle model and the constants c_i are defined as

$$\begin{aligned} & c_1 = -Mfg, \quad c_2 = -(C_f + C_r), \\ & c_3 = c_5 = -(aC_f - bC_r), \quad c_4 = -Mfh \\ & c_6 = -(a^2 C_f + b^2 C_r) \end{aligned}$$

with F_l and δ denote the control inputs of the nonlinear vehicle model.

3. Variables transformation

For technical reasons of computation and from Outbib and Rachid (2000), we can use the following transformation:

$$\tilde{V}_\psi = V_\psi \frac{I_z}{a} - MV_y \Leftrightarrow V_\psi = (\tilde{V}_\psi + MV_y)\frac{a}{I_z} \qquad (12)$$

Then the nonlinear model (11) becomes

$$\begin{cases} \dot{V}_x = \dfrac{1}{M}\Bigg(c_1 + \dfrac{Ma}{I_z}V_y\tilde{V}_\psi + \dfrac{aM^2}{I_z}V_y^2 - k_{ax}V_x^2 + F_l \\ \qquad + C_f\delta\left[\dfrac{V_y}{V_x} + \dfrac{a^2}{I_z}\dfrac{\tilde{V}_\psi}{V_x} + M\dfrac{a^2}{I_z}\dfrac{V_y}{V_x}\right]\Bigg) \\ \dot{V}_y = \dfrac{1}{M}\Bigg(-\dfrac{Ma}{I_z}V_x\tilde{V}_\psi - \dfrac{M^2a}{I_z}V_xV_y \\ \qquad + c_2\dfrac{V_y}{V_x} + c_3\dfrac{a}{I_z}\dfrac{\tilde{V}_\psi}{V_x} + c_3\dfrac{Ma}{I_z}\dfrac{V_y}{V_x} \\ \qquad - k_{ay}V_y^2 + F_l\delta + C_f\delta\Bigg) \\ \dot{\tilde{V}}_\psi = \dfrac{a}{I_z}\left(\dfrac{c_4}{a} + M\right)V_x\tilde{V}_\psi + \dfrac{aM}{I_z}\left(\dfrac{c_4}{a} + M\right)V_xV_y \\ \qquad + \left(\dfrac{c_5}{V_x} - c_2\right)\dfrac{V_y}{a} + \dfrac{a}{I_z}\left(\dfrac{c_6}{a} - c_3\right)\dfrac{\tilde{V}_\psi}{V_x} \\ \qquad + \dfrac{aM}{I_z}\left(\dfrac{c_6}{a} - c_3\right)\dfrac{V_y}{V_x} + k_{ay}V_y^2 \end{cases} \qquad (13)$$

The preliminary feedback is defined by

$$\sigma = F_l + C_f\delta\left(\frac{V_y}{V_x} + \frac{a^2}{I_z}\frac{\tilde{V}_\psi}{V_x} + M\frac{a^2}{I_z}\frac{V_y}{V_x}\right) \quad \text{and}$$
$$\gamma = F_l\delta + C_f\delta \qquad (14)$$

For the sake of simplicity, we use Θ to indicate the following quantity:

$$\Theta = \frac{V_y}{V_x} + \frac{a^2}{I_z}\frac{\tilde{V}_\psi}{V_x} + M\frac{a^2}{I_z}\frac{V_y}{V_x}$$

The equations of the system (14) become

$$\begin{cases} \sigma = F_l + C_f\delta\Theta \\ \gamma = F_l\delta + C_f\delta \end{cases} \qquad (15)$$

According to the second equation in Equation (15), we have

$$\gamma\Theta = \left(\frac{F_l}{C_f} + 1\right)C_f\delta\Theta \qquad (16)$$

By using the first equation of Equation (15), Equation (16) becomes

$$\gamma\Theta = \left(\frac{F_l}{C_f} + 1\right)(\sigma - F_l) \qquad (17)$$

or

$$\frac{F_l^2}{C_f} + F_l\left(1 - \frac{\sigma}{C_f}\right) + \gamma\Theta - \sigma = 0 \qquad (18)$$

By a similar computation, we obtain for δ

$$C_f\Theta\delta^2 - \delta(C_f + \sigma) + \gamma = 0 \qquad (19)$$

Finally, the control variables F_l and δ can be calculated according to σ and γ by using Equations (18) and (19). However, it will be necessary to consider only the operating ranges where Equations (18) and (19) have real solutions which respect physical considerations. Precisely, Equation (18) has a real solution if the following condition holds:

$$\Delta_1 = \left(1 - \frac{\sigma}{C_f}\right)^2 - \frac{4}{C_f}(\gamma\Theta - \sigma) > 0 \qquad (C1)$$

In other words, it is necessary for all the following study to ensure that the condition $(C1)$ is satisfied. Then, we have to choose among the two possible solutions according to physical criteria

$$F_l = \frac{\sigma - C_f}{2} \pm \frac{C_f}{2}\sqrt{\Delta_1} \qquad (S1)$$

Equation (19), considered as a second degree polynomial in δ, has a real solution if its discriminant is positive

$$\Delta_2 = (C_f + \sigma)^2 - 4\gamma C_f\Theta > 0 \qquad (C2)$$

The real solution will be given by

$$\delta = \begin{cases} \dfrac{(C_f + \sigma) - \mathrm{Sgn}\,(C_f + \sigma)\sqrt{\Delta_2}}{2C_f\Theta} & \text{if } \Theta \neq 0 \\ \dfrac{\gamma}{C_f + \sigma} & \text{elsewhere} \end{cases} \qquad (S2)$$

where Sgn indicates the sign function. It is introduced to keep only the acceptable solution according to the steering angle δ (when $\Theta \neq 0$).

It should be noted that δ is defined as a continuous function. To prove this, we can use the following approximation:

$$\sqrt{k - x} \approx \sqrt{k} - \frac{1}{2}\frac{x}{\sqrt{k}}$$

for x in a neighbourhood of zero and k one positive constant. Now we try to solve the initial problem of the stabilisation by a feedback control on the nonlinear system (13) where the control variables are given by (σ, γ) and after that, we use Equations $(S1)$ and $(S2)$ to deduce the values of the origin variables, F_l and δ. However, during the synthesis of the feedback control (σ, γ), it is necessary to check both conditions $(C1)$ and $(C2)$.

By using the expressions of F_l and δ involving σ and γ, the system (13) becomes

$$\begin{cases} \dot{V}_x = \frac{1}{M}\left(c_1 + \frac{Ma}{I_z}V_y\tilde{V}_\psi + \frac{aM^2}{I_z}V_y^2 - k_{ax}V_x^2 + \sigma\right) \\ \dot{V}_y = \frac{1}{M}\Big(-\frac{Ma}{I_z}V_x\tilde{V}_\psi - \frac{aM^2}{I_z}V_xV_y + c_2\frac{V_y}{V_x} \\ \quad + c_3\frac{a}{I_z}\frac{\tilde{V}_\psi}{V_x} + c_3\frac{Ma}{I_z}\frac{V_y}{V_x} - k_{ay}V_y^2 + \gamma\Big) \\ \dot{\tilde{V}}_\psi = \frac{a}{I_z}\left(\frac{c_4}{a} + M\right)V_x\tilde{V}_\psi + \frac{aM}{I_z}\left(\frac{c_4}{a} + M\right)V_xV_y \\ \quad + \left(\frac{c_5}{V_x} - c_2\right)\frac{V_y}{a} + \frac{a}{I_z}\left(\frac{c_6}{a} - c_3\right)\frac{\tilde{V}_\psi}{V_x} \\ \quad + \frac{aM}{I_z}\left(\frac{c_6}{a} - c_3\right)\frac{V_y}{V_x} + k_{ay}V_y^2 \end{cases} \tag{20}$$

Let

$$\sigma_1 = \frac{1}{M}\left(c_1 + \frac{Ma}{I_z}V_y\tilde{V}_\psi + \frac{aM^2}{I_z}V_y^2 - k_{ax}V_x^2 + \sigma\right)$$

and

$$\gamma_1 = \frac{1}{M}\Big(-\frac{Ma}{I_z}V_x\tilde{V}_\psi - \frac{M^2a}{I_z}V_xV_y + c_2\frac{V_y}{V_x} + c_3\frac{a}{I_z}\frac{\tilde{V}_\psi}{V_x} + c_3\frac{Ma}{I_z}\frac{V_y}{V_x} - k_{ay}V_y^2 + \gamma\Big)$$

the system (20) becomes

$$\begin{cases} \dot{V}_x = \sigma_1 \\ \dot{V}_y = \gamma_1 \\ \dot{\tilde{V}}_\psi = \frac{a}{I_z}\left(\frac{c_4}{a} + M\right)V_x\tilde{V}_\psi + \frac{aM}{I_z}\left(\frac{c_4}{a} + M\right)V_xV_y \\ \quad + \left(\frac{c_5}{V_x} - c_2\right)\frac{V_y}{a} + \frac{a}{I_z}\left(\frac{c_6}{a} - c_3\right)\frac{\tilde{V}_\psi}{V_x} \\ \quad + \frac{aM}{I_z}\left(\frac{c_6}{a} - c_3\right)\frac{V_y}{V_x} + k_{ay}V_y^2 \end{cases} \tag{21}$$

The method consists now in stabilising this system under the conditions $(C1)$ and $(C2)$. The procedure is declined in two steps. The first one consists of checking that for $\sigma_1 = \gamma_1 = 0$ or $\sigma = \sigma_0$ and $\gamma = \gamma_0$ with

$$\sigma_0 = -c_1 - \frac{Ma}{I_z}V_y\tilde{V}_\psi - \frac{aM^2}{I_z}V_y^2 + k_{ax}V_x^2$$

$$\gamma_0 = \frac{Ma}{I_z}V_x\tilde{V}_\psi + \frac{M^2a}{I_z}V_xV_y - c_2\frac{V_y}{V_x} - c_3\frac{a}{I_z}\frac{\tilde{V}_\psi}{V_x} - c_3\frac{Ma}{I_z}\frac{V_y}{V_x} + k_{ay}V_y^2$$

and by taking into account the considered operating range, both conditions $(C1)$ and $(C2)$ are satisfied. In the second step, a control law is synthesised with $|\sigma_1| \le \epsilon$ and $|\gamma_1| \le \epsilon$, where ϵ is a real positive number. Therefore, the conditions $(C1)$ and $(C2)$ remain satisfied for all $t > 0$.

4. Control of vehicle velocities

We are interested here in the stabilisation of the nonlinear system (21) which describes the vehicle longitudinal, lateral and yaw angular velocities around the set point $(V_x^0, 0, 0)$. V_x^0 is the reference value for longitudinal velocity. To this end, the third equation of the above nonlinear system can be transformed by introducing a function F defined as

$$F(V_x) = \frac{a}{I_z}\left(\frac{c_4}{a} + M\right)V_x + \frac{a}{I_z}\left(\frac{c_6}{a} - c_3\right)\frac{1}{V_x} \tag{22}$$

It is an increasing function of the longitudinal velocity V_x. A simple reasoning shows that the above nonlinear system (21) can be written under the following form:

$$\begin{cases} \dot{V}_x = \sigma_1 \\ \dot{V}_y = \gamma_1 \\ \dot{\tilde{V}}_\psi = G\left(V_x, V_y, \tilde{V}_\psi\right) \end{cases} \tag{23}$$

with

$$G\left(V_x, V_y, \tilde{V}_\psi\right) = \left[F(V_x) - F(V_x^0)\right]\left[\tilde{V}_\psi + MV_y\right] + F(V_x^0)\left[\tilde{V}_\psi + MV_y\right] + \left(\frac{c_5}{a} - c_2\right)\frac{V_y}{V_x} + k_{ay}V_y^2$$

The expression of the derivative $\dot{\tilde{V}}_\psi$ for $V_x = V_x^0$ is given by

$$\dot{\tilde{V}}_\psi = G\left(V_x^0, V_y, \tilde{V}_\psi\right) = \lambda_1\left[V_y^2 + \lambda_2 V_y + \lambda_3\tilde{V}_\psi\right]$$

with

$$\begin{cases} \lambda_1 = k_{ay} \\ \lambda_2 = \frac{1}{\lambda_1}\Big[\frac{aM}{I_z}\left(\frac{c_4}{a} + M\right)V_x^0 + \left(\frac{c_5}{a} - c_2\right)\frac{1}{V_x^0} \\ \quad + \frac{aM}{I_z}\left(\frac{c_6}{a} - c_3\right)\frac{1}{V_x^0}\Big] \\ \lambda_3 = \frac{1}{\lambda_1}\left[\frac{a}{I_z}\left(\frac{c_4}{a} + M\right)V_x^0 + \frac{a}{I_z}\left(\frac{c_6}{a} - c_3\right)\frac{1}{V_x^0}\right] \end{cases}$$

Let

$$\Delta_{V_x^0} = \lambda_2^2 - 4\lambda_3\tilde{V}_\psi$$

Then,

$$G\left(V_x^0, V_y, \tilde{V}_\psi\right) = \prod_{i=1}^{2} \lambda_1 \left(V_y + \frac{\lambda_2 + (-1)^{(i+1)}\sqrt{\Delta_{V_x^0}}}{2}\right)$$

Given the Lyapunov function W defined by

$$\begin{aligned} W\left(V_y, \tilde{V}_\psi\right) = \frac{1}{2}V_y^2 + \frac{\lambda_2 - \mathrm{Sgn}(\lambda_2)\sqrt{\Delta_{V_x^0}}}{2}V_y \\ + \left(\lambda_2 - \mathrm{Sgn}(\lambda_2)\sqrt{\Delta_{V_x^0}}\right)^2 \end{aligned}$$

where Sgn indicates the sign function. A simple computation shows that

$$G\left(V_x^0, V_y, \tilde{V}_\psi\right) = \lambda_1 \left(V_y + \frac{\lambda_2 + \mathrm{Sgn}(\lambda_2)\sqrt{\Delta_{V_x^0}}}{2}\right)\frac{\partial W}{\partial V_y}$$

Next, the system (23) becomes

$$\begin{cases} \dot{V}_x &= \sigma_1 \\ \dot{V}_y &= \gamma_1 \\ \dot{\tilde{V}}_\psi &= G\left(V_x^0, V_y, \tilde{V}_\psi\right) + G\left(V_x, V_y, \tilde{V}_\psi\right) \\ & \quad - G\left(V_x^0, V_y, \tilde{V}_\psi\right) \end{cases}$$

First, we will prove that

$$\begin{aligned} & G(V_x, V_y, \tilde{V}_\psi) - G\left(V_x^0, V_y, \tilde{V}_\psi\right) \\ & = G_1(V_x, V_y, \tilde{V}_\psi)\left(V_x - V_x^0\right) \end{aligned} \tag{24}$$

where G_1 is a regular function. Indeed, a simple reasoning shows that

$$G\left(V_x, V_y, \tilde{V}_\psi\right) - G\left(V_x^0, V_y, \tilde{V}_\psi\right) = \int_{V_x^0}^{V_x} \frac{\partial G}{\partial V_x}\left(\tau, V_y, \tilde{V}_\psi\right) d\tau$$

Let us consider the change of variable defined by

$$\tau = tV_x + (1-t)\,V_x^0$$

We have

$$d\tau = \left(V_x - V_x^0\right) dt$$

therefore,

$$\begin{aligned} & G\left(V_x, V_y, \tilde{V}_\psi\right) - G\left(V_x^0, V_y, \tilde{V}_\psi\right) = \left(V_x - V_x^0\right)\int_0^1 \frac{\partial G}{\partial V_x} \\ & \times \left(tV_x + (1-t)\,V_x^0, V_y, \tilde{V}_\psi\right) dt \end{aligned}$$

Hence, we obtain expression (24). Now, we have to compute explicitly

$$G_1\left(V_x, V_y, \tilde{V}_\psi\right) = \int_0^1 \frac{\partial G}{\partial V_x}\left(tV_x + (1-t)\,V_x^0, V_y, \tilde{V}_\psi\right) dt$$

Otherwise, G is given by the following expression:

$$\begin{aligned} G\left(V_x, V_y, \tilde{V}_\psi\right) = & \left[\frac{a}{I_z}\left(\frac{c_4}{a} + M\right)V_x + \frac{a}{I_z}\left(\frac{c_6}{a} - c_3\right)\frac{1}{V_x}\right]\tilde{V}_\psi \\ & + \left[\frac{aM}{I_z}\left(\frac{c_4}{a} + M\right)V_x + \left(\frac{c_5}{a} - c_3\right)\frac{1}{V_x}\right. \\ & \left. + \frac{aM}{I_z}\left(\frac{c_6}{a} - c_3\right)\frac{1}{V_x}\right]V_y + k_{ay}V_y^2 \end{aligned}$$

A simple computation gives

$$\frac{\partial G}{\partial V_x} = \eta_1 \frac{\tilde{V}_\psi}{V_x^2} + \eta_2 \frac{V_y}{V_x^2} + \eta_3 \tilde{V}_\psi + \eta_4 V_y \tag{25}$$

where the constants η_i, $i = 1, \ldots, 4$ are given by

$$\begin{cases} \eta_1 = -\dfrac{a}{I_z}\left(\dfrac{c_6}{a} - c_3\right) \\ \eta_2 = -\dfrac{c_5}{a} + c_3 - \dfrac{aM}{I_z}\left(\dfrac{c_6}{a} - c_3\right) \\ \eta_3 = \dfrac{a}{I_z}\left(\dfrac{c_4}{a} + M\right) \\ \eta_4 = \dfrac{aM}{I_z}\left(\dfrac{c_4}{a} + M\right) \end{cases}$$

Finally, by using expression (25), we obtain

$$\begin{aligned} G_1 &= \int_0^1 \frac{\partial G}{\partial V_x}\left(tV_x + (1-t)\,V_x^0, V_y, \tilde{V}_\psi\right) dt \\ &= \frac{\eta_1 \tilde{V}_\psi + \eta_2 V_y}{V_x V_x^0} + \eta_3 \tilde{V}_\psi + \eta_4 V_y \end{aligned}$$

Then, the system (23) can be rewritten as

$$\begin{cases} \dot{V}_x = \sigma_1 \\ \dot{V}_y = \gamma_1 \\ \dot{\tilde{V}}_\psi = G(V_x^0, V_y, \tilde{V}_\psi) + G_1(V_x, V_y, \tilde{V}_\psi)\left(V_x - V_x^0\right) \end{cases} \tag{26}$$

To demonstrate the asymptotic stability of the feedback nonlinear system, we introduce the following Lyapunov function:

$$W_1(V_x, V_y, \tilde{V}_\psi) = \frac{1}{2}(V_x - V_x^0)^2 + ln(W(V_y, \tilde{V}_\psi) + 1)$$

A simple computation shows that W_1 is a positive function on $\left(V_x^0, 0, 0\right)$.

Consider the nonlinear state feedback given by

$$\begin{cases} \sigma_1 = -\eta \dfrac{V_x - V_x^0}{1 + \left(V_x - V_x^0\right)^2} \\ \quad - G_1\left(V_x, V_y, \tilde{V}_\psi\right) \dfrac{\partial W}{\partial \tilde{V}_\psi} \dfrac{1}{W+1} \\ \gamma_1 = -\lambda_1 \left(V_y + \dfrac{\lambda_2 - \mathrm{Sgn}(\lambda_2)\sqrt{\sigma_{V_x^0}}}{2}\right) \dfrac{\partial W}{\partial \tilde{V}_\psi} \\ \quad - \eta \dfrac{\partial W}{\partial V_y} \dfrac{1}{1 + \left(\dfrac{\partial W}{\partial V_y}\right)^2} \end{cases} \tag{27}$$

where $\eta > 0$. The computation of the derivatives leads to

$$\begin{cases} \sigma_1 = -\eta \dfrac{V_x - V_x^0}{1 + \left(V_x - V_x^0\right)^2} + \dfrac{\lambda_3\left(V_y + 4\lambda_2 + 4\sqrt{\Delta_{V_x^0}}\right)}{(W+1)\sqrt{\Delta_{V_x^0}}} \\ \quad \times G_1\left(V_x, V_y, \tilde{V}_\psi\right) \\ \gamma_1 = \lambda_1\lambda_3 \left(V_y + \dfrac{\lambda_2 - \mathrm{Sgn}(\lambda_2)\sqrt{\Delta_{V_x^0}}}{2}\right) H\left(V_y\right) \\ \quad - \eta \dfrac{\left(V_y + \frac{\lambda_2 - \mathrm{Sgn}(\lambda_2)\sqrt{\Delta_{V_x^0}}}{2}\right)}{1 + \left(V_y + \frac{\lambda_2 - \mathrm{Sgn}(\lambda_2)\sqrt{\Delta_{V_x^0}}}{2}\right)^2} \end{cases} \tag{28}$$

where $H(V_y) = \left(\frac{V_y + 4\lambda_2 + 4\sqrt{\Delta_{V_x^0}}}{\sqrt{\Delta_{V_x^0}}}\right)$

The derivative of W_1 by considering the system (26) with the state feedback (27) is given by

$$\begin{aligned} \dot{W}_1\left(V_x, V_y, \tilde{V}_\psi\right) &= \frac{\partial W_1}{\partial V_x}\dot{V}_x + \frac{\partial W_1}{\partial V_y}\dot{V}_y + \frac{\partial W_1}{\partial \tilde{V}_\psi}\dot{\tilde{V}}_\psi \\ &= -\eta \frac{\left(V_x - V_x^0\right)^2}{1 + \left(V_x - V_x^0\right)^2} \\ &\quad - \eta \left(\frac{\partial W}{\partial V_y}\right)^2 \frac{1}{1 + \left(\frac{\partial W}{\partial V_y}\right)^2} \le 0 \end{aligned}$$

Then, the system is stable.

To prove that the system is attractive, we use the invariance principle of LaSalle and Lefschetz (1961). We consider a set Ω defined by

$$\Omega = \{\dot{W}_1(V_x, V_y, \tilde{V}_\psi) = 0\}$$

or

$$\Omega = \left\{ \dot{W}_1(V_x, V_y, \tilde{V}_\psi) = 0 : V_x = V_x^0 \text{ and } V_y + \frac{\lambda_2 - \mathrm{Sgn}(\lambda_2)\sqrt{\Delta_{V_x^0}}}{2} = 0 \right\}$$

On the set Ω, we have

$$\begin{aligned} &\frac{d}{dt}\left(V_y + \frac{\lambda_2 - \mathrm{Sgn}(\lambda_2)\sqrt{\sigma_{V_x^0}}}{2}\right) \\ &\quad = -\lambda_1 \left(V_y + \frac{\lambda_2 - \mathrm{Sgn}(\lambda_2)\sqrt{\sigma_{V_x^0}}}{2}\right) \frac{\partial W}{\partial \tilde{V}_\psi} \end{aligned}$$

The greatest invariant set checks the following equations:

$$\begin{cases} V_y + \dfrac{\lambda_2 - \mathrm{Sgn}(\lambda_2)\sqrt{\Delta_{V_x^0}}}{2} = 0 \\ \dfrac{\partial W}{\partial \tilde{V}_\psi}\left(V_y + \dfrac{\lambda_2 + \mathrm{Sgn}(\lambda_2)\sqrt{\Delta_{V_x^0}}}{2}\right) = 0 \end{cases}$$

or

$$\begin{cases} V_y + \dfrac{\lambda_2 - \mathrm{Sgn}(\lambda_2)\sqrt{\Delta_{V_x^0}}}{2} = 0 \\ \dfrac{-2\epsilon\lambda_3\left[V_y + 2(\lambda_2 - \mathrm{Sgn}(\lambda_2)\sqrt{\Delta_{V_x^0}})\right]}{\sqrt{\lambda_2^2 - 4\lambda_3\tilde{V}_\psi}} R\left(V_y, \lambda_2\right) = 0 \end{cases} \tag{29}$$

where

$$R\left(V_y, \lambda_2\right) = \left(V_y + \frac{\lambda_2 + \mathrm{Sgn}(\lambda_2)\sqrt{\Delta_{V_x^0}}}{2}\right)$$

Finally, and according to Equation (29), we have

$$\begin{cases} V_y + \dfrac{\lambda_2 - \mathrm{Sgn}(\lambda_2)\sqrt{\Delta_{V_x^0}}}{2} = 0 \\ V_y + \dfrac{\lambda_2 + \mathrm{Sgn}(\lambda_2)\sqrt{\Delta_{V_x^0}}}{2} = 0 \end{cases} \quad \text{or} \quad \begin{cases} V_y + \dfrac{\lambda_2 - \mathrm{Sgn}(\lambda_2)\sqrt{\Delta_{V_x^0}}}{2} = 0 \\ V_y + 2(\lambda_2 - \mathrm{Sgn}(\lambda_2)\sqrt{\Delta_{V_x^0}}) = 0 \end{cases} \tag{30}$$

The system (30) does not have a solution in a physical meaning. The above computation is done under the assumption that $\Delta_{V_x^0} > 0$, but it imposes a condition on $\tilde{V}_\psi$. The system (26) has $V_y = \tilde{V}_\psi = 0$ as a single solution. Thus, the greatest invariant set contained in Ω is reduced to $\{(V_x^0, 0, 0)\}$. Finally, the system is asymptotically stable. This finishes the proof of the following theorem.

Theorem 4.1: *The system* (26) *controlled by the state feedback* (28) *is asymptotically stable on* $\left(V_x^0, 0, 0\right)$.

5. Simulation results and discussion

To evaluate the performance of the proposed nonlinear feedback control, different simulations were performed by using the vehicle data given in the Appendix. The feedback control model simulations were thus performed under acceleration-in-a-turn manoeuvre. The main goal of the nonlinear feedback control is the stabilisation of the different vehicle velocities.

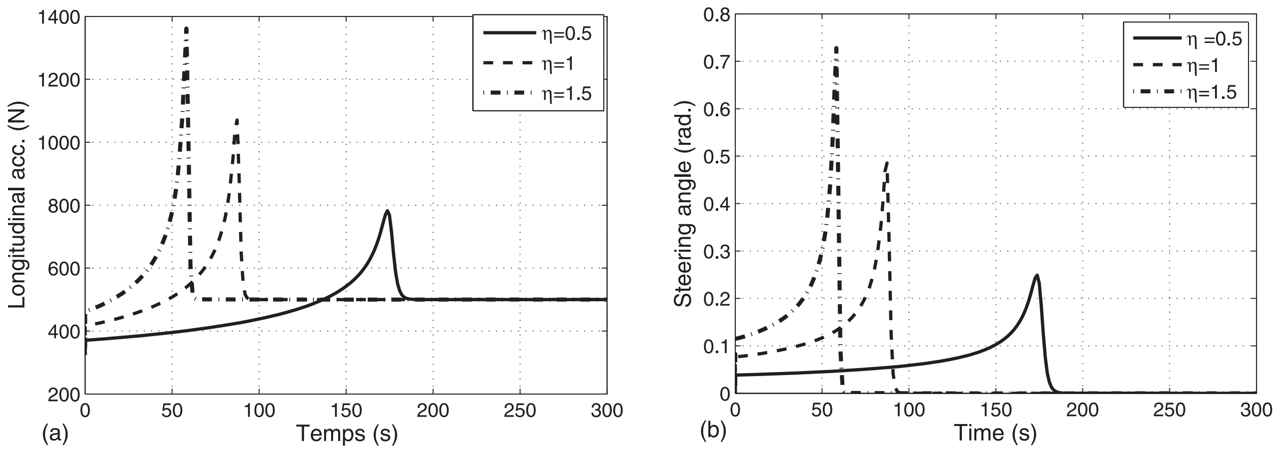


Figure 3. Acceleration and steering responses of the vehicle with different parameter η: (a) acceleration force input F_l and (b) steering angle input δ.

Figure 3(a) and 3(b) illustrates the responses of the acceleration force F_l and the steering angle δ input with different values for the parameter η. We can see that the vehicle acceleration and steering in the simulation increase slowly when the parameter η increases.

Figure 4(a) and 4(b) shows the longitudinal and lateral velocities responses to the acceleration-in-a-turn manoeuvre.

Figure 5 presents the vehicle stabilisation in terms of longitudinal velocity, acceleration and steering manoeuvre. The results show that the vehicle was stabilised at two desired longitudinal velocities: ${V_x}^0 = 25$ and ${V_x}^0 = 20$ m/s from an initial velocity 30 m/s with $\eta = 0.5$. Notice that peak of acceleration force (traction force) responses at ${V_x}^0 = 20$ m/s are considerably lower than those at ${V_x}^0 = 25$ m/s. The higher acceleration or traction force is also evident from the relatively higher longitudinal velocity. However, the steering angle responses at ${V_x}^0 = 25$ m/s are higher than those at ${V_x}^0 = 20$ m/s, which is partially attributed to the difference between the two longitudinal velocities. The parameter η could be selected for a soft acceleration/braking in which the driver does not accelerate too fast or brake too hard. The results thus further confirm the improved nonlinear feedback control which could provide directional stability and control performances.

Figure 6 illustrates the stabilisation of vehicle at the desired velocity: $V_x = 17$ m/s. The results thus show two cases to achieve the desired velocity: deceleration from 25 to 17 m/s and acceleration from 6 to 17 m/s. The simulation results further show that the vehicle during acceleration or deceleration has reached the desired velocity without being deflected.

6. Trajectory tracking

In this section, we propose a technical lemma on feedback stabilisation of nonlinear systems by adding an integrator and we apply this result to the problem of trajectory tracking of the vehicle. This lemma deals with class of nonlinear systems that contain an integrator in their structure (Outbib and Aggoune, 1999; Outbib and Jghima, 1998).

$$\begin{cases} \dot{x} = f(x, y, t) \\ \dot{y} = u \\ x \in \mathbb{R}^n, \quad y \in \mathbb{R}^m \end{cases} \tag{31}$$

where f is a function which is at least continuous.

Lemma 6.1: *Let us consider the system defined on $\mathbb{R}^n \times \mathbb{R}^n$ by*

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = A(t).x_1 \end{cases} \tag{32}$$

where $A(t)$ is such that[1]

$$\begin{cases} Det^1(A(t)) \neq 0 \\ \|A(t)\| < C_0 \\ \left\|A(t)^{-1}\right\| \leq C_1 \\ \left\|\dot{A}^{-1}(t)\right\| \leq C_2 \quad \textit{for all} \quad t \geq 0. \end{cases}$$

Let also $\Gamma(t)$ be a curve on $\mathbb{R}^n$ such that $\left\|\dot{\Gamma}(t)\right\| \leq C_3$ and $\left\|\ddot{\Gamma}(t)\right\| \leq C_4$, for all $t \geq 0$, where $C_i (i = 0, \ldots, 4)$ are positive constants. For all $\epsilon > 0$, there exists u_ϵ, such that $\|u_\epsilon\| < \epsilon + C$ where $C = C_2C_3 + C_1C_4$. The closed-loop system defined from Equation (32) with $u = u_\epsilon$ verifies

$$\|x_2(t) - \Gamma(t)\| \rightarrow 0 \textit{ when } t \rightarrow +\infty$$

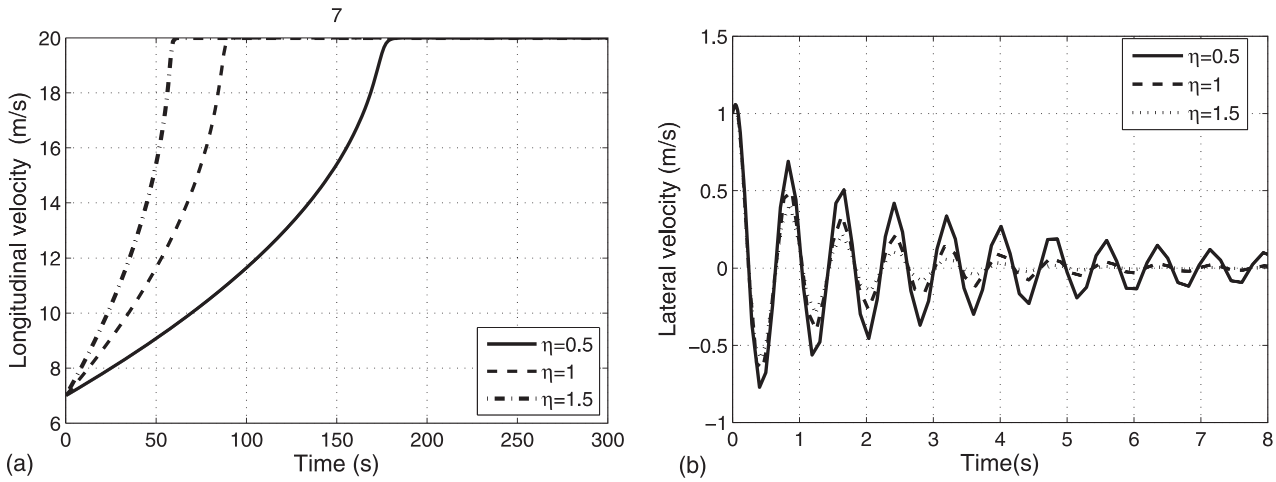


Figure 4. Longitudinal and lateral velocities corresponding to acceleration-in-a-turn manoeuvre with different parameter η: (a) longitudinal velocity V_x from 7 to 20 m/s and (b) lateral velocity V_y.

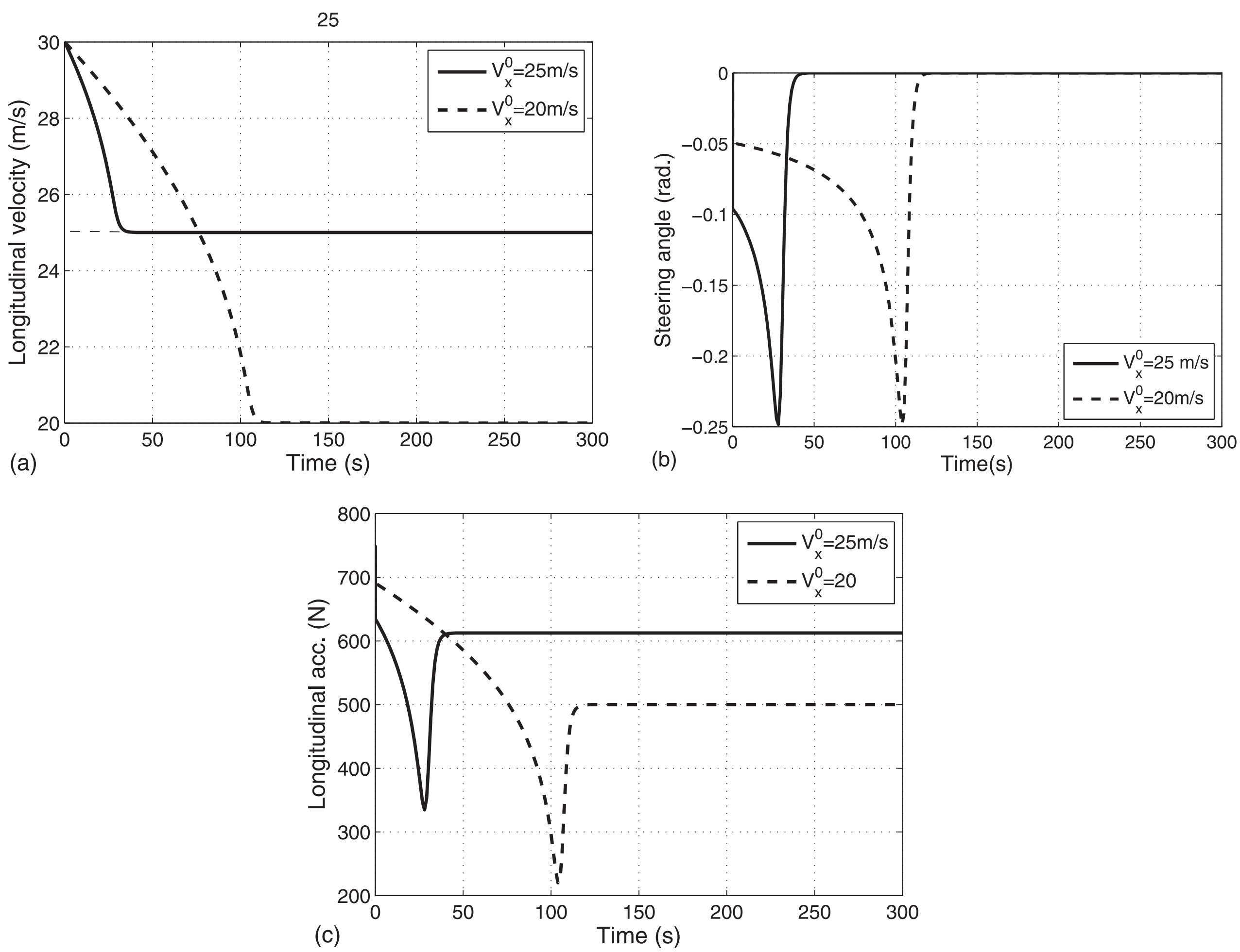


Figure 5. Stabilisation of vehicle at two desired velocities: $V_x{}^0 = 25$ and $V_x{}^0 = 20$ m/s: (a) longitudinal velocity V_x; (b) steering angle δ and (c) acceleration force F_l.

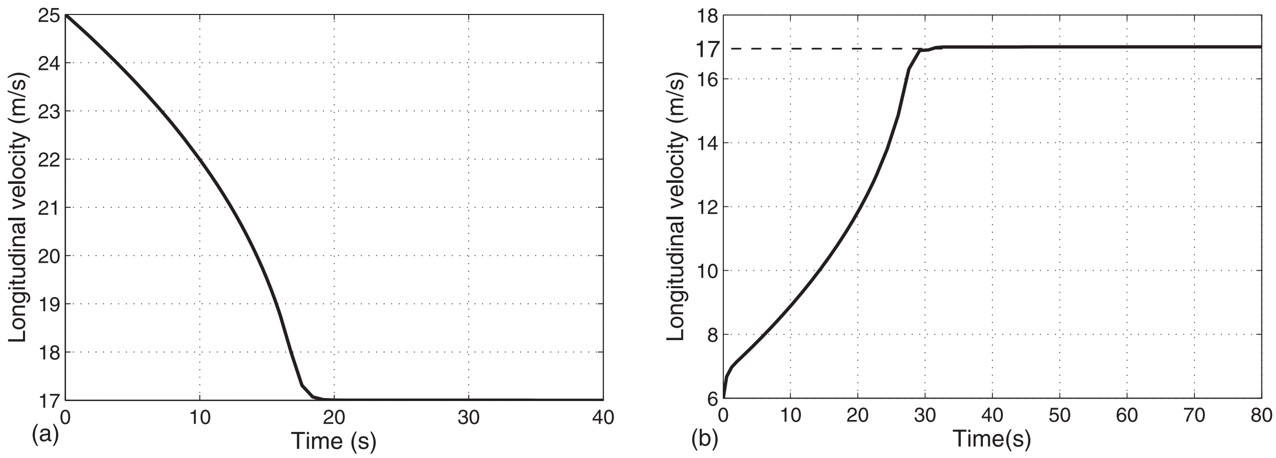


Figure 6. Stabilisation of vehicle at desired velocity $V_x = 17$ m/s: (a) from 25 to 17 m/s and (b) from 6 to 17 m/s.

Proof: Set

$$\xi(t) = A^{-1}(t)\left[\alpha e^{-\beta\|x_1\|^2}\frac{\Gamma(t) - x_2}{1 + \|\Gamma(t) - x_2\|^2} + \dot{\Gamma}(t)\right] \tag{33}$$

where $\alpha, \beta > 0$.

The system (32) can be written as

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = A(t)\xi(t) + A(t)(x_1 - \xi(t)) \end{cases} \tag{34}$$

For

$$\begin{aligned} u = {} & -\alpha A^T(t)\frac{x_2 - \Gamma(t)}{1 + \|\Gamma(t) - x_2\|^2} \\ & - \alpha\frac{x_1 - \xi(t)}{1 + \|x_1 - \xi(t)\|^2} + \dot{\xi}(t) \end{aligned} \tag{35}$$

Then, the system (34) becomes

$$\begin{cases} \dot{x}_1 = -\alpha A^T(t)\dfrac{x_2 - \Gamma(t)}{1 + \|\Gamma(t) - x_2\|^2} - \alpha\dfrac{x_1 - \xi(t)}{1 + \|x_1 - \xi(t)\|^2} + \dot{\xi}(t) \\ \dot{x}_2 = \alpha e^{-\beta\|x_1\|^2}\dfrac{\Gamma(t) - x_2}{1 + \|\Gamma(t) - x_2\|^2} + \dot{\Gamma}(t) + A(t)(x_1 - \xi(t)) \end{cases} \tag{36}$$

if we set

$$\begin{cases} e_1(t) = x_1(t) - \xi_(t) \\ e_2(t) = x_2(t) - \Gamma(t) \end{cases}$$

A simple reasoning shows that the dynamics of $e(t) = (e_1(t), e_2(t))$ can be directly given by

$$\begin{cases} \dot{e}_1(t) = -\alpha\dfrac{A^T(t)e_2}{1 + \|e_2\|^2} - \alpha\dfrac{e_1}{1 + \|e_1\|^2} \\ \dot{e}_2(t) = -\alpha e^{-\beta\|x_1\|^2}\dfrac{e_2}{1 + \|e_2\|^2} + A(t)e_1 \end{cases}$$

Consider the function of Lyapunov W_1 defined by

$$W_1(e) = \frac{1}{2}\|e_1\|^2 + \frac{\alpha}{2}ln(1 + \|e_2\|^2)$$

The simple computation yields to show that the derivatives of W_1 using the system (36) can be written as

$$\dot{W}_1(e) = -\alpha\frac{\|e_1\|^2}{1 + \|e_1\|^2} - \alpha e - \beta\|x_1\|^2\frac{\|e_2\|^2}{(1 + |e_2\|^2)^2} < 0 \tag{37}$$

for $\|e\| \neq 0$, the condition (37) proves $\|e\|^2$ is bounded. In addition, if $\xi(t)$ is bounded, then $\|x_1\|$ is bounded too. Finally, there is M_α, a positive constant that verifies

$$\dot{W}_3(e) < -M_\alpha\|e\|^2$$

This implies that $\|e(t)\| \to 0$ when $t \to +\infty$ and thus $\|\Gamma(t) - x_2(t)\| \to 0$ when $t \to +\infty$. It remains only to prove that for a suitable choice of α we have

$$\|u_\epsilon\| < \epsilon + C$$

By choosing

$$\bar{u} = -\alpha\left[A^T(t)\frac{x_2 - \Gamma}{1 + \|\Gamma - x_2\|^2} + \frac{x_1 - \xi(t)}{1 + \|x_1 - \xi\|^2}\right]$$

and since $\left\| A^T(t) \right\|$ is bounded, then

$$\|\bar{u}\| \leq \alpha(\beta + 1)$$

where β is the upper bound. For

$$\alpha \leq \frac{\epsilon}{3(\beta + 1)}$$

we can obtain

$$\|\bar{u}\| \leq \frac{\epsilon}{3} \tag{38}$$

Now, let us prove that

$$\left\| \dot{\xi}(t) \right\| < \frac{2\epsilon}{3} + C$$

The derivatives of Equation (33) can be expressed by

$$\begin{aligned}\dot{\xi}(t) = {} & \dot{A}^{-1}(t)[\alpha e^{-\beta\|x_1\|^2} K_1 + \dot{\Gamma}] \\ & + A^{-1}(t)[-2\beta\alpha \prec \dot{x}_1, x_1 \succ e^{-\beta\|x_1\|^2} K_1 \\ & + \alpha e^{-\beta\|x_1\|^2} K_2 - 2\alpha e^{-\beta\|x_1\|^2} \prec \dot{\Gamma} - A(t)x_1, \\ & \Gamma - x_2 \succ K_3 + \ddot{\Gamma}]\end{aligned} \tag{39}$$

where

$$K_1 = \frac{\Gamma - x_2}{1 + \|\Gamma - x_2\|^2}, \qquad K_2 = \frac{\dot{\Gamma} - A(t)x_1}{1 + \|\Gamma - x_2\|^2},$$
$$K_3 = \frac{\Gamma - x_2}{(1 + \|\Gamma - x_2\|^2)^2}$$

if we replace $\dot{x}_1$ by its expression

$$\dot{x}_1 = u = \bar{u} + \dot{\xi}$$

we obtain

$$\begin{aligned}\dot{\xi}(t) = {} & \dot{A}^{-1}(t)[\alpha e^{-\beta\|x_1\|^2} K_1 + \dot{\Gamma}] \\ & + A^{-1}(t)[-2\beta\alpha \prec \bar{u}, x_1 \succ e^{-\beta\|x_1\|^2} K_1 \\ & - 2\beta\alpha \prec \dot{\xi}, x_1 \succ e^{-\beta\|x_1\|^2} K_1 \\ & + \alpha e^{-\beta\|x_1\|^2} K_2 - 2\alpha e^{-\beta\|x_1\|^2} \\ & \prec \dot{\Gamma} - A(t)x_1, \Gamma - x_2 \succ K_3 + \ddot{\Gamma}] \\ = {} & \dot{A}^{-1}(t)\dot{\Gamma} + A^{-1}(t)\ddot{\Gamma} + \alpha H - J_\alpha \dot{\xi}\end{aligned} \tag{40}$$

where

$$\begin{aligned}H = {} & \dot{A}^{-1}(t) e^{-\beta\|x_1\|^2} K_1 + A^{-1}(t)[-2\beta \prec \bar{u}, x_1 \succ e^{-\beta\|x_1\|^2} K_1 \\ & + e^{-\beta\|x_1\|^2} K_2 - 2e^{-\beta\|x_1\|^2} \prec \dot{\Gamma} - A(t)x_1, \Gamma - x_2 \succ K_3]\end{aligned}$$

and

$$J_\alpha = 2\alpha\beta e^{-\beta\|x_1\|^2} A^{-1}(t) K_1 {x_1}^T$$

Finally, we obtain

$$(I + J_\alpha)\dot{\xi} = \dot{A}^{-1}(t)\dot{\Gamma} + A^{-1}(t)\ddot{\Gamma} + \alpha H$$

And we can have

$$|J_\alpha| \leq 2\alpha\beta e^{-\beta\|x_1\|^2} \|x_1\| \left\| A^{-1}(t) \right\| K_1$$

For β sufficiently small, we obtain

$$2\beta \left\| A^{-1}(t) \right\| e^{-\beta\|x_1\|^2} \|x_1\| \leq 1$$

Then, $\|J_\alpha\| \leq \alpha$

For α sufficiently small, $I + J_\alpha$ is thus invertible and

$$\dot{\xi}(t) = (I + J_\alpha)^{-1}[\dot{A}^{-1}(t)\dot{\Gamma} + A^{-1}(t)\ddot{\Gamma} + \alpha H] \tag{41}$$

or

$$\dot{\xi}(t) = (I + \bar{J}_\alpha)^{-1}[\dot{A}^{-1}(t)\dot{\Gamma} + A^{-1}(t)\ddot{\Gamma} + \alpha H]$$

where $\bar{J}_\alpha$ is such that $\|\bar{J}_\alpha\| \to 0$ when $\alpha \to 0$. Thus

$$\begin{aligned}\dot{\xi}(t) = {} & \dot{A}^{-1}(t)\dot{\Gamma} + A^{-1}(t)\ddot{\Gamma} + \alpha H + \bar{J}_\alpha[\dot{A}^{-1}(t)\dot{\Gamma} \\ & + A^{-1}(t)\ddot{\Gamma} + \alpha H]\end{aligned}$$

which implies

$$\|\dot{\xi}(t)\| \leq C + \alpha\|\widehat{J_\alpha}\| + (C + \alpha\,\|H\|)$$

Considering that

$$\left\| \frac{\Gamma - x_2}{1 + \|\Gamma - x_2\|^2} \right\| < 1 \text{ et } e^{-\beta\|x_1\|^2} < 1$$

Then, we have

$$\begin{aligned}\|H\| \leq {} & \|\dot{A}^{-1}(t)\| + \|\dot{A}^{-1}(t)\| \Bigg[2\beta| \prec \bar{u}, x_1 \succ | e^{-\beta|x_1\|^2} \\ & + \frac{3e^{-\beta\|x_1\|^2}(\|A(t)x_1\| + \|\dot{\Gamma}\|)}{1 + \|\Gamma - x_2\|^2} \Bigg] \\ \leq {} & C_2 + C_1(2\beta\alpha(\beta + 1)\|x_1\| e^{-\beta|x_1\|^2} \\ & + 3e^{-\beta|x_1\|^2}\|x_1\|\|A(t)\| + 3C_3)\end{aligned}$$

Therefore $\|H\|$ is bounded. Finally, for a sufficiently small, we have

$$\|\dot{\xi}(t)\| < C + \frac{\epsilon}{2}$$

$\square$

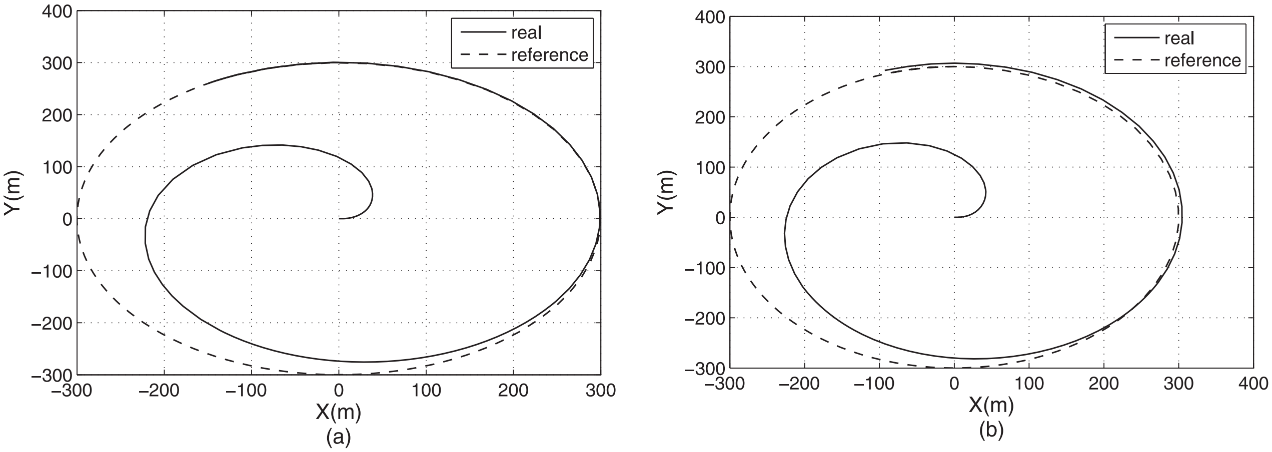


Figure 7. Vehicle trajectory tracking in XY plane: (a) with constant parameters $\alpha = 0.014$ and $\beta = 0.01$ and (b) constant parameters $\alpha = 0.02$ and $\beta = 0.015$.

6.1. Application

The result established by the lemma is proposed for application to the problem of vehicle trajectory tracking. It consists of a vehicle change maneuver and moves along a circular path according to the earth-fixed frame (X, Y) as shown in Figure 1. In this case, and where ψ may attain a large values, e.g., when moving along a circular path, it is preferred to use modified equations by introducing the longitudinal velocity Vx, the lateral velocity V_y and the yaw angular velocity V_ψ (Hans & Pacejka, 2005). The relations between these variables are

$$\begin{cases} \dot{X} = V_x \cos\psi - V_y \sin\psi \\ \dot{Y} = V_x \sin\psi + V_y \cos\psi \end{cases} \tag{42}$$

where X and Y are the Cartesian coordinates of the vehicle according to the earth fixed frame (X, Y).

Considering a given reference trajectory $\Gamma(t) = (\Gamma_1(t), \Gamma_2(t))$, the development of $\xi(t)$ gives

$$\begin{cases} \xi_1(t) = \dot{\Gamma}_1 \cos\psi + \dot{\Gamma}_2 \sin\psi + \dfrac{\alpha e^{-\beta\|x_1\|^2}}{\lambda_1} \\ \qquad \times \left[(\Gamma_1 - X)\cos\psi + (\Gamma_2 - Y)\sin\psi\right] \\ \xi_2(t) = \dot{\Gamma}_1 \sin\psi + \dot{\Gamma}_2 \cos\psi + \dfrac{\alpha e^{-\beta\|x_1\|^2}}{\lambda_1} \\ \qquad \times \left[(\Gamma_1 - X)\sin\psi + (\Gamma_2 - Y)\cos\psi\right] \end{cases} \tag{43}$$

with

$$\lambda_1 = 1 + (\Gamma_1 - X)^2 + (\Gamma_2 - Y)^2 \text{ and } \|x_1\|^2 = {V_x}^2 + {V_y}^2$$

Therefore, the system (36) becomes

$$\begin{cases} \dot{V}_x = -\alpha \dfrac{[(X - \Gamma_1)\cos\psi + (Y - \Gamma_2)\sin\psi]}{\lambda_1} \\ \qquad - \alpha \dfrac{(V_x - \xi_1)}{\lambda_2} \\ \dot{V}_y = -\alpha \dfrac{[-(X - \Gamma_1)\sin\psi + (Y - \Gamma_2)\cos\psi]}{\lambda_1} \\ \qquad - \alpha \dfrac{(V_y - \xi_2)}{\lambda_2} \\ \dot{X} = \alpha e^{-\beta\|x_1\|^2} \dfrac{(\Gamma_1 - X)}{\lambda_1} + (V_x - \xi_1)\cos\psi \\ \qquad - (V_y - \xi_2)\sin\psi + \dot{\Gamma}_1 \\ \dot{Y} = \alpha e^{-\beta\|x_1\|^2} \dfrac{(\Gamma_1 - X)}{\lambda_1} + (V_x - \xi_1)\sin\psi \\ \qquad - (V_y - \xi_2)\cos\psi + \dot{\Gamma}_2 \end{cases} \tag{44}$$

with

$$\lambda_2 = 1 + (V_x - \xi_1)^2 + (V_y - \xi_2)^2$$

6.1.1. Circular trajectory results

We now describe the simulation results that illustrate the performance of the proposed lemma for vehicle trajectory tracking in case of a circular path about the origin of the earth fixed frame, defined by

$$\Gamma(t) = \begin{cases} \Gamma_1(t) = R\sin(\omega t) \\ \Gamma_2(t) = R\cos(\omega t) \end{cases}$$

The simulation was performed for a radius $R = 300$ m and an angular velocity $\omega = 0.04$ rad/s. The initial vehicle velocity was selected as $V_{x0} = 12$ m/s and the initial lateral velocity as $V_{y0} = 0$, while the initial yaw angle was selected as $\psi_0 = \frac{\pi}{90}$ rad. This means that with these initial conditions, the vehicle was in skew position. The displacement of the vehicle was simulated during $t = 75$ s.

The corresponding vehicle trajectory is shown in Figure 7(a) and 7(b). The simulation results show that the vehicle converges quickly to the reference trajectory when the constant parameters α and β are fixed at 0.014 and 0.01, respectively. Thus, the assumptions fixed by the lemma are respected (α and β must be sufficiently small). When the parameters α and β are slightly increased, the vehicle converges to the reference system, but a little difference can be observed, while staying relatively close to the reference trajectory as shown in Figure 7(b).

7. Conclusions

A nonlinear approach method to build a nonlinear state feedback control that guarantees the stabilisation of vehicle velocities was presented. The stabilisation of the different velocities of the vehicle was chosen as an application. The method consisted first of a changing of the state variables according to the structure of the system. Thus, the stabilisation was declined in two steps. The first one consisted of a changing of variables to simplify the complexity of the system structure. The second step was the synthesis of the nonlinear feedback control based on the combined Lyapunov–LaSalle method. The transformations of the considered variables were introduced by a second-degree polynomial relation between the old and the new variables of control. The simulation results showed the ability of the proposed nonlinear controller to stabilise the vehicle in critical manoeuvre such as acceleration-in-a-turn. The results further demonstrated that the proposed method could considerably enhance vehicle speed stability. Furthermore, we proposed a lemma to solve the trajectory tracking and path-following problem for vehicles. The simulation results performed for the vehicle trajectory showed excellent tracking and a strong dependence on the reference trajectory. However, the result of the lemma is valid for any reference trajectory satisfying the corresponding assumption.

Note

1. Det(.) indicate the determinant and ||.|| represent the euclidean norm.

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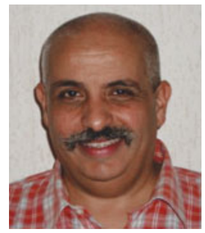

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Appendix. Vehicle data

$M = 1480$ kg; $I_z = 1950$; N·m/s^2; $C_f = 95{,}000$ N/rad; $C_r = 50{,}000$ N/rad; $h = 0.42$ m; $l = 0.751$ m; $a = 1.421$ m; $b = 1.029$ m; $k_{ax} = 0.41$ kg·m/s^2; $k_{ay} = 0.54$ kg·m/s^2.