Stabilization and Consensus of Linear Systems with Multiple Input Delays by Truncated Pseudo-Predictor Feedback

Bin Zhou*   Shen Cong†

Abstract

This paper provides an alternative approach referred to as pseudo-predictor feedback (PPF) for stabilization of linear systems with multiple input delays. Differently from the traditional predictor feedback which is from the model reduction appoint of view, the proposed PPF utilizes the idea of prediction by generalizing the corresponding results for linear systems with a single input delay to the case of multiple input delays. Since the PPF will generally lead to distributed controllers, a truncated pseudo-predictor feedback (TPPF) approach is established instead which gives finite dimensional controllers. It is shown that the TPPF can compensate arbitrarily large yet bounded delays as long as the open-loop system is only polynomially unstable. The proposed TPPF approach is then used to solve the consensus problems for multi-agent systems characterized by linear systems with multiple input delays. Numerical examples show the effectiveness of the proposed approach.

Keywords: Truncated pseudo-predictor feedback; Multiple input delays; Consensus; Multi-agent systems; Stabilization

1 Introduction

Stability analysis and stabilization of time-delay systems have received considerable attention during the past several decades since they are respectively the foundations of analysis and design for this class of systems [1, 2, 3]. For stability analysis, there are tools built in both frequency domain [4, 5] and time domain [6, 7, 8] and they are equivalently powerful in a sense. For stabilization of time-delay systems, there are also two kinds of approaches. The first kind of approaches searches for finite dimensional controllers which lead to an infinite dimensional closed-loop system whose stability is generally delay-dependent, namely, its stability cannot guaranteed for arbitrarily large delay in general. The second kind of approaches is to design an infinite dimensional controller, which may make the overall closed-loop system behave as a delay-free linear system and thus its stability is independent of the delay in general. One of the most efficient methods within the second category of approaches is the predictor feedback [9, 10, 11, 12, 13].

Consider a linear system with a pure input delay

$$\dot{x}(t) = Ax(t) + B_0 u(t - \tau_0),\tag{1}$$

where $A \in \mathbf{R}^{n \times n}$ and $B_0 \in \mathbf{R}^{n \times m}$ are constant matrices and $\tau_0 \geq 0$ is a constant scalar denoting the input delay. To use the predictor feedback approach to stabilize system (1), we can design

$$u(t) = Fx(t + \tau_0),\tag{2}$$

such that the closed-loop system is delay-free, namely,

$$\dot{x}(t) = (A + B_0 F_0) x,\tag{3}$$

which is asymptotically stable if $A + B_0 F_0$ is Hurwitz. However, controller (2) is acausal as it involves the future state vector $x(t + \tau_0)$. There are three prediction methods for making (2) be causal:

---

*Center for Control Theory and Guidance Technology, Harbin Institute of Technology, Harbin, 150001, China. Email:`binzhou@hit.edu.cn`, `binzhoulee@163.com`.

†School of Mechanical & Electrical Engineering, Heilongjiang University, Harbin, 150080, P. R. China. Email: `shen_tsong@163.com`

A. The traditional Predictor Feedback (PF): by the variation of the constant formula we predict $x(t + \tau_0)$ from the *open-loop* system (1) to get

$$u(t) = F_0 \left(e^{A\tau_0} x(t) + \int_t^{t+\tau_0} e^{A(t+\tau_0-s)} B_0 u(s-\tau_0) \, ds \right),$$ (4)

which is implementable. However, this controller is infinite dimensional and may suffice some implementation problems [14].

B. Truncated Predictor Feedback (TPF): to overcome the shortcoming existing in the PF (4), under the condition that (A, B) is controllable and all the eigenvalues of A are on the imaginary axis, we drop the distributed term in it to yield

$$u(t) = F_0 e^{A\tau_0} x(t),$$ (5)

which is a finite dimensional controller and will be referred to as TPF. We have shown in [13] that the closed-loop system consisting of (1) and (5) is asymptotically stable for arbitrarily large delay τ_0 if F_0 is well designed.

C. Pseudo-predictor Feedback (PPF): differently from the PF (4), we predict $x(t + \tau_0)$ from the *closed-loop* system (3) to get

$$u(t) = F_0 e^{(A+B_0 F_0)\tau_0} x(t),$$ (6)

which is also a finite dimensional controller and will be referred to as PPF. It is shown in [15] that the closed-loop system is asymptotically stable if and only if the following integral delay system is

$$\rho_0(t) = -F_0 \int_{t-\tau_0}^t e^{(A+B_0 F_0)(t-s)} B_0 \rho_0(s) \, ds,$$ (7)

which has been studied in [16]. Moreover, similar to the TPF (5), if (A, B) is controllable and all the eigenvalues of A are on the imaginary axis, the closed-loop system consisting of (1) and (6) is asymptotically stable for arbitrarily large yet bounded delay τ_0 if F_0 is well designed [15].

The aim of the present paper is to generalize the PPF approach for the linear system (1) with a single input delay to the following linear system with multiple input delays

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^p B_i u(t - \tau_i),$$ (8)

where $A \in \mathbf{R}^{n \times n}$ and $B_i \in \mathbf{R}^{n \times m}, i = 1, 2, \ldots, p$, are constant matrices, and $\tau_i, i = 1, 2, \ldots, p$, are constant scalars. The main contribution and significance of this paper are listed below:

1. Differently from the PPF design for linear system (1) with a single input delay, the PPF design for the linear system (8) with multiple input delays cannot be accomplished by using a virtual closed-loop system to predict the future state since the traditional predictor feedback (also known as model reduction) for system (8) does not use the future state at all. Hence we have to build an alternative traditional predictor feedback for system (8) based on which a reasonable PPF can be designed. It turns out that the designed PPF can indeed stabilize system (8) as long as an integral delay system in the form of (7) is stable, indicating that our PPF design for system (8) is indeed a generalization of the PPF design for system (1) to the case of multiple input delays.

2. However, differently from the PPF for system (1) with a single input delay, the PPF for system (8) with multiple input delays results in an infinite dimensional controller which can only be implemented by numerical approximation and may lead to instability to the overall closed-loop system. To overcome this shortcoming, under the condition that (A, B) is controllable and all the eigenvalues of A are on the imaginary axis, we provide a truncated pseudo-predictor feedback (TPPF) approach which leads to a finite dimensional controller that can stabilize system (8) for arbitrarily large yet bounded delays if the feedback gain is well designed. A numerical comparison of the proposed TPPF with our early proposed TPF shows that the TPPF can generally lead to much better control performances.

3. To show the effectiveness of the TPPF design for linear system (8) with multiple input delays, we use it to solve the consensus problem for multi-agent systems modeled by general high-order linear systems with multiple input delays. It is shown that as long as the open dynamics is not exponentially stable and the communication network contains a spanning tree, the TPPF based protocols can ensure the consensus of the concerned multi-agent systems for arbitrarily large yet bounded delays.

The reminder of this paper is organized as follows. After formulating the problem in Section 2, the PPF design and TPPF design are respectively carried out in Sections 3 and 4. Application of the TPPF design to the consensus of multi-agent systems is then discussed in Section 5. Two numerical examples are worked out in Section 6 to show the effectiveness of the proposed approaches and Section 7 concludes the paper.

Notation: The notation used in this paper is fairly standard. The symbol $\lambda(A)$ denotes the eigenvalue set of the square matrix A. For a positive scalar τ, let $\mathscr{C}_{n,\tau} = \mathscr{C}([-\tau, 0], \mathbf{R}^n)$ denote the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\mathbf{R}^n$ with the topology of uniform convergence, and let $x_t \in \mathscr{C}_{n,\tau}$ denote the restriction of $x(t)$ to the interval $[t-\tau, t]$ translated to $[-\tau, 0]$, that is, $x_t(\theta) = x(t+\theta), \theta \in [-\tau, 0]$. For two matrices A and B, we use $A \otimes B$ to denote their Kronecker product. For two integers p and q with $p \leq q$, the symbol $\mathbf{I}[p, q]$ refers to the set $\{p, p+1, \cdots, q\}$. Finally, for a complex number s, we use $\text{Re}\{s\}$ and $|s|$ to denote respectively its real part and model.

2 System Description and Problem Formulation

Consider the following linear system with three input delays

$$\dot{x}(t) = Ax(t) + B_1 u(t - \tau_1) + B_2 u(t - \tau_2) + B_3 u(t - \tau_3), \tag{9}$$

where $A \in \mathbf{R}^{n \times n}$ and $B_i \in \mathbf{R}^{n \times m}, i = 1, 2, 3$, are constant matrices, and $\tau_i, i = 1, 2, 3$, are constant scalars and such that

$$0 \leq \tau_1 \leq \tau_2 \leq \tau_3 < \infty. \tag{10}$$

For future use, we denote

$$\tau = \tau_1 + \tau_2 + \tau_3. \tag{11}$$

It is well-known that system (9) is stabilizable if and only if the following assumption is satisfied [9]:

Assumption 1 *The matrix pair (A, B) is stabilizable, where*

$$B = e^{-A\tau_3}B_3 + e^{-A\tau_2}B_2 + e^{-A\tau_1}B_1, \tag{12}$$

namely, there exists a matrix $F \in \mathbf{R}^{m \times n}$ such that $A + BF$ is Hurwitz.

In this paper, we are interested in the design of state feedback stabilizing controller $u(t)$ for system (9) under Assumption 1. One of the existing approaches for solving this problem is the so-called model reduction approach, which is described as follows. Let

$$z(t) = x(t) + \sum_{i=1}^{3} \int_{t-\tau_i}^{t} e^{A(t-\tau_i-s)} B_i u(s) \, ds. \tag{13}$$

Then direct computation gives (see, for example, [9] and [17])

$$\dot{z}(t) = Az(t) + Bu(t), \tag{14}$$

which is a delay-free linear system. Then, under Assumption 1, a stabilizing controller can be designed as

$$u(t) = Fz(t)$$

$$= F\left(x(t) + \sum_{i=1}^{3} \int_{t-\tau_i}^{t} e^{A(t-\tau_i-s)} B_i u(s) \, ds\right), \tag{15}$$

which is implementable.

However, as the model reduction controller (15) involves the integration of $u\left(t+s\right), s \leq 0$, it must be implemented via numerical approximation, which may lead to some stability problems (see, for example, [14]). To overcome this problem, we proposed in [12] and [17] a truncated version of the controller (15), namely,

$$u\left(t\right) = Fx\left(t\right),\tag{16}$$

which is a finite dimensional controller and is referred to as truncated predictor feedback (TPF) [12], under the following assumption:

Assumption 2 *The matrix pair $(A, B) \in \left(\mathbf{R}^{n \times n} \times \mathbf{R}^{n \times m}\right)$ is controllable with all the eigenvalues of A being on the imaginary axis.*

Under this assumption, the following result has been proven in [12] and [17].

Lemma 1 *Let (A, B) satisfy Assumption 2 and $F = -B^{\mathrm{T}}P$ where P is the unique positive definite solution to the following parametric algebraic Riccati equation (ARE) (see Lemma 4 given later)*

$$A^{\mathrm{T}}P + PA - PBB^{\mathrm{T}}P = -\gamma P.\tag{17}$$

Then there exists a $\gamma^* > 0$ such that the TPF (16) stabilizes system (9) for all $\gamma \in (0, \gamma^*)$.

In this paper, we will establish an alternative approach for stabilizing system (25) from the prediction point of view and will also build a class of finite dimensional stabilizing controllers under Assumption 2. This new approach provides an option for the designer since it may possess much better control performance than the TPF (16).

Remark 1 *For easy presentation, in this paper we have assumed that there are only three delays in the inputs. However, the reported results can be directly generalized to the more general system (8).*

3   Stabilization by Pseudo-Predictor Feedback

3.1   A Recursive Derivation of the Model Reduction Approach

We first consider the following new state vector

$$y_3\left(t\right) = \mathrm{e}^{A\tau_3}x\left(t\right) + \int_t^{t+\tau_3} \mathrm{e}^{A(t+\tau_3-s)}B_3 u\left(s - \tau_3\right)\mathrm{d}s\tag{18}$$

by which and some computation we can rewrite system (9) as

$$\dot{y}_3\left(t\right) = Ay_3\left(t\right) + B_3 u\left(t\right) + \mathrm{e}^{A\tau_3}B_1 u\left(t - \tau_1\right) + \mathrm{e}^{A\tau_3}B_2 u\left(t - \tau_2\right).\tag{19}$$

Now choose another new state vector $y_2\left(t\right)$ as

$$y_2\left(t\right) = \mathrm{e}^{A\tau_2}y_3\left(t\right) + \int_t^{t+\tau_2} \mathrm{e}^{A(t+\tau_2-s)}\left(\mathrm{e}^{A\tau_3}B_2 u\left(s - \tau_2\right)\right)\mathrm{d}s,\tag{20}$$

by which system (19) is transformed into

$$\dot{y}_2\left(t\right) = Ay_2\left(t\right) + \left(\mathrm{e}^{A\tau_2}B_3 + \mathrm{e}^{A\tau_3}B_2\right)u\left(t\right) + \mathrm{e}^{A(\tau_2+\tau_3)}B_1 u\left(t - \tau_1\right).\tag{21}$$

Finally, we choose the following new state vector $y_1\left(t\right)$:

$$y_1\left(t\right) = \mathrm{e}^{A\tau_1}y_2\left(t\right) + \int_t^{t+\tau_1} \mathrm{e}^{A(t+\tau_1-s)}\left(\mathrm{e}^{A(\tau_2+\tau_3)}B_1 u\left(s - \tau_1\right)\right)\mathrm{d}s,\tag{22}$$

4

and it follows from (21) that

$$\dot{y}_1(t) = Ay_1(t) + e^{A\tau}\left(e^{-A\tau_3}B_3 + e^{-A\tau_2}B_2 + e^{-A\tau_1}B_1\right)u(t).$$ (23)

Now we consider a state transformation

$$y(t) = e^{-A\tau}y_1(t),$$ (24)

by which system (23) can be expressed as

$$\dot{y}(t) = Ay(t) + Bu(t),$$ (25)

which is also in the form of (14).

Since (25) is a delay-free linear system, under Assumption 1, if we choose

$$u(t) = Fy(t),$$ (26)

then the closed-loop system is given by

$$\dot{y}(t) = (A + BF)y(t),$$ (27)

which is asymptotically stable if $A + BF$ is Hurwitz.

To write the controller (26) in terms of the original state vector $x(t)$, we present the following lemma.

Lemma 2 *Let $y(t)$ be defined through (18), (20), (22) and (24) and $z(t)$ be defined in (13). Then*

$$y(t) = z(t).$$ (28)

Hence the controller (26) is exactly the model reduction based controller (15). However, the aim of the recursive design of the model reduction described in the above is not to yield the controller (15) but is to link the new state vector $z(t)$ or $y(t)$ with the future state vector $x(t+\tau)$ and the future input $u(t+s), s > 0$, as stated in the following lemma whose proof is provided in the appendix for clarity.

Lemma 3 *Let $y(t)$ be defined through (18), (20), (22) and (24). Then*

$$y(t) = e^{-A\tau}x(t+\tau) - \sum_{i=1}^{3}\int_{t}^{t+\tau-\tau_i} e^{A(t-s-\tau_i)}B_i u(s)\,\mathrm{d}s.$$ (29)

The connection revealed in the above lemma helps us to derive a predictor based approach for stabilization of system (25) in the next subsection.

3.2   Pseudo-Predictor Feedback

Since the right-hand side of (29) is acausal, we consider

$$\begin{aligned}
y(t-\tau) &= e^{-A\tau}x(t) - \sum_{i=1}^{3}\int_{t-\tau}^{t-\tau_i} e^{A(t-\tau-s-\tau_i)}B_i u(s)\,\mathrm{d}s \\
&= e^{-A\tau}\left(x(t) - \sum_{i=1}^{3}\int_{t-\tau}^{t-\tau_i} e^{A(t-s-\tau_i)}B_i u(s)\,\mathrm{d}s\right) \\
&\triangleq e^{-A\tau}\left(x(t) - \varphi(t)\right),
\end{aligned}$$ (30)

and use the closed-loop system model (27) to yield

$$\begin{aligned}
u(t) &= Fy(t) \\
&= Fe^{(A+BF)\tau}y(t-\tau)
\end{aligned}$$

5

$$= Fe^{(A+BF)\tau}e^{-A\tau}\left(x\left(t\right)-\varphi\left(t\right)\right), \tag{31}$$

which will be referred to as pseudo-predictor feedback (PPF) since we have used the virtual closed-loop system (27) to predict $y\left(t\right)$ from $y\left(t-\tau\right)$ [15]. The controller (31) is now causal and is thus implementable.

Regarding the stability of the closed-loop system under the PPF (31), we have the following result.

Proposition 1 *The closed-loop system consisting of (1) and the PPF (31) is asymptotically stable if and only if the following integral delay system is*

$$\rho\left(t\right)=-F\int_{t-\tau}^{t}e^{(A+BF)(t-s)}B\rho\left(s\right)\mathrm{d}s. \tag{32}$$

Proof. By (31) and (25), the closed-lop system is given by

$$\begin{aligned}\dot{y}\left(t\right)&=Ay\left(t\right)+BFe^{(A+BF)\tau}e^{-A\tau}\left(x\left(t\right)-\varphi\left(t\right)\right)\\&=Ay\left(t\right)+BFe^{(A+BF)\tau}y\left(t-\tau\right)\\&=\left(A+BF\right)y\left(t\right)+B\rho\left(t\right),\end{aligned} \tag{33}$$

where

$$\rho\left(t\right)=F\left(e^{(A+BF)\tau}y\left(t-\tau\right)-y\left(t\right)\right). \tag{34}$$

On the other hand, it follows from (33) that

$$y\left(t\right)=e^{(A+BF)\tau}y\left(t-\tau\right)+\int_{t-\tau}^{t}e^{(A+BF)(t-s)}B\rho\left(s\right)\mathrm{d}s, \tag{35}$$

substituting of which into (33) gives the integral delay system (32). Hence if (32) is asymptotically stable, it follows from (33) that the closed-loop system is stable. The converse can be shown similarly as the proof of Theorem 1 in [15] and is omitted for brevity. ∎

Remark 2 *The PPF (31) will reduce to the PPF in [15] if there is only one delay in the system, namely, $B_1=B_2=0$ and $\tau_1=\tau_2=0$. In fact, in this case, (31) reduces to*

$$\begin{aligned}u\left(t\right)&=Fe^{(A+BF)\tau_3}e^{-A\tau_3}x\left(t\right)\\&=Fe^{\left(A+e^{-A\tau_3}B_3F\right)\tau_3}e^{-A\tau_3}x\left(t\right)\\&=Fe^{e^{-A\tau_3}\left[\left(A+B_3Fe^{-A\tau_3}\right)\tau_3\right]e^{A\tau_3}}e^{-A\tau_3}x\left(t\right)\\&=Fe^{-A\tau_3}e^{\left(A+B_3Fe^{-A\tau_3}\right)\tau_3}e^{A\tau_3}e^{-A\tau_3}x\left(t\right)\\&=F_3e^{(A+B_3F_3)\tau_3}x\left(t\right),\end{aligned} \tag{36}$$

where $F_3=Fe^{-A\tau_3}$ and is such that $A+B_3F_3$ is Hurwitz. This controller (36) is exactly in the form of the PPF (6) established in [15]. On the other hand, the integral delay system (32) is exactly in the form of (7). Hence Proposition 1 can be naturally considered a generalization of the results for the PPF established in [15] to the case of multiple input delays.

Differently from the PPF (6) for the linear system (1) with a single input delay, the PPF (31) for system (9) with multiple input delays also involves the integration of $u\left(t\right)$ in the past time and is thus an infinite dimensional controller which may suffice some implementation problems. To overcome this shortcoming, we will establish a truncated version of this class of controller in the next section.

4 Truncated Pseudo-Predictor Feedback

4.1 Derivation of the Truncated Pseudo-Predictor Feedback

Similarly to the development in [13], let the feedback gain F be parameterized as $F=F\left(\gamma\right):\mathbf{R}\to\mathbf{R}^{m\times n}$ and such that

$$\lim_{\gamma\downarrow0}\frac{1}{\gamma}\left\|F\left(\gamma\right)\right\|<\infty, \tag{37}$$

6

then $u(t)$ is also "of order 1" with respect to γ. Consequently, the vector $\varphi(t)$ is "of order 1" with respect to γ. As a result, $Fe^{(A+BF)\tau}e^{-A\tau}\varphi(t)$ is at least "of order 2" with respect to γ. Hence, the term $Fe^{(A+BF)\tau}e^{-A\tau}\varphi(t)$ is dominated by the term $Fe^{(A+BF)\tau}e^{-A\tau}x(t)$ and thus might be safely neglected in $u(t)$ when γ is sufficiently small [13]. As a result, the PPF (31) can be truncated as

$$u(t) = Fe^{(A+BF)\tau}e^{-A\tau}x(t), \tag{38}$$

which we refer to as the "truncated pseudo-predictor feedback (TPPF)." The main advantage of the TPPF (38) over the PPF (31) is that the former one is finite dimensional and is easy to implement.

According to the discussion in [13], to ensure that there exists a stabilizing F satisfying (37), we should assume that system (9) satisfies Assumption 2. In this case, we have the following lemma which is helpful for proving our main results.

Lemma 4 *[15] Assume that (A, B) satisfies Assumption 2. Then the ARE (17) admits a unique positive definite solution $P(\gamma) = W^{-1}(\gamma)$ for all $\gamma > 0$, where $W(\gamma)$ is the unique positive definite solution to the Lyapunov equation*

$$W\left(A + \frac{\gamma}{2}I_n\right)^{\mathrm{T}} + \left(A + \frac{\gamma}{2}I_n\right)W = BB^{\mathrm{T}}. \tag{39}$$

Moreover, $A + BF$ is Hurwitz, where $F = -B^{\mathrm{T}}P$, $\mathrm{tr}\left(B^{\mathrm{T}}PB\right) = n\gamma, PBB^{\mathrm{T}}P \leq n\gamma P(\gamma), \lim_{\gamma\downarrow 0} P(\gamma) = 0$, and

$$Pe^{(A+BF)} = e^{-(A+\gamma I_n)^{\mathrm{T}}}P. \tag{40}$$

In the next subsection we will show that the TPPF (38) can indeed stabilize system (9).

4.2 Stability of the Closed-Loop System

The aim of this subsection is to prove the following result.

Theorem 1 *Let $F = -B^{\mathrm{T}}P$ where P is the unique positive definite solution to the parametric ARE (17). Then there exits a scalar $\gamma^* > 0$ such that the closed-loop system consisting of (9) and the TPPF (38) is asymptotically stable for all $\gamma \in (0, \gamma^*)$.*

Proof. Consider the closed-loop system consisting of (25) and (38) given by

$$\begin{aligned}
\dot{y}(t) &= Ay(t) + BFe^{(A+BF)\tau}e^{-A\tau}x(t) \\
&= Ay(t) + BFe^{(A+BF)\tau}\left(y(t-\tau) + e^{-A\tau}\varphi(t)\right) \\
&= Ay(t) + BFe^{(A+BF)\tau}y(t-\tau) + BFe^{(A+BF)\tau}e^{-A\tau}\varphi(t) \\
&= (A+BF)y(t) + B\rho(t),
\end{aligned} \tag{41}$$

where

$$\begin{cases}
\rho(t) = F\left(e^{(A+BF)\tau}y(t-\tau) - y(t)\right) + \phi(t), \\
\phi(t) = Fe^{(A+BF)\tau}e^{-A\tau}\varphi(t).
\end{cases} \tag{42}$$

By using the variation of constant formula, we get from (41) that

$$y(t) = e^{(A+BF)\tau}y(t-\tau) + \int_{t-\tau}^{t} e^{(A+BF)(t-s)}B\rho(s)\,\mathrm{d}s, \tag{43}$$

by which $\rho(t)$ can be simplified as

$$\rho(t) = -F\int_{t-\tau}^{t} e^{(A+BF)(t-s)}B\rho(s)\,\mathrm{d}s + \phi(t). \tag{44}$$

7

For later use, we notice from the definition of $\varphi(t)$ in (30) that

$$
\begin{cases}
\phi(t) = F\mathrm{e}^{(A+BF)\tau}\mathrm{e}^{-A\tau}\sum_{i=1}^{3}\varphi_i(t), \\
\varphi_i(t) = \int_{t-\tau}^{t-\tau_i}\mathrm{e}^{A(t-s-\tau_i)}B_i u(s)\,\mathrm{d}s, \ i = 1, 2, 3.
\end{cases}
\tag{45}
$$

Moreover, by equation (30) we can write $u(t)$ as

$$
\begin{aligned}
u(t) &= F\mathrm{e}^{(A+BF)\tau}\left(y(t-\tau) + \mathrm{e}^{-A\tau}\varphi(t)\right) \\
&= F\mathrm{e}^{(A+BF)\tau}y(t-\tau) + \phi(t).
\end{aligned}
\tag{46}
$$

Consider a Lyapunov function $V_1(y(t)) = y^{\mathrm{T}}(t)Py(t)$ whose time-derivative along system (41) is given by

$$
\begin{aligned}
\dot{V}_1(y(t)) &= y^{\mathrm{T}}(t)\left((A+BF)^{\mathrm{T}}P + P(A+BF)\right)y(t) + 2y^{\mathrm{T}}(t)PB\rho(t) \\
&= -\gamma y^{\mathrm{T}}(t)Py(t) - y^{\mathrm{T}}(t)PBB^{\mathrm{T}}Py(t) + 2y^{\mathrm{T}}(t)PB\rho(t) \\
&\leq -\gamma y^{\mathrm{T}}(t)Py(t) + \|\rho(t)\|^2,
\end{aligned}
\tag{47}
$$

where we have used Lemma 4. By (44) and the Jensen inequality we can derive

$$
\begin{aligned}
\|\rho(t)\|^2 &\leq 2\left(\int_{t-\tau}^{t}\mathrm{e}^{(A+BF)(t-s)}B\rho(s)\,\mathrm{d}s\right)^{\mathrm{T}}F^{\mathrm{T}}F\int_{t-\tau}^{t}\mathrm{e}^{(A+BF)(t-s)}B\rho(s)\,\mathrm{d}s + 2\|\phi(t)\|^2 \\
&\leq 2n\gamma\|P\|\tau\int_{t-\tau}^{t}\rho^{\mathrm{T}}(s)B^{\mathrm{T}}\mathrm{e}^{(A+BF)^{\mathrm{T}}(t-s)}\mathrm{e}^{(A+BF)(t-s)}B\rho(s)\,\mathrm{d}s + 2\|\phi(t)\|^2 \\
&\leq 2n\gamma\|P\|\tau\|B\|^2\,\mathrm{e}^{2\left(\|A\|+\|B\|^2\|P\|\right)\tau}\int_{t-\tau}^{t}\|\rho(s)\|^2\,\mathrm{d}s + 2\|\phi(t)\|^2 \\
&= c_1\gamma\|P\|\int_{t-\tau}^{t}\|\rho(s)\|^2\,\mathrm{d}s + 2\|\phi(t)\|^2,
\end{aligned}
\tag{48}
$$

where

$$
c_1 = c_1(\gamma) = 2n\tau\|B\|^2\,\mathrm{e}^{2(\|A\|+\|B\|^2\|P\|)\tau}.
\tag{49}
$$

Let $\gamma_1^* > 0$ be such that

$$
c_1\gamma\|P\|\tau + \sqrt{\|P\|} \leq 1, \ \forall\gamma \in (0, \gamma_1^*).
\tag{50}
$$

Then the time-derivative of the second Lyapunov functional

$$
V_2(\rho(t)) = c_1\gamma\|P\|\int_{-\tau}^{0}\int_{t-s}^{t}\|\rho(\sigma)\|^2\,\mathrm{d}\sigma\mathrm{d}s
\tag{51}
$$

along (48) satisfies

$$
\begin{aligned}
\dot{V}_2(\rho(t)) &= c_1\gamma\|P\|\tau\|\rho(t)\|^2 - c_1\gamma\|P\|\int_{t-\tau}^{t}\|\rho(\sigma)\|^2\,\mathrm{d}\sigma \\
&\leq c_1\gamma\|P\|\tau\|\rho(t)\|^2 + 2\|\phi(t)\|^2 - \|\rho(t)\|^2 \\
&\leq 2\|\phi(t)\|^2 - \sqrt{\|P\|}\|\rho(t)\|^2.
\end{aligned}
\tag{52}
$$

Now by using (45) we can obtain

$$
\begin{aligned}
\|\phi(t)\|^2 &= \varphi^{\mathrm{T}}(t)\mathrm{e}^{-A^{\mathrm{T}}\tau}\mathrm{e}^{(A+BF)^{\mathrm{T}}\tau}F^{\mathrm{T}}F\mathrm{e}^{(A+BF)\tau}\mathrm{e}^{-A\tau}\varphi(t) \\
&\leq n\gamma\|P\|\mathrm{e}^{2\left(\|A\|+\|B\|^2\|P\|\right)\tau}\mathrm{e}^{2\|A\|\tau}\varphi^{\mathrm{T}}(t)\varphi(t) \\
&\leq 3n\gamma\|P\|\mathrm{e}^{2\left(\|A\|+\|B\|^2\|P\|\right)\tau}\mathrm{e}^{2\|A\|\tau}\sum_{i=1}^{3}\|\varphi_i(t)\|^2,
\end{aligned}
\tag{53}
$$

and, by using the Jensen inequality again, we get, for $i = 1, 2, 3$,

$$\|\varphi_i(t)\|^2 \leq \tau \int_{t-\tau}^{t-\tau_i} u^{\mathrm{T}}(s) B_i^{\mathrm{T}} e^{A^{\mathrm{T}}(t-s-\tau_i)} e^{A(t-s-\tau_i)} B_i u(s) \, ds$$

$$\leq \|B_i\|^2 e^{2\|A\|\tau} (\tau - \tau_i) \int_{t-\tau}^{t-\tau_i} \|u(s)\|^2 \, ds$$

$$\leq \sum_{i=1}^{3} \|B_i\|^2 e^{2\|A\|\tau} \tau \int_{t-\tau}^{t} \|u(s)\|^2 \, ds. \tag{54}$$

Moreover, it follows from (46) that

$$\int_{t-\tau}^{t} \|u(s)\|^2 \, ds \leq 2 \int_{t-\tau}^{t} \|\phi(s)\|^2 \, ds + 2 \int_{t-\tau}^{t} y^{\mathrm{T}}(s-\tau) e^{(A+BF)^{\mathrm{T}}\tau} F^{\mathrm{T}} F e^{(A+BF)\tau} y(s-\tau) \, ds$$

$$= 2 \int_{t-\tau}^{t} \|\phi(s)\|^2 \, ds + 2 \int_{t-2\tau}^{t-\tau} y^{\mathrm{T}}(s) P e^{-(A+\gamma I)\tau} B B^{\mathrm{T}} e^{-(A+\gamma I)^{\mathrm{T}}\tau} P y(s) \, ds$$

$$\leq 2 \int_{t-\tau}^{t} \|\phi(s)\|^2 \, ds + 2 \|P\| \left\|e^{-A\tau} B\right\|^2 \int_{t-2\tau}^{t-\tau} y^{\mathrm{T}}(s) P y(s) \, ds$$

$$\leq 2 \int_{t-\tau}^{t} \|\phi(s)\|^2 \, ds + 2 \|P\| \left\|e^{-A\tau} B\right\|^2 \int_{t-2\tau}^{t} y^{\mathrm{T}}(s) P y(s) \, ds, \tag{55}$$

where we have used the inequality (40). Inserting (55) into (54) and then (53) gives

$$\|\phi(t)\|^2 \leq 9n\gamma\tau \|P\| e^{2(\|A\|+\|B\|^2\|P\|)\tau} e^{4\|A\|\tau} \sum_{i=1}^{3} \|B_i\|^2 \int_{t-\tau}^{t} u^{\mathrm{T}}(s) u(s) \, ds$$

$$\leq c_2 \gamma \|P\| \int_{t-\tau}^{t} \|\phi(s)\|^2 \, ds + c_3 \gamma \|P\|^2 \int_{t-2\tau}^{t} y^{\mathrm{T}}(s) P y(s) \, ds, \tag{56}$$

where

$$\begin{cases} c_2 = c_2(\gamma) = 18n\tau e^{2(\|A\|+\|B\|^2\|P\|)\tau} e^{4\|A\|\tau} \sum_{i=1}^{3} \|B_i\|^2, \\ c_3 = c_3(\gamma) = 18n\tau e^{2(\|A\|+\|B\|^2\|P\|)\tau} e^{4\|A\|\tau} \sum_{i=1}^{3} \|B_i\|^2 \left\|e^{-A\tau} B\right\|^2. \end{cases} \tag{57}$$

Let $\gamma_2^* \in (0, \gamma_1^*)$ be such that

$$c_2 \gamma \|P\| \tau + \sqrt{\|P\|} \leq 1, \ \forall \gamma \in (0, \gamma_2^*). \tag{58}$$

Then the time-derivative of the Lyapunov functional

$$V_3(\phi_t) = c_2 \gamma \|P\| \int_{-\tau}^{0} \int_{t-s}^{t} \|\phi(\sigma)\|^2 \, d\sigma ds, \tag{59}$$

satisfies

$$\dot{V}_3(\phi_t) = c_2 \gamma \|P\| \tau \|\phi(t)\|^2 - c_2 \gamma \|P\| \int_{t-\tau}^{t} \|\phi(\sigma)\|^2 \, d\sigma$$

$$\leq c_2 \gamma \|P\| \tau \|\phi(t)\|^2 + c_3 \gamma \|P\|^2 \int_{t-2\tau}^{t} y^{\mathrm{T}}(s) P y(s) \, ds - \|\phi(t)\|^2$$

$$\leq c_3 \gamma \|P\|^2 \int_{t-2\tau}^{t} y^{\mathrm{T}}(s) P y(s) \, ds - \sqrt{\|P\|} \|\phi(t)\|^2. \tag{60}$$

Finally, we choose

$$V(y_t, \rho_t, \phi_t) = \frac{1}{8} \|P\| V_1(y(t)) + \frac{1}{4} \sqrt{\|P\|} V_2(\rho_t) + V_3(\phi_t)$$

9

$$+ c_3\gamma \|P\|^2 \int_{-2\tau}^{0} \int_{t-s}^{t} y^{\mathrm{T}}(\sigma) Py(\sigma)\, \mathrm{d}\sigma \mathrm{d}s, \tag{61}$$

and it follows from (47), (52) and (60) that

$$\begin{aligned}
\dot{V}(y_t, \rho_t, \phi_t) \leq {}& \frac{1}{8}\|P\|\left(-\gamma y^{\mathrm{T}}(t) Py(t) + \|\rho(t)\|^2\right)\\
& + \frac{1}{4}\left(\sqrt{\|P\|}\left(2\|\phi(t)\|^2 - \sqrt{\|P\|}\|\rho(t)\|^2\right)\right)\\
& + c_3\gamma\|P\|^2 \int_{t-2\tau}^{t} y^{\mathrm{T}}(s) Py(s)\, \mathrm{d}s - \sqrt{\|P\|}\|\phi(t)\|^2\\
& + 2\tau c_3\gamma\|P\|^2 y^{\mathrm{T}}(t) Py(t) - c_3\gamma\|P\|^2 \int_{t-2\tau}^{t} y^{\mathrm{T}}(s) Py(s)\, \mathrm{d}s\\
= {}& -\frac{1}{8}\|P\|\|\rho(t)\|^2 - \frac{1}{2}\sqrt{\|P\|}\|\phi(t)\|^2\\
& - \left(\frac{1}{8} - 2\tau c_3\|P\|\right)\|P\|\gamma y^{\mathrm{T}}(t) Py(t).
\end{aligned} \tag{62}$$

Hence there exists a $\gamma^* \in (0, \gamma_2^*)$ such that

$$\begin{aligned}
\dot{V}(y_t, \rho_t, \phi_t) \leq {}& -\frac{1}{8}\|P\|\|\rho(t)\|^2 - \frac{1}{2}\sqrt{\|P\|}\|\phi(t)\|^2\\
& - \frac{1}{16}\|P\|\gamma y^{\mathrm{T}}(t) Py(t), \ \forall \gamma \in (0, \gamma^*).
\end{aligned} \tag{63}$$

Then by a similar argument in [17], we can show that $\lim_{t\to\infty}\|\rho(t)\| = \lim_{t\to\infty}\|\phi(t)\| = \lim_{t\to\infty}\|y(t)\| = 0$. Consequently, by (30) we know $\lim_{t\to\infty}\|x(t)\| = 0$, which implies asymptotic stability. The proof is finished. ∎

Remark 3 *We point out that all the plant parameters and the delays in system (1) can be time-varying as in [15] and similar results can be obtained. We here however only consider time-invariant parameters for the sake of brevity.*

5 Applications to the Consensus of Multi-Agent Systems

5.1 Problem Formulation and Assumptions

We consider a continuous-time high-order multi-agent system described by

$$\dot{x}_i(t) = Ax_i(t) + \sum_{j=1}^{3} B_j u_i(t - \tau_j), \ i \in \mathbf{I}[1, N], \tag{64}$$

where $x_i \in \mathbf{R}^n$ and $u_i \in \mathbf{R}^m$ are the state and the control of Agent i, respectively, $N \geq 1$ is a given integer denoting the number of agents, $\tau_j, j = 1, 2, 3$ are the input delays satisfying (10), and A, B_i are given matrices. Let the communication topology among these agents be characterized by a weighted directed graph $\mathcal{G}(\mathcal{N}, \mathcal{E}, \mathcal{A})$, where $\mathcal{N}$ is the node set, $\mathcal{E}$ is the edge set, and $\mathcal{A} = [\alpha_{ij}] \in \mathbf{R}^{N \times N}$ is the weighted adjacency matrix. Denote the corresponding Laplacian by $L = [l_{ij}] \in \mathbf{R}^{N \times N}$ (see [18] for a detailed introduction on graph theory).

We assume that Agent i collects the state information of its neighboring agents by the rule [19]

$$\begin{aligned}
z_i(t) = {}& \sum_{j \in N_i} \alpha_{ij}(x_i(t) - x_j(t))\\
= {}& \sum_{j=1}^{N} l_{ij} x_j(t), \ i \in \mathbf{I}[1, N],
\end{aligned} \tag{65}$$

where $N_i = \{j : \alpha_{ij} \neq 0\}$. In this section, we are interested in the design of the state feedback protocol $u_i(t) = u_i(z_i), i \in \mathbf{I}[1, N]$ that achieves consensus of the multi-agent system (64). To this end, we first introduce the concept of consensus.

Definition 1 *The linear multi-agent system (64) achieves consensus if* $\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0, \forall i, j \in \mathbf{I}[1, N]$.

Remark 4 *By definition, if the high-order multi-agent system (64) achieves consensus, then there exists a signal $s(t) \in \mathbf{R}^n$, which may be unbounded, such that $\lim_{t \to \infty} \|x_i(t) - s(t)\| = 0, \forall i \in \mathbf{I}[1, N]$. The signal $s(t)$ is referred to as the reference trajectory.*

According to the discussions in [20] and [19], to ensure that the consensus value reached by the agents will not tend to infinity exponentially, namely, the reference trajectory $s(t)$ defined in Remark 4 is not exponentially diverging, the matrix A should not contain eigenvalues in the open right-half plane [20]. Hence, without loss of generality, we assume that Assumption 2 is satisfied for (64) [19].

The following assumption is also necessary for the solvability of the consensus problem (see, for example, [19]).

Assumption 3 *The communication topology $\mathcal{G}(\mathcal{N}, \mathcal{E}, \mathcal{A})$ contains a directed spanning tree.*

By this assumption we know that $\lambda(L) = \{0, \lambda_2, \ldots, \lambda_N\}$ with $\mathrm{Re}\{\lambda_i\} > 0, i \in \mathbf{I}[2, N]$ (see, for example, [18]).

5.2 Consensus by Truncated Pseudo-Predictor Feedback Protocols

In this section we present our solutions to the consensus problem by using the TPPF approach developed in Section 4 for stabilization of the linear system (9).

Theorem 2 *Let Assumptions 2 and 3 be satisfied, $F = -B^{\mathrm{T}}P$ where $P > 0$ is the unique positive definite solution to the parametric ARE (17) and μ be any positive number such that*

$$\mu \geq \max_{i \in \mathbf{I}[2, N]} \left\{ \frac{1}{\mathrm{Re}\{\lambda_i\}} \right\}. \tag{66}$$

Then, for any delay τ that is exactly known and can be arbitrarily large yet bounded, there exists a number $\gamma^* = \gamma^*(\mu, \tau, \{\lambda_i\}_{i=2}^N)$ such that the consensus of the multi-agent system (64) is achieved by the protocol

$$u_i(t) = \mu F \mathrm{e}^{(A+BF)\tau} \mathrm{e}^{-A\tau} z_i(t), \ i \in \mathbf{I}[1, N], \ \gamma \in (0, \gamma^*). \tag{67}$$

Moreover, the reference signal $s(t)$ defined in Remark 4 satisfies

$$\|s(t)\| \leq k \left(1 + t^{N^*-1} \right), \ \forall t \geq 0, \tag{68}$$

where N^* is the maximal algebraic multiplicity of the eigenvalues of A and k is a positive function of the initial conditions.

Proof. Under Assumption 3, there exists a nonsingular matrix $U \in \mathbf{C}^{N \times N}$, whose first column is $\mathbf{1}_N \triangleq [1, 1, \ldots, 1]^{\mathsf{T}} \in \mathbf{R}^N$, such that [18]

$$U^{-1}LU = \begin{bmatrix} 0 & & & & \\ & \lambda_2 & \delta_2 & & \\ & & \ddots & \ddots & \\ & & & \lambda_{N-1} & \delta_{N-1} \\ & & & & \lambda_N \end{bmatrix} \triangleq J, \tag{69}$$

11

where λ_i are such that $\text{Re}\{\lambda_i\} > 0, i \in \mathbf{I}[2, N]$, and $\delta_i, i \in \mathbf{I}[2, N-1]$, equals either 1 or 0. For future use, we define $\delta_N = 0$.

Let $\mathscr{F} = \mu F e^{(A+BF)\tau} e^{-A\tau}$. The closed-loop system consisting of (64) and (67) is given by

$$
\begin{aligned}
\dot{x}_i(t) &= A x_i(t) + \sum_{k=1}^{3} B_k \mathscr{F} z_i(t - \tau_k) \\
&= A x_i(t) + \sum_{k=1}^{3} B_k \mathscr{F} \sum_{j=1}^{N} l_{ij} x_j(t - \tau_k) \\
&= A x_i(t) + \sum_{k=1}^{3} \sum_{j=1}^{N} l_{ij} B_k \mathscr{F} x_j(t - \tau_k), \ i \in \mathbf{I}[1, N].
\end{aligned}
\tag{70}
$$

By letting $x = [x_1^{\mathsf{T}}, x_2^{\mathsf{T}}, \cdots, x_N^{\mathsf{T}}]^{\mathsf{T}}$, the above system can be expressed as

$$
\begin{aligned}
\dot{x}(t) &= (I_N \otimes A) x(t) + \sum_{k=1}^{3} (L \otimes B_k \mathscr{F}) x(t - \tau_k) \\
&= (U \otimes I_n)(I_N \otimes A)(U^{-1} \otimes I_n) x(t) \\
&\quad + (U \otimes I_n) \sum_{k=1}^{3} (J \otimes B_k \mathscr{F})(U^{-1} \otimes I_n) x(t - \tau_k),
\end{aligned}
\tag{71}
$$

namely,

$$
\dot{\chi}(t) = (I_N \otimes A) \chi(t) + \sum_{k=1}^{3} (J \otimes B_k \mathscr{F}) \chi(t - \tau_k),
\tag{72}
$$

where $\chi = (U^{-1} \otimes I_n) x = [\chi_1^{\mathsf{T}}, \chi_2^{\mathsf{T}}, \cdots, \chi_N^{\mathsf{T}}]^{\mathsf{T}}$. In view of (56), system (72) is equivalent to

$$
\begin{cases}
\dot{\chi}_i(t) = A \chi_i(t), & i = 1, \\
\dot{\chi}_i(t) = A \chi_i(t) + \lambda_i \sum_{k=1}^{3} B_k \mathscr{F} \chi_i(t - \tau_k) + \delta_i \chi_{i+1}(t - \tau_k), & i \in \mathbf{I}[2, N],
\end{cases}
\tag{73}
$$

where $\chi_{N+1}(t) \equiv 0$. Since $x = (U \otimes I_n) \chi$, it follows from the special structure of U that the consensus is achieved if

$$
\lim_{t \to \infty} \|\chi_i(t)\| = 0, \ i \in \mathbf{I}[2, N],
\tag{74}
$$

and in this case

$$
x_i(t) \xrightarrow{t \to \infty} \chi_1(t), \ i \in \mathbf{I}[1, N],
\tag{75}
$$

which implies (68) in view of the first equation of (73). It is easy to see that (74) is true if and only if

$$
\dot{\chi}_i(t) = A \chi_i(t) + \lambda_i \sum_{k=1}^{3} B_k \mathscr{F} \chi_i(t - \tau_k), \ i \in \mathbf{I}[2, N],
\tag{76}
$$

are all asymptotically stable. In the following we will show that (76) are all indeed asymptotically stable if (66) is satisfied.

We consider the following linear time-delay system

$$
\begin{cases}
\dot{\varkappa}(t) = A \varkappa(t) + \sum_{k=1}^{3} B_k u(t - \tau_k), \\
u(t) = \mathcal{F} e^{(A+BF)\tau} e^{-A\tau} \varkappa(t),
\end{cases}
\tag{77}
$$

where

$$
\mathcal{F} = \lambda \mu F, \ \mu \geq \frac{1}{\text{Re}\{\lambda\}}, \ \text{Re}\{\lambda\} > 0.
\tag{78}
$$

By comparing (76) with (77)-(78), it suffices to show that there exists a $\gamma^{\#}(\lambda)$ such that (77) is asymptotically stable for all $\gamma \in (0, \gamma^{\#}(\lambda))$. However, system (77) is very similar to the closed-loop system consisting of (9) and the TPPF (38). Hence we need only to prove the stability of

$$\dot{y}(t) = (A + B\mathcal{F}) y(t) + B\rho(t), \tag{79}$$

where

$$\begin{cases} \rho(t) = \mathcal{F}\left(e^{(A+BF)\tau} y(t-\tau) - y(t)\right) + \phi(t), \\ \phi(t) = \mathcal{F}e^{(A+BF)\tau} e^{-A\tau} \varphi(t), \\ \varphi(t) = \displaystyle\sum_{i=1}^{3} \int_{t-\tau}^{t-\tau_i} e^{A(t-s-\tau_i)} B_i u(s) \, ds. \end{cases} \tag{80}$$

The proof is almost the same as the proof of Theorem 1 where the constants $c_i, i = 1, 2, 3$, are redefined accordingly. Particularly, the inequality in (47) can be obtained since

$$\begin{aligned} (A + B\mathcal{F})^{\mathrm{H}} P + P(A + B\mathcal{F}) &= A^{\mathrm{T}} P + PA - 2\operatorname{Re}\{\lambda\} \mu PBB^{\mathrm{T}} P \\ &\leq A^{\mathrm{T}} P + PA - 2PBB^{\mathrm{T}} P \\ &= -\gamma P - PBB^{\mathrm{T}} P, \end{aligned} \tag{81}$$

where we have used (78).

Based on the above discussion, the proof is finished by choosing $\gamma^* = \min_{i \in \mathbf{I}[2,N]} \left\{\gamma^{\#}(\lambda_i)\right\}$. $\blacksquare$

The TPPF protocols can also be generalized to the observer based output feedback case. The details are omitted for brevity.

6 Two Illustrative Examples

6.1 Stabilization by Truncated Pseudo-Predictor Feedback

Consider a linear system in the following form

$$\dot{x}(t) = Ax(t) + B_1 u(t) + B_2 u(t - \tau), \tag{82}$$

where $\tau = \frac{\pi}{2}$ and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}. \tag{83}$$

It follows that $\lambda(A) = \{\pm i, \pm i\}$ and (A, B) is controllable, where B is found to be

$$B = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} - \pi & -1 & 0 \end{bmatrix}^{\mathrm{T}}. \tag{84}$$

By Lemma 1 and Theorem 1, both the TPF and the TPPF controllers take the form

$$u(t) = \mathscr{F}(\gamma) x(t), \tag{85}$$

where $\mathscr{F}(\gamma) = F_{\mathrm{TPF}}$ for the TPF and $\mathscr{F} = F_{\mathrm{TPPF}}$ for the TPPF with F_{TPF} and F_{TPPF} given by

$$F_{\mathrm{TPF}} = \begin{bmatrix} 4\gamma^3 & -\gamma^4 + 4\gamma^2 & 4\gamma + \gamma^4 - 2\gamma^3 - 4\gamma^2 - \frac{1}{4}\pi\gamma^4 + \pi\gamma^2 & -\frac{1}{2}\gamma^4 + 4\gamma^3 - 4\gamma^2 - \gamma^3\pi \end{bmatrix}, \tag{86}$$

$$F_{\mathrm{TPPF}} = \frac{1}{4}\gamma e^{-\frac{1}{2}\gamma\pi} \begin{bmatrix} \gamma^2 \left(4\gamma\pi - 3\gamma^3\pi + 2\gamma^4 + 8\gamma^2 - 16\right) \\ 2\gamma \left(-4\gamma\pi - 5\gamma^3\pi + 3\gamma^4 + 18\gamma^2 + 8\right) \\ 16 + 6\gamma^4 - 36\gamma^3 + 40\gamma^2 - 16\gamma - 6\gamma^5 - 4\gamma\pi + 10\pi\gamma^4 - 11\gamma^3\pi + 8\pi\gamma^2 \\ \gamma \left(8\gamma^3 + 2\gamma^2 - 16\gamma + 16 + 2\gamma^5 + 4\pi\gamma^2 - 4\gamma\pi - 3\pi\gamma^4\right) \end{bmatrix}^{\mathrm{T}}. \tag{87}$$

The closed-loop system consisting of (82) and (85) is given by

$$\dot{x}(t) = (A + B_1 \mathscr{F}(\gamma)) x(t) + B_2 \mathscr{F}(\gamma) x(t - \tau), \tag{88}$$
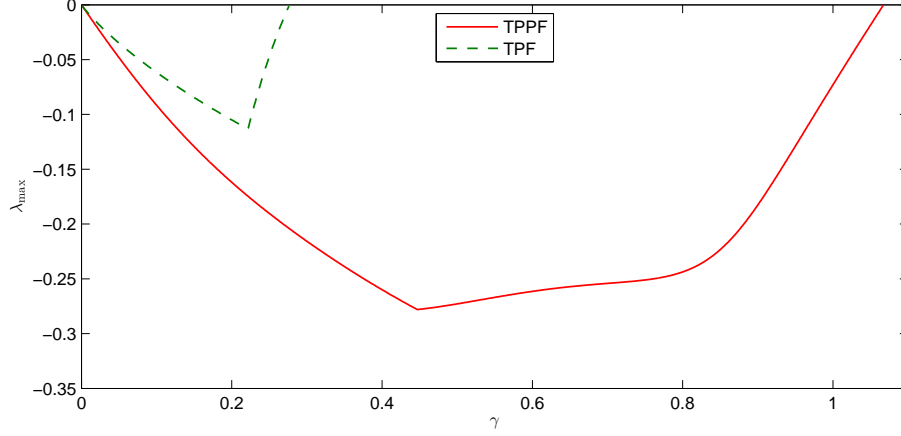
13

Figure 1: The functions $\lambda_{\max}(\gamma)$ for the closed-loop system (88) under the TPF and the TPPF

which is a linear time-delay system whose stability is completely determined by the right-most roots of its characteristic equation $c(s, \gamma) = \det(sI_4 - (A + B_1\mathscr{F}(\gamma)) - B_2\mathscr{F}(\gamma)\,\mathrm{e}^{-\tau s})$ [21, 13]. Let [15]

$$\lambda_{\max}(\gamma) = \max\{\mathrm{Re}\{s\} : c(s, \gamma) = 0\}, \qquad (89)$$

which can be efficiently computed by the software package DDE-BIFTOOL [22] for a fixed γ. Then the closed-loop system (88) is asymptotically stable if and only if $\lambda_{\max}(\gamma) < 0$ and its convergence rate is completely determined by $\lambda_{\max}(\gamma)$ [21, 13]. Let $\gamma_{\mathrm{sup}} > 0$ be the minimal number such that $\lambda_{\max}(\gamma) = 0$. Then clearly there exists an optimal value $\gamma_{\mathrm{opt}} \in (0, \gamma_{\mathrm{sup}})$ such that $\lambda_{\max}(\gamma)$ is minimized at $\gamma = \gamma_{\mathrm{opt}}$ with the minimal value $\lambda_{\max\,\mathrm{min}}$. It is thus clear that $\lambda_{\max\,\mathrm{min}}$ is the maximal convergence rate that the controller (85) can achieve [12, 13, 15].

The functions $\lambda_{\max}(\gamma)$ associated with both $\mathscr{F}(\gamma) = F_{\mathrm{TPF}}$ and $\mathscr{F}(\gamma) = F_{\mathrm{TPPF}}$ are shown in Fig. 1 from which we clearly see that

A. The closed-loop system under the TPF is asymptotically stable if and only if $\gamma \in (0, \gamma_{\mathrm{sup}}^{\mathrm{TPF}})$ where $\gamma_{\mathrm{sup}}^{\mathrm{TPF}} = 0.276$, and its convergence rate is maximized with $\gamma = \gamma_{\mathrm{opt}}^{\mathrm{TPF}} = 0.221$ and $\lambda_{\max\,\mathrm{min}}^{\mathrm{TPF}} = \lambda_{\max}(\gamma_{\mathrm{opt}}^{\mathrm{TPF}}) = -0.113$, which is the maximal convergence rate that the TPF can achieve.

B. The closed-loop system under the TPPF is asymptotically stable if and only if $\gamma \in (0, \gamma_{\mathrm{sup}}^{\mathrm{TPPF}})$ where $\gamma_{\mathrm{sup}}^{\mathrm{TPPF}} = 1.067$ and its convergence rate is maximized with $\gamma = \gamma_{\mathrm{opt}}^{\mathrm{TPPF}} = 0.447$ and $\lambda_{\max\,\mathrm{min}}^{\mathrm{TPPF}} = \lambda_{\max}(\gamma_{\mathrm{opt}}^{\mathrm{TPPF}}) = -0.278$, which is the maximal convergence rate that the TPPF can achieve.

We thus conclude that the TPPF can not only allow a larger range of γ ensuring the stability than the TPF (namely, $\gamma_{\mathrm{sup}}^{\mathrm{TPPF}} > \gamma_{\mathrm{sup}}^{\mathrm{TPF}}$) but also lead to a much better control performance than the TPF (namely, $\lambda_{\max\,\mathrm{min}}^{\mathrm{TPPF}} < \lambda_{\max\,\mathrm{min}}^{\mathrm{TPF}}$). For the comparison purpose, both the state responses of the closed-loop system by TPF with $\gamma = \gamma_{\mathrm{opt}}^{\mathrm{TPF}}$ and the closed-loop system by the TPPF with $\gamma = \gamma_{\mathrm{opt}}^{\mathrm{TPPF}}$ are recorded in Fig. 2 from which we can see that the later one indeed converges faster than the former one.

6.2 Consensus by Truncated Pseudo-Predictor Feedback

We assume that there are six agents with each agent being characterized by the linear system (82)–(83). Let these agents be connected by a communication network as shown in Fig. 3. This network can be
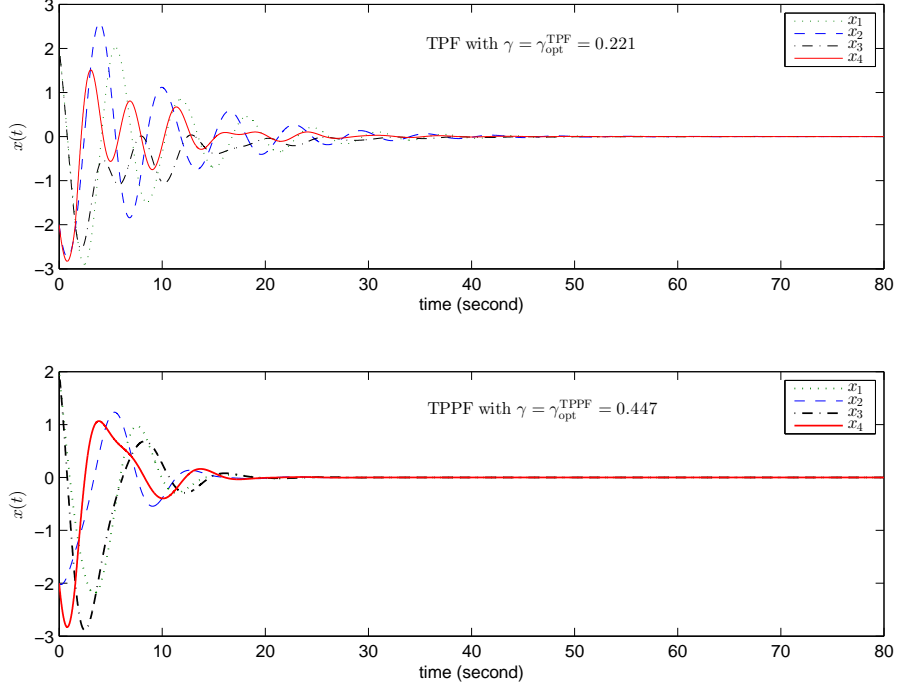
Figure 2: The state responses of the closed-loop systems (88) by the TPF with $\gamma = \gamma_{\text{opt}}^{\text{TPF}}$ and by the TPPF with $\gamma = \gamma_{\text{opt}}^{\text{TPPF}}$

characterized by the Laplacian

$$L = \begin{bmatrix} 2 & -2 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}. \tag{90}$$

As $\lambda(L) = \{0, 1, 1, 2, 2, 3\}$, Assumption 3 is fulfilled. We can thus choose $\mu = 1$. By Theorem 2, the protocol can then be designed as

$$u_i(t) = \mathscr{F}(\gamma) z_i(t), \ i \in \mathbf{I}[1, 6], \tag{91}$$

where $\mathscr{F}(\gamma)$ is given by (87) and $\gamma = 0.1$. For some specified initial conditions, the state vector x_1 of the first agent and the differences between the states x_i of the other five agents and x_1 are recorded in Fig. 4. From this figure, we clearly see that the consensus is achieved.

7 Conclusion

This paper has studied the stabilization problem for linear systems with multiple input delays. We first establish a pseudo-predictor feedback controller, which is infinite dimensional, and then build a truncated version of this pseudo-predictor feedback controller, which is finite dimensional and is termed as truncated pseudo-predictor feedback (TPPF). It is shown that the TPPF can stabilize the concerned systems for arbitrarily large yet bounded delay if the open-loop system is not exponentially stable. Comparison of the TPPF with the existing results shows that the former one can generally lead to better control performance. The TPPF approach has also been utilized to solve the consensus problem for linear multi-agent systems with multiple input delays.
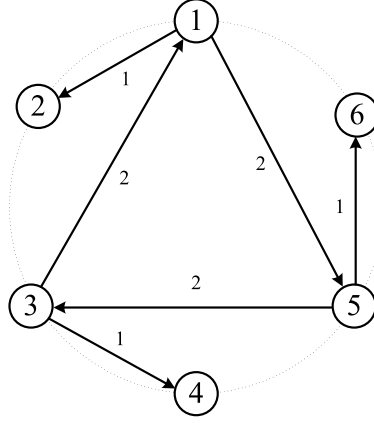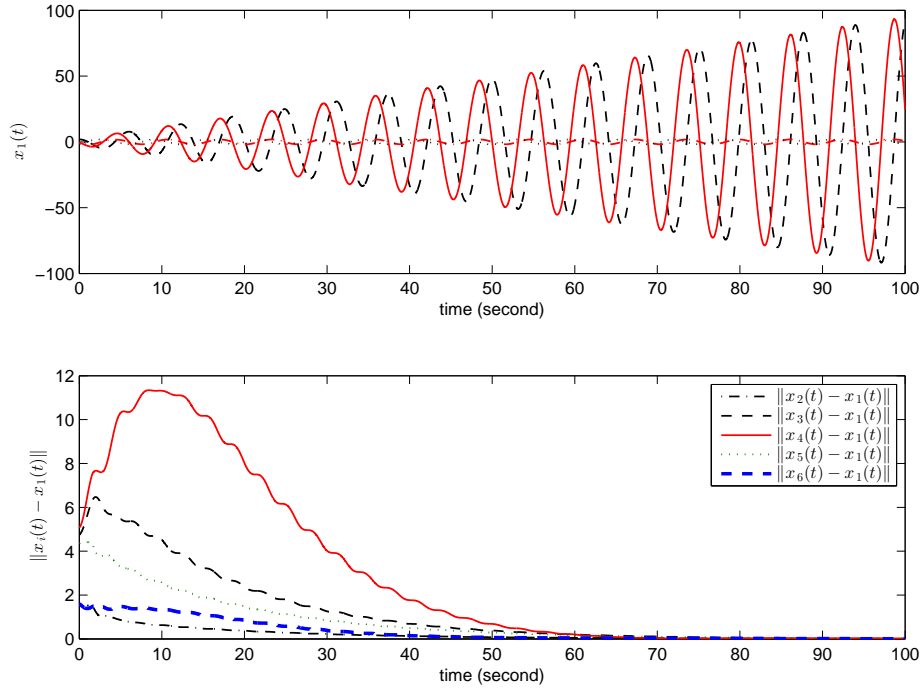
Figure 3: The communication network

Figure 4: The state vector $x_1(t)$ and $\|x_i(t) - x_1(t)\|$, $i \in \mathbf{I}[2, 6]$

Appendix

A1. Proof of Lemma 2

We compute from (18), (20), (22) and (24) that

$$
\begin{aligned}
y(t) &= e^{-A\tau} y_1(t) \\
&= e^{-A\tau} \left(e^{A\tau_1} y_2(t) + \int_t^{t+\tau_1} e^{A(t+\tau_1-s)} \left(e^{A(\tau_2+\tau_3)} B_1 u(s-\tau_1) \right) ds \right) \\
&= e^{-A(\tau_2+\tau_3)} y_2(t) + \int_t^{t+\tau_1} e^{A(t-s)} B_1 u(s-\tau_1) ds \\
&= e^{-A(\tau_2+\tau_3)} \left(e^{A\tau_2} y_3(t) + \int_t^{t+\tau_2} e^{A(t+\tau_2-s)} \left(e^{A\tau_3} B_2 u(s-\tau_2) \right) ds \right) + \int_t^{t+\tau_1} e^{A(t-s)} B_1 u(s-\tau_1) ds \\
&= e^{-A\tau_3} y_3(t) + \int_t^{t+\tau_2} e^{A(t-s)} B_2 u(s-\tau_2) ds + \int_t^{t+\tau_1} e^{A(t-s)} B_1 u(s-\tau_1) ds \\
&= e^{-A\tau_3} \left(e^{A\tau_3} x(t) + \int_t^{t+\tau_3} e^{A(t+\tau_3-s)} B_3 u(s-\tau_3) ds \right) \\
&\quad + \int_t^{t+\tau_2} e^{A(t-s)} B_2 u(s-\tau_2) ds + \int_t^{t+\tau_1} e^{A(t-s)} B_1 u(s-\tau_1) ds \\
&= x(t) + \sum_{i=1}^3 \int_t^{t+\tau_i} e^{A(t-s)} B_i u(s-\tau_i) ds \\
&= x(t) + \sum_{i=1}^3 \int_{t-\tau_i}^t e^{A(t-\tau_i-s)} B_i u(s) ds \\
&= z(t),
\end{aligned}
\tag{92}
$$

which completes the proof.

A2. Proof of Lemma 3

In view of systems (9), (19) and (21), we can write

$$
\begin{cases}
y_3(t) = x(t+\tau_3) - \int_t^{t+\tau_3} e^{A(t+\tau_3-s)} \left(B_1 u(s-\tau_1) + B_2 u(s-\tau_2) \right) ds, \\
y_2(t) = y_3(t+\tau_2) - \int_t^{t+\tau_2} e^{A(t+\tau_2-s)} \left(B_3 u(s) + e^{A\tau_3} B_1 u(s-\tau_1) \right) ds, \\
y_1(t) = y_2(t+\tau_1) - \int_t^{t+\tau_1} e^{A(t+\tau_1-s)} \left(e^{A\tau_2} B_3 + e^{A\tau_3} B_2 \right) u(s) ds,
\end{cases}
\tag{93}
$$

from which it follows that

$$
\begin{aligned}
y(t) &= e^{-A\tau} y_1(t) \\
&= e^{-A\tau} y_2(t+\tau_1) - \int_t^{t+\tau_1} e^{-A\tau} e^{A(t+\tau_1-s)} \left(e^{A\tau_2} B_3 + e^{A\tau_3} B_2 \right) u(s) ds \\
&= e^{-A\tau} y_3(t+\tau_1+\tau_2) \\
&\quad - \int_{t+\tau_1}^{t+\tau_1+\tau_2} e^{-A\tau} e^{A(t+\tau_1+\tau_2-s)} \left(B_3 u(s) + e^{A\tau_3} B_1 u(s-\tau_1) \right) ds \\
&\quad - \int_t^{t+\tau_1} e^{-A\tau} e^{A(t+\tau_1-s)} \left(e^{A\tau_2} B_3 + e^{A\tau_3} B_2 \right) u(s) ds \\
&= e^{-A\tau} x(t+\tau_1+\tau_2+\tau_3) \\
&\quad - \int_{t+\tau_1+\tau_2}^{t+\tau_1+\tau_2+\tau_3} e^{-A\tau} e^{A(t+\tau_1+\tau_2+\tau_3-s)} \left(B_1 u(s-\tau_1) + B_2 u(s-\tau_2) \right) ds \\
&\quad - \int_{t+\tau_1}^{t+\tau_1+\tau_2} e^{-A\tau} e^{A(t+\tau_1+\tau_2-s)} \left(B_3 u(s) + e^{A\tau_3} B_1 u(s-\tau_1) \right) ds
\end{aligned}
$$

$$-\int_{t}^{t+\tau_1} \mathrm{e}^{-A\tau}\mathrm{e}^{A(t+\tau_1-s)}\left(\mathrm{e}^{A\tau_2}B_3 + \mathrm{e}^{A\tau_3}B_2\right)u(s)\,\mathrm{d}s$$

$$= \mathrm{e}^{-A\tau}x\left(t+\tau_1+\tau_2+\tau_3\right)$$

$$-\int_{t+\tau_1+\tau_2}^{t+\tau_1+\tau_2+\tau_3} \mathrm{e}^{A(t-s)}B_1 u\left(s-\tau_1\right) + \mathrm{e}^{A(t-s)}B_2 u\left(s-\tau_2\right)\mathrm{d}s$$

$$-\int_{t+\tau_1}^{t+\tau_1+\tau_2} \mathrm{e}^{A(t-\tau_3-s)}B_3 u\left(s\right) + \mathrm{e}^{A(t-s)}B_1 u\left(s-\tau_1\right)\mathrm{d}s$$

$$-\int_{t}^{t+\tau_1} \mathrm{e}^{A(t-\tau_3-s)}B_3 u\left(s\right) + \mathrm{e}^{A(t-\tau_2-s)}B_2 u\left(s\right)\mathrm{d}s$$

$$= \mathrm{e}^{-A\tau}x\left(t+\tau_1+\tau_2+\tau_3\right) - \sum_{i=1}^{3}\int_{t}^{t+\tau_1+\tau_2+\tau_3-\tau_i} \mathrm{e}^{A(t-s-\tau_i)}B_i u\left(s\right)\mathrm{d}s, \tag{94}$$

which completes the proof.

References

[1] W.-H. Chen and W. X. Zheng. Exponential stability of nonlinear time-delay systems with delayed impulse effects. *Automatica*, 47(5):1075–1083, 2011.

[2] Z.-G. Wu, P. Shi, H. Su, and J. Chu. Stochastic synchronization of markovian jump neural networks with time-varying delay using sampled data. *IEEE Transactions on Cybernetics*, 43(6):1796–1806, 2013.

[3] Z.-G. Wu, P. Shi, H. Su, and J. Chu. Passivity analysis for discrete-time stochastic markovian jump neural networks with mixed time delays. *IEEE Transactions on Neural Networks*, 22(10):1566–1575, 2011.

[4] K. Gu, J. Chen, and V. L. Kharitonov. *Stability of Time-Delay Systems*. Springer, 2003.

[5] N. Olgac and A. S. Kammer. Stabilisation of open-loop unstable plants under feedback control with distributed delays. *IET Control Theory & Applications*, 8(10):813–820, 2014.

[6] B. Du, J. Lam, Z. Shu, and Z. Wang. A delay-partitioning projection approach to stability analysis of continuous systems with multiple delay components. *IET Control Theory & Applications*, 3(4):383–390, 2009.

[7] J. Lam, S. Xu, D. W. C. Ho, and Y. Zou. On global asymptotic stability for a class of delayed neural networks. *International Journal of Circuit Theory and Applications*, 40(11):1165–1174, 2012.

[8] F. O. Souza. Further improvement in stability criteria for linear systems with interval time-varying delay. *IET Control Theory & Applications*, 7(3):440–446, 2013.

[9] Z. Artstein. Linear systems with delayed controls: a reduction. *IEEE Transactions on Automatic Control*, 27(4):869–879, 1982.

[10] M. Krstic. Lyapunov stability of linear predictor feedback for time-varying input delay. *IEEE Transactions on Automatic Control*, 55(2):554–559, 2010.

[11] A. Manitius and A. W. Olbrot. Finite spectrum assignment problem for systems with delays. *IEEE Transactions on Automatic Control*, 24(4):541–552, 1979.

[12] B. Zhou. *Truncated Predictor Feedback for Time-Delay Systems*. Springer, Heidelberg, Germany, 2014.

[13] B. Zhou, Z. Lin, and G. Duan. Truncated predictor feedback for linear systems with long time-varying input delays. *Automatica*, 48(10):2387–2399, 2012.

[14] V Van Assche, M Dambrine, J. F. Lafay, and J. P. Richard. Some problems arising in the implementation of distributed-delay control laws. In *Proceedings of the 38th IEEE Conference on Decision and Control*, volume 5, pages 4668–4672. IEEE, 1999.

[15] B. Zhou. Pseudo-predictor feedback stabilization of linear systems with time-varying input delays. *Automatica*, (to appear), 2014.

[16] Z.-Y. Li, B. Zhou, and Z. Lin. On exponential stability of integral delay systems. *Automatica*, 49(11):3368–3376, 2013.

[17] B. Zhou, Z. Lin, and G. Duan. Global and semi-global stabilization of linear systems with multiple delays and saturations in the input. *SIAM Journal on Control and Optimization*, 48(8):5294–5332, 2010.

[18] W. Ren and Y. Cao. *Distributed Coordination of Multi-agent Networks: Emergent Problems, Models, and Issues.* Springer, 2010.

[19] B. Zhou and Z. Lin. Consensus of high-order multi-agent systems with large input and communication delays. *Automatica*, 50(2):452–464, 2014.

[20] Z. Li, Z. Duan, G. Chen, and L. Huang. Consensus of multiagent systems and synchronization of complex networks: a unified viewpoint. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 57(1):213–224, 2010.

[21] J. K. Hale. *Functional Differential Equations.* Springer, 1971.

[22] K. Engelborghs, T. Luzyanina, and G. Samaey. Dde-biftool v. 2.00: a matlab package for bifurcation analysis of delay differential equations. *Dept. Comput. Sci., K.U. Leuven, TW Reports*, page 61, 2001.