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An inventory model for nonperishable items with warehouse mode selection and partial backlogging under trapezoidal-type demand

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ABSTRACT
Considering a nonperishable product which may be stored either in an own warehouse or in both the own and the rented warehouse, this paper deals with the ordering decisions under a generalized trapezoidal-type demand rate in an inventory system. Shortages are allowed and the unsatisfied demand is assumed to be partially backlogged. Furthermore, the existence and uniqueness of the optimal solution to each warehouse mode is proved and used in an easy-to-use algorithm, and a decision-making theorem for measuring whether to adopt a rented warehouse is developed. Finally, numerical examples and a case study are presented to illustrate the feasibility and efficiency of the proposed model and algorithm. The results show that, the storage capacity of the own warehouse and the unit rental cost have remarkable impact on determining whether to use the rented warehouse. When both the unit rental cost and the unit opportunity cost are higher in the external market, the profitability of the inventory system mainly relies on the storage capacity of the own warehouse. Meanwhile, the optimal profit performance is sensitive to the selling price and the purchasing cost, and the optimal rented warehouse's ordering quantity is sensitive to the order cycle length. But overall, the proposed model is basically robust.

KEYWORDS
Inventory; warehouse mode selection; trapezoidal-type demand; time-varying; two-warehouse; partial backlogging

1. Introduction

In traditional deterministic inventory replenishment issues, the market demand for items is generally considered to be either increasing or decreasing over time. However, the demand for items may not always continue to keep in a certain state within the inventory cycle in practice, since global business competition and new information technology advances have dramatically shortened the lifespan of products in today's market environment. It was observed by Micheal, Rochford, & Wotruba (2013) and Cheng & Wang (2009) that in some industries such as apparel, electronics and food, the demand for seasonal or fashion products increases with time initially when potential consumers are attracted by the style and quality, then holds steady when this product type is accepted in the market, and finally decreases with time. Later, it was also

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found by Lin (2013) and Glock & Grosse (2015) that the vast majority of products in their life cycle basically follow this kind of time-varying demand characteristic, which is referred to the trapezoidal-type demand pattern. The trapezoidal-type time-varying demand is more general, since it can capture some common demand patterns such as increasing demand, constant demand, decreasing demand and ramp-type demand. Thus, inventory models with the trapezoidal-type time-varying demand have attracted much attention from academica in recent years (Cheng, Zhang, & Wang, 2011; Lin, 2013; N. Shah, Shah, & Patel, 2015; N. Singh, Vaish, & Singh, 2010; Uthayakumar & Rameswari, 2012).

It is assumed that the inventory system has a full storage space in the aforementioned researches adopting the trapezoidal-type demand. However, the retailer's warehouse for holding items has only a limited capacity in reality, due to some restrictions such as capital, land investment and labor input. Usually, in order to take full advantage of attractive price discounts provided on bulk purchasing of the product or in anticipation of growth in the customer's demand over time, retailers have incentives to order more. Under a limited warehouse capacity, they often face two different warehouse modes: the single warehouse mode and the two-warehouse mode. While they may only use their own warehouse to keep items in the single warehouse mode, retailers in the two-warehouse mode may employ an extra warehouse to hold the surplus items over the capacity of their own warehouse (Hartley, 1976; Yang, 2004; Lee & Hsu, 2009). As observed in China, with the rapid development of the third party storage industries, this renting trade becomes a common phenomenon in today's business practice (Alibaba, 2014). However, the above two modes often make the retailers place in a dilemma position. On one hand, the single warehouse mode can save operation costs of the inventory system, but it may lead to the risk of shortage and eventually lose potential customers (Abad, 1996; Verhoef & Sloot, 2010; Ghosh, 2011). On the other hand, although the two-warehouse mode may avoid some shortages, it may not be profitable for them to employ another warehouse, since the higher rent incurred by employing warehouse will affect the performance of the inventory system (Hartley, 1976). As far as we know, there is no research about inventory models that focused on the warehouse mode selection issues with the trapezoidal-type demand from academia until now. This paper aims to address this gap in the field of inventory researches on the trapezoidal-type demand by exploring the following questions:

1. Facing the trapezoidal-type demand products, how do the retailers make tradeoffs between the single warehouse mode and the two-warehouse mode?

2. What's more, if the two-warehouse mode is adopted and shortages are also allowed, how many orders should be allocated to the rented warehouse?

3. What are the impacts of some key inventory characteristics such as the holding cost of the own warehouse, the own warehouse's capacity, and the holding cost of the rented warehouse on the performance of the inventory system?

To answer these questions, in a fixed inventory cycle, this paper will focus on a new inventory model with warehouse mode selection for nonperishable items under a general trapezoidal-type demand, in which shortages are allowed and unsatisfied demands are assumed to be partially backlogged during the stock-out period. The analysis aims to gain the optimal replenishment policy and the warehouse mode. This paper will also examine the effects of trapezoidal-type demand, warehouse mode and shortages on the downstream retailer's optimal solution, and reveal how to balance the relations between the shortages and the rented orders when some exogenous inventory parameters vary.

The contribution of this paper mainly includes the following three aspects. First, theoretically, this study makes the first attempt to incorporate the warehouse mode selection into inventory issues with trapezoidal-type time-varying demand, and an inventory decision model is developed for measuring whether or not to adopt a rented warehouse in the inventory system. Second, the existence and uniqueness of the optimal solutions to the proposed models are proved, and then the corresponding easy-to-use algorithms are presented for searching the optimal solutions. Third, the utilization of a generalized time-varying demand rate and warehouse mode selection makes the application scope of the models broader. For example, by setting the relevant model parameters, the proposed models can be applied to some particular inventory circumstances such as linear demand, exponential demand, ramp-type demand, single warehouse, two-warehouse, no storage limit, no shortage, full backlogging, and so on.

The remainder of this paper is organized as follows. The related literature is reviewed in Section 2. In Section 3, the model descriptions, assumptions, and notations used throughout this study are listed. Section 4 explores inventory issues under two different warehouse modes, and then provides the optimal replenishment policies and gives the corresponding algorithm for each warehouse mode. In section 5, some numerical examples and sensitivity analysis are presented to illustrate the proposed model. Section 6 summarizes our conclusions and discusses future directions. All proofs are presented in the online appendix.

2. Literature review

To highlight our contribution, we mainly review the following three streams of inventory literature relevant to this article: (1) trapezoidal-type demand in inventory research, (2) inventory system with shortages consideration, and (3) two-warehouse inventory research.

2.1. Trapezoidal-type demand in inventory research

In today's time-based business competition, the demand for products may be time-varying rather than only constant. Concerning inventory research on time-varying demand, the study on linear time-varying demand is pioneered by Donaldson (1977). Subsequent time-varying demand types include power demand (San-José, Sicilia, & Alcaide-López-de-Pablo, 2018), exponential demand (Datta & Pal, 1988), quadratic demand (Sarkar, Ghosh, & Chaudhuri, 2012), ramp type demand (Skouri, Konstantaras, Papachristos, & Ganas, 2009), and trapezoidal-type demand (Micheal et al., 2013). Among them, trapezoidal-type time-varying demand can fully describe the demand trajectory of all-life-cycle products in the retail market (Cheng & Wang, 2009). As a result, in recent years, many researchers have devoted considerable attention to the inventory replenishment policies for items with the trapezoidal-type demand rate. For example, Panda, Senapati, & Basu (2008) firstly developed an inventory model in a finite time horizon, in which the demand rate for a perishable season product is assumed to follow a time-dependent function, and an optimal order policy is derived by minimizing the total inventory system cost in the entire inventory cycle. Thereafter, under the assumption that all the replenishment cycles are limited to be of a fixed length, Cheng & Wang (2009) generalized the work of Hill (1995) to an inventory system with the trapezoidal-type demand rate, which is a piecewise linear function of time, and then they discussed an inventory replenishment policy. N. Singh, et al. (2010) also analyzed an EOQ model by considering that the trapezoidal-type demand rate is a piecewise linear time-varying function. However, they mainly focused on the effects of the random deterioration rate and trade credit policy on the inventory performance. Furthermore, under the manufacturing inventory environment, an economic production quantity (EPQ) model for trapezoidal-type demand with defective items was established by Uthayakumar & Rameswari (2012), who studied the effects of production rate and reliability on the imperfect manufacturing inventory system. Very recently, considering that the supplier offers two different choices such as price discount and credit period, N. Shah et al. (2015) investigated an integrated supplier-buyer inventory issue with price-sensitive demand, in which a joint average profit is maximized to gain the optimal procurement quantity, procurement price, and number of transfers from the upstream supplier. Under the condition of the time-varying deterioration rate and partial backlogging, Wu, Skouri, Teng, & Hu (2016) investigated two inventory systems beginning with and without shortages by adopting the theory of the net present value, in which the total profit is maximized to gain the optimal inventory order policies. The aforementioned researches have incorporated the trapezoidal-type demand into inventory issues, and they discussed the effect of this demand-type on the inventory performance in own warehouse frame as well. However, most of them failed to investigate the effect of the rented warehouse on inventory replenishment policies and did not consider warehouse mode selection under trapezoidal-type demand.

2.2. Inventory system with the shortages consideration

The shortages consideration in the inventory research is another stream that makes a significant impact on the performance of the inventory system. In the past three decades, many literatures published in the journal have investigated inventory issues with shortages during the finite planning horizon. For instance, some classical issues of inventory shortages start with an instantaneous replenishment and finish with a zero inventory (Teng, Chern, & Yang, 1997; Yang, Teng, & Chern, 2001; Dye & Teng, 2006). The previous literatures on inventory shortages also start with an instantaneous replenishment but end with shortages (Zhou, Lau, & Yang, 2003; Dye, 2007; Abad, 2008). More recently, the inventory issues with shortages have been further extended to the supply chain environment (Roy, Sana, & Chaudhuri, 2012, 2018; Eduardo & Sana, 2014; Sana, 2016; Li, Liu, Teng, & Tsao, 2019).

However, most of them often assumed that the shortages are either completely lost or completely backlogged during the stock-out period. But in reality, some loyal potential customers are willing to wait for these shortages if the wait time is longer, others become more impatient and go elsewhere. In the all-units discount retail context, Taleizadeh & Pentico (2014) studied a model with a constant partial backlogging rate. Later, using the vendor-managed inventory policy, Taleizadeh (2017) explored two models for evaporating chemical raw materials in the supply chain frame. Abad (2001) also investigated the pricing and lot-sizing inventory problem for perishable items with a general partial backlogging rate. Dye (2007) developed a deterministic inventory model under a time-dependent backlogging rate. More recently, some related literatures focused on this inventory type (Ghosh, 2011; Salehi, Taleizadeh, & Tavakkoli-Moghaddam, 2016; Taleizadeh, Khanbaglo, & Eduardo, 2016; Lashgari, Taleizadeh, & Sadjadi, 2017; Xu, Bisi, & Dada, 2017; Taleizadeh (2018); Khan, Shaikh, Panda, Konstantaras, & Taleizadeh, 2019). However, none of them investigated the two-warehouse inventory scenario for non-perishable items with the trapezoidal-type time-varying demand.

2.3. Two-warehouse inventory research

On the warehouse mode selection, the first concern behind the efficient inventory replenishment is the trade-off between the storage space of the own warehouse and the order quantity when the rental cost is included in the performance of the inventory system. In general, as upstream suppliers provide seasonal products or attractive price discounts for bulk purchase, the downstream retailers may buy more items than can be stored in their own warehouse (OW), and then they often choose a rented warehouse (RW) for storing the surplus items over the capacity of the OW (Goswami & Chaudhuri, 1992; Zhou & Yang, 2005). However, RW can offer a higher service level as compared to OW, and thus the unit cost of holding items in the RW is much higher than that in the OW (Hartley, 1976). For economic reasons, the goods of RW are provided firstly, and the goods stored in OW will be consumed until exhausting all in RW. In recent years, many studies have been done in the field of the two-warehouse inventory. For example, Hartley (1976) firstly discussed an inventory issue in a two-warehouse framework. Based on a deterministic two-warehouse environment, Sarma (1987) also studied an inventory replenishment issue with the deteriorating item. Considering the finite replenishment rate, Pakkala & Achary (1992) analyzed the effect of the deterioration factor in their model on the optimal two-warehouse replenishment strategy. Taking into account the linear time-varying demand, Goswami & Chaudhuri (1992) formulated an EOQ model with backlogging under two-warehouse. Later, Bhunia & Maiti (1998) also considered a two-warehouse inventory model for deteriorating items with linear increasing demand. Yang (2004) further presented an inventory model where each order cycle starts with shortages and ends with surplus goods, and discussed an optimal replenishment policy with constant demand by incorporating inflation. Zhou & Yang (2005) developed a two-warehouse inventory model with inventory-dependent demand rate, and focused on investigating whether to employ RW. Using the theory of the net present value in finance, a deterministic two-warehouse issue for deteriorating items with complete shortages was investigated by Hsieh, Dye, & Ouyang (2008). Lee & Hsu (2009) established a general two-warehouse model with a finite replenishment rate and time-dependent demands in a finite planning horizon. More recently, Jaggi, Khanna, & Verma (2011) considered a two-warehouse model for deteriorating items and focused merely on the impacts of the time value of money and inflation on the optimal inventory replenishment policy. Based on the ramp-type time-varying demand rate, Agrawal, Banerjee, & Papachristos (2013) also investigated a two-warehouse system problem with partial backlogging and deteriorating items. Sett, Sarkar, & Goswami (2012) studied a two-warehouse model with the quadratically increasing demand and time-varying deterioration rate. Using the particle swarm optimisation, Bhunia, Shaikh, & Gupta (2015) studied a deterministic problem with inflation and deterioration in a two-warehouse inventory system. Under the permissible delay in payments, Chakraborty, Jana, & Roy (2018) explored a two-warehouse scenario with the ramp time-varying demand rate and three-parameter Weibull distribution deterioration.

The inventory models with a warehouse mode in existing studies have been summarised in Table 1. Similar to this research, Cheng et al. (2011) developed an inventory model with partial backlogging, where the trapezoidal-type demand rate is defined as a general time-varying function. However, it is assumed that OW's storage space is unlimited, only a single warehouse is considered, and warehouse mode selection is not investigated. Agrawal & Banerjee (2011) also examined a two-warehouse issue with ramp-type demand, where the partial backlogging is assumed to be a constant during the stock-out period. Different from the above researches, considering the partial backlogging rate depends on the customer's waiting time, this paper makes the first attempt to integrate the warehouse mode selection into inventory ordering decisions with a general trapezoidal-type demand rate. Furthermore, we prove the existence and uniqueness of the optimal solutions to the two different warehouse modes. Through numerical examples and sensitivity analysis, we explore the effects of the exogenous inventory parameters on the system performance. Finally, a real inventory case is presented to illustrate the applicability of the model.

3. Model descriptions

In this study, in order to formulate an inventory model for the trapezoidal-type demand with warehouse mode selection, the inventory problem descriptions, assumptions and notations are listed as follows.

- Inventory problem descriptions
  we consider a continuous review inventory system running in finite planning period T. Inventory replenishment is instantaneous, and lead time is zero. The items ordered are nonperishable in the inventory. OW's capacity is limited. The retailers are able to forecast future customer demand and make ordering decisions either via the single warehouse mode or the two-warehouse mode. Once the inventory system starts, the items from stock are depleted due to customer demand until the net inventory level is zero. Shortages are allowed. Considering some customers are unwilling to wait backlogged order during the stock-out period, the unsatisfied demand will only be backlogged partially in the next replenishment. Thus, to describe this phenomenon, the backlogging rate is described as $\lambda(\tau) = e^{-\delta\tau}$, where $\delta > 0$, and τ is the unsatisfied customer's waiting time during the stock-out period. That is, the backlogging rate is a decreasing function of the waiting time, which implies that, the more cumulative unsatisfied customers in the waiting line, the more the amount of lost sales. This backlogging rate function has been widely adopted in the literature (Abad, 1996).
- Assumptions
  (1) Demand assumption: The demand rate (also known as **Trapezoidal-type demand**) is a piecewise time-dependent function, which can be described as

$$D(t) = \begin{cases} a(t), & 0 \leq t \leq \mu, \\ d_0, & \mu \leq t \leq \gamma, \\ b(t), & \gamma \leq t \leq T, \end{cases}$$

  where $a(t)$ is a positive, differentiable, and increasing function of time $t \in [0, \mu]$, $b(t)$ is a positive, differentiable, and decreasing function of time $t \in [\gamma, T]$, and d_0 is a constant(i.e., $d_0 = a(\mu) = b(\gamma)$) in the interval $[\mu, \gamma]$. Since the above mentioned demand rate captures common demand patterns by varying some relevant parameters, it can be also considered as a generalized trapezoidal-type demand rate. For example, when $\mu = \gamma = T$, the demand $D(t)$ becomes **Increasing demand**; when $\mu = \gamma = 0$, it reduces to **Decreasing demand**; when $\mu \neq \gamma$, $\mu = 0$, and $\gamma = T$, it is **Constant demand**; when $\mu < \gamma = T$, it reduces to **Ramp-type demand**, and so on.

(2) Assumptions related to the two-warehouse mode: The unit holding cost in RW is greater than that in OW. The goods in RW are depleted firstly, and the goods stored in OW will be consumed until exhausting all in RW.

(3) In addition, the transporting time and the cost of transporting goods from RW to OW are ignored. And meanwhile, to avoid some trivial cases, it is assumed that the following relationship is satisfied: $\delta\tau < 1$.

The following fundamental notations are used throughout the study.

- Model parameters

A	the fixed cost per order;
C	the purchasing cost per unit item;
p	the selling price per unit item (i.e., $p > C$);
h	the holding cost per unit item per unit time in OW;
H	the holding cost per unit item per unit time in RW, and $H > h$;
B	the backlogging cost per unit item per unit time during the shortage period;
L	the opportunity cost per unit item due to lost sales;
T	the fixed length of the order cycle;
W	the maximal limited capacity of OW;
S_{1i}	the total quantity of lost sales for case i in the single inventory mode, where $i = 1, 2, 3$;
S_{2j}	the total quantity of lost sales for case j in the two-warehouse mode, where $j = 1, 2, \ldots, 5$;
$I_r(t)$	the inventory level in RW at time t;
$I_o(t)$	the inventory level in OW at time t;

- Decision variables

t_1	the time point at which the inventory level of OW reaches 0 in the single warehouse mode;
t_0	the time point at which the inventory level of RW reaches 0 in the two-warehouse mode;
T_1	the time point at which the inventory level of OW reaches 0 in the two-warehouse mode;
Q	the total ordering quantity per cycle in the entire inventory system;
Q^o	the maximal inventory level of OW in the entire inventory system;
Q^r	the maximal inventory level of RW in the entire inventory system;
Q^B	the maximal shortages in the entire inventory system;
Q^o_{1i}	the maximal inventory level of OW for case i in the single warehouse mode, where $i = 1, 2, 3$;
Q^o_{2j}	the maximal inventory level of OW for case j in the two-warehouse mode, where $j = 1, 2, \ldots, 5$;
Q^r_j	the maximal inventory level of RW for case j in the two-warehouse mode, where $j = 1, 2, \ldots, 5$;
Q^B_{1i}	the maximal shortages for case i in the single warehouse mode, where $i = 1, 2, 3$;
Q^B_{2j}	the maximal shortages for case j in the two-warehouse mode, where $j = 1, 2, \ldots, 5$.

- Other variables

$\Pi(\cdot)$	the average net profit operator in the entire inventory system;
$\Pi_1(\cdot)$	the average net profit operator in the single warehouse mode;
$\Pi_2(\cdot)$	the average net profit operator in the two-warehouse mode.

This retail system with the above inventory characteristics could be observed in reality. For instance, we often see some seasonal or new products displayed either in a limited shelf space or in a rented warehouse, and this product type usually has a short life cycle. When potential consumers are attracted by its style and quality, the demand increases with time initially; but once the type of product is accepted, the demand keeps steady, and finally it decreases with time till it is replaced by another new product. This indicates that the customer demand feature can be described by the trapezoidal time-varying function. When shortages happen, some impatient customers will not wait for goods if there is a longer waiting line. In fact, this retail system can be described in essential by the inventory model with the above assumptions. Thus, inventory managers need to decide the lot size ordered in a finite planning period in order to enhance competitiveness, and if the two-warehouse mode is adopted, they further need to decide the maximal inventory level of RW. Therefore, facing a trapezoidal-type product, this study will help inventory decision-makers decide how to order and whether to adopt a two-warehouse mode.

4. Model and model analysis

In this section, we will consider the following warehouse modes based on the above model descriptions: 1) the single warehouse mode; 2) the two-warehouse mode. To be specific, in the single warehouse mode, we will investigate three inventory cases based on possible values of μ, γ, t_1, and T; while in the two-warehouse mode, we will investigate five inventory cases based on possible values of μ, γ, t_0, T_1, and T.[1] The decision-making problem that we need to solve is to measure whether or not to rent warehouse. The warehouse modes are summarized as follows:

- Single warehouse mode.
  - ∗ case 1: $0 \le t_1 \le \mu$;
  - ∗ case 2: $\mu \le t_1 \le \gamma$;
  - ∗ case 3: $\gamma \le t_1 \le T$.
- Two-warehouse mode.
  - ∗ case 1: $\mu < \gamma \le t_0 < T_1 \le T$;
  - ∗ case 2: $\mu \le t_0 \le \gamma \le T_1 \le T$;
  - ∗ case 3: $t_0 \le \mu < \gamma \le T_1 \le T$;
  - ∗ case 4: $t_0 \le \mu \le T_1 \le \gamma \le T$;
  - ∗ case 5: $t_0 < T_1 \le \mu < \gamma \le T$.

4.1. Single warehouse mode

In this subsection, we mainly consider that the initial ordering quantity does not exceed the capacity of OW (i.e. $Q^o \le W$), i.e., there is only an OW in the inventory system. Therefore, the inventory level of OW at any time $t \in [0, T]$ goes like this:

At the beginning of the replenishment cycle, Q^o units enter the inventory system, the current inventory level $I_o(t)$ reaches its maximum at time $t = 0$. Due to the effect of the trapezoidal-type demand, the inventory level in OW is depleted gradually in the interval $[0, t_1]$, and it achieves zero at time $t = t_1$. Shortages occur in the interval $[t_1, T]$, and the unsatisfied customer's demand is partial backlogged at the rate of $\lambda(\tau) = e^{-\delta(T-t)}$. Hence, the inventory level $I_o(t)$ at any time $t \in [0, T]$ in the OW can be described by the following differential equations:

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -D(t), 0 \leq t \leq t_1 \tag{1}$$

and

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -e^{-\delta(T-t)}D(t), t_1 \leq t \leq T. \tag{2}$$

Based on the possible values of μ, γ, t_1, and T, the following three different situations are considered.

4.1.1. Case with $0 \leq t_1 \leq \mu$

[Position of Figure 1]

As shown in Figure 1, the relationship $0 \leq t_1 \leq \mu$ can also be described in detail as $0 \leq t_1 \leq \mu < \gamma < T$. It can be observed from Figure 1 that the inventory level in OW decreases due to the effect of the increasing demand $a(t)$ in the interval $[0, t_1]$, and hence, the behavior of the inventory in $[0, t_1]$ can be described by

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -a(t), 0 \leq t \leq t_1. \tag{3}$$

Solving Eq.(3) with the boundary condition $I_o(t_1) = 0$, we get

$$I_o(t) = \int_t^{t_1} a(x)\mathrm{d}x, 0 \leq t \leq t_1. \tag{4}$$

In the interval $[t_1, T]$, the depletion of inventory level happens due to the customer demand backlogged. Thus, the behavior of the inventory in $[t_1, T]$ can be described, respectively, by

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -e^{-\delta(T-t)}a(t), t_1 \leq t \leq \mu, \tag{5}$$

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -e^{-\delta(T-t)}d_0, \mu \leq t \leq \gamma, \tag{6}$$

and

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -e^{-\delta(T-t)}b(t), \gamma \leq t \leq T. \tag{7}$$

Solving Eqs.(5), (6) and (7) with the boundary conditions of $I_o(t_1) = 0$, $I_o(\mu^-) = I_o(\mu^+)$ and $I_o(\gamma^-) = I_o(\gamma^+)$, we have

$$I_o(t) = \int_t^{t_1} e^{-\delta(T-x)} a(x)\mathrm{d}x, t_1 \le t \le \mu, \tag{8}$$

$$I_o(t) = \int_t^{\mu} e^{-\delta(T-x)} d_0\mathrm{d}x - \int_{t_1}^{\mu} e^{-\delta(T-x)} a(x)\mathrm{d}x, \mu \le t \le \gamma, \tag{9}$$

and

$$I_o(t) = \int_t^{\gamma} e^{-\delta(T-x)} b(x)\mathrm{d}x - \int_{\mu}^{\gamma} e^{-\delta(T-x)} d_0\mathrm{d}x - \int_{t_1}^{\mu} e^{-\delta(T-x)} a(x)\mathrm{d}x, \gamma \le t \le T, \tag{10}$$

respectively.

From Eqs.(4) and (10), the ordering quantity per cycle can be computed as

$$\begin{aligned} Q_{11} &= Q_{11}^o + Q_{11}^B = I_o(0) - I_o(T) = \int_0^{t_1} a(x)\mathrm{d}x + \int_{\gamma}^{T} e^{-\delta(T-x)} b(x)\mathrm{d}x \\ &+ \int_{\mu}^{\gamma} e^{-\delta(T-x)} d_0\mathrm{d}x + \int_{t_1}^{\mu} e^{-\delta(T-x)} a(x)\mathrm{d}x. \end{aligned} \tag{11}$$

The total quantity of lost sales due to shortages in the interval $[t_1, T]$ can be expressed as

$$\begin{aligned} S_{11} &= \textstyle\int_{t_1}^{\mu} (1 - e^{-\delta(T-x)}) a(x)\mathrm{d}x + \int_{\mu}^{\gamma} (1 - e^{-\delta(T-x)}) d_0\mathrm{d}x \\ &+ \textstyle\int_{\gamma}^{T} (1 - e^{-\delta(T-x)}) b(x)\mathrm{d}x. \end{aligned} \tag{12}$$

The related cost and revenue of the inventory system per cycle can be expressed as:

(a) The sales revenue per cycle: pQ_{11};

(b) The fixed set up cost per cycle: A;

(c) The purchasing cost per cycle: CQ_{11};

(d) The holding cost per cycle: $h \int_0^{t_1} I_o(t)\mathrm{d}t$;

(e) The shortage cost per cycle: $B \int_{t_1}^{T} -I_o(t)\mathrm{d}t$;

(f) The opportunity cost due to lost sales per cycle: LS_{11}.

Therefore, the total profit per unit time under the condition $0 < t_1 \le \mu$ is determined by

$$\begin{aligned} \Pi_{11}(t_1) &= \tfrac{1}{T}\{(p-C)Q_{11} - A - h\textstyle\int_0^{t_1} I_o(t)\mathrm{d}t - B\int_{t_1}^{T} -I_o(t)\mathrm{d}t - LS_{11}\} \\ &= \tfrac{1}{T}\{(p-C)[\textstyle\int_0^{t_1} a(x)\mathrm{d}x + \int_{\gamma}^{T} e^{-\delta(T-x)} b(x)\mathrm{d}x] \\ &+ (p-C)[\textstyle\int_{\mu}^{\gamma} e^{-\delta(T-x)} d_0\mathrm{d}x + \int_{t_1}^{\mu} e^{-\delta(T-x)} a(x)\mathrm{d}x] \\ &- A - h\textstyle\int_0^{t_1}\int_t^{t_1} a(x)\mathrm{d}x\mathrm{d}t + B\int_{t_1}^{\mu}\int_t^{t_1} a(x)\mathrm{d}x\mathrm{d}t \\ &+ B\textstyle\int_{\mu}^{\gamma}[\int_t^{\mu} e^{-\delta(T-x)} d_0 dx - \int_{t_1}^{\mu} e^{-\delta(T-x)} a(x)\mathrm{d}x]\mathrm{d}t \\ &+ B\textstyle\int_{\gamma}^{T}[\int_t^{\gamma} e^{-\delta(T-x)} b(x)dx - \int_{\mu}^{\gamma} e^{-\delta(T-x)} d_0 dx - \int_{t_1}^{\mu} e^{-\delta(T-x)} a(x)dx]\mathrm{d}t \\ &- L[\textstyle\int_{t_1}^{\mu}(1 - e^{-\delta(T-x)}) a(x)\mathrm{d}x + \int_{\mu}^{\gamma}(1 - e^{-\delta(T-x)}) d_0\mathrm{d}x] \\ &+ L\textstyle\int_{\gamma}^{T}(1 - e^{-\delta(T-x)}) b(x)\mathrm{d}x\}. \end{aligned} \tag{13}$$

4.1.2. Cases with $\mu \le t_1 \le \gamma$ and $\gamma \le t_1 \le T$

Similar to case with $0 \le t_1 \le \mu$, the total average profits for cases with $\mu \le t_1 \le \gamma$ and $\gamma \le t_1 \le T$ are obtained, respectively, by

$$
\begin{aligned}
\Pi_{12}(t_1) &= \tfrac{1}{T}\{(p-C)Q_{12} - A - h\int_0^{t_1} I_o(t)\mathrm{d}t - B\int_{t_1}^{T} -I_o(t)\mathrm{d}t - LS_{12}\} \\
&= \tfrac{1}{T}\{(p-C)[\int_0^{\mu} a(x)\mathrm{d}x + d_0(t_1-\mu) + \int_{\gamma}^{T} e^{-\delta(T-x)} b(x)\mathrm{d}x + \int_{t_1}^{\gamma} e^{-\delta(T-x)} d_0\mathrm{d}x] \\
&- A - h\int_0^{\mu}[\int_t^{\mu} a(x)\mathrm{d}x + d_0(t_1-\mu)]\mathrm{d}t - h\int_{\mu}^{t_1}[d_0(t_1-t)]\mathrm{d}t \\
&+ B\int_{t_1}^{\gamma}\int_t^{t_1} e^{-\delta(T-x)} d_0\mathrm{d}x\mathrm{d}t + B\int_{\gamma}^{T}[\int_t^{\gamma} e^{-\delta(T-x)} b(x)\mathrm{d}x - \int_{t_1}^{\gamma} e^{-\delta(T-x)} d_0\mathrm{d}x]\mathrm{d}t \\
&- L[\int_{t_1}^{\gamma}(1-e^{-\delta(T-x)})d_0\mathrm{d}x + \int_{\gamma}^{T}(1-e^{-\delta(T-x)})b(x)\mathrm{d}x]\}
\end{aligned} \tag{14}
$$

and

$$
\begin{aligned}
\Pi_{13}(t_1) &= \tfrac{1}{T}\{(p-C)Q_{13} - A - h\int_0^{t_1} I_o(t)\mathrm{d}t - B\int_{t_1}^{T} -I_o(t)\mathrm{d}t - LS_{13}\} \\
&= \tfrac{1}{T}\{(p-C)[\int_0^{\mu} a(x)\mathrm{d}x + d_0(\gamma-\mu) + \int_{\gamma}^{t_1} b(x)\mathrm{d}x + \int_{t_1}^{T} e^{-\delta(T-x)} b(x)\mathrm{d}x] - A \\
&- h\int_0^{\mu}[\int_t^{\mu} a(x)\mathrm{d}x + d_0(\gamma-\mu) + \int_{\gamma}^{t_1} b(x)\mathrm{d}x]\mathrm{d}t - h\int_{\mu}^{\gamma}[d_0(\gamma-t) + \int_{\gamma}^{t_1} b(x)\mathrm{d}x]\mathrm{d}t \\
&- h\int_{\gamma}^{t_1}\int_t^{t_1} b(x)\mathrm{d}x\mathrm{d}t + B\int_{t_1}^{T}\int_t^{t_1} e^{-\delta(T-x)} b(x)\mathrm{d}x\mathrm{d}t - L\int_{t_1}^{T}[1-e^{-\delta(T-x)}]b(x)\mathrm{d}x\}.
\end{aligned} \tag{15}
$$

Detailed analyses of the above two scenarios are provided in the online appendix.

Based on the above analysis, the total profit per unit time $\Pi_1(t_1)$ in the interval $[0,T]$ under the single warehouse setting is

$$
\Pi_1(t_1) = \begin{cases} \Pi_{11}(t_1), & 0 \le t_1 \le \mu; \\ \Pi_{12}(t_1), & \mu \le t_1 \le \gamma; \\ \Pi_{13}(t_1), & \gamma \le t_1 \le T, \end{cases}
$$

where $\Pi_{11}(t_1)$, $\Pi_{12}(t_1)$ and $\Pi_{13}(t_1)$ can be obtained from Eqs. (13), (14) and (15), respectively.

Therefore, the optimization problem addressed in the single warehouse mode is given by

M1: max $\quad \Pi_1(t_1)$, $\quad$ s.t. $\quad Q^o \le W$ and $0 \le t_1 \le T$.

In the next section, our aim is to search for the optimal solution of M1. Combining with the possible objective behaviors and constraint conditions, the M1 can also be explored by separating it into three optimization subproblems as below:

M1-1: max $\quad \Pi_{11}(t_1)$, $\quad$ s.t. $\quad Q_{11}^o \le W$ and $0 \le t_1 \le \mu$.

M1-2: max $\quad \Pi_{12}(t_1)$, $\quad$ s.t. $\quad Q_{12}^o \le W$ and $\mu \le t_1 \le \gamma$.

M1-3: max $\quad \Pi_{13}(t_1)$, $\quad$ s.t. $\quad Q_{13}^o \le W$ and $\gamma \le t_1 \le T$.

In order to derive the optimal results for the above three models, we give the following theorems:

Theorem 4.1. *For* M1-i, $i = 1,2,3$, *the first-order necessary condition for the optimality of* $\Pi_{1i}(t_1)$ *is equivalent to the condition that* $z(t_1) = 0$, *where* $z(t_1) = (p-C+L)(1-e^{-\delta(T-t_1)}) - ht_1 + Be^{-\delta(T-t_1)}(T-t_1)$.

Proof. Proof available in the online appendix. □

Theorem 4.1 shows that, under no considering inventory capacity constraints, the optimal solution of the single warehouse mode does not depend on the fixed set up cost and the form of the trapezoidal-type demand. This finding is consistent with Cheng & Wang (2009), Cheng et al. (2011), Lin (2013) and Wu et al. (2016).

Proposition 4.2. *$z(t_1)$ is a strictly decreasing function with respect to t_1.*

Proof. Proof available in the online appendix. □

Theorem 4.3. *For M1-i, $i = 1, 2, 3$, let $t_1^{ub_i}$ and $t_1^{lb_i}$ be the upper feasible value for t_1^i and the lower feasible value for t_1^i respectively, and we have*

(1) if $z(t_1^{lb_i}) > 0$ and $z(t_1^{ub_i}) < 0$, then there exists a unique value $(t_1^i)^* \in (t_1^{lb_i}, t_1^{ub_i})$, which is the optimal solution of M1-i;

(2) if $z(t_1^{ub_i}) > 0$, then there exists a unique value $(t_1^i)^* = t_1^{ub_i}$, which is the optimal solution of M1-i;

(3) if $z(t_1^{lb_i}) < 0$, then there exists a unique value $(t_1^i)^* = t_1^{lb_i}$, which is the optimal solution of M1-i.

Proof. Proof available in the online appendix. □

Remark 1. Theorem 4.3 indicates that, for M1-i, the optimal value $(t_1^i)^*$ is judged by the signs of $z(t_1^{ub_i})$ and $z(t_1^{lb_i})$, and it may occur either at an interior point or at boundary points of the feasible interval. Therefore, for the former, the interior value can be given by solving the equation $z(t_1) = 0$; for the latter, let $Q_{1i}^o(t_w^i) = W$, the boundary values for case i can be easily determined as follows: $t_1^{lb_1} = 0, t_1^{ub_1} = \min\{t_w^1, \mu\}$, $t_1^{lb_2} = \mu, t_1^{ub_2} = \min\{t_w^2, \gamma\}$, $t_1^{lb_3} = \gamma$, and $t_1^{ub_3} = \min\{t_w^3, T\}$. These show that, the optimal solution to the single warehouse mode with a limited inventory capacity is not only dependent on selling price, purchasing cost, holding cost, backlogging cost, and opportunity cost due to lost sales, but also subjected to the maximal limited capacity of OW and trapezoidal-type demand time point (i.e, μ and γ), which are essentially different from existing models with a full storage space in OW under the trapezoidal-type demand.

In the following theorem, we will give the optimal solution of M1.

Theorem 4.4. *Let t_{1S}^* be a unique value of t_1 that is the optimal solution of M1, then t_{1S}^* can be represented as*

$$t_{1S}^* = \arg\max\{\Pi_{11}(t_1^{1*}), \Pi_{12}(t_1^{2*}), \Pi_{13}(t_1^{3*})\}.$$

Noting that, once t_{1S}^* is determined in the single warehouse mode, the corresponding optimal total average profit and order quantity are also obtained. From the previous analysis, a solution procedure for the optimal policy of the single warehouse mode can be summarised as the following algorithm.

Algorithm 4.1

Step1.1 Input all the parameters;

Step1.2 For case i, $i = 1, 2, 3$, compute $t_1^{lb_i}$ and $t_1^{ub_i}$;

Step1.3 Judge the signs of $z(t_1^{lb_i})$ and $z(t_1^{ub_i})$, and there are three possible cases as follows:

Step1.3.1 If $z(t_1^{lb_i}) < 0$, then $t_1^{i_*} = t_1^{lb_i}$ and go to **Step1.4**. Otherwise go to **Step1.3.2**;

Step1.3.2 If $z(t_1^{lb_i}) > 0$ and $z(t_1^{ub_i}) < 0$, use a linear search method to obtain $t_1^{i_*}$, and then go to **Step1.4**. Otherwise go to **Step1.3.3**;

Step1.3.3 If $z(t_1^{ub_i}) > 0$, then $t_1^{i_*} = t_1^{ub_i}$ and go to **Step1.4**;

Step1.4 Let t_{1S}^{*}=arg max$\{\Pi_{11}(t_1^{1*}), \Pi_{12}(t_1^{2*}), \Pi_{13}(t_1^{3*})\}$, output t_{1S}^{*} and $\Pi_1(t_{1S}^{*})$, and then get Q^o and Q^B by the above corresponding formulas, stop.

4.2. Two-warehouse mode

Different from the single warehouse mode, at time $t = 0$, some unit items are kept in the inventory system, where $Q^o = W$ units are stored in OW, and the rest Q^r units are kept in RW. The items of OW will be met after consuming all in RW. In the time interval $[0, t_0]$, due to the effect of the demand, the inventory in RW gradually decreases and achieves zero at time $t = t_0$. While in $[0, t_0]$, the OW always maintains the maximal inventory level (i.e., $Q^o = W$); In the next time interval $[t_0, T_1]$, the inventory in OW is depleted gradually and it vanishes at time $t = T_1$; the customer demand in shortage time interval $[T_1, T]$ is partially backlogged at the rate of $\lambda(\tau) = e^{-\delta(T-t)}$, but the total quantity of shortages backlogged is met by the next replenishment. Hence, the two-warehouse inventory behavior at any time $t \in [0, T]$ can be described as follows:

In RW,

$$\frac{\mathrm{d}I_r(t)}{\mathrm{d}t} = -D(t), 0 \le t \le t_0, \tag{16}$$

with the boundary condition $I_r(t_0) = 0$. In OW,

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -D(t), t_0 \le t \le T_1, \tag{17}$$

and

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -e^{-\delta(T-t)}D(t), T_1 \le t \le T, \tag{18}$$

with the boundary condition $I_o(T_1) = 0$.

Based on the possible values of μ, γ, t_0, T_1 and T, the following five distinct cases are considered.

Case 4.2.1. Case with $\mu < \gamma \le t_0 < T_1 \le T$

[Position of Figure 2]

As shown in Figure 2, in RW, because of the increasing demand $a(t)$ in $[0, \mu]$, the constant demand d_0 in $[\mu, \gamma]$, and the decreasing demand $b(t)$ in $[\gamma, t_0]$, the current inventory level $I_r(t)$ gradually decreases and reduces to zero at $t = t_0$. Then, from Eq.(16), we have

$$\frac{\mathrm{d}I_r(t)}{\mathrm{d}t} = -a(t), 0 \le t \le \mu, \tag{19}$$

$$\frac{\mathrm{d}I_r(t)}{\mathrm{d}t} = -d_0, \mu \le t \le \gamma, \tag{20}$$

and

$$\frac{\mathrm{d}I_r(t)}{\mathrm{d}t} = -b(t), \gamma \le t \le t_0, \tag{21}$$

respectively. Using the boundary conditions $I_r(\mu^-) = I_r(\mu^+)$, $I_r(\gamma^-) = I_r(\gamma^+)$ and $I_r(t_0) = 0$, the solutions of Eqs.(19), (20) and (21) are

$$I_r(t) = \int_t^{\mu} a(x)\mathrm{d}x + d_0(\gamma - \mu) + \int_{\gamma}^{t_0} b(x)\mathrm{d}x, 0 \leq t \leq \mu, \tag{22}$$

$$I_r(t) = d_0(\gamma - t) + \int_{\gamma}^{t_0} b(x)\mathrm{d}x, \mu \leq t \leq \gamma, \tag{23}$$

and

$$I_r(t) = \int_t^{t_0} b(x)\mathrm{d}x, \gamma \leq t \leq t_0, \tag{24}$$

respectively.

In OW, due to the decreasing demand $b(t)$ in $[t_0, T_1]$ and the shortages in $[T_1, T]$, from Eqs.(17) and (18), we have

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -b(t), t_0 \leq t \leq T_1, \tag{25}$$

and

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -e^{-\delta(T-t)}b(t), T_1 \leq t \leq T, \tag{26}$$

respectively. Using the boundary condition $I_o(T_1) = 0$, the solutions to Eqs.(25) and (26) can be obtained, respectively, by

$$I_o(t) = \int_t^{T_1} b(x)\mathrm{d}x, t_0 \leq t \leq T_1, \tag{27}$$

and

$$I_o(t) = \int_t^{T_1} e^{-\delta(T-x)}b(x)\mathrm{d}x, T_1 \leq t \leq T. \tag{28}$$

From $I_o(t_0) = W$, we easily get $\int_{t_0}^{T_1} b(x)\mathrm{d}x = W$, which implies that T_1 can be expressed as a function of t_0. Let $G(x) = \int b(x)\mathrm{d}x$. Then t_0 can be simplified as the following relation:

$$t_0 = G^{-1}((G(T_1) - W) = t_0^1(T_1). \tag{29}$$

Next, taking the first derivative of $t_0^1(T_1)$, we have

$$\frac{\mathrm{d}t_0^1(T_1)}{\mathrm{d}T_1} = \frac{b(T_1)}{b(t_0)} > 0. \tag{30}$$

From Eqs.(22) and (28), we also attain the ordering quantity per cycle as

$$\begin{aligned} Q_{21} &= I_r(0) + W - I_o(T) = \int_0^{\mu} a(x)\mathrm{d}x + d_0(\mu - \gamma) \\ &+ \int_{\gamma}^{t_0} b(x)\mathrm{d}x + W + \int_{T_1}^{T} e^{-\delta(T-x)} b(x)\mathrm{d}x, \end{aligned} \tag{31}$$

and the total quantity of lost sales during the shortage period $[T_1, T]$ as

$$S_{21} = \int_{T_1}^{T} (1 - e^{-\delta(T-x)}) b(x)\mathrm{d}x. \tag{32}$$

Based on Eqs.(22), (23), (24), (27), (28), (31) and (32), the profit function per cycle includes the following seven elements:

(a) The sales revenue per cycle: pQ_{21};
(b) The fixed set up cost per cycle: A;
(c) The purchasing cost per cycle: CQ_{21};
(d) The holding cost in RW per cycle: $H \int_0^{t_0} I_r(t)\mathrm{d}t$;
(e) The holding cost in OW per cycle: $h \int_0^{T_1} I_o(t)\mathrm{d}t$;
(f) The shortage cost per cycle: $B \int_{T_1}^{T} -I_o(t)\mathrm{d}t$;
(g) The opportunity cost due to lost sales per cycle: LS_{21}.

Therefore, the total profit per unit time under the condition $\mu < \gamma \leq t_0 < T_1 \leq T$ is given by

$$\begin{aligned} \Pi_{21}(T_1) &= \tfrac{1}{T}\{(p - C)Q_{21} - A - H \textstyle\int_0^{t_0} I_r(t)\mathrm{d}t - h \int_0^{T_1} I_o(t)\mathrm{d}t - B \int_{T_1}^{T} -I_o(t)\mathrm{d}t - LS_{21}\} \\ &= \tfrac{1}{T}\{(p - C)[\textstyle\int_0^{\mu} a(x)\mathrm{d}x + d_0(\gamma - \mu) + \int_{\gamma}^{t_0} b(x)\mathrm{d}x] \\ &+ (p - C)[W + \textstyle\int_{T_1}^{T} e^{-\delta(T-x)} b(x)\mathrm{d}x] \\ &- A - H[\textstyle\int_0^{\mu}\int_t^{\mu} a(x)\mathrm{d}x\mathrm{d}t + \int_0^{\mu} d_0(\gamma - \mu)dt + \int_0^{\mu}\int_{\gamma}^{t_0} b(x)\mathrm{d}x\mathrm{d}t] \\ &- H[\textstyle\int_{\mu}^{\gamma} d_0(\gamma - t)\mathrm{d}t + \int_{\mu}^{\gamma}\int_{\gamma}^{t_0} b(x)\mathrm{d}x\mathrm{d}t + \int_{\gamma}^{t_0}\int_t^{t_0} b(x)\mathrm{d}x\mathrm{d}t] \\ &- h[Wt_0 + \textstyle\int_{t_0}^{T_1}\int_t^{T_1} b(x)\mathrm{d}x\mathrm{d}t] + B \int_{T_1}^{T}\int_t^{T_1} e^{-\delta(T-x)} b(x)\mathrm{d}x\mathrm{d}t \\ &- L \textstyle\int_{T_1}^{T}(1 - e^{-\delta(T-x)}) b(x)\mathrm{d}x\}. \end{aligned} \tag{33}$$

4.2.2. The other four cases

Similar to case with $\mu < \gamma \leq t_0 < T_1 \leq T$, the total average profits for the cases with $\mu \leq t_0 \leq \gamma \leq T_1 \leq T$, $t_0 \leq \mu < \gamma \leq T_1 \leq T$, $t_0 \leq \mu \leq T_1 \leq \gamma \leq T$ and $t_0 < T_1 \leq \mu < \gamma \leq T$ are obtained, respectively, by

$$\begin{aligned} \Pi_{22}(T_1) &= \tfrac{1}{T}\{(p - C)Q_{22} - A - H \textstyle\int_0^{t_0} I_r(t)\mathrm{d}t - h \int_0^{T_1} I_o(t)\mathrm{d}t - B \int_{T_1}^{T} -I_o(t)\mathrm{d}t - LS_{22}\} \\ &= \tfrac{1}{T}\{(p - C)[\textstyle\int_0^{\mu} a(x)\mathrm{d}x + d_0(t_0 - \mu) + W + \int_{T_1}^{T} e^{-\delta(T-x)} b(x)\mathrm{d}x] \\ &- A - H[\textstyle\int_0^{\mu}\int_t^{\mu} a(x)\mathrm{d}x\mathrm{d}t + \int_0^{\mu} d_0(t_0 - \mu)\mathrm{d}t + \int_{\mu}^{t_0} d_0(t_0 - t)\mathrm{d}t] \\ &- h[Wt_0 + \textstyle\int_{t_0}^{\gamma} d_0(\gamma - t)\mathrm{d}t + \int_{t_0}^{\gamma}\int_{\gamma}^{T_1} b(x)\mathrm{d}x\mathrm{d}t + \int_{\gamma}^{T_1}\int_t^{T_1} b(x)\mathrm{d}x\mathrm{d}t] \\ &+ B \textstyle\int_{T_1}^{T}\int_t^{T_1} e^{-\delta(T-x)} b(x)\mathrm{d}x\mathrm{d}t - L \int_{T_1}^{T}(1 - e^{-\delta(T-x)}) b(x)\mathrm{d}x\}, \end{aligned} \tag{34}$$

$$
\begin{aligned}
\Pi_{23}(T_1) &= \tfrac{1}{T}\{(p-C)Q_{23} - A - H\int_0^{t_0} I_r(t)\mathrm{d}t - h\int_0^{T_1} I_o(t)\mathrm{d}t - B\int_{T_1}^{T} -I_o(t)\mathrm{d}t - LS_{23}\}\\
&= \tfrac{1}{T}\{(p-C)[\int_0^{t_0} a(x)\mathrm{d}x + W + \int_{T_1}^{T} e^{-\delta(T-x)}b(x)\mathrm{d}x] - A\\
&- H\int_0^{t_0}\int_t^{t_0} a(x)\mathrm{d}x\mathrm{d}t - h[Wt_0 + \int_{t_0}^{\mu}\int_t^{\mu} a(x)\mathrm{d}x\mathrm{d}t + \int_{t_0}^{\mu}\int_{\mu}^{\gamma} d_0\mathrm{d}x\mathrm{d}t]\\
&- h[\int_{t_0}^{\mu}\int_{\gamma}^{T_1} b(x)\mathrm{d}x\mathrm{d}t + \int_{\mu}^{\gamma}\int_t^{\gamma} d_0\mathrm{d}x\mathrm{d}t + \int_{\mu}^{\gamma}\int_{\gamma}^{T_1} b(x)\mathrm{d}x\mathrm{d}t + \int_{\gamma}^{T_1}\int_t^{T_1} b(x)\mathrm{d}x\mathrm{d}t]\\
&+ B\int_{T_1}^{T}\int_t^{T_1} e^{-\delta(T-x)}b(x)\mathrm{d}x\mathrm{d}t - L\int_{T_1}^{T}(1-e^{-\delta(T-x)})b(x)\mathrm{d}x\},
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
\Pi_{24}(T_1) &= \tfrac{1}{T}\{(p-C)Q_{24} - A - H\int_0^{t_0} I_r(t)\mathrm{d}t - h\int_0^{T_1} I_o(t)\mathrm{d}t - B\int_{T_1}^{T} -I_o(t)\mathrm{d}t - LS_{24}\}\\
&= \tfrac{1}{T}\{(p-C)[\int_0^{t_0} a(x)\mathrm{d}x + W + \int_{\gamma}^{T} e^{-\delta(T-x)}b(x)\mathrm{d}x + \int_{T_1}^{\gamma} e^{-\delta(T-x)}d_0\mathrm{d}x] - A\\
&- H\int_0^{t_0}\int_t^{t_0} a(x)\mathrm{d}x\mathrm{d}t - h[Wt_0 + \int_{t_0}^{\mu}\int_t^{\mu} a(x)\mathrm{d}x\mathrm{d}t + \int_{t_0}^{\mu}\int_{\mu}^{T_1} d_0\mathrm{d}x\mathrm{d}t]\\
&- h\int_{\mu}^{T_1}\int_t^{T_1} d_0\mathrm{d}x\mathrm{d}t + B[\int_{T_1}^{\gamma}\int_t^{t_1} e^{-\delta(T-x)}d_0\mathrm{d}x\mathrm{d}t + \int_{\gamma}^{T}\int_t^{\gamma} e^{-\delta(T-x)}b(x)\mathrm{d}x\mathrm{d}t]\\
&- B\int_{\gamma}^{T}\int_{T_1}^{\gamma} e^{-\delta(T-x)}d_0\mathrm{d}x\mathrm{d}t - L[\int_{T_1}^{\gamma}(1-e^{-\delta(T-x)})d_0\mathrm{d}x + \int_{\gamma}^{T}(1-e^{-\delta(T-x)})b(x)\mathrm{d}x]\}
\end{aligned}
\tag{36}
$$

and

$$
\begin{aligned}
\Pi_{25}(T_1) &= \tfrac{1}{T}\{(p-C)Q_{25} - A - H\int_0^{t_0} I_r(t)\mathrm{d}t - h\int_0^{T_1} I_o(t)\mathrm{d}t - B\int_{T_1}^{T} -I_o(t)\mathrm{d}t - LS_{25}\}\\
&= \tfrac{1}{T}\{(p-C)[\int_0^{t_0} a(x)\mathrm{d}x + W + \int_{\gamma}^{T} e^{-\delta(T-x)}b(x)\mathrm{d}x + \int_{\mu}^{\gamma} e^{-\delta(T-x)}d_0\mathrm{d}x]\\
&+ (p-C)[\int_{T_1}^{\mu} e^{-\delta(T-x)}a(x)\mathrm{d}x] - A - H[\int_0^{t_0}\int_t^{t_0} a(x)dxdt]\\
&- h[Wt_0 + \int_{t_0}^{T_1}\int_t^{T_1} a(x)\mathrm{d}x\mathrm{d}t] + B[\int_{T_1}^{\mu}\int_t^{T_1} e^{-\delta(T-x)}a(x)\mathrm{d}x\mathrm{d}t]\\
&+ B[\int_{\mu}^{\gamma}\int_t^{\mu} e^{-\delta(T-x)}d_0\mathrm{d}x\mathrm{d}t - \int_{\mu}^{\gamma}\int_{T_1}^{\mu} e^{-\delta(T-x)}a(x)\mathrm{d}x\mathrm{d}t]\\
&+ B[\int_{\gamma}^{T}\int_t^{\gamma} e^{-\delta(T-x)}b(x)\mathrm{d}x\mathrm{d}t - \int_{\gamma}^{T}\int_{\mu}^{\gamma} e^{-\delta(T-x)}d_0\mathrm{d}x\mathrm{d}t]\\
&- B[\int_{\gamma}^{T}\int_{T_1}^{\mu} e^{-\delta(T-x)}a(x)\mathrm{d}x\mathrm{d}t] - L[\int_{T_1}^{\mu}(1-e^{-\delta(T-x)})a(x)\mathrm{d}x]\\
&- L[\int_{\mu}^{\gamma}(1-e^{-\delta(T-x)})d_0\mathrm{d}x + \int_{\gamma}^{T}(1-e^{-\delta(T-x)})b(x)\mathrm{d}x]\}.
\end{aligned}
\tag{37}
$$

Detailed analyses of the above four scenarios are provided in the online appendix.

Based on the above analysis, the total profit per unit time $\Pi_2(T_1)$ in the interval $[0, T]$ under the two-warehouse inventory mode can be summarized as

$$
\Pi_2(T_1) = \begin{cases}
\Pi_{21}(T_1), & \mu < \gamma \le t_0 < T_1 \le T;\\
\Pi_{22}(T_1), & \mu \le t_0 \le \gamma \le T_1 \le T;\\
\Pi_{23}(T_1), & t_0 \le \mu < \gamma \le T_1 \le T;\\
\Pi_{24}(T_1), & t_0 \le \mu \le T_1 \le \gamma \le T;\\
\Pi_{25}(T_1), & t_0 < T_1 \le \mu < \gamma \le T.
\end{cases}
\tag{38}
$$

where $\Pi_{21}(T_1)$, $\Pi_{22}(T_1)$, $\Pi_{23}(T_1)$, $\Pi_{24}(T_1)$ and $\Pi_{25}(T_1)$ can be obtained from (33), (34), (35), (36) and (37), respectively.

Therefore, this optimization problem addressed under the two-warehouse mode is simply described as

M2: max $\quad \Pi_2(T_1)$, $\quad$ s.t. $\quad Q^o = W$, $Q^r \ge 0$ and $0 \le T_1 \le T$.

In order to search for the optimal solution of M2, combining with the possible objective behaviors and constraint conditions mentioned above, the M2 can also be investigated by separating it into five optimization subproblems as follows:

M2-1: max $\quad \Pi_{21}(T_1)$, $\quad$ s.t. $\quad Q^o_{21} = W$, $Q^r_1 \ge 0$, and $\mu < \gamma \le t_0 < T_1 \le T$.

M2-2: max $\Pi_{22}(T_1)$, s.t. $Q_{22}^o = W$, $Q_2^r \geq 0$, and $\mu \leq t_0 \leq \gamma \leq T_1 \leq T$.
M2-3: max $\Pi_{23}(T_1)$, s.t. $Q_{23}^o = W$, $Q_3^r \geq 0$, and $t_0 \leq \mu < \gamma \leq T_1 \leq T$.
M2-4: max $\Pi_{24}(T_1)$, s.t. $Q_{24}^o = W$, $Q_4^r \geq 0$, and $t_0 \leq \mu \leq T_1 \leq \gamma \leq T$.
M2-5: max $\Pi_{25}(T_1)$, s.t. $Q_{25}^o = W$, $Q_5^r \geq 0$, and $t_0 < T_1 \leq \mu < \gamma \leq T$.

In what follows, our aim is to derive the optimal results under the two-warehouse inventory mode, and we firstly give the following theorem.

Theorem 4.5. *For case j, $j = 1, 2, \ldots, 5$, the first-order necessary condition for the optimality of $\Pi_{2j}(T_1)$ is equivalent to the condition that $Z_j(T_1) = 0$, where $Z_j(T_1) = (p - C + L)(1 - e^{-\delta(T-T_1)}) - Ht_0^j(T_1) - h(T_1 - t_0^j(T_1)) + Be^{-\delta(T-T_1)}(T - T_1)$.*

Proof. Proof available in the online appendix. □

Furthermore, the proposition can be gained as follow.

Proposition 4.6. *For case j, $j = 1, 2, \ldots, 5$, $Z_j(T_1)$ is a strictly decreasing function with respect to T_1.*

Proof. Proof available in the online appendix. □

Combining with Proposition 4.6, the following theorem is obtained.

Theorem 4.7. *For given case j, $j = 1, 2, \ldots, 5$, let $T_1^{Ub_j}$ and $T_1^{Lb_j}$ be the upper feasible value for T_1^j and the lower feasible value for T_1^j, and we have*

(1) if $Z_j(T_1^{Lb_j}) > 0$ and $Z_j(T_1^{Ub_j}) < 0$, then there exists a unique value $(T_1^j)^* \in (T_1^{Lb_j}, T_1^{Ub_j})$, which is the optimal solution of M2-j;

(2) if $Z_j(T_1^{Ub_j}) > 0$, then there exists a unique value $(T_1^j)^* = T_1^{Ub_j}$, which is the optimal solution of M2-j;

(3) if $Z_j(T_1^{Lb_j}) < 0$, then there exists a unique value $(T_1^j)^* = T_1^{Lb_j}$, which is the optimal solution of M2-j.

Proof. Proof available in the online appendix. □

Remark 2. From Theorems 4.5 and 4.7, it is clear that, for case j, $j = 1, 2, \ldots, 5$, the optimal value $(T_1^j)^*$ for maximizing $\Pi_{2j}(T_1)$ is unique, and it is mainly affected by some factors such as rented cost in RW, own warehouse capacity, trapezoidal-type demand time point, and so on. The effects of these factors on the inventory performance will be further discussed in the sensitivity analysis section.

In addition, from the technique of solving the model, for case j, $j = 1, 2, \ldots, 5$, Theorem 4.7 indicates that the optimal value $(T_1^j)^*$ may happen either at an interior point or at boundary points of the feasible interval. For the former, the interior optimal value can be obtained by solving the equation $Z_j(T_1) = 0$; while for the latter, it is worth noting that, the boundary values of $T_1^{Ub_j}$ and $T_1^{Lb_j}$ for each case j can play an important role in the process of searching for the optimal solution. Thus, we attempt to further investigate their possible values by the following procedures. For example, in Case 4.2.1, using the constraint conditions in M2-1, it is clear that the upper feasible value of T_1 is T (i.e., $T_1^{Ub_1} = T$). Also, from Eq.(29), we can further get $T_1 = G^{-1}(G(t_0) + W)$, and $G(x)$ is a monotone increasing function (i.e., $G'(x) = b(x) > 0$). Thus, the lower limit value of T_1 occurs at $t_0 = \gamma$, that is, $T_1^{Lb_1} = G^{-1}(G(\gamma) + W)$. The other results are shown in Table 2 by adopting the similar procedure in Case 4.2.1.

[Position of Table 2]

Theorem 4.8. *Let T_{1T}^* be a unique value of T_1 that maximizes the total average profit $\Pi_2(T_1)$, then T_{1T}^* can be represented as*

$$T_{1T}^* = \arg\max\{\Pi_{21}(T_1^{1*}), \Pi_{22}(T_1^{2*}), \ldots, \Pi_{25}(T_1^{5*})\}.$$

Based on the above arguments, a solution procedure for the optimal policies of the two-warehouse inventory mode can be summarised as the following algorithm.

Algorithm 4.2

Step2.1 Input all the parameters;

Step2.2 For case j, $j = 1, 2, \ldots, 5$, compute $T_1^{Lb_j}$ and $T_1^{Ub_j}$;

Step2.3 Judge the signs of $Z_j(T_1^{Lb_j})$ and $Z_j(T_1^{Ub_j})$, and there are three possible cases as follows:

Step2.3.1 If $Z_j(T_1^{Lb_j}) < 0$, then $T_1^{j*} = T_1^{Lb_j}$ and go to **Step4**. Otherwise go to **Step2.3.2**;

Step2.3.2 If $Z_j(T_1^{Lb_j}) > 0$ and $Z_j(T_1^{Ub_j}) < 0$, use a linear search method to obtain T_1^{j*}, and then go to **Step2.4**. Otherwise go to **Step2.3.3**;

Step2.3.3 If $Z_j(T_1^{Ub_j}) > 0$, then $T_1^{j*} = T_1^{Ub_j}$ and go to **Step2.4**;

Step2.4 Let T_{1T}^*=$\arg\max\{\Pi_{21}(T_1^{1*}), \Pi_{22}(T_1^{2*}), \ldots, \Pi_{25}(T_1^{5*})\}$, output t_{1T}^* and $\Pi_2(t_{1T}^*)$, and then obtain t_0^*, Q^r and Q^B by the above corresponding formulas, stop.

Remark 3. Constructing the single warehouse mode and the two-warehouse mode mentioned above, it can be also found that if $t_0 = 0$, $T_1 = t_1$ and $H = h$, then $\Pi_{23}(T_1)$ reduces to the same as $\Pi_{13}(t_1)$, $\Pi_{24}(T_1)$ is equivalent to $\Pi_{12}(t_1)$, and $\Pi_{25}(T_1)$ is simplified to the same as $\Pi_{11}(t_1)$. In fact, the relationship between the above modes is very clear from the perspective of management. This is because, when the marginal holding cost in OW is equal to the unit external rental cost (i.e., $H = h$), there is no significant difference between the above two modes, which implies that, the two-warehouse mode can be considered as the single warehouse mode with an unlimited inventory storage space. However, when this marginal holding cost in RW is larger than that in OW (i.e., $H > h$), the comparisons between the above two modes become more complicated in operation performance. Therefore, it is critical for inventory managers to make the final warehouse mode selection after weighing the above two modes.

In the next, combining with Theorems (4.3), (4.4), (4.7), and (4.8), we propose the following theorem to guide whether to adopt the two-warehouse mode.

Theorem 4.9. *Let $\Pi(t_1^*) = \max\{\Pi_1(t_{1S}^*), \Pi_2(T_{1T}^*)\}$, and further let t_1^* be the optimal time point that maximizes the total average profit in the entire inventory system, we have*

(1) if $\Pi(t_1^*) = \Pi_1(t_{1S}^*)$, then the single warehouse mode is adopted and $t_1^* = t_{1S}^*$;

(2) if $\Pi(t_1^*) = \Pi_2(T_{1T}^*)$, then the two-warehouse mode is adopted and $t_1^* = T_{1T}^*$.

From the above discussions, a flowchart for searching the optimal behavior of $\Pi(t_1)$ in the inventory system is summarized as in the online appendix.

Special case. Notice that, if we ignore some retail parameters such as the selling price, the purchasing cost and fixed order cost, and a particular type of demand and a constant partial backlogging rate are considered, namely, when $p = 0, C = 0, A = 0$, $\mu < \gamma = T$ and $e^{-\delta\tau} \to B$, the above model can be simplified to the model in Agrawal & Banerjee (2011).

5. Numerical examples and sensitivity analysis

5.1. Numerical examples

In order to illustrate the solution procedure numerically, the following examples are presented. The algorithms are coded in Matlab R2013a for obtaining the optimal solution to the proposed model.

[Position of Table 3]

Example 5.1. Consider the following parameter values: $A = \$50$/order, $C = \$5$, $p = \$12$, $h = \$1$/unit/month, $H = \$1.5$/unit/month, $B = \$2$/unit/month, $L = \$3$/unit, $T = 2$ months, $\mu = 0.5$ months, $\gamma = 0.8$ months, $W = 50$ units, $\delta = 0.01$, the demand $a(t) = Me^{mt}$ and $b(t) = Ne^{-nt}$, where $M = 100$, $m = 0.1$ and $N = 200$.

Based on the above assumption, $d_0 = a(\mu) = b(\gamma)$, we have $d_0 = 100e^{0.05}$ and $n = (\ln 2 - 0.05)/0.8$. Under the single warehouse case, using **Algorithm 4.1**, we firstly obtain $\Pi_1(t_{1S}^*) = \Pi_{11}(t_1^{1*})$=\$438.9612 and $t_{1S}^* = t_1^{1*} = 0.4879$ months, respectively; using **Algorithm 4.2**, the computed results of the two-warehouse mode can be obtained in Table 3. From Table 3, the optimal profit in the two-warehouse mode is $\Pi_2(T_{1T}^*) = \Pi_{22}(T_1^{2*}) = \$487.1265 > \$438.9612$. Therefore, the two-warehouse mode is adopted in the inventory system, the maximum total average profit of $\Pi(t_1)$ in the inventory system is $\Pi(t_1^*) = \Pi_2(T_{1T}^*) = \$487.1265$, and the other optimal values are $t_1^* = T_{1T}^* = 1.2390$ months, $t_0^* = t_0^{2*} = 0.6943$ months, $Q^r = Q_2^r = 71.6982$ units and $Q^B = Q_{22}^B = 41.8677$ units.

Example 5.2. Consider $H = \$1.5$/unit/month, $W = 150$ units and $L = \$3$/unit, the other parameter values are the same as those of Example 5.1. Using the similar solution procedure, we also obtain $\Pi_1(t_{1S}^*)$=\$494.1105 $> \Pi_2(T_{1T}^*) = \$487.1265$. Thus, the single warehouse mode is adopted in the inventory system, and the other optimal values are $t_1^* = t_{1S}^* = 1.3521$ months, $Q^o = 129.6789$ units and $Q^B = 33.9430$ units.

In what follows, to get more comparisons, we will further discuss the effects of the unit holding cost in RW, OW's capacity and the unit opportunity cost on the warehouse mode selection and profitability of the inventory system.

Example 5.3. In this example, we use the same data as those of Example 5.1 except the three parameters: H, W and L , where the values of the parameters H, L and W are assumed to be $H \in \{1.5, 3, 4.5\}$, $W \in \{50, 100, 150\}$ and $L \in \{3, 4\}$. The computational results obtained are shown in Table 4.

[Position of Table 4]

From Table 4, some interesting findings are listed as follows.

1. As the unit rented cost H increases, the optimal average profit $\Pi(t_1^*)$ and the optimal Q^r decrease, but the optimal Q^B increases. It can be found that the larger the value of H, the more beneficial to the retailer the single warehouse mode will be. For example, when $W = 50$, $L = 3$ and $H \in \{1.5, 3, 4.5\}$, $\Pi(t_1^*)$ and Q^r will decrease, and especially for Q^r, it decreases from 71.6982 units to 0 unit. This shows that, if the unit rent in RW is larger in the retail context, it will send a strong signal for the retailer to cut down this cost. Facing this case, the retailer may reduce the rental risk by ordering less or even giving up this rented behavior, and raising appropriate shortages.

2. As the unit opportunity cost L increases, the optimal Q^r increases, but the optimal average profit $\Pi(t_1^*)$ and the total Q^B decrease. But when $L = 4$, $W = 150$ and $H = 4.5$, most of the total orders are stored in OW, and the quantity of shortages is only a small proportion. It implies that, when the opportunity cost is larger in the shortage period, excessive shortages will lead to more losses of profit on the sales. Moreover, when both the unit opportunity cost and the unit rented cost are larger in the external market, if OW has a full storage space, then only OW is adopted and the level of the shortages is also relatively low. But if OW only has a very smaller storage space (e.g., $L \in \{3, 4\}$, $W = 50$ and $H = 4.5$), then it can be observed that the optimal order decisions have no changes except for the loss of profit. In fact, once this situation occurs, it is very unfavorable to the retailer. Thus, it is recommended that retailer should seek to cooperate with rental companies by the contractual mechanisms, or raise the customers' brand loyalty of the products by some market ways such as advertisements and shopping experience, and so on.

3. As the OW's capacity W increases, the optimal average profit $\Pi(t_1^*)$ increases, but the optimal Q^r and the optimal Q^B will decrease. In particular, when OW has sufficient capacity, there will be no need for renting an extra warehouse. Interestingly, when the unit opportunity cost is lower, then even if OW has a full storage space, it is not completely occupied by the optimal Q^o, but more shortages Q^B are allowed in the inventory system (e.g., $W = 150$ and $L = 3$). This shows that, if the shortages are operated properly, it can also improve the inventory efficiency. Therefore, it is critical for the retailer to estimate the parameters such as W, H and L in the real world, accurately.

5.2. Sensitivity analysis and managerial insights

In this subsection, to assess the robustness of the model as well as to obtain managerial insights, we use the same parameter values as those in Example 5.1, and sensitivity analyses are performed by changing one parameter at a time and keeping the others unchanged. Under the two-warehouse environment, the computational results obtained are given in Table 5.

[Position of Table 5]

From Table 5, we can observe the following phenomena.

1. The optimal total average profit $\Pi(t_1^*)$ will increase as one of parameters p, M, μ and γ increases, while it will decrease as one of the parameters C, A, h, B, δ, N and T increases. This means that the increment in p, M, μ or γ will bring a positive effect on $\Pi(t_1^*)$, while the increment in C, A, h, B, δ, N or T will affect on $\Pi(t_1^*)$ negatively.

2. The optimal Q^r increases as one of the parameters $p, B, \delta, M, \mu, \gamma$ and T increases, it decreases as one of the parameters C, h and N increases, whereas it keeps unchanged as parameter A increases. It means that the increment in $p, B, \delta, M, \mu, \gamma$ or T affect on Q^r positively, and the increment in C, h or N will bring a negative effect on Q^r, while the increment in A has no any impact on Q^r.

3. The optimal total shortages Q^B increases as one of the parameters C, h, M, μ and γ increases, it decreases as one of the parameters p, B, δ, N and T increases, whereas it remains unchanged as parameter A increases. It means that the increment in C, h, M, μ or γ will bring a positive effect on Q^B, the increment in p, B, δ, N or T will bring a negative effect on Q^B, whereas the increment in A has no any impact on Q^B.

4. The optimal total average profit is more sensitive to parameters p, C and M than other parameters, the optimal rented order Q^r is more sensitive to parameters M and T than other parameters, and the optimal total shortages Q^B is more sensitive to parameter M than other parameters(see the bold numbers in Table 5). However, it can be concluded that the proposed inventory model is basically robust.[2]

From the sensitivity analyses, the following observations and management insights can be obtained.

The above sensitivity analyses demonstrate that increasing the unit holding cost of OW results in decreasing the total average profit of inventory system, but increasing the total shortages. Thus, the inventory manager should make order adjustment by adopting more shortages and less initial ordering quantity from the supplier. Moreover, as the unit holding cost of OW increases, the ordering quantity in RW also decreases. This finding implies that it is crucial for the inventory manager to choose a reliable partner who offers a relatively lower rent during the storage period.

Further analyses reveal that increasing the unit shortage cost, will lead to decreasing the total average profit of inventory system, but increasing the ordering quantity in RW. Therefore, the results suggest that the inventory manager should try to avoid more losses by adding more ordering quantity in RW. In addition, as the coefficient of the customer's waiting time increases, the total average profit of inventory system decreases, but the ordering quantity in RW increases. The result implies that, when the end customers are more sensitive to the waiting time during the storage period, more shortages will result in more losses of profit. Therefore, in order to avoid this situation, the inventory managers should focus on expanding the influence of products by improving service quality in the sales or after-sale stage, so as to increase the customers' brand loyalty of the products during the shortage period.

From the sensitivity analyses, it can be also observed that, as the demand factor parameter M increases in the product's expansion period, the total average profit of inventory system increases. Especially in this period, it is possible to raise the value of the parameter M by the market means such as advertisements investment, multi-channel selling, product promotion, and so on. Therefore, the inventory manager may take the relevant cost-benefit analysis as a decision-making reference to judge whether to increase the value of the parameter.

From the robustness of the model, considering the high sensitivity of total annual profit to the selling price p, the purchasing cost C, demand parameters M and the inventory cycle T, it is recommended that the inventory manager should pay particular attention to the values of those parameters when making the market demand forecasting.

In addition, from the application of the model, this study provides a novel idea for the inventory managers to capture inventory issues with the warehouse warehouse mode selection in the real world. And meanwhile, this study incorporates the trapezoidal-type demand into the inventory system, and the utilization of a generalized time-varying factor makes the application scope of the models broader. Using the model frame, it is convenient for the inventory manager to handle the problems with particular inventory circumstances such as linear demand, exponential demand, ramp-type demand, single warehouse, two-warehouse, no storage limit, no shortage, full backordering, and so on.

5.3. A real inventory case

With more than 4684 stores including 11 overseas stores, Heilan Home (HLA) is one of the well-known retail store chains in China's garment and textile industries. Since its establishment in 2002, HLA has been leading the Chinese fashion industry. As a matter of fact, HLA's booming development is attributed partly to its excellent management ability, powerful brand, shopping experience and inventory control.

In order to illustrate the applicability of the model in practice, an inventory case study was investigated at a local retail store of HLA in Tianjin. After coming in contact with HLA's store staffs and on-site visit, it is found that the main product type in their store is menswear and the demand features of the menswear fit well into the trapezoidal-type time distribution. Especially, most of the menswear products on sale in their store are seasonal, and as the new products enter the retail store, the demand for them increases with time initially when potential consumers are attracted by the style, then holds steady once this style is accepted, and finally decreases with time till the products retreat from the market. And meanwhile, we further knew from talking with them that most of the products in their store are nonperishable and have a relatively fixed sales cycle. Sometimes when the new type menswear is first introduced to the market, the store managers employ a RW for holding more items in anticipation of growth in customers' demand over time. When the products ordered are sold out, some loyal customers are willing to wait for products, but more impatient customers will go elsewhere if the wait time is longer. Therefore, the proposed model for nonperishable items with warehouse mode selection and partial backlogging under the trapezoidal-type demand is applicable in their store.

The following data on retail inventory operations were collected from HLA staffs' interviews. The HLA manager gave rough estimates for the customer demand data and sales data. The customer demand rate at time t is approximated by the following piecewise function

$$D(t) = \begin{cases} 130 + 7.5t, & 0 \le t \le 2, \\ 145, & 2 \le t \le 7.5, \\ 220 - 10t, & 7.5 \le t \le 15. \end{cases}$$

The backlogging rate with respect to the waiting time τ could be fitted by $\lambda(\tau) = e^{-0.01\tau}$, where $\tau > 0$. The store's maximum storage space W is 1300 units, the fixed sales cycle T is about 15 weeks, the fixed cost per order A is about 500 Chinese Yuan (CNY), and the holding cost in OW h is 3.5 CNY per unit item per unit time. The manager purchases the items from the upstream partners with $C = 30$ CNY per unit and sells to the end customer with $p = 90$ CNY per unit. If RW is adopted, the holding cost in RW H is 6 CNY per unit item per unit time. Moreover, from the accounting data, the backlogging cost and goodwill cost can be estimated roughly. The backlogging cost B is 20 CNY per unit item per unit time and the opportunity cost is $L = 25$ CNY per unit due to lost sales. Thus, facing this retail scenario, how does the manager make the optimal replenishment strategies to control retail inventory in HLA store?

Using the model and its solution procedures provided in this study, we have $\Pi_1(t_{1S}^*) = \Pi_{11}(t_1^{3*})$=3613.9918 and $\Pi_2(T_{1T}^*) = \Pi_{22}(T_1^{2*}) = 4701.9135 > 3613.9918$. Thus, it is a better choice for the HLA manager to adopt the two-warehouse mode, and the maximum average profit is 4701.9135. The other optimal results can be gained as follows: the optimal time point $t_1^* = T_1^{2*} = 12.5173$ weeks, the optimal total ordering quantity $Q = 1876.1051$ units, and the optimal ordering level of RW $Q^r = 374.1452$ units. Moreover, the corresponding inventory observations are easily obtained by using a similar approach in Section 5.2.

6. Conclusions and future research

Taking both a general trapezoidal-type demand and rented warehouse into consideration, this paper explores an inventory model for nonperishable items with warehouse mode selection in a fixed inventory cycle, in which the shortages are allowed and partial backlogging rate is assumed to be dependent on customer's waiting time. We prove the existence and uniqueness of the optimal inventory replenishment strategy for each of the warehouse modes, and present it in an easy-to-use algorithm. Furthermore, a theorem is used to make a tradeoff between the two modes from the perspective of maximizing the total average profit. Finally, numerical examples, sensitivity analysis and a case study are used to illustrate the robustness and applicability of the model.

Main conclusions of this paper include:

(1) Following our investigation, it is found that the optimal solutions of the model mainly depend on some key parameters such as the maximal limited capacity of OW, the trapezoidal-type demand time point, and so on. But if the marginal holding cost in OW is equal to the unit external rental cost, the optimal solutions of the model are independent of any time-varying demand type. This finding is consistent with Cheng & Wang (2009), Cheng et al. (2011), Lin (2013) and Wu et al. (2016), who investigated inventory models with an unlimited storage capacity under the single warehouse mode.

(2) Following the analysis, it is observed that as the storage space of OW increases or the unit rented cost increases, it will actually be economical for the retailers to keep more items in OW, and meanwhile, these two parameters have a marked impact on determining whether or not to use the rented warehouse. When the end customers are unwilling to wait backlogged order, more shortages would actually lead to more losses of profit. But increasing the ordering quantity in RW can reduce the losses due to lost sales. Moreover, when RW is adopted, the optimal total average profit is more sensitive to the unit selling price and the unit purchasing cost than other model parameters; the optimal rented order is more sensitive to the order cycle length than other model parameters, but overall, the proposed inventory model is basically robust. A case study also shows that the inventory managers are able to control inventory with the help of the model established in this study.

However, there are some limitations in this study. As mentioned in the previous section, when both the opportunity cost due to lost sales and the unit rented cost are larger in the external market and their own warehouse capacity is very small, the retailers may avoid this predicament by cooperating with third-party rental companies. This paper does not discuss the optimal inventory policy after third-party companies cooperation. Thus, in the future study, it is hoped to further incorporate third-party companies cooperation into this inventory issue under contractual mechanisms. In addition, some more realistic inventory features such as deteriorating items, variable inventory cycle, trade credit, quantity discount and environmental consideration, can also be incorporated in the proposed model.

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Notes

1. In order to simplify the model, the case that the time points t_0 and T_1 fall in the interval (μ, γ) at the same time is not considered, but by setting the relevant model parameters, the proposed model may reduce to the particular case. Thus, we here omit this case.
2. In order to investigate the impact of over or under estimation of the model parameters, we here adopt $(\Pi' - \Pi)/\Pi$, $(Q^{r'} - Q^r)/Q^r$ and $(Q^{B'} - Q^B)/Q^B$ as measures of parameter's sensitivity, where Π, Q^r and Q^B denote the true values, and Π', $Q^{r'}$ and $Q^{B'}$ denote estimated ones. As the bold numbers shown in Table 5, when p changes from -33% to 33%, the varying range of Π is from 67% to -67%; when c changes from -40% to 40%, the varying range of Π is from 34% to -34%; when M changes from -20% to 20%, the varying ranges of Π, Q^B and Q^r are from -25% to 28%, -34% to 35%, and -38% to 47%; when T changes from -11% to 11%, the varying range of Q^B is from -17% to 15%. Except for the previous parameters, the effects of other ones on the Π, Q^r and Q^B are not very significant.

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Online Appendix to accompany "An inventory model for nonperishable items with warehouse mode selection and partial backlogging under trapezoidal-type demand"

OA1 Single warehouse mode

1.1. Case with $\mu \le t_1 \le \gamma$

[Position of Figure 3]

In this case, as shown in Figure 3, the relationship $\mu \le t_1 \le \gamma$ can also be described in detail $0 < \mu \le t_1 \le \gamma < T$, the inventory in OW is depleted due to the effects of $a(t)$ and d_0, and hence, the changes of inventory level $I_o(t)$ in OW during the interval $[0, t_1]$ can be described by the following equations:

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -a(t), 0 \le t \le \mu \tag{39}$$

and

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -d_0, \mu \le t \le t_1. \tag{40}$$

Solving Eqs.(39) and (40) with boundary conditions $I_o(\mu^-) = I_o(\mu^+)$ and $I_o(t_1) = 0$, we have

$$I_o(t) = \int_t^{\mu} a(x)\mathrm{d}x + d_0(t_1 - \mu), 0 \le t \le \mu \tag{41}$$

and

$$I_o(t) = d_0(t_1 - t), \mu \le t \le t_1, \tag{42}$$

respectively.

In the interval $[t_1, T]$, the customer demand is partial backlogged due to shortages. The behavior of the inventory in $[t_1, T]$ can be described by the following differential equations:

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -e^{-\delta(T-t)}d_0, t_1 \le t \le \gamma \tag{43}$$

and

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -e^{-\delta(T-t)}b(t), \gamma \le t \le T. \tag{44}$$

Combining with the boundary conditions $I_o(t_1) = 0$ and $I_o(\gamma^-) = I_o(\gamma^+)$, the solutions of Eqs.(43) and (44) are

$$I_o(t) = \int_t^{t_1} e^{-\delta(T-x)}d_0\mathrm{d}x, t_1 \le t \le \gamma \tag{45}$$

and

$$I_o(t) = \int_t^{\gamma} e^{-\delta(T-x)} b(x)\mathrm{d}x - \int_{t_1}^{\gamma} e^{-\delta(T-x)} d_0 \mathrm{d}x, \gamma \le t \le T, \tag{46}$$

respectively.

From Eqs.(41) and (46), we gain the ordering quantity per cycle

$$\begin{aligned} Q_{12} &= Q_{12}^o + Q_{12}^B = I_o(0) - I_o(T) = \int_0^{\mu} a(x)\mathrm{d}x + d_0(t_1 - \mu) \\ &+ \int_{\gamma}^{T} e^{-\delta(T-x)} b(x)\mathrm{d}x + \int_{t_1}^{\gamma} e^{-\delta(T-x)} d_0 \mathrm{d}x. \end{aligned} \tag{47}$$

The total quantity of lost sales due to shortages as

$$S_{12} = \int_{t_1}^{\gamma} (1 - e^{-\delta(T-x)}) d_0 \mathrm{d}x + \int_{\gamma}^{T} (1 - e^{-\delta(T-x)}) b(x)\mathrm{d}x. \tag{48}$$

Therefore, the total profit per unit time under the condition $\mu \le t_1 \le \gamma$ is determined by

$$\begin{aligned} \Pi_{12}(t_1) &= \tfrac{1}{T}\{(p-C)Q_{12} - A - h\textstyle\int_0^{t_1} I_o(t)\mathrm{d}t - B\int_{t_1}^{T} -I_o(t)\mathrm{d}t - LS_{12}\} \\ &= \tfrac{1}{T}\{(p-C)[\textstyle\int_0^{\mu} a(x)\mathrm{d}x + d_0(t_1-\mu) + \int_{\gamma}^{T} e^{-\delta(T-x)} b(x)\mathrm{d}x + \int_{t_1}^{\gamma} e^{-\delta(T-x)} d_0 \mathrm{d}x] \\ &- A - h\textstyle\int_0^{\mu}[\int_t^{\mu} a(x)\mathrm{d}x + d_0(t_1-\mu)]\mathrm{d}t - h\int_{\mu}^{t_1}[d_0(t_1-t)]\mathrm{d}t \\ &+ B\textstyle\int_{t_1}^{\gamma}\int_t^{t_1} e^{-\delta(T-x)} d_0 \mathrm{d}x\mathrm{d}t + B\int_{\gamma}^{T}[\int_t^{\gamma} e^{-\delta(T-x)} b(x)\mathrm{d}x - \int_{t_1}^{\gamma} e^{-\delta(T-x)} d_0\mathrm{d}x]\mathrm{d}t \\ &- L[\textstyle\int_{t_1}^{\gamma}(1-e^{-\delta(T-x)})d_0\mathrm{d}x + \int_{\gamma}^{T}(1-e^{-\delta(T-x)})b(x)\mathrm{d}x]\}. \end{aligned} \tag{49}$$

[Position of Figure 4]

1.2. Case with $\gamma \le t_1 \le T$

Likewise, the condition $\gamma \le t_1 \le T$ can be rewritten in detail as $0 < \mu < \gamma \le t_1 \le T$. From Figure 4, the inventory in OW is depleted due to the effects of $a(t)$, d_0 and $b(t)$. Hence, the variation of the inventory level $I_o(t)$ with respect to time t in the interval $[0, t_1]$ can be described by

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -a(t), 0 \le t \le \mu, \tag{50}$$

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -d_0, \mu \le t \le \gamma, \tag{51}$$

and

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -b(t), \gamma \le t \le t_1, \tag{52}$$

respectively.

Using the boundary conditions $I_o(\mu^-) = I_o(\mu^+)$, $I_o(\gamma^-) = I_o(\gamma^+)$, and $I_o(t_1) = 0$, the solutions of Eqs.(50), (51) and (52) are given, respectively, by

$$I_o(t) = \int_t^{\mu} a(x)\mathrm{d}x + d_0(\gamma - \mu) + \int_{\gamma}^{t_1} b(x)\mathrm{d}x, 0 \le t \le \mu, \tag{53}$$

$$I_o(t) = d_0(\gamma - t) + \int_{\gamma}^{t_1} b(x)\mathrm{d}x, \mu \le t \le \gamma, \tag{54}$$

and

$$I_o(t) = \int_t^{t_1} b(x)\mathrm{d}x, \gamma \le t \le t_1. \tag{55}$$

In the shortage interval $[t_1, T]$, the customer demand is partial backlogged due to shortages. Thus, the behavior of the inventory in $[t_1, T]$ can be described by the following differential equation:

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -e^{-\delta(T-t)}b(t), t_1 \le t \le T. \tag{56}$$

Using the boundary condition $I_o(t_1) = 0$, the solution to Eq.(56) is given by

$$I_o(t) = \int_t^{t_1} e^{-\delta(T-x)}b(x)\mathrm{d}x, t_1 \le t \le T. \tag{57}$$

From Eqs.(53) and (57), we get the ordering quantity per cycle as

$$\begin{aligned} Q_{13} &= Q_{13}^o + Q_{13}^B = I_o(0) - I_o(T) = \int_0^{\mu} a(x)\mathrm{d}x + d_0(\gamma - \mu) \\ &+ \int_{\gamma}^{t_1} b(x)\mathrm{d}x + \int_{t_1}^{T} e^{-\delta(T-x)}b(x)\mathrm{d}x, \end{aligned} \tag{58}$$

and the total quantity of lost sales during the shortage period $[t_1, T]$ as

$$S_{13} = \int_{t_1}^{T} (1 - e^{-\delta(T-t)})b(t)\mathrm{d}t. \tag{59}$$

Therefore, the total profit per unit time under the condition $\gamma \le t_1 \le T$ is determined by

$$\begin{aligned} \Pi_{13}(t_1) &= \tfrac{1}{T}\{(p - C)Q_{13} - A - h\int_0^{t_1} I_o(t)\mathrm{d}t - B\int_{t_1}^{T} -I_o(t)\mathrm{d}t - LS_{13}\} \\ &= \tfrac{1}{T}\{(p - C)[\int_0^{\mu} a(x)\mathrm{d}x + d_0(\gamma - \mu) + \int_{\gamma}^{t_1} b(x)\mathrm{d}x + \int_{t_1}^{T} e^{-\delta(T-x)}b(x)\mathrm{d}x] - A \\ &- h\int_0^{\mu}[\int_t^{\mu} a(x)\mathrm{d}x + d_0(\gamma - \mu) + \int_{\gamma}^{t_1} b(x)\mathrm{d}x]\mathrm{d}t - h\int_{\mu}^{\gamma}[d_0(\gamma - t) + \int_{\gamma}^{t_1} b(x)\mathrm{d}x]\mathrm{d}t \\ &- h\int_{\gamma}^{t_1}\int_t^{t_1} b(x)\mathrm{d}x\mathrm{d}t + B\int_{t_1}^{T}\int_t^{t_1} e^{-\delta(T-x)}b(x)\mathrm{d}x\mathrm{d}t - L\int_{t_1}^{T}[1 - e^{-\delta(T-x)}]b(x)\mathrm{d}x\}. \end{aligned} \tag{60}$$

OA2 Two-warehouse mode

2.1. Case with $\mu \le t_0 \le \gamma \le T_1 \le T$

[Position of Figure 5]

Similarly, as shown in Figure 5, in RW, the inventory level $I_r(t)$ in $[0, t_0]$ with respect to time t can be described by

$$\frac{\mathrm{d}I_r(t)}{\mathrm{d}t} = -a(t), 0 \le t \le \mu, \tag{61}$$

and

$$\frac{\mathrm{d}I_r(t)}{\mathrm{d}t} = -d_0, \mu \le t \le t_0. \tag{62}$$

Using the boundary conditions $I_r(\mu^-) = I_r(\mu^+)$ and $I_r(t_0) = 0$, the solutions of Eqs.(61) and (62) can be given, respectively, by

$$I_r(t) = \int_t^{\mu} a(x)\mathrm{d}x + d_0(t_0 - \mu), 0 \le t \le \mu, \tag{63}$$

and

$$I_r(t) = d_0(t_0 - t), \mu \le t \le t_0. \tag{64}$$

In OW, the instantaneous change of $I_o(t)$ in $[t_0, T]$ with respect to time t can be described, respectively, by

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -d_0, t_0 \le t \le \gamma, \tag{65}$$

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -b(t), \gamma \le t \le T_1, \tag{66}$$

and

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -e^{-\delta(T-t)}b(t), T_1 \le t \le T. \tag{67}$$

Using the boundary conditions $I_o(\gamma^-) = I_o(\gamma^+)$ and $I_o(T_1) = 0$, the solutions of Eqs.(65), (66) and (67) can be given, respectively, by

$$I_o(t) = d_0(\gamma - t) + \int_{\gamma}^{T_1} b(x)\mathrm{d}x, t_0 \le t \le \gamma, \tag{68}$$

$$I_o(t) = \int_t^{T_1} b(x)\mathrm{d}x, \gamma \le t \le T_1, \tag{69}$$

and

$$I_o(t) = \int_t^{T_1} e^{-\delta(T-x)} b(x)\mathrm{d}x, T_1 \le t \le T. \tag{70}$$

From $I_o(t_0) = W$, we easily get $d_0(\gamma - t_0) + \int_\gamma^{T_1} b(x)\mathrm{d}x = W$. Similarly, t_0 can be simplified as

$$t_0 = \gamma_0 + \frac{1}{d_0}(\int_\gamma^{T_1} b(x)\mathrm{d}x - W) = t_0^2(T_1). \tag{71}$$

Then, taking the first derivative of $t_0^2(T_1)$, we have

$$\frac{\mathrm{d}t_0^2(T_1)}{\mathrm{d}T_1} = \frac{b(T_1)}{d_0} > 0. \tag{72}$$

From Eqs.(63) and (70), we get the ordering quantity per cycle as

$$Q_{22} = I_r(0) + W - I_o(T) = \int_0^\mu a(x)\mathrm{d}x + d_0(t_0 - \mu) + W + \int_{T_1}^T e^{-\delta(T-x)} b(x)\mathrm{d}x. \tag{73}$$

The total quantity of lost sales during the shortage period $[T_1, T]$ as

$$S_{22} = \int_{T_1}^T (1 - e^{-\delta(T-x)}) b(x)\mathrm{d}x. \tag{74}$$

Therefore, the total profit per unit time under the condition $\mu \le t_0 \le \gamma \le T_1 \le T$ is determined by

$$\begin{aligned}
\Pi_{22}(T_1) &= \tfrac{1}{T}\{(p-C)Q_{22} - A - H\textstyle\int_0^{t_0} I_r(t)\mathrm{d}t - h\int_0^{T_1} I_o(t)\mathrm{d}t - B\int_{T_1}^T -I_o(t)\mathrm{d}t - LS_{22}\} \\
&= \tfrac{1}{T}\{(p-C)[\textstyle\int_0^\mu a(x)\mathrm{d}x + d_0(t_0-\mu) + W + \int_{T_1}^T e^{-\delta(T-x)} b(x)\mathrm{d}x] \\
&- A - H[\textstyle\int_0^\mu \int_t^\mu a(x)\mathrm{d}x\mathrm{d}t + \int_0^\mu d_0(t_0-\mu)\mathrm{d}t + \int_\mu^{t_0} d_0(t_0 - t)\mathrm{d}t] \\
&- h[Wt_0 + \textstyle\int_{t_0}^\gamma d_0(\gamma - t)\mathrm{d}t + \int_{t_0}^\gamma \int_\gamma^{T_1} b(x)\mathrm{d}x\mathrm{d}t + \int_\gamma^{T_1}\int_t^{T_1} b(x)\mathrm{d}x\mathrm{d}t] \\
&+ B\textstyle\int_{T_1}^T \int_t^{T_1} e^{-\delta(T-x)} b(x)\mathrm{d}x\mathrm{d}t - L\int_{T_1}^T (1 - e^{-\delta(T-x)}) b(x)\mathrm{d}x\}.
\end{aligned} \tag{75}$$

2.2. Case with $t_0 \le \mu < \gamma \le T_1 \le T$

[Position of Figure 6]

Similar to the above cases, as shown in Figure 6, in RW, the inventory level $I_r(t)$ with respect to time t can be described by

$$\frac{\mathrm{d}I_r(t)}{\mathrm{d}t} = -a(t), 0 \le t \le t_0. \tag{76}$$

Using the boundary condition $I_r(t_0) = 0$, we have

$$I_r(t) = \int_t^{t_0} a(x)\mathrm{d}x, 0 \le t \le t_0. \tag{77}$$

In OW, the changes of $I_o(t)$ in $[t_0, T]$ with respect to time t can be given, respectively, by

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -a(t), t_0 \leq t \leq \mu, \tag{78}$$

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -d_0, \mu \leq t \leq \gamma, \tag{79}$$

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -b(t), \gamma \leq t \leq T_1, \tag{80}$$

and

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -e^{-\delta(T-t)}b(t), T_1 \leq t \leq T. \tag{81}$$

Using the boundary conditions $I_o(\mu^-) = I_o(\mu^+)$, $I_o(\gamma^-) = I_o(\gamma^+)$ and $I_o(T_1) = 0$, the solutions of Eqs.(78), (79), (80) and (81) are

$$I_o(t) = \int_t^{\mu} a(x)\mathrm{d}x + d_0(\gamma - \mu) + \int_{\gamma}^{T_1} b(x)\mathrm{d}x, t_0 \leq t \leq \mu, \tag{82}$$

$$I_o(t) = d_0(\gamma - t) + \int_{\gamma}^{T_1} b(x)\mathrm{d}x, \mu \leq t \leq \gamma, \tag{83}$$

$$I_o(t) = \int_t^{T_1} b(x)\mathrm{d}x, \gamma \leq t \leq T_1, \tag{84}$$

and

$$I_o(t) = \int_t^{T_1} e^{-\delta(T-x)}b(x)\mathrm{d}x, T_1 \leq t \leq T, \tag{85}$$

respectively.

From $I_o(t_0) = W$, we can easily get $\int_{t_0}^{\mu} a(x)\mathrm{d}x + d_0(\gamma - \mu) + \int_{\gamma}^{T_1} b(x)\mathrm{d}x = W$. Let $F(x) = \int a(x)\mathrm{d}x$. Then t_0 can also be simplified as the following relation:

$$t_0 = F^{-1}(F(\mu) - W + d_0(\gamma - \mu) + \int_{\gamma}^{T_1} b(x)\mathrm{d}x) = t_0^3(T_1). \tag{86}$$

Next, taking the first derivative of $t_0^3(T_1)$, we have

$$\frac{\mathrm{d}t_0^3(T_1)}{\mathrm{d}T_1} = \frac{b(T_1)}{a(t_0)} > 0. \tag{87}$$

From Eqs.(77) and (85), we get the ordering quantity per cycle as

$$Q_{23} = I_r(0) + W - I_o(T) = \int_0^{t_0} a(x)\mathrm{d}x + W + \int_{T_1}^{T} e^{-\delta(T-x)} b(x)\mathrm{d}x. \tag{88}$$

The total quantity of lost sales during the shortage period $[T_1, T]$ as

$$S_{23} = \int_{T_1}^{T} (1 - e^{-\delta(T-x)}) b(x)\mathrm{d}x. \tag{89}$$

Therefore, the total profit per unit time under the condition $t_0 \le \mu < \gamma \le T_1 \le T$ is determined by

$$\begin{aligned}
\Pi_{23}(T_1) &= \tfrac{1}{T}\{(p-C)Q_{23} - A - H\textstyle\int_0^{t_0} I_r(t)\mathrm{d}t - h\int_0^{T_1} I_o(t)\mathrm{d}t - B\int_{T_1}^{T} -I_o(t)\mathrm{d}t - LS_{23}\} \\
&= \tfrac{1}{T}\{(p-C)[\textstyle\int_0^{t_0} a(x)\mathrm{d}x + W + \int_{T_1}^{T} e^{-\delta(T-x)} b(x)\mathrm{d}x] - A \\
&- H\textstyle\int_0^{t_0}\int_t^{t_0} a(x)\mathrm{d}x\mathrm{d}t - h[Wt_0 + \int_{t_0}^{\mu}\int_t^{\mu} a(x)\mathrm{d}x\mathrm{d}t + \int_{t_0}^{\mu}\int_{\mu}^{\gamma} d_0\mathrm{d}x\mathrm{d}t] \\
&- h[\textstyle\int_{t_0}^{\mu}\int_{\gamma}^{T_1} b(x)\mathrm{d}x\mathrm{d}t + \int_{\mu}^{\gamma}\int_t^{\gamma} d_0\mathrm{d}x\mathrm{d}t + \int_{\mu}^{\gamma}\int_{\gamma}^{T_1} b(x)\mathrm{d}x\mathrm{d}t + \int_{\gamma}^{T_1}\int_t^{T_1} b(x)\mathrm{d}x\mathrm{d}t] \\
&+ B\textstyle\int_{T_1}^{T}\int_t^{T_1} e^{-\delta(T-x)} b(x)\mathrm{d}x\mathrm{d}t - L\int_{T_1}^{T}(1 - e^{-\delta(T-x)}) b(x)\mathrm{d}x\}.
\end{aligned} \tag{90}$$

2.3. Case with $t_0 \le \mu \le T_1 \le \gamma \le T$

[Position of Figure 7]

Likewise, as shown in Figure 7, in RW, the inventory level $I_r(t)$ in $[0, t_0]$ with respect to time t can be described by

$$\frac{\mathrm{d}I_r(t)}{\mathrm{d}t} = -a(t), 0 \le t \le t_0. \tag{91}$$

Using the boundary condition $I_r(t_0) = 0$, we have

$$I_r(t) = \int_t^{t_0} a(x)\mathrm{d}x, 0 \le t \le t_0. \tag{92}$$

In OW, the changes of $I_o(t)$ in $[t_0, T]$ with respect to time t can be described by

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -a(t), t_0 \le t \le \mu, \tag{93}$$

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -d_0, \mu \le t \le T_1, \tag{94}$$

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -e^{-\delta(T-t)} d_0, T_1 \le t \le \gamma, \tag{95}$$

and

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -e^{-\delta(T-t)}b(t), \gamma \le t \le T, \tag{96}$$

respectively. Using the boundary conditions $I_o(\mu^-) = I_o(\mu^+)$, $I_o(T_1) = 0$ and $I_o(\gamma^-) = I_o(\gamma^+)$, the solutions of Eqs.(93), (94), (95) and (96) can be given, respectively, by

$$I_o(t) = \int_t^{\mu} a(x)\mathrm{d}x + d_0(T_1 - \mu), t_0 \le t \le \mu, \tag{97}$$

$$I_o(t) = d_0(T_1 - t), \mu \le t \le T_1, \tag{98}$$

$$I_o(t) = \int_t^{T_1} e^{-\delta(T-x)} d_0 \mathrm{d}x, T_1 \le t \le \gamma, \tag{99}$$

and

$$I_o(t) = \int_t^{\gamma} e^{-\delta(T-x)} b(x)\mathrm{d}x - \int_{T_1}^{\gamma} e^{-\delta(T-x)} d_0 \mathrm{d}x, \gamma \le t \le T. \tag{100}$$

From $I_o(t_0) = W$, we can easily get $\int_{t_0}^{\mu} a(x)\mathrm{d}x + d_0(T_1 - \mu) = W$. Similarly, t_0 can also be simplified as the following relation:

$$t_0 = F^{-1}(F(\mu) - W + d_0(T_1 - \mu)) = t_0^4(T_1), \tag{101}$$

and furthermore, taking the first derivative of $t_0^4(T_1)$, we have

$$\frac{\mathrm{d}t_0^4(T_1)}{\mathrm{d}T_1} = \frac{d_0}{a(t_0)} > 0. \tag{102}$$

From Eqs.(92) and (100), we get the ordering quantity per cycle as

$$\begin{aligned} Q_{24} &= I_r(0) + W - I_o(T) = \int_0^{t_0} a(x)\mathrm{d}x + W + \int_{\gamma}^{T} e^{-\delta(T-x)} b(x)\mathrm{d}x \\ &\quad + \int_{T_1}^{\gamma} e^{-\delta(T-x)} d_0 \mathrm{d}x, \end{aligned} \tag{103}$$

and the total quantity of lost sales during the shortage period $[T_1, T]$ as

$$S_{24} = \int_{T_1}^{\gamma} (1 - e^{-\delta(T-x)}) d_0 \mathrm{d}x + \int_{\gamma}^{T} (1 - e^{-\delta(T-x)}) b(x)\mathrm{d}x. \tag{104}$$

Therefore, the total profit per unit time under the condition $t_0 \le \mu \le T_1 \le \gamma \le T$ is determined by

$$
\begin{aligned}
\Pi_{24}(T_1) &= \tfrac{1}{T}\{(p-C)Q_{24} - A - H\int_0^{t_0} I_r(t)\mathrm{d}t - h\int_0^{T_1} I_o(t)\mathrm{d}t - B\int_{T_1}^{T} -I_o(t)\mathrm{d}t - LS_{24}\} \\
&= \tfrac{1}{T}\{(p-C)[\int_0^{t_0} a(x)\mathrm{d}x + W + \int_\gamma^T e^{-\delta(T-x)}b(x)\mathrm{d}x + \int_{T_1}^{\gamma} e^{-\delta(T-x)}d_0\mathrm{d}x] - A \\
&- H\int_0^{t_0}\int_t^{t_0} a(x)\mathrm{d}x\mathrm{d}t - h[Wt_0 + \int_{t_0}^{\mu}\int_t^{\mu} a(x)\mathrm{d}x\mathrm{d}t + \int_{t_0}^{\mu}\int_{\mu}^{T_1} d_0\mathrm{d}x\mathrm{d}t] \\
&- h\int_{\mu}^{T_1}\int_t^{T_1} d_0\mathrm{d}x\mathrm{d}t + B[\int_{T_1}^{\gamma}\int_t^{t_1} e^{-\delta(T-x)}d_0\mathrm{d}x\mathrm{d}t + \int_\gamma^T\int_t^{\gamma} e^{-\delta(T-x)}b(x)\mathrm{d}x\mathrm{d}t] \\
&- B\int_\gamma^T\int_{T_1}^{\gamma} e^{-\delta(T-x)}d_0\mathrm{d}x\mathrm{d}t - L[\int_{T_1}^{\gamma}(1-e^{-\delta(T-x)})d_0\mathrm{d}x + \int_\gamma^T(1-e^{-\delta(T-x)})b(x)\mathrm{d}x]\}.
\end{aligned} \tag{105}
$$

[Position of Figure 8]

2.4. Case with $t_0 < T_1 \le \mu < \gamma \le T$

As shown in Figure 8, in RW, the inventory level $I_r(t)$ in $[0, t_0]$ with respect to time t can be described by

$$\frac{\mathrm{d}I_r(t)}{\mathrm{d}t} = -a(t), 0 \le t \le t_0. \tag{106}$$

Using the boundary condition $I_r(t_0) = 0$, we have

$$I_r(t) = \int_t^{t_0} a(x)\mathrm{d}x, 0 \le t \le t_0. \tag{107}$$

In OW, the changes of $I_o(t)$ in $[t_0, T]$ with respect to time t can be given, respectively, by

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -a(t), t_0 \le t \le T_1, \tag{108}$$

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -e^{-\delta(T-t)}a(t), T_1 \le t \le \mu, \tag{109}$$

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -e^{-\delta(T-t)}d_0, \mu \le t \le \gamma, \tag{110}$$

and

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -e^{-\delta(T-t)}b(t), \gamma \le t \le T. \tag{111}$$

Using the boundary conditions $I_o(T_1) = 0$, $I_o(\mu^-) = I_o(\mu^+)$ and $I_o(\gamma^-) = I_o(\gamma^+)$, the solutions of Eqs.(108), (109), (110) and (111) are

$$I_o(t) = \int_t^{T_1} a(x)\mathrm{d}x, t_0 \le t \le T_1, \tag{112}$$

$$I_o(t) = \int_t^{T_1} e^{-\delta(T-x)}a(x)\mathrm{d}x, T_1 \le t \le \mu, \tag{113}$$

$$I_o(t) = \int_t^{\mu} e^{-\delta(T-x)} d_0 \mathrm{d}x - \int_{T_1}^{\mu} e^{-\delta(T-x)} a(x)\mathrm{d}x, \mu \le t \le \gamma, \tag{114}$$

and

$$I_o(t) = \int_t^{\gamma} e^{-\delta(T-x)} b(x)\mathrm{d}x - \int_{\mu}^{\gamma} e^{-\delta(T-x)} d_0 \mathrm{d}x - \int_{T_1}^{\mu} e^{-\delta(T-x)} a(x)\mathrm{d}x, \gamma \le t \le T, \tag{115}$$

respectively.

From $I_o(t_0) = W$, we easily get $\int_{t_0}^{T_1} a(x)\mathrm{d}x = W$. Similarly, t_0 can also be simplified as the following relation:

$$t_0 = F^{-1}(F(T_1) - W) = t_0^5(T_1). \tag{116}$$

Then, taking the first derivative of $t_0^5(T_1)$, we have

$$\frac{\mathrm{d}t_0^5(T_1)}{\mathrm{d}T_1} = \frac{a(T_1)}{a(t_0)} > 0. \tag{117}$$

From Eqs.(107) and (115), we can get the ordering quantity per cycle as

$$\begin{aligned} Q_{25} &= I_o(0) + W - I_o(T) = \int_0^{t_0} a(x)\mathrm{d}x + W + \int_{\gamma}^{T} e^{-\delta(T-x)} b(x)\mathrm{d}x \\ & \int_{\mu}^{\gamma} e^{-\delta(T-x)} d_0 \mathrm{d}x + \int_{T_1}^{\mu} e^{-\delta(T-x)} a(x)\mathrm{d}x, \end{aligned} \tag{118}$$

and the total quantity of lost sales during the shortage period $[t_1, T]$ as

$$S_{25} = \int_{T_1}^{\mu} (1 - e^{-\delta(T-x)}) a(x)\mathrm{d}x + \int_{\mu}^{\gamma} (1 - e^{-\delta(T-x)}) d_0 \mathrm{d}x + \int_{\gamma}^{T} (1 - e^{-\delta(T-x)}) b(x)\mathrm{d}x. \tag{119}$$

Therefore, the total profit per unit time under the condition $t_0 < T_1 \le \mu < \gamma \le T$ is determined by

$$\begin{aligned} \Pi_{25}(T_1) &= \tfrac{1}{T}\{(p-C)Q_{25} - A - H\int_0^{t_0} I_r(t)\mathrm{d}t - h\int_0^{T_1} I_o(t)\mathrm{d}t - B\int_{T_1}^{T} -I_o(t)\mathrm{d}t - LS_{25}\} \\ &= \tfrac{1}{T}\{(p-C)[\int_0^{t_0} a(x)\mathrm{d}x + W + \int_{\gamma}^{T} e^{-\delta(T-x)} b(x)\mathrm{d}x + \int_{\mu}^{\gamma} e^{-\delta(T-x)} d_0 \mathrm{d}x] \\ &+ (p-C)[\int_{T_1}^{\mu} e^{-\delta(T-x)} a(x)\mathrm{d}x] - A - H[\int_0^{t_0}\int_t^{t_0} a(x)dxdt] \\ &- h[Wt_0 + \int_{t_0}^{T_1}\int_t^{T_1} a(x)\mathrm{d}x\mathrm{d}t] + B[\int_{T_1}^{\mu}\int_t^{T_1} e^{-\delta(T-x)} a(x)\mathrm{d}x\mathrm{d}t] \\ &+ B[\int_{\mu}^{\gamma}\int_t^{\mu} e^{-\delta(T-x)} d_0 \mathrm{d}x\mathrm{d}t - \int_{\mu}^{\gamma}\int_{T_1}^{\mu} e^{-\delta(T-x)} a(x)\mathrm{d}x\mathrm{d}t] \\ &+ B[\int_{\gamma}^{T}\int_t^{\gamma} e^{-\delta(T-x)} b(x)\mathrm{d}x\mathrm{d}t - \int_{\gamma}^{T}\int_{\mu}^{\gamma} e^{-\delta(T-x)} d_0 \mathrm{d}x\mathrm{d}t] \\ &- B[\int_{\gamma}^{T}\int_{T_1}^{\mu} e^{-\delta(T-x)} a(x)\mathrm{d}x\mathrm{d}t] - L[\int_{T_1}^{\mu}(1 - e^{-\delta(T-x)}) a(x)\mathrm{d}x] \\ &- L[\int_{\mu}^{\gamma}(1 - e^{-\delta(T-x)}) d_0 \mathrm{d}x + \int_{\gamma}^{T}(1 - e^{-\delta(T-x)}) b(x)\mathrm{d}x]\}. \end{aligned} \tag{120}$$

OA3 The proofs of theorem and proposition

3.1. Single warehouse mode

3.1.1. The proof of Theorem 4.1

Proof. For case i, $i = 1, 2, 3$, taking the first-order differential of $\Pi_{1i}(t_1)$ with respect to t_1, we have

$$\frac{d\Pi_{11}(t_1)}{dt_1} = \frac{1}{T}a(t_1)z(t_1), \tag{121}$$

$$\frac{d\Pi_{12}(t_1)}{dt_1} = \frac{1}{T}d_0 z(t_1), \tag{122}$$

and

$$\frac{d\Pi_{13}(t_1)}{dt_1} = \frac{1}{T}b(t_1)z(t_1), \tag{123}$$

where $z(t_1) = (p - C + L)(1 - e^{-\delta(T-t_1)}) - ht_1 + Be^{-\delta(T-t_1)}(T - t_1)$.

The necessary condition for $\Pi_{1i}(t_1)$ to be maximized is $\frac{d\Pi_{1i}(t_1)}{dt_1} = 0$. Since $a(t)$, d_0, and $b(t)$ are positive for any t, it implies from Eqs.(121)-(123) that the necessary condition $\frac{d\Pi_{1i}(t_1)}{dt_1} = 0$ can be satisfied if $z(t_1) = 0$, $i = 1, 2, 3$. □

3.1.2. The proof of Proposition 4.2

Proof. Taking the first derivative of $z(t_1)$ with respect to t_1, we attain that $z'(t_1) = -\delta(p - C + L)e^{\delta(t_1-T)} - h - B(1 - \delta(T - t_1))e^{\delta(t_1-T)}$. From the assumption before, we easily get $(p - C + L) > 0$ and $1 - \delta(T - t_1) > 0$. Thus, we have $z'(t_1) < 0$, which implies that $z(t_1)$ is a strictly decreasing function with respect to t_1. □

3.1.3. The proof of Theorem 4.3

Proof. From the demand assumption, we have $d_0 = a(\mu) = b(\gamma)$. Furthermore, Eqs.(121)-(123) imply that

$$\frac{d\Pi_{11}(t_1)}{dt_1}|_{t_1\to\mu^-} = \frac{d\Pi_{12}(t_1)}{dt_1}|_{t_1\to\mu^+}$$

and

$$\frac{d\Pi_{12}(t_1)}{dt_1}|_{t_1\to\gamma^-} = \frac{d\Pi_{13}(t_1)}{dt_1}|_{t_1\to\gamma^+}.$$

Thus, $\Pi_1(t_1)$ is differentiable on $[0, T]$. For case i, $i = 1, 2, 3$, combining the functional behaviors of $\Pi_1(t_1)$ with Eqs.(121)-(123), the first-order derivative of $\Pi_{1i}(t_1)$ with respect to t_1 can also be summarized as follow:

$$\frac{d\Pi_{1i}(t_1)}{dt_1} = \frac{1}{T}D(t_1)z(t_1), t_1 \in [t_1^{lb_i}, t_1^{ub_i}]. \tag{124}$$

Since $D(t_1)$ is positive in $[0, T]$, and from Proposition 4.2, $z(t_1)$ is a strictly decreasing function with respect to t_1, thus if $z(t_1^{lb_i}) > 0$ and $z(t_1^{ub_i}) < 0$, then the Intermediate Value Theorem implies that there exists a unique value $t_1^{i*} \in (t_1^{lb_i}, t_1^{ub_i})$ satisfying $z(t_1) = 0$. Furthermore, taking the second derivative of $\Pi_{1i}(t_1)$ with respect to t_1, we have $\frac{d^2\Pi_{1i}(t_1)}{dt_1^2}|_{t_1=t_1^{i*}} = \frac{1}{T}z'(t_1^{i*})D(t_1^{i*}) < 0$. Thus, t_1^{i*} is the unique maximum solution of $\Pi_{1i}(t_1)$.

From Eq.(124), for any $t_1 \in [t_1^{lb_i}, t_1^{ub_i}]$, where $i = 1, 2, 3$, if $z(t_1^{ub_i}) > 0$, then $\frac{\mathrm{d}\Pi_{1i}(t_1)}{\mathrm{d}t_1} = \frac{1}{T}z(t_1)D(t_1) > \frac{1}{T}z(t_1^{ub_i})D(t_1) > 0$, which indicates that $\Pi_{1i}(t_1)$ is strictly increasing on $[t_1^{lb_i}, t_1^{ub_i}]$. Thus, $t_1^{i*} = t_1^{ub_i}$ is the optimal solution of $\Pi_{2i}(t_1)$.

Similarly, for any $t_1 \in [t_1^{lb_i}, t_1^{ub_i}]$, where $i = 1, 2, 3$, if $z(t_1^{lb_i}) < 0$, then $\frac{\mathrm{d}\Pi_{1i}(t_1)}{\mathrm{d}t_1} = \frac{1}{T}z(t_1)D(t_1) < \frac{1}{T}z(t_1^{lb_i})D(t_1) < 0$, which implies that $\Pi_{1i}(t_1)$ is strictly decreasing on $[t_1^{lb_i}, t_1^{ub_i}]$. Thus, $t_1^{i*} = t_i^{lb_i}$ is the optimal solution of $\Pi_{1i}(t_1)$. □

3.2. Two-warehouse mode

3.2.1. The proof of Theorem 4.5

Proof. Combing with Eqs. (30), (72), (87), (102) and (117), for case j, $j = 1, 2, \ldots, 5$, the first-order differential of $\Pi_{2j}(T_1)$ with respect to T_1 can be written as follows:

$$\frac{d\Pi_{21}(T_1)}{dT_1} = \frac{\partial\Pi_{21}(T_1)}{\partial T_1} + \frac{\partial\Pi_{21}(T_1)}{\partial t_0}\frac{dt_0^1(T_1)}{dT_1} = \frac{1}{T}Z_1(T_1)b(T_1) = 0. \tag{125}$$

$$\frac{\mathrm{d}\Pi_{22}(T_1)}{\mathrm{d}T_1} = \frac{\partial\Pi_{22}(T_1)}{\partial T_1} + \frac{\partial\Pi_{22}(T_1)}{\partial t_0}\frac{dt_0^2(T_1)}{dT_1} = \frac{1}{T}Z_2(T_1)b(T_1) = 0, \tag{126}$$

$$\frac{\mathrm{d}\Pi_{23}(T_1)}{\mathrm{d}T_1} = \frac{\partial\Pi_{23}(T_1)}{\partial T_1} + \frac{\partial\Pi_{23}(T_1)}{\partial t_0}\frac{dt_0^3(T_1)}{dT_1} = \frac{1}{T}Z_3(T_1)b(T_1) = 0, \tag{127}$$

$$\frac{\mathrm{d}\Pi_{24}(T_1)}{\mathrm{d}T_1} = \frac{\partial\Pi_{24}(T_1)}{\partial T_1} + \frac{\partial\Pi_{24}(T_1)}{\partial t_0}\frac{dt_0^4(T_1)}{dT_1} = \frac{1}{T}Z_4(T_1)d_0 = 0, \tag{128}$$

and

$$\frac{\mathrm{d}\Pi_{25}(t_1)}{\mathrm{d}T_1} = \frac{\partial\Pi_{25}(T_1)}{\partial T_1} + \frac{\partial\Pi_{25}(T_1)}{\partial t_0}\frac{dt_0^5(T_1)}{dT_1} = \frac{1}{T}Z_5(T_1)a(T_1) = 0. \tag{129}$$

Since $a(T_1)$, d_0, and $b(T_1)$ are positive, for case j, it implies from Eqs.(125)-(129) that the necessary condition $\frac{\mathrm{d}\Pi_{2j}(T_1)}{\mathrm{d}T_1} = 0$ can be satisfied if $Z_j(T_1) = 0$, where $j = 1, 2, \ldots, 5$. □

3.2.2. The proof of Proposition 4.6

Proof. For case j, $j = 1, 2, \ldots, 5$, taking the first derivative of $Z_j(T_1)$ with respect to T_1, we get $Z_j'(T_1) = -\delta(p-C+L)e^{\delta(T_1-T)} - h - (H-h)\frac{dt_0^j(T_1)}{dT_1} - B(1-\delta(T-T_1))e^{\delta(T_1-T)}$. From Eqs. (30), (72), (87), (102) and (117), we have $\frac{\mathrm{d}t_0^j(T_1)}{\mathrm{d}T_1} > 0$, and according to the assumption before, we have $p - C + L > 0, H > h$, and $1 - \delta(T - T_1) > 0$. Thus, we gain $Z_j'(T_1) < 0$, which implies that $Z_j(T_1)$ is a strictly decreasing function with respect to T_1. □

3.2.3. The proof of Theorem 4.7

Proof. Similarly, for case j, $j = 1, 2, \ldots, 5$, since $D(t_1)$ is positive in $[0, T]$, and from Proposition 4.6, $Z_j(T_1)$ is a strictly decreasing function with respect to T_1. Thus, if $Z_j(T_1^{Lb_j}) > 0$ and $Z_j(T_1^{Ub_j}) < 0$, then the Intermediate Value Theorem implies that there exists a unique value $T_1^{j*} \in (T_1^{Lb_j}, T_1^{Ub_j})$ satisfying $Z_j(T_1) = 0$. Furthermore, taking the second derivative of $\Pi_{2j}(T_1)$ with respect to T_1, we have $\frac{d^2\Pi_{2j}(T_1)}{dT_1^2}|_{T_1 = T_1^{j*}} = \frac{1}{T} Z_j'(T_1^{j*}) D(T_1^{j*}) < 0$. Thus, T_1^{j*} is the unique maximum solution of $\Pi_{2j}(T_1)$.

From Eqs.(125)-(129), the first derivative of $\Pi_{2j}(T_1)$ with respect to T_1 can also be formulated as $\frac{\mathrm{d}\Pi_{2j}(T_1)}{\mathrm{d}T_1} = \frac{1}{T} Z_j(T_1) D(T_1)$, so for any $T_1 \in [T_1^{Lb_j}, T_1^{Ub_j}]$, if $Z_j(T_1^{Ub_j}) > 0$, then $\frac{\mathrm{d}\Pi_{2j}(T_1)}{\mathrm{d}t_1} = \frac{1}{T} Z_j(T_1) D(T_1) > \frac{1}{T} Z_j(T_1^{Ub_j}) D(T_1) > 0$, which indicates that $\Pi_{2j}(T_1)$ is strictly increasing on $[T_1^{Lb_j}, T_1^{Ub_j}]$. Thus, $T_1^{j*} = T_1^{Ub_j}$ is the optimal solution of $\Pi_{2j}(T_1)$, where $j = 1, 2, \ldots, 5$.

For any $T_1 \in [T_1^{Lb_j}, T_1^{Ub_j}]$, if $Z_j(T_1^{Lb_j}) < 0$, then $\frac{\mathrm{d}\Pi_{2j}(T_1)}{\mathrm{d}T_1} = \frac{1}{T} Z_j(T_1) D(T_1) < \frac{1}{T} Z_j(T_1^{Lb_j}) D(T_1) < 0$, which implies that $\Pi_{2j}(T_1)$ is strictly decreasing on $[T_1^{Lb_j}, T_1^{Ub_j}]$. Thus, $T_1^{j*} = T_1^{Lb_j}$ is the optimal solution of $\Pi_{2j}(T_1)$, where $j = 1, 2, \ldots, 5$. □

OA4 The flowchart for the solution algorithm in the inventory system

[Position of Figure 9]

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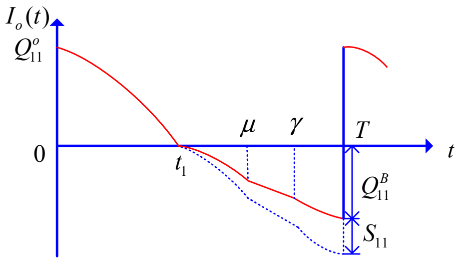

Figure 1. The inventory behavior graph with time in case with $0 \leq t_1 \leq \mu$

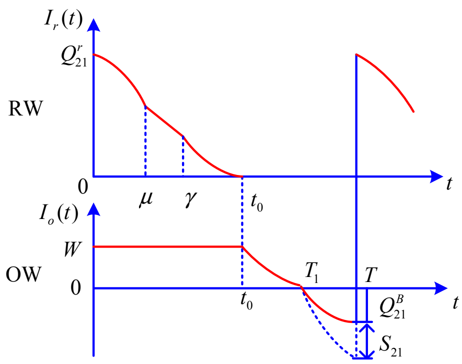

Figure 2. The inventory behavior graph with time in case with $\mu < \gamma \leq t_0 < T_1 \leq T$

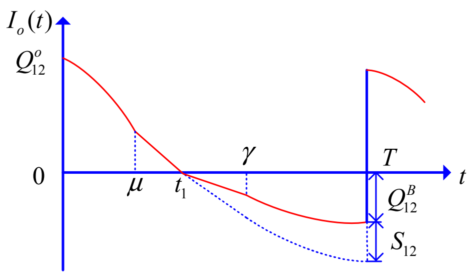

Figure 3. The inventory behavior graph with time in case with $\mu \leq t_1 \leq \gamma$

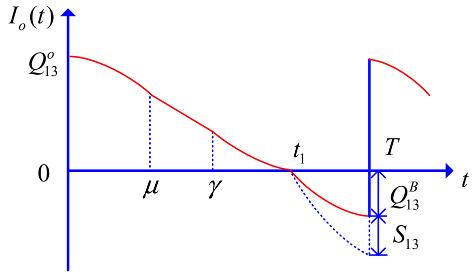

Figure 4. The inventory behavior graph with time in case with $\gamma \le t_1 \le T$

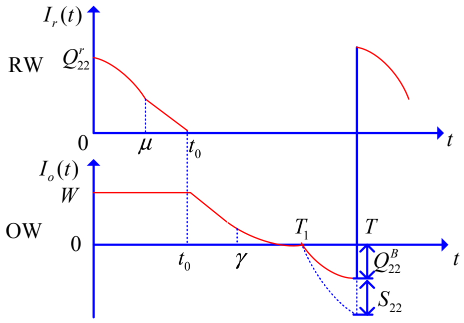

Figure 5. The inventory behavior graph with time in case with $\mu \le t_0 \le \gamma \le T_1 \le T$

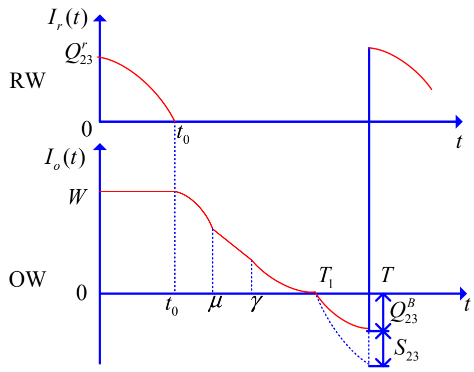

Figure 6. The inventory behavior graph with time in case with $t_0 \le \mu < \gamma \le T_1 \le T$

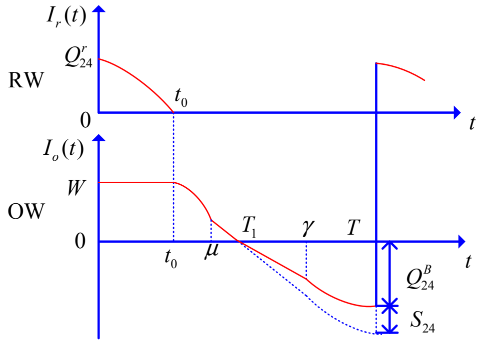

Figure 7. The inventory behavior graph with time in case with $t_0 \leq \mu \leq T_1 \leq \gamma \leq T$

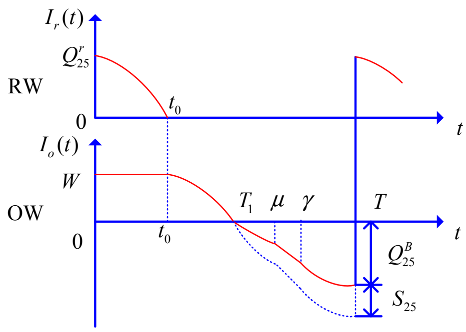

Figure 8. The inventory behavior graph with time in case with $t_0 < T_1 \leq \mu < \gamma \leq T$

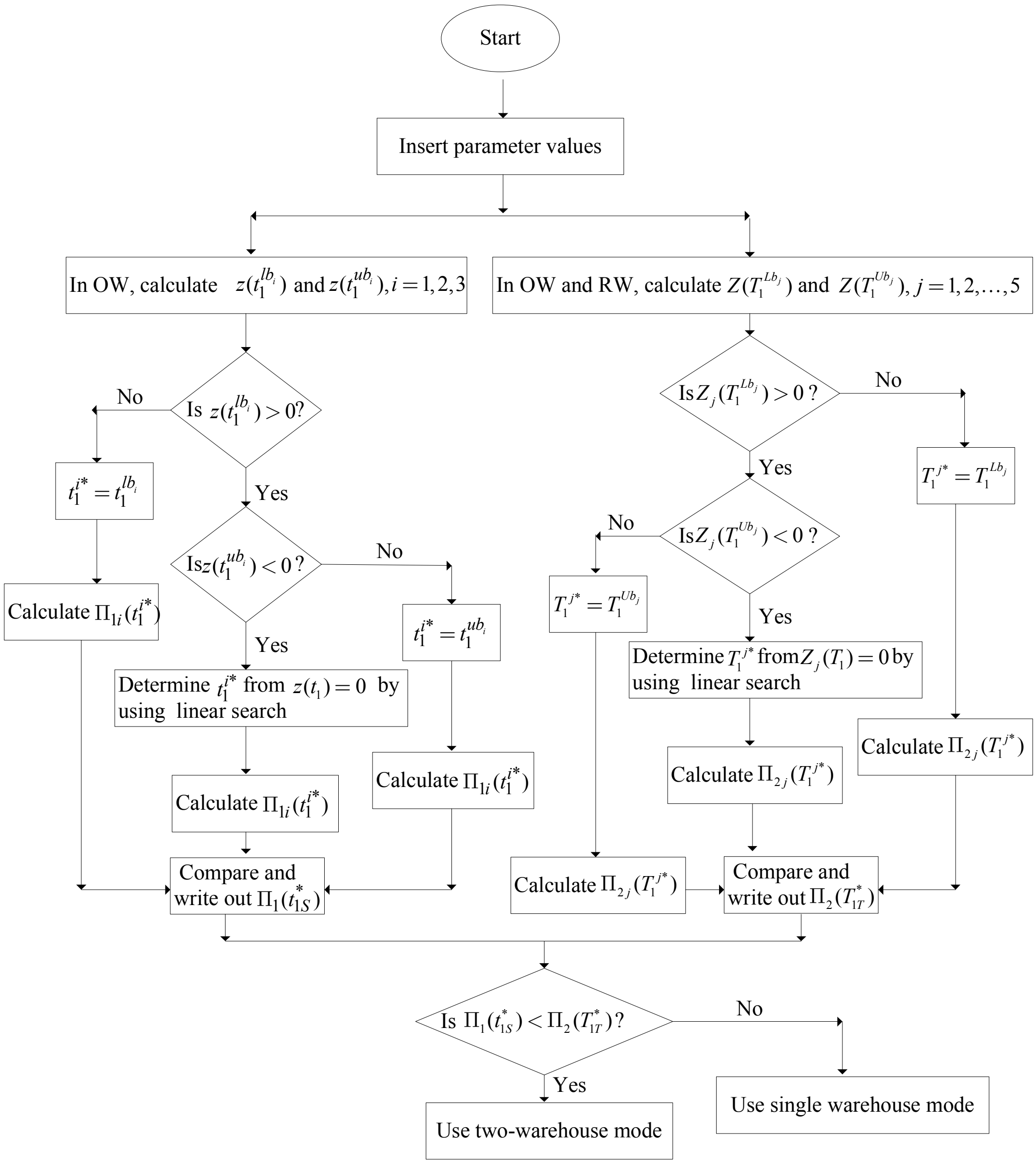

Figure 9. The flowchart for the solution algorithm in the inventory system

Table 1. Summary of related assumptions in previous studies.

Author(s) published(year)	Demand type	Two or single warehouse	Warehouse mode selection	Shortages	Partial backlogging rate
Bhunia & Maiti (1998)	linear	Both	Yes	Yes	No
Datta & Pal (1988)	Power	Single warhouse	No	Yes	No
Yang (2004)	Constant	Two-warehouse	No	Yes	No
Cheng & Wang (2009)	Trapezoidal	Single warhouse	No	Yes	No
Cheng et al. (2011)	Trapezoidal	Single warhouse	No	Yes	Time-varying
Skouri et al. (2009)	Ramp	Single warhouse	No	Yes	Time-varying
Uthayakumar & Rameswari (2012)	Trapezoidal	Single warhouse	No	No	No
N. Singh, et al. (2010)	Trapezoidal	Single warhouse	No	Yes	No
Lin (2013)	Trapezoidal	Single warhouse	No	Yes	Time-varying
Agrawal & Banerjee (2011)	Ramp	Both	Yes	Yes	Constant
N. Shah et al. (2015)	Trapezoidal	Single warhouse	No	No	No
Sarkar et al. (2012)	Quadratic	Single warhouse	No	Yes	Time-varying
Wu et al. (2016)	Trapezoidal	Single warhouse	No	Yes	Time-varying
Taleizadeh et al. (2016)	Constant	Single warhouse	No	Yes	Constant
Chakraborty et al. (2018)	Ramp	Two-warehouse	No	Yes	Time-varying
San-José et al. (2018)	Power	Single warhouse	No	Yes	No
Present	Trapezoidal	Both	Yes	Yes	Time-varying

Table 2. The values of $T_1^{Lb_j}$ and $T_1^{Ub_j}$ for each case j.

Case j	$T_1^{Lb_j}$	$T_1^{Ub_j}$
1	$G^{-1}(G(\gamma)+W)^{(i)}$	T
2	$\max\{\gamma, G^{-1}(G(\gamma)+W-d_0(\gamma-\mu))\}$	$\min\{G^{-1}(G(\gamma)+W), T\}$
3	$\max\{\gamma, G^{-1}(G(\gamma)+W-\int_0^{\mu} a(x)dx-d_0(\gamma-\mu))\}$	$\min\{G^{-1}(G(\gamma)+W-d_0(\gamma-\mu)), T\}$
4	$\max\{\mu, (W-\int_0^{\mu} a(x)dx+d_0\mu)/d_0\}$	$\min\{\gamma, (W+d_0\mu)/d_0\}$
5	$F^{-1}(W+F(0))^{(ii)}$	μ

Note: (i) $G(x)=\int b(x)\mathrm{d}x$; (ii) $F(x)=\int a(x)\mathrm{d}x$.

Table 3. Optimal solutions for different situations in the two-warehouse case.

Case j	$T_1^{Lb_j}$	$T_1^{Ub_j}$	$\Gamma_j(T_1^{Lb_j})$	$\Gamma_j(T_1^{Ub_j})$	t_0^{j*}	T_1^{j*}	$\Pi_{2i}(T_1^{j*})$	Q_j^r	Q_{2j}^B
1	1.3994	2.0000	<0	<0	0.8000	1.3994	485.6413	82.8092	30.8322
2	0.9893	1.3994	<0	>0	0.6943	1.2390	487.1265	71.6982	41.8677
3	0.8000	0.9893	<0	<0	0.5000	0.9893	482.5902	57.2711	62.1138
4	0.5000	0.8000	<0	<0	0.3228	0.8000	471.4847	32.8092	80.3723
5	0.4879	0.5000	<0	<0	0.0127	0.5000	440.6261	01.2711	111.4875

Table 4. Effects of changes in the model parameters.

H	W	L	t_0^*	t_1^*	$\Pi(t_1^*)$	Q^o	Q^r	Q^B	Use RW?
1.5	50	3	0.6943	1.2390	487.1265	50	71.6982	41.8677	Yes
		4	0.6958	1.2412	487.0389	50	71.8612	41.7059	Yes
	100	3	0.2638	1.3091	493.1273	100	26.7329	36.8694	Yes
		4	0.2652	1.3112	493.0579	100	26.8733	36.7299	Yes
	150	3	0	1.3521	494.1105	129.6789	0	33.9430	No
		4	0	1.3541	494.0509	129.8201	0	33.8028	No
3.0	50	3	0.5201	1.0129	472.9561	50	53.3804	60.0255	Yes
		4	0.5218	1.0150	472.7887	50	53.5633	59.8444	Yes
	100	3	0.2014	1.2209	491.0977	100	20.3474	43.2081	Yes
		4	0.2027	1.2226	491.0048	100	20.4752	43.0813	Yes
	150	3	0	1.3521	494.1105	129.6789	0	33.9430	No
		4	0	1.3541	494.0509	129.8201	0	33.8028	No
4.5	50	3	0	0.4879	438.9612	50	0	112.7397	No
		4	0	0.4879	436.7800	50	0	112.7397	No
	100	3	0	0.9753	485.1008	100	0	63.3720	No
		4	0	0.9753	484.9161	100	0	63.3720	No
	150	3	0	1.3521	494.1105	129.6789	0	33.9430	No
		4	0	1.3541	494.0509	129.8201	0	33.8028	No

Table 5. Effects of changes in the model parameters for Example 5.1

Parameters		t_0^*	t_1^*	$\Pi(t_1^*)$	Q^r	Q^B
p	8	0.6880	1.2301	**159.9998**	71.0334	42.5274
	10	0.6912	1.2346	**323.5619**	71.3683	42.1951
	12	0.6943	1.2390	**487.1265**	71.6982	41.8677
	14	0.6974	1.2434	**650.6936**	72.0231	41.5452
	16	0.7004	1.2478	**814.2631**	72.3433	41.2273
C	3	0.6974	1.2434	**650.6936**	72.0231	41.5452
	4	0.6959	1.2412	**568.9097**	71.8612	41.7059
	5	0.6943	1.2390	**487.1265**	71.6982	41.8677
	6	0.6927	1.2368	**405.3439**	71.5339	42.0308
	7	0.6912	1.2346	**323.5619**	71.3683	42.1950
A	30	0.6943	1.2390	497.1265	71.6982	41.8677
	50	0.6943	1.2390	487.1265	71.6982	41.8677
	70	0.6943	1.2390	477.1265	71.6982	41.8677
	90	0.6943	1.2390	467.1265	71.6982	41.8677
	110	0.6943	1.2390	457.1265	71.6982	41.8677
h	0.8	0.7168	1.2715	491.9361	74.0656	39.5178
	0.9	0.7055	1.2551	489.5150	72.8780	40.0967
	1.0	0.6943	1.2390	487.1265	71.6982	41.8677
	1.1	0.6831	1.2233	484.7700	70.5250	43.0319
	1.2	0.6720	1.2077	482.4451	69.3576	44.1901
B	1.6	0.6216	1.1396	491.2017	64.0551	49.4490
	1.8	0.6608	1.1923	489.0111	68.1775	45.3609
	2.0	0.6943	1.2390	487.1265	71.6982	41.8677
	2.2	0.7232	1.2809	485.4893	74.7352	38.8531
	2.4	0.7483	1.3184	484.0549	77.3786	36.2281
δ	0.01	0.6943	1.2390	487.1265	71.6982	41.8677
	0.02	0.7071	1.2573	486.3670	73.0396	40.3715
	0.03	0.7191	1.2748	485.6518	74.3031	38.9715
	0.04	0.7304	1.2915	484.9773	75.4950	37.6593
	0.05	0.7411	1.3075	484.3403	76.6208	36.4272
M	80	0.5676	1.2597	**361.0694**	**46.7015**	**25.9247**
	90	0.6377	1.2483	**422.6320**	**59.1698**	**33.4378**
	100	0.6943	1.2390	**487.1265**	**71.6982**	**41.8677**
	110	0.7411	1.2314	**554.5791**	**84.2841**	**51.2151**
	120	0.7806	1.2250	**625.0117**	**96.9252**	**61.4817**
N	160	0.7140	1.2358	525.9714	73.7636	52.4262
	180	0.7035	1.2375	504.4350	72.6598	46.5410
	200	0.6943	1.2390	487.1265	71.6982	41.8677
	220	0.6862	1.2404	472.8800	70.8490	38.0671
	240	0.6790	1.2415	460.9300	70.0909	34.9156
μ	0.46	0.6923	1.2394	485.1467	71.3932	41.5136
	0.48	0.6932	1.2392	486.1416	71.5476	41.6903
	0.50	0.6943	1.2390	487.1265	71.6982	41.8677
	0.52	0.6953	1.2389	488.1014	71.8448	42.0459
	0.54	0.6964	1.2387	489.0663	71.9875	42.2248
γ	0.76	0.6798	1.2414	473.9453	70.1722	38.9952
	0.78	0.6873	1.2402	480.6589	70.9642	40.4441
	0.80	0.6943	1.2390	487.1265	71.6982	41.8677
	0.82	0.7008	1.2380	493.3570	72.3773	43.2658
	0.84	0.7067	1.2370	499.3589	73.0045	44.6381
T	1.6	0.5087	0.9995	548.4042	**52.1822**	42.5197
	1.7	0.5581	1.0589	532.7641	**57.3796**	42.6213
	1.8	0.6055	1.1186	517.2675	**62.3602**	42.5304
	1.9	0.6509	1.1787	502.0282	**67.1307**	42.2718
	2.0	0.6943	1.2390	487.1265	**71.6982**	41.8677