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Journal:	*Communications in Statistics - Simulation and Computation*
Manuscript ID:	LSSP-2007-0004.R1
Manuscript Type:	Original Paper
Date Submitted by the Author:	05-Jun-2007
Complete List of Authors:	Antzoulakos, Demetrios; University of Piraeus, Statistics and Insurance Science Rakitzis, Athanasios; University of Piraeus, Statistics and Insurance Science
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It is well-known that the Shewhart $\overline{X}$ control chart is relative insensitive to detect small process average shifts. In this article we introduce and investigate the basic features of the modified r out of m control chart. The new control chart outperforms the Shewhart $\overline{X}$ control chart for small to moderate process average shifts and the corresponding r out of m control chart which has been recently appeared in the literature.

Keywords Shewhart control chart; Run length distribution; Average run length; Standard deviation; Percentiles; Markov chain imbedding technique; Compound pattern.

1. Introduction

The most efficient procedure in statistical process control for monitoring a manufacturing process is the control chart. The Shewhart $\overline{X}$ control chart is the most popular control chart for monitoring the mean of the distribution of a quality characteristic of items produced by a certain process. The standard Shewhart control chart utilizes three-sigma control limits and indicates an out-of-control signal if a single point falls beyond the control limits.

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[2] Research supported by the State Scholarship Foundation of Greece.

It is well-known that while Shewhart control charts detect large process average shifts quickly, they are relative insensitive to small shifts. To enhance the effectiveness of the Shewhart control charts to detect small shifts various runs rules have been suggested and studied by several authors (Page (1955), Western Electric Company (1956), Roberts (1958), Bissell (1978), Wheeler (1983), Nelson (1984), Champ and Woodall (1987), Palm (1990), Shmueli and Cohen (2003) and references therein). Traditionally, the performance of a control chart is evaluated by the average run length (ARL). For a specific control chart and a given process average shift, ARL is the average number of points plotted on the chart until an out-of-control signal is obtained. The ARL value associated with a zero (non-zero) process average shift is called in-control (out-of-control) ARL. A disadvantage of the use of supplementary runs rules in a Shewhart $\overline{X}$ control chart is the reduction of the in-control ARL and hence the increase of the rate of false alarms. For instance, the simultaneous use of the four popular Western Electric rules results in an in-control ARL of 91.75 (see, e.g., Champ and Woodall (1987)) which is significantly lower from the in-control ARL value of 370.4 corresponding to the standard Shewhart $\overline{X}$ control chart.

To overcome this disadvantage, Klein (2000) suggested two alternatives to the standard Shewhart $\overline{X}$ control chart: the two of two $(2/2)$ and the two of three $(2/3)$ control charts which have symmetric upper and lower control limits. Both control charts are easily implemented and have better ARL performance than the standard Shewhart $\overline{X}$ control chart for process average shifts up to 2.6 standard deviation. Khoo (2004), extended the work of Klein (2000) by proceeding to a simulation study of the ARL performance of the $2/2$, $2/3$, $2/4$, $3/3$ and $3/4$ control charts. He suggested the use of the $3/4$ and 2/2 control charts for detecting small and moderate process average shifts, respectively.

In the present paper we propose a modified version of the *r*/*m* (*r* out of *m*) control chart studied by Klein (2000) and Khoo (2004). The new control chart, which we call modified *r* out of m control chart and denote it by $M : r / m$, is presented in Section 2. In Section 3, we provide the ARL performance of the $M : r / m$ control chart for in-control ARL value equal to 370.4. The $M : r / m$ chart uniformly outperforms the corresponding r / m control chart for every process average shift. For process average shifts up to 2.6 standard deviations the $M : r / m$ control chart outperforms the standard Shewhart $\overline{X}$ control chart. Additional characteristics of the run length distribution of the $M : r / m$ control chart, such as standard deviation and percentile points, are also presented. In Section 4, we summarize the results of the article and draw some conclusions. In the Appendix, we present a Markov chain approach for the study of the waiting time for the first appearance of a pattern in a sequence of independent and identically distributed trials. In the sequel, we demonstrate how this technique can be used to capture the study of the run length distribution of the $M : r / m$ control chart and to obtain the numerical results of the article.

2. The modified r out of m control chart

Klein (2000), motivated by Derman and Ross (1997), proposed the following two runs rules schemes alternative to the standard Shewhart $\overline{X}$ control chart: (a) the two of two scheme which gives an out-of-control signal if either two successive points are plotted above an upper control limit or two successive points are plotted below a lower control limit, and (b) the two of three scheme which gives an out-of-control signal if two out of three successive points are plotted above an upper control limit or two out of three successive points are plotted below a lower control limit. Both control schemes have better ARL profiles than the standard Shewhart scheme for detecting process average shifts up to about 2.6 standard deviations.

The two aforementioned schemes are special cases of the r out of m scheme ($1 \le r \le m$), to be denoted by r/m scheme, which gives an out-of-control signal if either r out of m successive points are plotted above an upper control limit (UCL) or r out of m successive points are plotted below a lower control limit (LCL). The (a) and (b) schemes correspond to the $2/2$ and $2/3$ schemes, respectively, while the Shewhart $\overline{X}$ control chart corresponds to the $1/1$ scheme. Khoo (2004), proceeded to a detailed study of the $2/2$, $2/3$, $2/4$, $3/3$ and $3/4$ schemes and concluded that the $3/4$ scheme is the most sensitive scheme for detecting small process average shifts.

In the r/m scheme with $r < m$, the set of points of length at most m which causes an out-of-control signal can be written as a union of two sets of points. The first set, say A, includes the r points which are all above (below) the UCL (LCL), and the second set, say B, includes at most $(m-r)$ points which are placed between the points of set A.

In Figure 1 we provide a few processes and corresponding r/m schemes used for the detection of a shift in the process average.

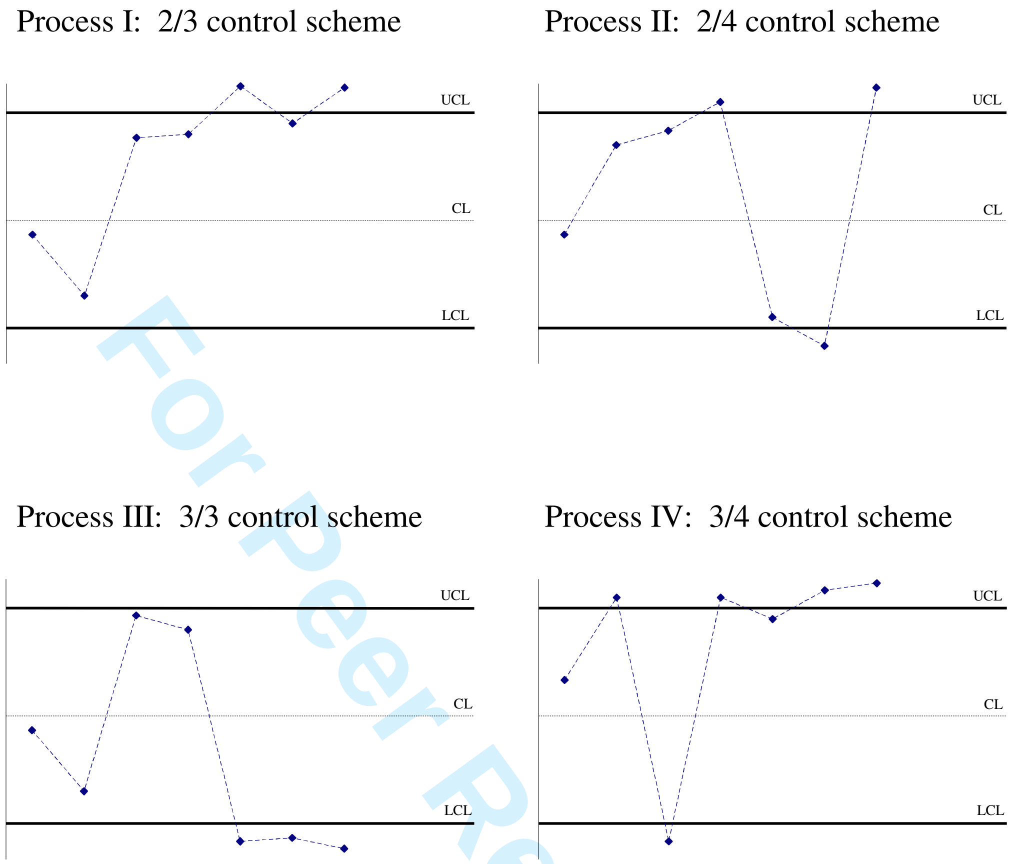

Figure 1. Illustrations of r/m control schemes.

For processes I, II and IV we have indications that the process average has shifted to a higher level, while for process III we have indications that the process average has shifted to a lower level. However, for process II, there are reasonable doubts about the shift of the process average to a higher level since between the $r(=2)$ points which are located above the UCL (elements of set A) there are $m-r\,(=2)$ points far away from them (elements of set B). Thus, it seems plausible to take into account the location of the points of set B relative to the location of the points of set A in order to obtain an out-of-control signal. A reasonable approach is to demand for the points of set B to be "close enough" to the points of set A.

Therefore, in the case where $r<m$, we introduce a modified control chart which gives an out-of-control signal if either r points are plotted above an upper control limit which are

separated by at most $(m-r)$ points placed between the center line and the upper control limit or r points are plotted below a lower control limit which are separated by at most $(m-r)$ points placed between the center line and the lower control limit. We call the new control chart as modified r out of m control chart and we denote it by $M:r/m$. The performance of the new control chart is investigated in the following section.

3. Performance of the modified r out of m control chart

In the present section we proceed to a detailed study of the $M:r/m$ control chart. For all calculations, we assume that the random variables giving rise to the points plotted on the control chart are independent and normally distributed with a standard deviation equal to one ($\sigma=1$). The process is considered to be in-control when the process mean is zero ($\mu=0$). The method used to obtain the numerical results of the present section is described in the Appendix.

In Table 1, we provide ARL values and control limits of the $M:r/m$ scheme ($m=3,\ 4,\ 5$ and $2 \le r < m$) for in-control ARL value equal to 370.4. For comparison reasons we have also included the ARL values of corresponding r/m schemes studied by Khoo (2004) and Klein (2000) (the standard Shewhart $\bar{X}$ control chart corresponds to the column labeled 1/1). Process average shifts vary from zero to out-of-control values up to four-sigma. The lowest ARL value for every process average shift is indicated by a boldfaced value. Since the ARL, as a single parameter, is not necessarily a very "typical" value of the run length distribution, the standard deviation (SD) of the run length distribution is also given in parentheses.

Table 1. ARL and SD values for r/r and $M:r/m$ schemes ($\text{ARL}_{in} = 370.40$)

	1/1	2/2	*M*: 2/3	2/3	3/3	*M*: 2/4	2/4	*M*: 3/4	3/4	4/4	*M*: 2/5	*M*: 3/5	*M*: 4/5	5/5
	Control Limits													
Shift	± 3	± 1.781	± 1.866	± 1.929	± 1.2	± 1.897	± 2.011	± 1.312	± 1.393	± 0.832	± 1.91	± 1.358	± 0.949	± 0.568
0.0	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40
	(369.90)	(368.94)	(368.63	(368.47)	(368.03)	(368.04)	(368.43)	(367.61)	(367.44)	(367.13)	(368.28)	(367.30)	(366.68)	(366.27)
0.2	308.43	276.67	264.79	270.10	259.30	257.81	266.96	243.10	248.65	248.54	253.39	233.55	**231.24**	241.32
	(307.93)	(275.22)	(263.03)	(268.20)	(256.96)	(264.64)	(255.82)	(240.35)	(245.76)	(245.34)	(251.24)	(230.48)	(227.61)	(237.28)
0.4	200.10	150.25	134.92	141.61	129.55	126.61	137.81	112.01	117.78	118.70	121.52	102.82	**101.68**	112.26
	(199.58)	(148.82)	(133.18)	(139.78)	(127.26)	(135.58)	(124.63)	(109.34)	(115.01)	(115.96)	(119.35)	(99.83)	(98.18)	(108.37)
0.6	119.67	78.91	67.89	72.64	65.25	62.24	70.12	53.79	57.48	58.99	58.85	**48.26**	48.34	55.71
	(119.16)	(77.51)	(66.18)	(70.86)	(63.02)	(67.99)	(60.29)	(51.21)	(54.83)	(55.98)	(56.70)	(45.37)	(44.98)	(51.95)
0.8	71.55	43.63	36.64	39.64	35.76	33.22	38.18	28.83	31.04	32.63	31.21	**25.71**	26.28	31.28
	(71.05)	(42.25)	(34.97)	(37.92)	(33.59)	(36.15)	(31.33)	(26.34)	(28.49)	(29.71)	(29.12)	(22.93)	(23.03)	(27.63)
1.0	43.90	25.78	21.44	23.30	21.45	19.42	22.50	17.23	18.57	20.06	18.26	**15.46**	16.18	19.72
	(43.39)	(24.42)	(18.82)	(21.64)	(19.34)	(20.57)	(17.59)	(14.82)	(16.11)	(17.20)	(16.25)	(12.78)	(13.03)	(16.13)
1.2	27.82	16.28	13.56	14.73	14.00	12.37	14.30	11.36	12.18	13.54	11.70	**10.32**	11.09	13.72
	(27.32)	(19.94)	(11.99)	(13.12)	(11.92)	(12.45)	(10.60)	(9.00)	(9.80)	(10.73)	(9.77)	(7.72)	(7.98)	(10.18)
1.4	18.25	10.94	9.21	9.96	9.85	8.49	9.74	8.14	8.67	9.91	8.11	**7.53**	8.30	10.37
	(17.74)	(9.62)	(7.67)	(8.40)	(7.79)	(7.97)	(6.78)	(5.82)	(6.33)	(7.11)	(6.25)	(4.98)	(5.20)	(6.82)
1.6	12.38	7.79	6.67	7.16	7.41	6.23	7.06	6.26	6.62	7.77	6.02	**5.90**	6.67	8.39
	(11.87)	(6.48)	(5.15)	(5.63)	(5.35)	(5.34)	(4.56)	(3.95)	(4.28)	(4.95)	(4.21)	(3.37)	(3.55)	(4.80)
1.8	8.70	5.85	5.10	5.43	5.89	4.84	5.40	5.11	5.35	6.44	**4.72**	4.91	5.69	7.16
	(8.18)	(4.54)	(3.60)	(3.92)	(3.82)	(3.72)	(3.19)	(2.78)	(3.01)	(3.58)	(2.96)	(2.36)	(2.50)	(3.49)
2.0	6.30	4.61	4.10	4.33	4.92	3.95	4.33	4.38	4.55	5.59	**3.89**	4.27	5.07	6.38
	(5.78)	(3.29)	(2.60)	(2.82)	(2.81)	(2.67)	(2.31)	(2.01)	(2.17)	(2.66)	(2.15)	(1.70)	(1.80)	(2.60)
2.2	4.70	3.79	3.44	3.60	4.28	3.35	3.62	3.91	4.02	5.03	**3.33**	3.85	4.67	5.87
	(4.19)	(2.45)	(1.93)	(2.09)	(2.12)	(1.97)	(1.71)	(1.47)	(1.59)	(2.01)	(1.61)	(1.26)	(1.31)	(1.97)
2.4	3.65	3.23	2.99	3.10	3.85	2.95	3.14	3.59	3.68	4.66	**2.94**	3.57	4.42	5.54
	(3.11)	(1.87)	(1.45)	(1.57)	(1.63)	(1.49)	(1.30)	(1.10)	(1.18)	(1.54)	(1.24)	(0.95)	(0.96)	(1.50)
2.6	2.90	2.85	2.68	2.76	3.56	2.66	2.80	3.39	3.44	4.42	**2.66**	3.38	4.26	5.33
	(2.35)	(1.45)	(1.11)	(1.20)	(1.26)	(1.14)	(1.00)	(0.82)	(0.88)	(1.19)	(0.97)	(0.72)	(0.70)	(1.15)
2.8	**2.38**	2.58	2.47	2.52	3.36	2.46	2.56	3.25	3.29	4.26	2.46	3.25	4.16	5.20
	(1.81)	(1.13)	(0.86)	(0.93)	(0.98)	(0.89)	(0.79)	(0.62)	(0.66)	(0.91)	(0.77)	(0.56)	(0.52)	(0.87)
3.0	**2.00**	2.39	2.32	2.36	3.23	2.32	2.39	3.16	3.18	4.16	2.32	3.16	4.09	5.11
	(1.41)	(0.89)	(0.67)	(0.72)	(0.76)	(0.70)	(0.62)	(0.46)	(0.50)	(0.70)	(0.62)	(0.44)	(0.38)	(0.66)
3.5	**1.45**	2.14	2.11	2.13	3.07	2.12	2.15	3.05	3.05	4.04	2.12	3.05	4.02	5.03
	(0.80)	(0.24)	(0.36)	(0.39)	(0.40)	(0.35)	(0.40)	(0.23)	(0.25)	(0.34)	(0.36)	(0.23)	(0.17)	(0.31)
4.0	**1.19**	2.04	2.03	2.04	3.02	2.04	2.05	3.01	3.01	4.01	2.04	3.01	4.00	5.00
	(0.47)	(0.07)	(0.19)	(0.20)	(0.19)	(0.19)	(0.22)	(0.11)	(0.12)	(0.15)	(0.20)	(0.11)	(0.07)	(0.13)

It follows from Table 1 that the $M:r/m$ scheme uniformly outperforms the corresponding r/m scheme for every process average shift. Table 1 also reveals that for process average shifts up to 2.6 standard deviations the $M:r/5$ scheme has the best ARL performance ($r = 2, 3, 4$). However, for process average shifts greater than 2.6 standard deviations, the standard Shewhart $\overline{X}$ control chart has the best ARL performance. The choice of the optimal modified control scheme depends on the magnitude of the process average shift we wish to detect. In Table 2 a practical guidance is provided for the selection of the proper $M:r/m$ scheme as an alternative to the standard Shewhart $\overline{X}$ control chart for the detection of a given shift in the process mean.

Table 2. Selection of the optimal $M:r/5$ control chart ($\text{ARL}_{in} = 370.40$).

Shift	ARL of the 1/1 scheme	ARL of the $M:r/m$ scheme	Control chart	Control limits
0.0	370.40	370.40		
0.2	308.43	231.24	$M:4/5$	± 0.949
0.4	200.10	101.68		
0.6	119.67	48.26	$M:3/5$	± 1.358
0.8	71.55	25.71		
1.0	43.90	15.46		
1.2	27.82	10.32		
1.4	18.25	7.53		
1.6	12.38	5.90		
1.8	8.70	4.72	$M:2/5$	± 1.910
2.0	6.30	3.89		
2.2	4.70	3.33		
2.4	3.65	2.94		
2.6	2.90	2.66		

The run length distribution associated with a $M:r/m$ control chart is a highly skewed distribution with a right tail which decreases slowly for small process average shifts. In such cases, practitioners are more interested in percentile values of the run length distribution as a measure of the performance of the manufacturing process than in a single ARL value (see, e.g., Montgomery (2005), Palm (1990) and Shmueli and Cohen (2003)). Therefore, in Table 3 we give percentiles of the run length distribution for the $M:2/5$, $M:3/5$ and $M:4/5$ schemes. We mention that Palm (1990) reported several examples showing the use of percentiles of the run length distribution.

Table 3. Percentiles and ARL values for the $M:r/5$, $r=2, 3, 4$ scheme ($\text{ARL}_{in}=370.4$).

	ARL			Percentile Points								
				25th			50th			75th		
Shift	2/5	3/5	4/5	2/5	3/5	4/5	2/5	3/5	4/5	2/5	3/5	4/5
0.0	370.40	370.40	370.40	108	109	109	257	258	258	513	512	512
0.2	253.30	233.55	231.24	74	69	69	176	163	161	350	323	319
0.4	121.52	102.82	101.68	37	32	32	85	72	72	168	141	140
0.6	58.85	48.26	48.34	18	16	16	41	34	35	81	66	66
0.8	31.21	25.71	26.28	10	9	10	22	19	19	42	35	35
1.0	18.26	15.46	16.18	7	6	7	13	11	12	25	20	21
1.2	11.70	10.32	11.09	5	5	5	9	8	9	15	13	14
1.4	8.11	7.53	8.30	4	4	5	6	6	6	11	9	10
1.6	6.02	5.90	6.67	3	4	4	5	5	5	8	7	8
1.8	4.72	4.91	5.69	3	3	4	4	4	5	6	5	6
2.0	3.89	4.27	5.07	2	3	4	3	4	4	5	5	5
2.2	3.33	3.85	4.67	2	3	4	3	3	4	4	4	5
2.4	2.94	3.57	4.42	2	3	4	3	3	4	3	4	5
2.6	2.66	3.38	4.26	2	3	4	2	3	4	3	4	4
2.8	2.46	3.25	4.16	2	3	4	2	3	4	3	3	4
3.0	2.32	3.16	4.09	2	3	4	2	3	4	3	3	4
3.5	2.12	3.05	4.02	2	3	4	2	3	4	2	3	4
4.0	2.04	3.01	4.00	2	3	4	2	3	4	2	3	4

4. Conclusions

In this article, the $M:r/m$ control chart has been introduced and studied. For in-control ARL value equal to 370.4, it has been shown that the $M:r/m$ control chart is uniformly better than the corresponding r/m control chart in terms of the out-of-control ARL value. For process average shifts up to 2.6 standard deviations the $M:r/5$ scheme outperforms the standard Shewhart $\overline{X}$ control chart. These features present solid evidence in favour of the $M:r/5$ control chart as a viable alternative to the standard Shewhart $\overline{X}$ control chart for small to moderate process average shifts. For the practitioners interested in the detection of small process average shifts we recommend the use of the $M:4/5$ scheme, while for moderate shifts the $M:3/5$ and the $M:2/5$ schemes are more suitable. For shifts larger than 2.6 standard deviations the standard Shewhart scheme should be used.

It is worth mentioning that our extensive numerical experimentation revealed that the aforementioned conclusions are still valid for various choices of the in control ARL value.

Furthermore, standard deviation values and percentile points of the run length distribution of $M:r/m$ schemes have been presented offering to practitioners an in-depth analysis of the effectiveness of each scheme. Finally, the Markov chain approach employed for the derivation of the results of the present article provides a general theoretical framework for the study of run length distributions of control charts that are based on runs rules.

In closing we mention that CUSUM or EWMA charts are more effective than the Shewhart-type charts in detecting small to moderate process average shifts. However they have not been widely adopted for use, possibly due to their intricate statistical basis.

Appendix

The Appendix is separated into two major parts. In the first part (A1, A2) we present in brief a finite Markov chain imbedding technique suitable for the study of the waiting time distribution of a pattern $\mathcal{E}$ (simple or compound). In the second part (A3, A4) we utilize this technique for the study of the $M:r/m$ and r/r control schemes. For articles dealing with the waiting time distribution for a pattern $\mathcal{E}$ we refer to Fu (1986, 1996), Koutras and Alexandrou (1997), Antzoulakos (2001) and references there in. For further details the interested reader may consult Balakrishnan and Koutras (2002) and Fu and Lou (2003). Computer programs that produce the values given in the tables of Section 3 are available for any interested reader from the authors upon request.

*A*1. *Waiting time for the first appearance of a simple pattern*

Let $\{X_t,\ t\geq 1\}$ be a sequence of i.i.d. trials taking values in a set $A=\{a_1,a_2,...,a_\lambda\}$, $\lambda\geq 2$, and let $P(X_j=a_i)=p_i$ ($j\geq 1,\ 1\leq i\leq\lambda,\ \Sigma p_i=1$). Let $\mathcal{E}=a_{i_1}a_{i_2}...a_{i_k}$ be a simple pattern of length k ($1\leq i_n\leq\lambda,\ 1\leq n\leq k$) and denote by T the waiting time for the first occurrence of $\mathcal{E}$. Decompose the pattern $\mathcal{E}$ into k blocks (sub-patterns) labeled $"2"=a_{i_1}$, $"3"=a_{i_1}a_{i_2}$, …, $"k+1"=a_{i_1}a_{i_2}...a_{i_k}$, and let the label $"1"$ stands for the set of sub-patterns $A-\{a_{i_1}\}$. We define a Markov chain $\{Y_t,\ t\geq 1\}$ with state space $\Omega=\{1,2,...,k+1\}$ operating on $\{X_t,\ t\geq 1\}$ as follows: (a) the state $k+1$ is an absorbing state and (b) we assign to Y_n the value j ($2\leq j\leq k+1$) if the maximum ending block of the first n trials $X_1,X_2,...,X_n$ (counting backward) is identified to be the block corresponding to the label j, otherwise we assign to Y_n the value 1.

The above definitions establish a time homogeneous Markov chain on Ω with initial probability vector

$$\boldsymbol{\pi} = [\pi_1, \pi_2, ..., \pi_{k+1}] = [\Pr[Y_1 = 1], \Pr[Y_1 = 2], ..., \Pr[Y_1 = k+1]] = [1 - p_{i_1}, p_{i_1}, 0, ..., 0]$$

and transition probability matrix $\mathbf{P} = [p_{ij}]_{(k+1)\times(k+1)}$ defined by

$$p_{ij} = \Pr[Y_n = j \mid Y_{n-1} = i] = \sum\nolimits_{(i,j)} p_m, \quad i, j \in \Omega, \; n \geq 2$$

where $\sum_{(i,j)} p_m$ means sum over all p_m corresponding to a_m for which the ending block i is changed to the ending block j. We note that $\pi_m = p_{1m}$ for $1 \leq m \leq k+1$, which implies that $\boldsymbol{\pi} = \mathbf{e}_1\mathbf{P}$ ($\mathbf{e}_m$ denotes the m-th unit row vector of R^{k+1}).

The event $\{Y_n = k+1\}$ implies that the pattern $\mathcal{E}$ has occurred on or before the n-th trial. Therefore, for $n \geq 1$ we have that

$$\Pr[T \leq n] = \Pr[Y_n = k+1] = \boldsymbol{\pi}\mathbf{P}^{n-1}\mathbf{e}'_{k+1} = \mathbf{e}_1\mathbf{P}^n\mathbf{e}'_{k+1}$$

and

$$\Pr[T > n] = \Pr[Y_n \in \{1,2,...,k\}] = \sum_{i=1}^{k} \Pr[Y_n = i] = \sum_{i=1}^{k} \boldsymbol{\pi}\mathbf{P}^{n-1}\mathbf{e}'_i = \mathbf{e}_1\mathbf{P}^n(\mathbf{1}' - \mathbf{e}'_{k+1})$$

($\mathbf{1}$ denotes the row vector of R^{k+1} with all its entries being equal to 1). The transition probability matrix $\mathbf{P}$ can always be written in the following form

$$\mathbf{P} = \left[\begin{array}{c|c} \mathbf{R} & (\mathbf{I}-\mathbf{R})\mathbf{1}' \\ \hline \mathbf{0} & 1 \end{array}\right]$$

and hence

$$\mathbf{P}^n = \left[\begin{array}{c|c} \mathbf{R}^n & (\mathbf{I}-\mathbf{R}^n)\mathbf{1}' \\ \hline \mathbf{0} & 1 \end{array}\right], \quad n = 1, 2, ...$$

($\mathbf{I}$ denotes the $k \times k$ identity matrix and $\mathbf{1}$ denotes here the row vector of R^k with all its entries being equal to 1). Then, for $n \geq 1$ it is true that

$$\Pr[T \leq n] = \mathbf{e}_1(\mathbf{I} - \mathbf{R}^n)\mathbf{1}', \qquad \Pr[T > n] = \mathbf{e}_1\mathbf{R}^n\mathbf{1}'$$

and

$$\Pr[T=n]=\mathbf{e}_1\mathbf{R}^{n-1}(\mathbf{I}-\mathbf{R})\mathbf{1}'.$$

The probability generating function $G(s)$ of T is given by

$$G(s)=E(s^T)=\sum_{n=1}^{\infty}\Pr[T=n]s^n=s\mathbf{e}_1\left(\sum_{n=1}^{\infty}(sR)^{n-1}\right)(\mathbf{I}-\mathbf{R})\mathbf{1}'=s\mathbf{e}_1(\mathbf{I}-s\mathbf{R})^{-1}(\mathbf{I}-\mathbf{R})\mathbf{1}'$$

and the m-th (descending) factorial moment of T is given by

$$E[T(T-1)\cdots(T-m+1)]=\left[\frac{d^mG(s)}{ds^m}\right]_{s=1}=m!\mathbf{e}_1\mathbf{R}^{m-1}(\mathbf{I}-\mathbf{R})^{-m}\mathbf{1}'.$$

The above formula implies that

$$E(T)=\mathbf{e}_1(\mathbf{I}-\mathbf{R})^{-1}\mathbf{1}',\qquad E[T(T-1)]=2\mathbf{e}_1\mathbf{R}(\mathbf{I}-\mathbf{R})^{-2}\mathbf{1}'.$$

A2. *Waiting time for the first appearance of a compound pattern*

The method for studying the waiting time of a simple pattern can be routinely extended for the study of the waiting time of a compound pattern $\mathcal{E}$. A compound pattern $\mathcal{E}$ is a union of m distinct simple patterns (m is a fixed integer). The basic idea in handling this situation is to decompose each simple pattern into blocks and to remove any duplications. Furthermore, in order to reduce the dimension of the transition probability matrix $\mathbf{P}$, the m absorbing states which signal the entrance of the Markov chain to each one of the m distinct simple patterns may be substituted by a single absorbing state.

A3. *Development of the r/r scheme*

We assume that the random variables giving rise to the points plotted on the control chart are independent and normally distributed with mean μ and standard deviation equal to one which remains constant. The process is considered to be in-control (out-of-control) when the process mean is zero (non-zero).

In a r/r control chart three regions are defined: one above the UCL (region 1), one below the LCL (region 2) and a central region extending between the two control limits (region 0). The CL of the chart is taken to be zero and we use symmetrical control limits, that is $\mathrm{UCL} = d$ and $\mathrm{LCL} = -d$ ($d > 0$). The probability that a single point falls in regions 1, 2, 0 will be denoted by p_U, p_L and q, respectively. Hence

$$p_U = p_U(\mu) = 1 - \Phi(d-\mu), \quad p_L = p_L(\mu) = 1 - \Phi(d+\mu), \quad q = q(\mu) = 1 - p_U - p_L$$

where $\Phi(\cdot)$ denotes the distribution function of a standard normal distribution.

Let $\{X_t,\ t \geq 1\}$ be a sequence of i.i.d. trials taking values in the set $A = \{0, 1, 2\}$, and let $P(X_t = 0) = q$, $P(X_t = 1) = p_U$ and $P(X_t = 2) = p_L$. Consider the compound pattern

$$\mathcal{E} = \{\underbrace{1\,1\,....\,1}_{r},\ \underbrace{2\,2\,....\,2}_{r}\}$$

and denote by T the waiting time for the first occurrence of $\mathcal{E}$.

From the above set-up it is easy to realize that the run length distribution of the r/r control chart coincides with the waiting time distribution T of the compound pattern $\mathcal{E}$. Decomposing the pattern $\mathcal{E}$ to the following $2r$ blocks

$$"1" = 0, \quad "2" = 1, \quad "3" = 1\,1, \ \ldots, \quad "r" = \underbrace{1\,1\,...\,1}_{r-1},$$

$$"r+1" = 2, \quad "r+2" = 2\,2, \ldots, \quad "2r-1" = \underbrace{2\,2\,...\,2}_{r-1}, \quad "2r" = \{\underbrace{1\,1\,....\,1}_{r},\ \underbrace{2\,2\,....\,2}_{r}\}$$

we obtain that

$$\mathbf{P} = \left[\begin{array}{c|c} \mathbf{R} & (\mathbf{I}-\mathbf{R})\mathbf{1}' \\ \hline \mathbf{0} & 1 \end{array}\right] = \left(\begin{array}{cccccccccc|c} q & p_U & 0 & \cdots & 0 & p_L & 0 & \cdots & 0 & 0 \\ q & 0 & p_U & \cdots & 0 & p_L & 0 & \cdots & 0 & 0 \\ q & 0 & 0 & \cdots & 0 & p_L & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \ddots & \vdots & \vdots \\ q & 0 & 0 & \cdots & 0 & p_L & 0 & \cdots & 0 & p_U \\ q & p_U & 0 & \cdots & 0 & 0 & p_L & \cdots & 0 & 0 \\ q & p_U & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q & p_U & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & p_L \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right)_{(2r)\times(2r)} .$$

Carrying out some algebra we get that

$$E(T) = E(T \mid \mu) = \mathbf{e}_1(\mathbf{I}-\mathbf{R})^{-1}\mathbf{1}' = \left(p_U + p_L + \frac{p_U - p_U^r}{1-p_U^r} + \frac{p_L - p_L^r}{1-p_L^r} \right)^{-1}.$$

For an in-control process ($\mu = 0$) we have that $p_U = p_L = p$, and

$$E(T \mid \mu = 0) = \frac{1-p^r}{2p^r(1-p)}, \quad p = 1 - \Phi(d).$$

Therefore, the control limits of the r/r scheme can be found in the following four steps:

Step 1. Choose a positive integer r.

Step 2. Choose the desired in control ARL value c.

Step 3. Calculate the unique root of the equation $c = E(T \mid \mu = 0)$ in the interval (0,1), say p^*.

Step 4. Calculate the $\mathrm{UCL} = d$ from the formula $d = \Phi^{-1}(1-p^*)$.

Thus, all the necessary tools for the study of the run length distribution of the r/r scheme are available.

A4. Development of the $M:r/m$ scheme

In a $M:r/m$ scheme ($2 \le r < m$) four regions are defined: one above the UCL (region 1), one extending between the CL and the UCL (region 2), one extending between the CL and the LCL (region 3), and one below the LCL (region 4). The CL of the chart is taken to be zero and we use symmetrical control limits, that is $\mathrm{UCL} = d$ and $\mathrm{LCL} = -d$ ($d > 0$). The probability that a single point falls in regions 1, 2, 3, 4 will be denoted by p_U, q_U, q_L and p_L, respectively. Hence, following the set-up of Section A3 we have that

$$p_U = p_U(\mu) = 1 - \Phi(d - \mu), \quad q_U = q_U(\mu) = \Phi(d - \mu) + \Phi(\mu) - 1$$

$$p_L = p_L(\mu) = 1 - \Phi(d + \mu), \quad q_L = q_L(\mu) = \Phi(d + \mu) - \Phi(\mu).$$

The derivation of general formulae for the study of the $M : r/m$ scheme seems to be a very difficult task. Therefore, in the sequel we restrict ourselves to the study of the $M : 3/4$ scheme as a typical example of the whole class of $M : r/m$ schemes.

Let $\{X_t,\ t \geq 1\}$ be a sequence of i.i.d. trials taking values in the set $A = \{1, 2, 3, 4\}$, and let $P(X_t = 1) = p_U$, $P(X_t = 2) = q_U$, $P(X_t = 3) = q_L$ and $P(X_t = 4) = p_L$. Consider the compound pattern

$$\mathcal{E} = \{111,\ 1211,\ 1121,\ 444,\ 4344,\ 4434\}$$

and denote by T the waiting time for the first occurrence of $\mathcal{E}$. The run length distribution corresponding to the $M : 3/4$ control chart coincides with the waiting time distribution T of the compound pattern $\mathcal{E}$. Decomposing the pattern $\mathcal{E}$ to the following 12 blocks

$$"1" = \{2,3\}, \quad "2" = 1, \quad "3" = 11, \quad "4" = 12, \quad "5" = 121, \quad "6" = 112,$$

$$"7" = 4, \quad "8" = 44, \quad "9" = 43, \quad "10" = 434, \quad "11" = 443,$$

$$"12" = \{111,\ 1211,\ 1121,\ 444,\ 4344,\ 4434\},$$

we get that

$$
\mathbf{P}=\left(\begin{array}{ccccccccccc|c}
q_U+q_L & p_U & 0 & 0 & 0 & 0 & p_L & 0 & 0 & 0 & 0 & 0 \\
q_L & 0 & p_U & q_U & 0 & 0 & p_L & 0 & 0 & 0 & 0 & 0 \\
q_L & 0 & 0 & 0 & 0 & q_U & p_L & 0 & 0 & 0 & 0 & p_U \\
q_U+q_L & 0 & 0 & 0 & p_U & 0 & p_L & 0 & 0 & 0 & 0 & 0 \\
q_L & 0 & 0 & q_U & 0 & 0 & p_L & 0 & 0 & 0 & 0 & p_U \\
q_U+q_L & 0 & 0 & 0 & 0 & 0 & p_L & 0 & 0 & 0 & 0 & p_U \\
q_U & p_U & 0 & 0 & 0 & 0 & 0 & p_L & q_L & 0 & 0 & 0 \\
q_U & p_U & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_L & p_L \\
q_U+q_L & p_U & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_L & 0 & 0 \\
q_U & p_U & 0 & 0 & 0 & 0 & 0 & 0 & q_L & 0 & 0 & p_L \\
q_U+q_L & p_U & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_L \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right).
$$

For an in-control process ($\mu=0$) we have that $p_U=p_L=p$, $q_U=q_L=(1-2p)/2$ and it may be checked that

$$
E(T \mid \mu=0)=\frac{4p^5-8p^4+7p^3-6p^2-4p-4}{2p^3(4p^3-8p^2+11p-8)}, \quad p=1-\Phi(d).
$$

The above formula and Steps 2-4 of Section A3 may be used for the determination of the control limits of the $M:3/4$ scheme. The study of the run length distribution of the $M:3/4$ scheme may be performed through the general results presented in Section A1.

Acknowledgements

The work of Athanasios C. Rakitzis is supported by the State Scholarship Foundation of Greece.

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