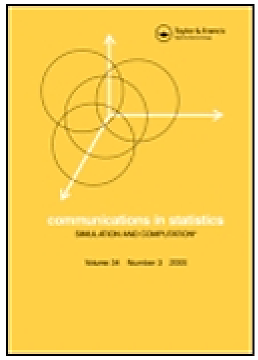

PORT Hill and moment estimators for heavy-tailed models

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Note: The following files were submitted by the author for peer review, but cannot be converted to PDF. You must view these files (e.g. movies) online.	
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PORT Hill and moment estimators for heavy-tailed models

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Abstract: In this paper we use the *peaks over random threshold* (PORT)-methodology, and consider Hill and moment PORT-classes of *extreme value index* estimators. These classes of estimators are invariant not only to changes in scale, like the classical Hill and moment estimators, but also to changes in location. They are based on the sample of excesses over a random threshold, the order statistic $X_{[np]+1:n}$, $0 \leq p < 1$, being p a *tuning parameter*, which makes them highly flexible. Under convenient restrictions on the underlying model, these classes of estimators are consistent and asymptotically normal for adequate values of k, the number of top order statistics used in the semi-parametric estimation of the *extreme value index* γ. In practice, there may however appear a stability around a value distant from the target γ when the minimum is chosen for the random threshold, and attention is drawn for the danger of transforming the original data through the subtraction of the minimum. A new bias-corrected moment estimator is introduced. The exact performance of the new extreme value index PORT-estimators is compared, through a large-scale Monte-Carlo simulation study, with the original Hill and moment estimators, the bias-corrected moment estimator and one of the *minimum-variance reduced-bias* (MVRB) extreme value index estimators recently introduced in the literature. As an empirical example we estimate the tail index associated to a set of real data from the field of finance.

Keywords: Statistics of extremes; Monte Carlo simulation; semi-parametric estimation; extreme value index; reduced-bias estimation; sample of excesses.

1 Introduction

The *extreme value index* (or *tail index*) γ is the shape parameter in the *extreme value* (*EV*) distribution function (d.f.), with the functional form

$$EV_\gamma(x) = \begin{cases} \exp\left\{-\left(1+\gamma x\right)^{-1/\gamma}\right\},\ 1+\gamma x>0 & \text{if} \quad \gamma \neq 0 \\ \exp\left(-e^{-x}\right),\ x\in\mathbb{R} & \text{if} \quad \gamma = 0. \end{cases} \tag{1.1}$$

This d.f. appears as the limiting d.f., as $n\to\infty$, of the linearly normalised maximum $X_{n:n}$ of an independent, identically distributed (i.i.d.), or even weakly dependent stationary sample of size n, $(X_1,\cdots,X_n)$.

We shall work in a context of heavy-tailed models, i.e., we shall consider that $\gamma>0$ in (1.1). Let us denote $F^{\leftarrow}(t):=\inf\{x: F(x)\geq t\}$, the generalized inverse function of F, $U(t):=F^{\leftarrow}(1-1/t)$ and RV_α the class of regularly functions at infinity with index of regular variation α, i.e., positive measurable functions h such that $\lim_{t\to\infty} h(tx)/h(t)=x^\alpha$, for all $x>0$. We shall work here with models F that are in the domain of attraction for maxima of EV_γ with $\gamma>0$, denoted $\mathcal{D}_{\mathcal{M}}(EV_{\gamma>0})$, i.e., with models F such that

$$1-F\in RV_{-1/\gamma} \quad \text{or equivalently} \quad U\in RV_\gamma. \tag{1.2}$$

For the estimation of the right tail we consider two classical estimators of the *extreme value index* γ based on the $k+1$ top order statistics (o.s.), denoted $\underline{X}_k := \left(X_{n:n}\geq\cdots\geq X_{n-k:n}\right)$, where $X_{n-k:n}$ is an intermediate o.s., i.e., k is an *intermediate* sequence of integers in $[1,n)$, i.e., a sequence of integers such that

$$k=k_n\to\infty \quad \text{and} \quad k_n/n\to 0, \qquad \text{as } n\to\infty. \tag{1.3}$$

Those estimators are the Hill estimator (Hill, 1975), with the functional expression

$$\hat\gamma^H_{n,k}=\hat\gamma^H_n(\underline{X}_k):=\frac{1}{k}\sum_{j=1}^k V_{jk}, \quad V_{jk}:=\ln X_{n-j+1:n}-\ln X_{n-k:n}, \tag{1.4}$$

and the moment estimator (Dekkers, Einmhal and de Haan, 1989),

$$\hat\gamma^M_{n,k}=\hat\gamma^M_n(\underline{X}_k):=M^{(1)}_{n,k}+1-\tfrac{1}{2}\left\{1-\left(M^{(1)}_{n,k}\right)^2/M^{(2)}_{n,k}\right\}^{-1}, \tag{1.5}$$

with

$$M^{(r)}_{n,k}=M^{(r)}_n(\underline{X}_k)=\frac{1}{k}\sum_{j=1}^k\left\{V_{jk}\right\}^r,\ r=1,2. \tag{1.6}$$

It is a well-known result in the field of *statistics of extremes* that the estimator in (1.4) is valid only for $\gamma \geq 0$, whereas the estimator in (1.5) is valid for all $\gamma \in \mathbb{R}$. They are both scale invariant, but not location invariant. Indeed the associated estimates, particularly the Hill estimates, may suffer drastic changes when we induce an arbitrary shift in the data.

Apart from the classical Hill and moment estimators, often simply denoted H and M, respectively, we shall also consider one of the three classes of second-order reduced-bias extreme value index estimators recently introduced in Gomes, de Haan and Henriques Rodrigues (2005), Caeiro, Gomes and Pestana (2005) and Gomes, Martins and Neves (2007). These classes are based on the adequate estimation of a "scale" and a "shape" second order parameters, β and ρ, respectively, are valid for a large class of heavy-tailed models and are appealing in the sense that we are able to reduce the asymptotic bias of the Hill estimator in (1.4) without increasing the asymptotic variance, which is kept at the value γ^2, the asymptotic variance of Hill's estimator. We shall call these estimators "*minimum-variance reduced-bias*" (MVRB) estimators. These MVRB-estimators are also non-invariant for changes in location. However, they are much less sensitive to changes in location than the classical Hill estimator in (1.4). The simplest one, and the one used here, is the class provided in Caeiro *et al.* (2005), denoted $\overline{H}$ for the sake of simplicity, with the functional form,

$$\hat{\gamma}^{\overline{H}}_{n,k,\hat{\beta},\hat{\rho}} = \hat{\gamma}^{\overline{H}}_{n,k,\hat{\beta},\hat{\rho}}(\underline{X}) := \hat{\gamma}^{H}_{n}\left(1 - \frac{\hat{\beta}}{1-\hat{\rho}}\left(\frac{n}{k}\right)^{\hat{\rho}}\right), \tag{1.7}$$

where $\hat{\beta}$ and $\hat{\rho}$ are adequate consistent estimators of the second order parameters β and ρ, respectively, to be specified later on in sub-section 3.2. We shall also consider a bias-corrected moment estimator, given by

$$\hat{\gamma}^{\overline{M}}_{n,k,\hat{\beta},\hat{\rho}} = \hat{\gamma}^{\overline{M}}_{n,k,\hat{\beta},\hat{\rho}}(\underline{X}) := \hat{\gamma}^{M}_{n}\left(1 - \frac{\hat{\beta}}{1-\hat{\rho}}\left(\frac{n}{k}\right)^{\hat{\rho}}\right) - \frac{\hat{\beta}\,\hat{\rho}}{(1-\hat{\rho})^2}\left(\frac{n}{k}\right)^{\hat{\rho}}. \tag{1.8}$$

However, the main classes of estimators considered in this paper are, just as the quantile estimators in Araújo Santos, Fraga Alves and Gomes (2006), functionals of a sample of excesses over a random threshold $X_{[np]+1:n}$, i.e., functionals of

$$\underline{X}^{(p)}_{k} := \left(X_{n:n} - X_{[np]+1:n}, X_{n-1:n} - X_{[np]+1:n}, \cdots, X_{n-k:n} - X_{[np]+1:n}\right), \tag{1.9}$$

with $1 \leq k < n - [np] - 1$, and where

- $0 < p < 1$, for any $F \in \mathcal{D}_{\mathcal{M}}(EV_{\gamma>0})$ (*the random threshold is an empirical quantile*);
- $p = 0$, for d.f.'s with finite left endpoint $x_F := \inf\{x : F(x) > 0\}$ (*the random threshold is the minimum*).

These new classes of extreme value index estimators are the so-called PORT-Hill estimators, also denoted $H(p)$, and the PORT-moment estimators, also denoted $M(p)$, theoretically studied, for heavy tails, in Araújo Santos *et al.* (2006). They are denoted

$$\hat{\gamma}_{n,k}^{T(p)} := \hat{\gamma}_n^T(\underline{X}_k^{(p)}) \quad 0 \le p < 1, \text{ with } T = H \text{ and } M, \tag{1.10}$$

where $\hat{\gamma}_{n,k}^H$, $\hat{\gamma}_{n,k}^M$ and $\underline{X}_k^{(p)}$ are provided in (1.4), (1.5) and (1.9), respectively. The estimators in (1.10) are now invariant for both changes of scale and location in the data, and depend on the *tuning parameter* p, that provides a highly flexible class of extreme value index estimators, which may even compare favorably with the MVRB extreme value index estimators, provided that we adequately choose the tuning parameter p. The choice $p = 0$ is appealing in practice, but should be used with care, as it can induce a problem of sub-estimation (see Figures 3.7–3.10).

In Section 2, and to motivate the need of new estimation procedures like the above mentioned *PORT* methodology, we study the behavior of the classical tail index estimators in the presence of shifts, for data generated from the Pareto Model. As expected, we easily come to the *Hill horror plots* associated not only to slowly varying functions $L_U(t) = t^{-\gamma}U(t)$ that go towards infinity or zero (Resnick, 1997), but also to shifts in the strict Pareto model. In Section 3, we provide the asymptotic properties of the estimators under study and, through simulation experiments, we compare the exact performance of the new estimators in (1.10) with the classical Hill and moment estimators in (1.4) and (1.5), respectively, as well as with the reduced-bias extreme value index estimators in (1.7) and in (1.8). Finally, in Section 4 we provide an illustration of the behavior of the estimators for a set of real data in the field of finance.

2 A simple motivation for the $PORT$-methodology: shifts in the Pareto model

It is worth looking at the special case of Pareto$(\gamma, \delta)\equiv$ Pareto$(\gamma, \lambda = 0, \delta)$ random variables (r.v.'s) X, with d.f.

$$F_X(x; \gamma, \delta) = 1 - (x/\delta)^{-1/\gamma}, \; x > \delta, \; \delta > 0. \tag{2.1}$$

For this underlying d.f., the Hill estimator $H|0 \equiv \hat{\gamma}_{n,k}^{H|0} \equiv \hat{\gamma}_{n,k}^H|\lambda = 0$ is unbiased for any sequence of integers $k = k_n$, $1 \le k_n < n$, and the quantile function is $U(t) = \delta t^\gamma$, $t \ge 1$. So, for this special model, the Hill estimator in (1.4) performs very well. Indeed, for the model in (2.1), the two errors due to the estimation of γ and to the approximation taken for $U(t)$ do not intervene. However, a small shift in the data may lead to disastrous results, even in this simple and specific case, as we shall see in the following.

For any r.v. X, with quantile function $U_X(t)$, the transformed r.v. $Z = X + \lambda$ has an associated quantile function given by $U_Z(t) = U_X(t) + \lambda$. This means that if X comes from the d.f. $F_X(x; \gamma, \delta)$ in (2.1), the r.v. $Z = X + \lambda$ is a Pareto$(\gamma, \lambda, \delta)$, with d.f.

$$F_Z(z; \gamma, \lambda, \delta) = 1 - \left(\frac{z - \lambda}{\delta}\right)^{-1/\gamma}, \; z > \lambda + \delta, \; \delta > 0. \tag{2.2}$$

The quantile function of the r.v. Z is then

$$U_Z(t) = \delta t^\gamma + \lambda = \delta t^\gamma (1 + \lambda t^\rho / \delta), \quad \rho := -\gamma.$$

The extra term $\lambda t^\rho/\delta$, in the slowly varying function $1 + \lambda t^\rho/\delta$, affects the first approximation $U_Z(t) \sim \delta t^\gamma$, with severe consequences in the two above mentioned errors, as illustrated in the sequel by simulation. We have first generated a sample of size $n = 1000$ from an i.i.d. Pareto$(1, 0, 1)$ model. At a second stage, we introduce a positive shift, i.e., we consider a random generation of $Z = X + \lambda$, from a Pareto$(1, \lambda, 1)$ model, $\lambda = 10$, and a similar analysis has been done.

On the left side of Figure 2.1, for the Pareto$(1, 0, 1)$ parent, we compare the two classical semi-parametric procedures for the estimation of $\gamma = 1$, providing sample paths of the Hill estimator $H \equiv H|0$ in (1.4) and the moment estimator $M \equiv M|0$ in (1.5), together with the PORT-Hill $H(0) \equiv H(0)|0$ and the PORT-moment $M(0) \equiv M(0)|0$ in (1.10), associated thus to the sample $X_{i:n} - X_{1:n}$ ($p = 0$ in (1.9)), $2 \le i \le n$. On the right side of Figure 2.1 we picture equivalent results, but for a Pareto$(1, \lambda, 1)$ underlying parent, with $\lambda = 10$.

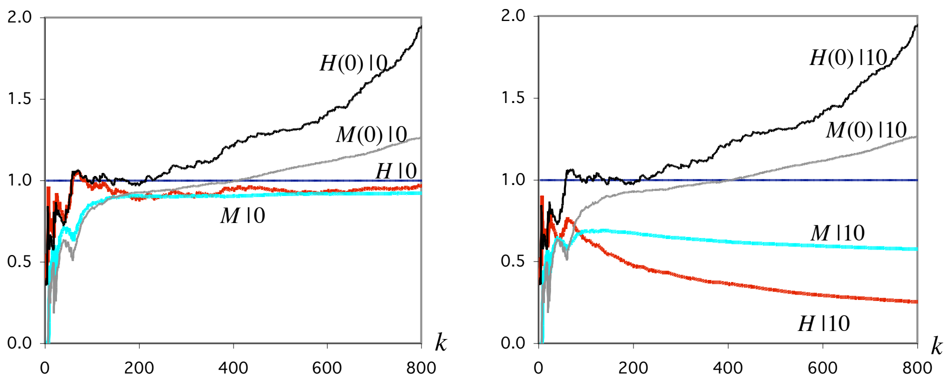

Figure 2.1: Sample paths of $\hat\gamma^{H|\lambda}$, $\hat\gamma^{M|\lambda}$, $\hat\gamma^{H(0)|\lambda}$ and $\hat\gamma^{M(0)|\lambda}$, for sample size $n = 1000$, from the Pareto model in (2.2) with $\gamma = 1$, $\lambda = 0$, $\delta = 1$ (*left*) and with $\gamma = 1$, $\lambda = 10$, $\delta = 1$ (*right*).

Note first that with estimators like the ones in (1.10) we get always the same sample path for any shift on the data. Next, it is evident from the left graphic of Figure 2.1 the unbiased property of the Hill estimator for the Pareto model in (2.1). On the other hand, the right graphic of this figure also illustrates the disastrous results we may achieve with shifted data and the classical estimates. In particular, we enhance the fact that "flat" stable zones of the graphs, based on the shifted data, may lead us to dangerous misleading conclusions: wrong underestimated tail index, for instance. Indeed, "stable zones" in the sample path of any estimator of a parameter of extreme events, need to be carefully identified. The above mentioned results clearly indicate that in practice we should take care with the Pareto approximation $U(t) \sim \delta t^\gamma$ and that the estimators under study in this paper, i.e., the estimators in (1.10), are of high practical importance.

3 Distributional behaviour of the estimators under comparison

3.1 A brief reference to their asymptotic behaviour

In order to obtain a non-degenerate behaviour for any extreme value index estimator, under a semi-parametric framework, it is conveniente to assume a second order condition, measuring the rate of convergence in the first order condition in (1.2). Such a condition involves a non-positive parameter ρ, and can be given by

$$\lim_{t\to\infty} \frac{U(tx)/U(t) - x^\gamma}{A(t)} = x^\gamma \frac{x^\rho - 1}{\rho}, \tag{3.1}$$

for all $x > 0$, where $A(\cdot)$ is a suitably chosen function of constant sign near infinity. Then, $|A| \in RV_\rho$ and ρ is a second order parameter (Geluk and de Haan, 1987). For the strict Pareto model, with tail function $\overline{F}(x) := 1 - F(x) = (x/\delta)^{-1/\gamma}$ and quantile function $U(t) = \delta t^\gamma$, $U(tx)/U(t) - x^\gamma \equiv 0$. We may then say that (3.1) holds with $A(t) \equiv 0$.

Here, and mainly because of the reduced-bias estimators in (1.7) and (1.8), we shall more restrictively assume that F belongs to the wide class of Hall (1982), that is, the associated quantile function U satisfies

$$U(t) = \delta t^\gamma (1 + \gamma\ \beta t^\rho/\rho + o(t^\rho)), \quad \rho < 0, \quad \gamma, \delta > 0, \quad \beta \in \mathbb{R}, \quad \text{as} \quad t \to \infty, \tag{3.2}$$

or equivalently, (3.1) holds, with $A(t) = \gamma\beta t^\rho$. The strict Pareto model appears when both β and the remainder term $o(t^\rho)$ are null. For the classical H and M estimators, generally denoted T, we know that for any intermediate sequence k as in (1.3) and under the validity of the second order condition in (3.1),

$$\hat\gamma^T_{n,k} \overset{d}{=} \gamma + \frac{\sigma_{_T} P^T_k}{\sqrt{k}} + c_{_T} A(n/k)\big(1 + o_p(1)\big), \tag{3.3}$$

where

$$\sigma_{_H} = \gamma,\ c_{_H} = \frac{1}{1-\rho}, \qquad \sigma_{_M} = \sqrt{\gamma^2+1},\ c_{_M} = \frac{\gamma(1-\rho)+\rho}{\gamma(1-\rho)^2} = \frac{1}{1-\rho} + \frac{\rho}{\gamma(1-\rho)^2}, \tag{3.4}$$

being P_k^T ($T = H$ or M) asymptotically standard normal r.v.'s (de Haan and Peng, 1998).

We may now generalize Theorem 3.1 in Caeiro *et al.* (2005), where it is possible to find a proof of the following theorem for the estimator $\hat{\gamma}^{\overline{H}}_{n,k,\hat{\beta},\hat{\rho}}$ in (1.7). Let T generically denote either H or M.

Theorem 3.1. *For any intermediate sequence k as in* (1.3)*, for models in* (3.2)*, for any $(\hat{\beta},\hat{\rho})$, consistent for the estimation of (β,ρ) and such that $(\hat{\rho}-\rho)\ln(n/k) = o_p(1)$, the asymptotic distributional representation*

$$\hat{\gamma}^{\overline{T}}_{n,k,\hat{\beta},\hat{\rho}} \stackrel{d}{=} \gamma + \frac{\sigma_{_T}\, P_k^T}{\sqrt{k}} + o_p(A(n/k))$$

holds both for $\hat{\gamma}^{\overline{H}}_n$ in (1.7) *as well as for $\hat{\gamma}^{\overline{M}}_n$ in* (1.8)*, where $\left(P_k^T, \sigma_{_T}\right)$ with $T = H$ and $T = M$ are given in* (3.3) *and* (3.4)*.*

Proof: If we estimate consistently β and ρ through the estimators $\widehat{\beta}$ and $\widehat{\rho}$ in the conditions of the theorem, we may use Cramer's delta-method, and write,

$$\begin{aligned}
\hat{\gamma}^{\overline{M}}_{n,k,\hat{\beta},\hat{\rho}}(k) \;&=\; \hat{\gamma}^{M}_{n,k}(k) \;\times\Big(1 - \frac{\beta}{1-\rho}\left(\frac{n}{k}\right)^{\rho} - (\widehat{\beta}-\beta)\frac{1}{1-\rho}\left(\frac{n}{k}\right)^{\rho}(1+o_p(1)) \\
&\quad - \frac{\beta}{1-\rho}(\widehat{\rho}-\rho)\left(\frac{n}{k}\right)^{\rho}\Big(\frac{1}{1-\rho} + \ln(n/k)\Big)(1+o_p(1))\Big) - \frac{\beta\,\rho}{(1-\rho)^2}\left(\frac{n}{k}\right)^{\rho} \\
&\quad - \Big\{(\hat{\beta}-\beta)\frac{\rho}{(1-\rho)^2}\left(\frac{n}{k}\right)^{\rho} + \frac{\beta(\hat{\rho}-\rho)}{1-\rho}\left(\frac{n}{k}\right)^{\rho}\Big(\frac{\rho\ln(n/k)}{1-\rho} + 3 - \rho\Big)\Big\}(1+o_p(1)) \\
&\stackrel{d}{=} \hat{\gamma}^{\overline{M}}_{n,k,\beta,\rho}(k) - \frac{A(n/k)}{1-\rho}\Big(\hat{\gamma}^{M}_{n,k}(k) - \frac{\rho}{1-\rho}\Big)\Big(\frac{\widehat{\beta}-\beta}{\beta} + (\widehat{\rho}-\rho)\;\ln(n/k)\Big)(1+o_p(1)).
\end{aligned}$$

The reasoning is then quite similar to the one used in Caeiro *et al.* (2005) for the $\overline{H}$-estimator. Since $\widehat{\beta}$ and $\widehat{\rho}$ are consistent for the estimation of β and ρ, respectively, and $(\widehat{\rho}-\rho)\ln(n/k) = o_p(1)$, the last summand is $o_p(A(n/k))$, and the result in the theorem, related to the $\overline{M}$-estimator, follows immediately. □

Finally, for the PORT-Hill and PORT-moment estimators in (1.10):

Theorem 3.2 (Araújo Santos *et al.*, 2006)**.** *For any intermediate sequence k as in* (1.3)*, under the validity of the second order condition in* (3.1) *and for any real p, $0 < p < 1$ or $p = 0$ provided that $x_F := F^{\leftarrow}(0)$ is finite, the asymptotic distributional representation*

$$\hat{\gamma}^{T(p)}_{n,k} \stackrel{d}{=} \gamma + \frac{\sigma_{_T} P_k^T}{\sqrt{k}} + \Big(c_{_T} A(n/k) + d_{_T}\frac{\chi_p}{U(n/k)}\Big)\Big(1 + o_p(1)\Big) \tag{3.5}$$

holds, where $\big(P_k^T, \sigma_{_T}, c_{_T}\big)$ *with* $T = H$ *and* $T = M$ *are given in* (3.3) *and* (3.4), *being* $\chi_p = F^{\leftarrow}(p) = U(1/p)$ $(\chi_0 = x_{_F})$. *Moreover,*

$$d_{_H} := \frac{\gamma}{\gamma+1} \qquad \textit{and} \qquad d_{_M} := \Big(\frac{\gamma}{\gamma+1}\Big)^2. \tag{3.6}$$

Corollary 3.1. *For the strict Pareto model in* (2.1), *the distributional representations* (3.5) *holds with* $A(t)$ *replaced by 0.*

Remark 3.1. *Note that as both* $d_{_H}$ *and* $d_{_M}$ *in* (3.6), *as well as* $U(t)$, *are positive, the dominant component of the bias of* $\hat\gamma_{n,k}^{T(p)}$, *given in* (3.5), *is increasing as a function of* p.

Remark 3.2. *Note also that if we induce a deterministic shift* λ *to data* X, *considering* $X+\lambda$, *i.e., if instead of working with data from a model* $F =: F_0$, *we work with the new model* $F_\lambda(x) := F_0(x-\lambda)$, *the associated* U*-quantile function changes to* $U_\lambda(t) = \lambda + \delta U_0(t) \equiv \lambda + U(t)$. *Then, if the second order condition* (3.1) *holds for* $F \equiv F_0$, *with an auxiliary function* $A(t) \equiv A_0(t)$, *we straightforwardly get*

$$\frac{U_\lambda(tx)}{U_\lambda(t)} = \frac{U(tx)}{U(t)}\left\{1 - \frac{\lambda\gamma}{U(t)}\Big(\frac{x^{-\gamma}-1}{-\gamma}\Big) + o\Big(\frac{1}{U(t)}\Big)\right\}.$$

Consequently,

$$\frac{U_\lambda(tx)}{U_\lambda(t)} - x^\gamma = x^\gamma\Big(A(t)\Big(\frac{x^\rho-1}{\rho}\Big) - \frac{\lambda\gamma}{U(t)}\Big(\frac{x^{-\gamma}-1}{-\gamma}\Big) + o(A(t)) + o(1/U(t))\Big),$$

and we get, for instance for the Hill estimator associated to this shift λ, *denoted* $\hat\gamma_{n,k}^{H|\lambda}$ *or* $H|\lambda$ *for the sake of simplicity, the distributional representation*

$$\hat\gamma_{n,k}^{H|\lambda} \stackrel{d}{=} \gamma + \frac{\sigma_{_H}}{\sqrt{k}} P_k^H + \Big(c_{_H} A(n/k) - d_{_H}\frac{\lambda}{U(n/k)}\Big)\big(1 + o_p(1)\big), \tag{3.7}$$

i.e., as expected, (3.5) *holds whenever we replace* $\hat\gamma_n^{H(p)}$ *by* $\hat\gamma_n^{H|\lambda}$, *provided that we replace* χ_p *by* $-\lambda$. *For details, see* Gomes and Oliveira (2003), *where the shift* λ *is regarded as a tuning parameter of the statistical procedure that leads to the tail index estimates. On the basis of the bias term associated with the Hill functional applied to shifted data, these authors have found easily a justification for some kind of "magic numbers", like* $\lambda = 0.5$, *appearing for a Fréchet model, with tail function* $1-F(x) = 1-\exp\big(-x^{-1/\gamma}\big)$, $x>0$, *and* $\lambda = 1/\gamma$, *appearing for a generalized Pareto (GP) model, with tail function* $1-F(x) = (1+\gamma x)^{-1/\gamma}$, $x>0$ $(\gamma>0)$. *Indeed, from a theoretical point of view, let us assume we are working in Hall's class of distributions, where*

$$1 - F(x) = Cx^{-1/\gamma}\Big(1 + Dx^{\rho/\gamma}(1+o(1))\Big), \ \textit{as } x \to \infty.$$

Then, regular variation theory (Bingham, Goldie and Teugels, 1987) *enables us to obtain the asymptotic inverse of* F*,*

$$U(t) := F^{\leftarrow}(1-1/t) = (Ct)^{\gamma}\big(1+\gamma D(Ct)^{\rho}(1+o(1))\big), \ \text{as } t\to\infty,$$

and we may choose any A *function, such that* $A(t) \sim \gamma\rho D(Ct)^{\rho}$, *as* $t \to \infty$.

Whenever $\rho = -\gamma$*, we may thus choose* $A(t)$ *such that*

$$A(t)U(t) = -\gamma^2 D, \quad i.e. \quad 1/U(t) = -A(t)/(\gamma^2 D).$$

If we look at (3.7) *we see that the dominant component of asymptotic bias is then given by* $(A(n/k) - \lambda\gamma/U(n/k))/(1+\gamma) = A(n/k)(1+\lambda/(\gamma D))/(1+\gamma)$*. Such a component is thus null whenever* $\lambda = -\gamma D$*.*

The Fréchet model belongs to Hall's class, with $C = 1$*,* $D = -1/2$ *and* $\rho = -1$*. Then, for* $\gamma = 1$*,* $\lambda = 0.5$ *enables us to remove the main component of asymptotic bias. If we think on a GP model, we are again in Hall's class of models with* $C = \gamma^{-1/\gamma}$*,* $D = -1/\gamma^2$ *and* $\rho = -\gamma$*. Then, for every* γ *if we induce in the data a shift* $\lambda = -\gamma D = 1/\gamma = -1/\rho$ *we are able to remove the dominant component of asymptotic bias.*

Remark 3.3. *The comments in Remark 3.2 are also true for the classical moment estimator, i.e., if we induce a shift* λ *to the data,* (3.5) *holds whenever we replace* $\hat{\gamma}_n^{M(p)}$ *by* $\hat{\gamma}_n^{M|\lambda}$*, provided that we replace* χ_p *by* $-\lambda$*. Moreover, also for the moment estimator the dominant component of asymptotic bias is null whenever in Hall's class of models, we have* $\rho = -\gamma$ *and we induce a shift* $\lambda = \rho D = -\gamma D$*.*

We still add the following:

Remark 3.4. *Let us now consider the general* EV_γ *model in* (1.1)*. Then, we may write*

$$1-F(x) = (\gamma x)^{-1/\gamma}\begin{cases} \big(1-\frac{1}{\gamma^2 x}+o(x^{-1})\big) & \text{if} \quad 0<\gamma<1 \\ \big(1-\frac{3}{2x}+o(x^{-1})\big) & \text{if} \quad \gamma=1 \\ \big(1-\frac{(\gamma x)^{-1/\gamma}}{2}+o(x^{-1/\gamma})\big) & \text{if} \quad \gamma>1, \end{cases}$$

i.e.

$$C=\gamma^{-1/\gamma}, \quad \rho = \begin{cases} -\gamma & \text{if} \quad 0<\gamma\le 1 \\ -1 & \text{if} \quad \gamma>1 \end{cases}, \quad D = \begin{cases} -1/\gamma^2 & \text{if} \quad 0<\gamma<1 \\ -3/2 & \text{if} \quad \gamma=1 \\ -\gamma^{-1/\gamma}/2 & \text{if} \quad \gamma>1. \end{cases}$$

For the EV_γ *model, with* $\gamma \le 1$*, we may thus get a second-order reduced-bias extreme value index estimator, on the basis of both the Hill and the moment functionals, in* (1.4) *and* (1.5)*, respectively, provided that we induce the deterministic shift*

$$\lambda = \begin{cases} 1/\gamma & \text{if} \quad 0<\gamma<1 \\ 3/2 & \text{if} \quad \gamma=1. \end{cases}$$

Note however that, with a deterministic shift, as suggested in Gomes and Oliveira (2003)*, the estimators loose even the scale invariance property.*

3.2 The estimation of second order parameters

For the estimation of the second order parameters, needed for the estimators in (1.7) and in (1.8), we suggest here an algorithm similar to the ones in Gomes *et al.* (2005), Gomes, Henriques Rodrigues, Vandewalle and Viseu (2006) and Gomes and Pestana (2007a, 2007b):

1. Given a sample $(X_1, X_2, \cdots, X_n)$, with the notation $a^{b\tau} = b \ln a$ whenever $\tau = 0$, and $M_{n,k}^{(r)}$ given in (1.6), plot, for $\tau = 0, 1$, the estimates

$$\hat\rho_\tau(k) := -\left|\frac{3\left(T_{n,k}^{(\tau)} - 1\right)}{\left(T_{n,k}^{(\tau)} - 3\right)}\right|, \quad \text{with } T_{n,k}^{(\tau)} := \frac{\left(M_{n,k}^{(1)}\right)^\tau - \left(M_{n,k}^{(2)}/2\right)^{\tau/2}}{\left(M_{n,k}^{(2)}/2\right)^{\tau/2} - \left(M_{n,k}^{(3)}/6\right)^{\tau/3}}, \tag{3.8}$$

2. Consider $\{\hat\rho_\tau(k)\}_{k\in\mathcal{K}}$, for integer values $k \in \mathcal{K} = \left(\left[n^{0.995}\right],\ \left[n^{0.999}\right]\right)$, and compute their median, denoted χ_τ, $\tau = 0, 1$. Choose

$$\tau^* := \begin{cases} 0 & \text{if} \quad \sum_{k\in\mathcal{K}} \left(\hat\rho_0(k) - \chi_0\right)^2 \le \sum_{k\in\mathcal{K}} \left(\hat\rho_1(k) - \chi_1\right)^2 \\ 1 & \quad\text{otherwise;} \end{cases}$$

3. Compute, for $k_1 = \left[n^{0.995}\right]$, $\hat\rho^* = \hat\rho(k_1; \tau^*)$ and $\hat\beta^* := \hat\beta(k_1;\ \hat\rho^*)$,

$$\hat\beta(k; r) := \left(\frac{k}{n}\right)^r \frac{d_k(-r) \times D_k(0) - D_k(-r)}{d_k(r) \times D_k(-r) - D_k(-2r)}, \tag{3.9}$$

where for any $\alpha \ge 0$, and with $W_i := i\left\{\ln X_{n-i+1,n} - \ln X_{n-i,n}\right\},\ 1 \le i \le k$,

$$D_k(\alpha) := \frac{1}{k}\sum_{i=1}^k (i/k)^\alpha\ W_i, \qquad d_k(\alpha) := \frac{1}{k}\sum_{i=1}^k (i/k)^\alpha. \tag{3.10}$$

Remark 3.5. *The implementation of this algorithm in practice leads often to $\tau^* = 0$ whenever $|\rho| \le 1$ and $\tau^* = 1$ whenever $|\rho| > 1$ (see Gomes and Pestana, 2007b). This is the reason why we are going to use such a rule in the simulations. The choices of $\mathcal{K}$ in step 2. and k_1 in step 3. are not crucial, provided that we restrict ourselves to reasonably large values of k, the number of o.s. used.*

Regarding the reduced-bias extreme value index estimators in (1.7) and (1.8), the estimators $(\widehat\beta_\tau, \widehat\rho_\tau)$ of (β, ρ), $\tau = 0, 1$, have been used, leading to

$$\overline{H}_\tau \equiv \overline{H}_\tau(k) \equiv \hat\gamma^{\overline{H}}_{n,k,\widehat\beta_\tau,\widehat\rho_\tau}, \quad \overline{M}_\tau \equiv \overline{M}_\tau(k) \equiv \hat\gamma^{\overline{M}}_{n,k,\widehat\beta_\tau,\widehat\rho_\tau}, \qquad \tau = 0, 1.$$

The simulations in Caeiro *et al.* (2005) and Gomes and Pestana (2007b) show that the tail index estimators $\overline{H}_\tau$, with τ equal to either 0 or 1, according as $|\rho| \le 1$ or $|\rho| > 1$, work quite well. The use of $\tau = 1$ always enables us to achieve a better performance than the one we get with the Hill estimator H. In a "blind" way, we might thus advise such a choice, and we shall do it for the reduced-bias moment estimator $\overline{M}_\tau$. But for $\overline{H}_\tau$, $\tau = 0$ provides much better results than $\tau = 1$ whenever $|\rho|$, unknown, is smaller than or equal to 1.

3.3 Simulated behaviour of the tail index estimators

We have implemented multi-sample Monte Carlo simulation experiments of size 5000×10 for the extreme value index estimators under study.

3.3.1 Mean values and mean squared error patterns of the tail index estimators

In Figure 3.1, for samples of size $n = 1000$ from a Fréchet(γ), with $\gamma = 1$, we show the simulated patterns of the mean values, $E[\bullet]$, and mean squared errors, $MSE[\bullet]$, of the Hill estimator H in (1.4) and its location invariant versions $H(p)$, $p = 0,\ 0.25$ and 0.5, in (1.10), together with the ones of the MVRB estimators $\overline{H}_0$ in (1.7). Figure 3.2 is similar to Figure 1, but for the moment estimator M in (1.5), its location invariant versions $M(p)$, $p = 0,\ 0.25$ and 0.5, in (1.10) and the MRVB estimator $\overline{M}_1$ in (1.8). The mean values and mean squared errors of the estimators are based on the first replicate, with a run of size 5000.

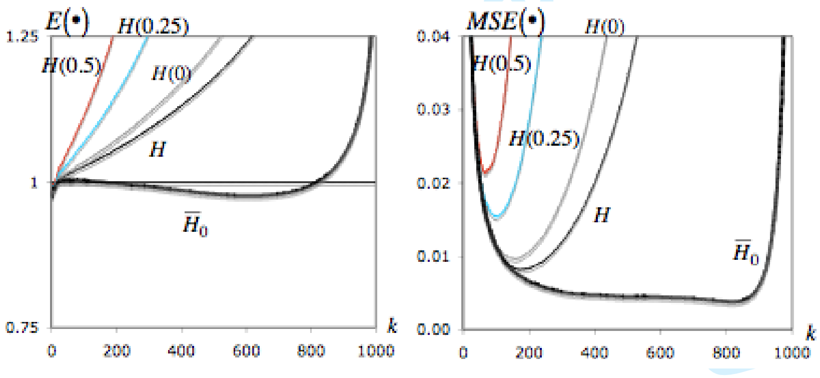

Figure 3.1: Simulated mean values (*left*) and mean squared errors (*right*) of the Hill estimator H in (1.4) and $H(p)$ $p = 0, 0.25$ and 0.5 in (1.10), together with $\overline{H}_0$ in (1.7), for samples of size $n = 1000$ from a Fréchet parent with $\gamma = 1$ ($\rho = -1$).

Figures 3.3 and 3.4 are equivalent to Figures 3.1 and 3.2, respectively, but for the EV_γ model in (1.1), with $\gamma = 0.25$. Similar comment applies to Figures 3.5 and 3.6, where we consider the underlying parent EV_γ, with $\gamma = 1$.

Finally, the pairs of Figures 3.7, 3.8 and Figures 3.9, 3.10 are equivalent to the pair of Figures 3.1, 3.2, but for Student t_ν, with $\nu = 4$ and $\nu = 2$, respectively. The Student t_ν probability

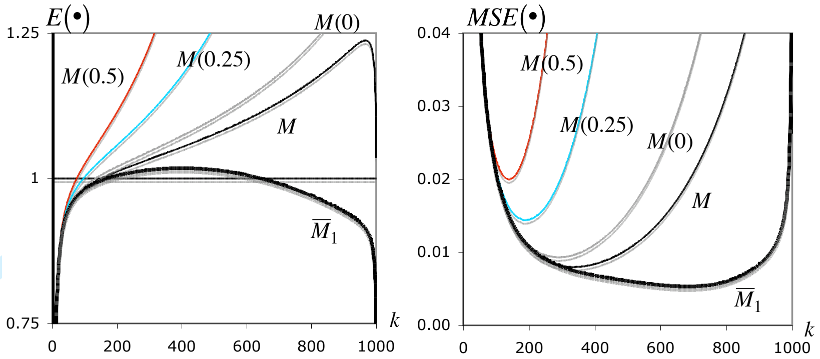

Figure 3.2: Simulated mean values (*left*) and mean squared errors (*right*) of M and $M(p)$ $p = 0, 0.25$ and 0.5 in (1.10), together with $\overline{M}_1$ in (1.8), for samples of size $n = 1000$ from a Fréchet parent with $\gamma = 1$ ($\rho = -1$).

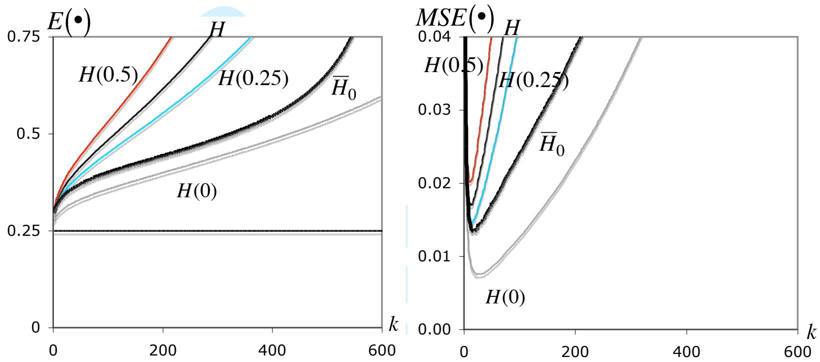

Figure 3.3: Simulated mean values (*left*) and mean squared errors (*right*) of H and $H(p)$ $p = 0, 0.25$ and 0.5, together with $\overline{H}_0$, for samples of size $n = 1000$ from a EV_γ parent with $\gamma = 0.25$ ($\rho = -0.25$).

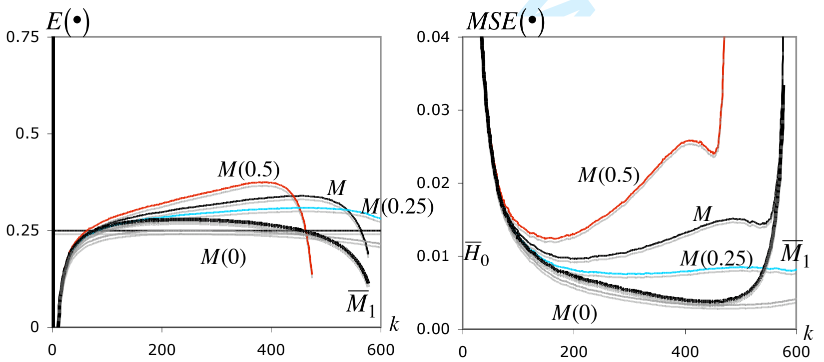

Figure 3.4: Simulated mean values (*left*) and mean squared errors (*right*) of M and $M(p)$ $p = 0, 0.25$ and 0.5, together with $\overline{M}_1$, for samples of size $n = 1000$ from a EV_γ parent with $\gamma = 0.25$ ($\rho = -0.25$).

density function (p.d.f.) is

$$f_\nu(x) = \Gamma((\nu+1)/2)\left[1 + x^2/(\nu-2)\right]^{-(\nu+1)/2} \Big/ \left(\sqrt{\pi(\nu-2)}\;\Gamma(\nu/2)\right),\ x \in \mathbb{R}.$$

For the Student t_ν model, we get $\gamma = 1/\nu$ and $\rho = -2/\nu$.

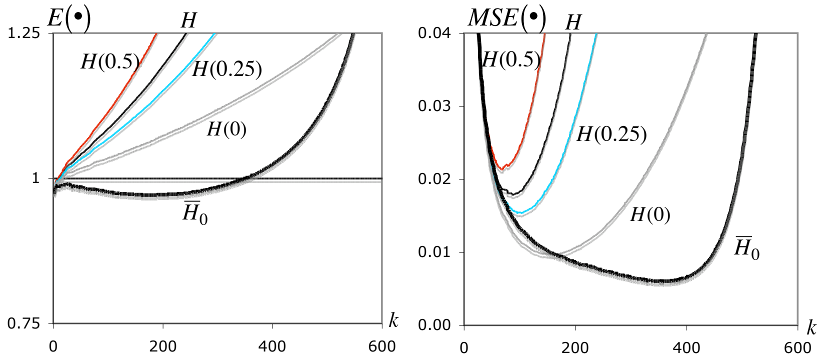

Figure 3.5: Simulated mean values (*left*) and mean squared errors (*right*) of H and $H(p)$ $p = 0, 0.25$ and 0.5, together with $\overline{H}_0$, for samples of size $n = 1000$ from a EV_γ parent with $\gamma = 1$ $(\rho = -1)$.

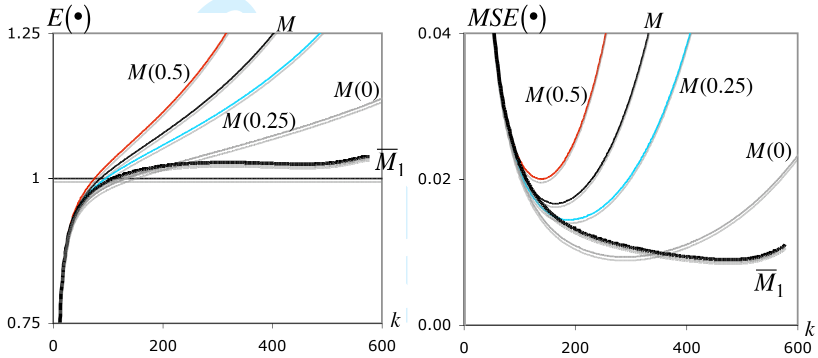

Figure 3.6: Simulated mean values (*left*) and mean squared errors (*right*) of M and $M(p)$ $p = 0, 0.25$ and 0.5, together with $\overline{M}_1$, for samples of size $n = 1000$ from a EV_γ parent with $\gamma = 1$ $(\rho = -1)$.

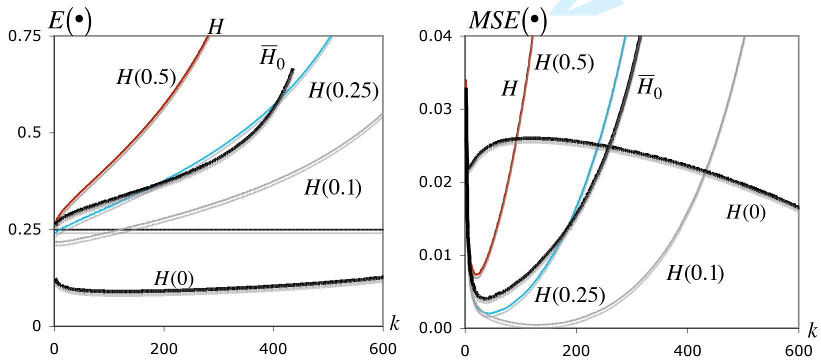

Figure 3.7: Simulated mean values (*left*) and mean squared errors (*right*) of H and $H(p)$ $p = 0, 0.1, 0.25$ and 0.5, together with $\overline{H}_0$, for samples of size $n = 1000$ from a t_4 parent with $\gamma = 0.25$ $(\rho = -0.5)$.

We may draw the following specific comments:

- As expected, on the basis of Remark 3.1, $H(p)$ and $M(p)$ are increasing in p. However,

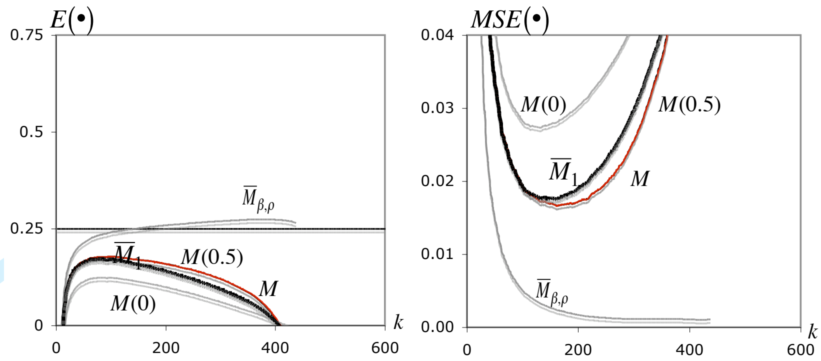

Figure 3.8: Simulated mean values (*left*) and mean squared errors (*right*) of M and $M(p)$ $p = 0$ and 0.5, together with $\overline{M}_1$ and the r.v. $\overline{M}_{\beta,\rho}$, for samples of size $n = 1000$ from a t_4 parent with $\gamma = 0.25$ ($\rho = -0.5$).

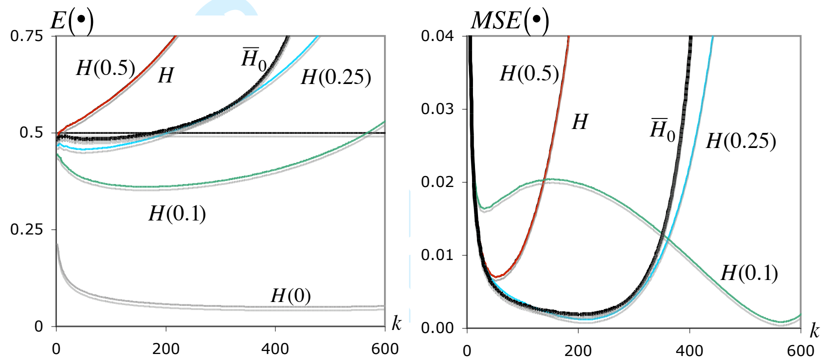

Figure 3.9: Simulated mean values (*left*) and mean squared errors (*right*) of H and $H(p)$ $p = 0, 0.1, 0.25$ and 0.5, together with $\overline{H}_0$, for samples of size $n = 1000$ from a t_2 parent with $\gamma = 0.5$ ($\rho = -1$).

and with T generally denoting either H or M, we expect to have $T < T(0)$ if the left endpoint x_F of the underlying model F is zero, but things work the other way round, i.e., $T(0) < T$ if $x_F \neq 0$.

- For a Fréchet model, and perhaps as expected more generally, if we induce a shift (random shift) through a central o.s. (or even the minimum, equal to 0), applying the Hill or the moment functionals to $X_i - X_{[np]+1:n}$, $1 \leq i \leq n$, $0 \leq p < 1$, we get worse results than when we work with either the Hill or the moment estimators, respectively. This result is not astonishing in the sense that we are replacing estimators that are only scale invariant by scale and location invariant estimators. Indeed, from the results in Gomes and Oliveira (2003), we know that, concerning the Hill estimator, we should shift our data from X towards $X + 0.5$ in order to remove the dominant component of bias of the Hill estimator, and $-0.5 < x_F = 0$. But then, we are working with estimators that are neither invariant for changes in scale nor location.

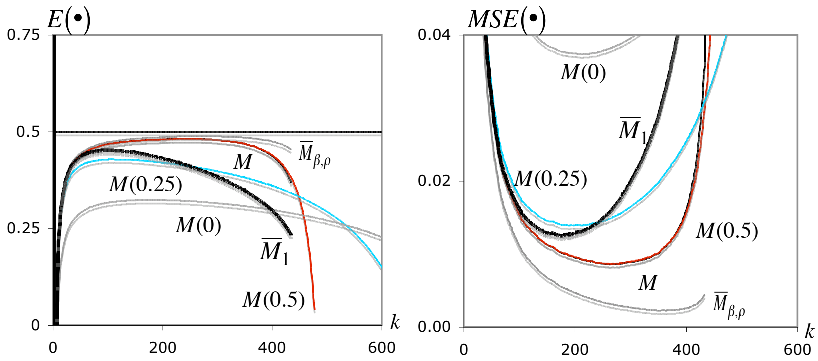

Figure 3.10: Simulated mean values (*left*) and mean squared errors (*right*) of M and $M(p)$ $p = 0, 0.25$ and 0.5, together with $\overline{M}_1$ and the r.v. $\overline{M}_{\beta,\rho}$, for samples of size $n = 1000$ from a t_2 parent with $\gamma = 0.5 (\rho = -1)$.

- As mentioned before, for the EV_γ model, with $0 < \gamma < 1$, we have $\rho = -\gamma$, and with a shift $\lambda = 1/\gamma = -1/\rho$ we would remove the dominant component of bias of Hill's as well as moment's estimators. This means that we should apply the Hill or the moment functionals to $X - x_{_F} = X + 1/\gamma$. Given that $X_{1:n} \to x_{_F} = -1/\gamma$, we expect to be reasonably close to a reduced-bias extreme value index estimator whenever we apply Hill's or moment's functionals to $X_i - X_{1:n}$, $1 \le i \le n$. If we look at Figures 3.3 and 3.4, we see that $H(0)$ and $M(0)$ behave even better than the corresponding MVRB-estimators.

  For the EV_1 model, the shift that would reduce the dominant component of bias would be induced by $\lambda = 3/2$. We should thus go below the minimum, given that $x_{_F} = -1$, and our estimator would no longer be location invariant (nor scale invariant). The statistics $H(0)$ and $M(0)$ are the best ones among the non reduced-bias estimators, but the corresponding MVRB estimators behave better than either $H(0)$ or $M(0)$.

  For the EV_γ model with $\gamma > 1$, although we have $\rho \neq -\gamma$, the relative behaviour of the PORT-estimators is quite similar to the one appearing when $\gamma = 1$. The location invariant estimators $H(p)$, $p \le 0.25$, behave better than the Hill, although not better than the MVRB-estimator $\overline{H}_0$.

- We have decided to consider also Student t_ν parents with ν degrees of freedom. Then, we have $\rho \neq -\gamma$. These parents have infinite left and right endpoints, and consequently, it is no longer sensible to consider $p = 0$ in the PORT-estimators, because of the possible non-consistency of the associated PORT-statistics. We did it merely to draw the attention for the erroneous conclusions we may take from a quite common behaviour in data analysis practice. Indeed, a usual solution to take care of the Pareto approximation $U(t) \sim \delta t^\gamma$ is to make statistical inference only after a suitable shift of the data. In the literature, it has been sometimes suggested to subtract a random quantity, usually the minimum of the sample. This shifted data set has the advantage of working out with usually more non-negative values, a desirable property for classical semi-parametric estimators of a positive tail index. An extensive discussion about this type of shifted procedures can be found for instance in Drees (2003). Therein, it is studied the effect of subtracting the minimum of the sample, previously to the subsequent analysis of the Nasdaq Composite index log-returns data set, in the context of VaR estimation. In fact, for that particular data, it is therein observed that this procedure constitutes a considerable improvement, arising for the Hill γ-estimates a larger flat zone in the associated sample path, after transforming the original data through the subtraction of the smallest observation. However, if we look at Figures 3.7 and 3.9, we easily see that “flat” zones in the sample path of the shifted-Hill (by the minimum) estimator can lead to serious underestimation of the extreme value index.

3.3.2 Mean values of the tail index estimators at optimal levels

Tables 1, 2 and 3 are related to underlying models with $|\rho| < 1$, $\rho = -1$ and $|\rho| > 1$, respectively. We shall there present, for $n = 200,\ 500,\ 1000,\ 2000$ and 5000, the simulated mean values at optimal levels (levels where mean squared errors are minima as functions of k) of the Hill estimator H in (1.4), the moment estimator M in (1.5), the $MVRB$-estimators, $\overline{H}_0$, $\overline{M}_1$, in (1.7) , (1.8), respectively, and the PORT-Hill and moment estimators in (1.10) associated with $p = 0,\ 0.1,\ 0.25$ and 0.5. Information on 95% confidence intervals, computed on the basis of the 10 replicates with 5000 runs each, is also provided. Among the estimators considered, the one providing the smallest squared bias is underlined and in **bold**.

3.3.3 Mean squared errors and relative efficiency indicators at optimal levels

We shall compute Hill's estimator at the simulated value of $k_0^H := \arg\min_k MSE\left[\hat{\gamma}_{n,k}^H\right]$, the simulated optimal k in the sense of minimum mean squared error, not relevant in practice, but providing an indication of the best possible performance of Hill's estimator. Such an estimator will be denoted H_0.

Let us generically denoted T any of the extreme value index estimators under study. We shall now compute T_0, the estimator T computed at its simulated optimal level, again in the sense of minimum mean squared error. The simulated indicators are

$$REFF_{T|H} := \sqrt{\frac{MSE\left[H_0\right]}{MSE\left[T_0\right]}}. \tag{3.11}$$

Remark 3.6. *An indicator higher than one means a better performance than the Hill estimator. Consequently, the higher these indicators are, the better the T_0-estimators perform, comparatively to H_0.*

Table 1: Simulated mean values, at optimal levels, of H, $\overline{H}_0$, $\overline{M}_1$, $H(p)$ and $M(p)$, $p = 0, 0.1, 0.25$ and 0.5 for parents with $|\rho| < 1$.

n	200	500	1000	2000	5000
Student $\mathbf{t_4}$ ($\gamma = \mathbf{0.25}$, $\rho = -\mathbf{0.5}$)					
H	0.3402 ± 0.0792	0.3409 ± 0.0636	0.3205 ± 0.0357	0.3062 ± 0.0486	0.2856 ± 0.0253
M	0.0845 ± 0.1027	0.1538 ± 0.0855	0.1630 ± 0.0495	0.1821 ± 0.0496	0.1916 ± 0.0301
$\overline{H}_0$	0.3231 ± 0.0521	0.3010 ± 0.0453	0.2876 ± 0.0276	0.2919 ± 0.0350	0.2862 ± 0.0205
$\overline{M}_1$	0.0799 ± 0.0722	0.1313 ± 0.0774	0.1595 ± 0.0592	0.1704 ± 0.0533	0.2019 ± 0.0223
$H(0)$	0.2735 ± 0.0504	0.1892 ± 0.0569	0.1889 ± 0.0298	0.1534 ± 0.0227	0.1040 ± 0.0145
$H(0.1)$	**0.2639** ± 0.0216	**0.2645** ± 0.0145	**0.2561** ± 0.0136	**0.2608** ± 0.0060	**0.2576** ± 0.0070
$H(0.25)$	0.2937 ± 0.0328	0.2766 ± 0.0345	0.2645 ± 0.0195	0.2721 ± 0.0241	0.2633 ± 0.0146
$H(0.5)$	0.3450 ± 0.0814	0.3410 ± 0.0664	0.3186 ± 0.0370	0.3059 ± 0.0477	0.2853 ± 0.0253
$M(0)$	0.0389 ± 0.0899	0.0978 ± 0.0720	0.1154 ± 0.0558	0.1385 ± 0.0603	0.1643 ± 0.0315
$M(.1)$	0.0474 ± 0.0908	0.1208 ± 0.0714	0.1311 ± 0.0586	0.1589 ± 0.0606	0.1816 ± 0.0302
$M(.25)$	0.0635 ± 0.0949	0.1210 ± 0.0794	0.1497 ± 0.0652	0.1593 ± 0.0666	0.1943 ± 0.0249
$M(.5)$	0.0888 ± 0.1075	0.1549 ± 0.0872	0.1623 ± 0.0496	0.1816 ± 0.0498	0.1914 ± 0.0299
EV$_\gamma$ ($\gamma = \mathbf{0.25}$) ($\rho = -\mathbf{0.25}$)					
H	0.3754 ± 0.0806	0.3910 ± 0.0951	0.3370 ± 0.0585	0.3909 ± 0.0801	0.3237 ± 0.0333
M	0.3473 ± 0.0957	0.2489 ± 0.0956	0.2923 ± 0.0718	0.3077 ± 0.0499	0.2957 ± 0.0350
$\overline{H}_0$	0.4026 ± 0.0903	0.3396 ± 0.0522	0.3648 ± 0.05970.	0.3884 ± 0.0768	0.3230 ± 0.0394
$\overline{M}_1$	0.2449 ± 0.0722	0.2012 ± 0.0955	0.2618 ± 0.0411	0.2834 ± 0.0338	0.2617 ± 0.0182
$H(0)$	0.3710 ± 0.0692	0.3120 ± 0.0553	0.3242 ± 0.0479	0.3434 ± 0.0431	0.2990 ± 0.0290
$H(0.1)$	0.3808 ± 0.0750	0.3335 ± 0.0716	0.3606 ± 0.0576	0.3772 ± 0.0760	0.3218 ± 0.0326
$H(0.25)$	0.3842 ± 0.0806	0.3739 ± 0.0870	0.3562 ± 0.0617	0.3904 ± 0.0849	0.3206 ± 0.0337
$H(0.5)$	0.3847 ± 0.0765	0.4274 ± 0.1251	0.3722 ± 0.0745	0.3848 ± 0.0691	0.3255 ± 0.0496
$M(0)$	0.2088 ± 0.0581	0.2223 ± 0.0595	**0.2471** ± 0.0423	**0.2650** ± 0.0342	**0.2514** ± 0.0107
$M(.1)$	**0.2406** ± 0.0649	0.2700 ± 0.0628	0.2946 ± 0.0453	0.2875 ± 0.0429	0.2652 ± 0.0191
$M(.25)$	0.3136 ± 0.0742	**0.2469** ± 0.1100	0.3042 ± 0.0633	0.3064 ± 0.0545	0.2732 ± 0.0273
$M(.5)$	0.3820 ± 0.0993	0.2822 ± 0.0750	0.3066 ± 0.0729	0.3226 ± 0.0597	0.3067 ± 0.0405
GP$_\gamma$ ($\gamma = \mathbf{0.5}$) ($\rho = -\mathbf{0.5}$)					
H	0.5938 ± 0.1056	0.6289 ± 0.0777	0.5993 ± 0.0553	0.5690 ± 0.0392	0.5366 ± 0.0463
M	0.5559 ± 0.1575	0.5814 ± 0.0916	0.5805 ± 0.0551	0.5693 ± 0.0371	0.5245 ± 0.0384
$\overline{H}_0$	0.5889 ± 0.0805	0.6004 ± 0.0648	0.5897 ± 0.0458	0.5864 ± 0.0227	0.5339 ± 0.0251
$\overline{M}_1$	0.5769 ± 0.1488	0.5908 ± 0.0877	0.5891 ± 0.0491	0.5794 ± 0.0319	0.5270 ± 0.0367
$H(0)$	0.5941 ± 0.1057	0.6290 ± 0.0777	0.5993 ± 0.0553	0.5690 ± 0.0391	0.5366 ± 0.0463
$H(0.1)$	0.5928 ± 0.1163	0.6344 ± 0.0846	0.5991 ± 0.0533	0.5745 ± 0.0421	0.5420 ± 0.0467
$H(0.25)$	0.6266 ± 0.1150	0.6270 ± 0.0919	0.6148 ± 0.0547	0.5643 ± 0.0329	0.5452 ± 0.0679
$H(0.5)$	0.6474 ± 0.1651	0.6280 ± 0.0854	0.6003 ± 0.0663	0.5910 ± 0.0529	0.5466 ± 0.0798
$M(0)$	0.5564 ± 0.1577	**0.5814** ± 0.0916	**0.5805** ± 0.0551	**0.5681** ± 0.0381	0.5245 ± 0.0384
$M(.1)$	0.5518 ± 0.1502	0.5861 ± 0.0889	0.5836 ± 0.0552	0.5722 ± 0.0385	0.5220 ± 0.0409
$M(.25)$	**0.5266** ± 0.1700	0.5924 ± 0.890	0.5828 ± 0.0593	0.5773 ± 0.0379	0.5213 ± 0.0466
$M(.5)$	0.5302 ± 0.1998	0.6043 ± 0.0946	0.5829 ± 0.0523	0.5803 ± 0.0502	**0.5152** ± 0.0612

In Tables from 4 until 11, we present in the first row, the mean squared error of H_0, so that we can easily recover the mean squared errors of all other estimators T_0. The following rows provide the $REFF$ indicators, $REFF_{T|H}$ in (3.11), for the different extreme value index estimators under study. Again, the estimator providing the highest $REFF$ indicator (minimum mean squared error at optimal level) is underlined and in **bold**.

Some comments regarding the $REFF$ indicators

Table 2: Simulated mean values, at optimal levels, of H, $\overline{H}_0$, $\overline{M}_1$, $H(p)$ and $M(p)$, $p = 0, 0.1, 0.25$ and 0.5 for parents with $\rho = -1$.

n	200	500	1000	2000	5000
Frechet$(\gamma = \mathbf{1})$ $(\rho = -\mathbf{1})$					
H	1.0498 ± 0.1085	1.0750 ± 0.0624	1.0657 ± 0.0463	1.0775 ± 0.0487	1.0356 ± 0.0268
M	1.0612 ± 0.1197	1.0709 ± 0.0809	1.0656 ± 0.0489	1.0697 ± 0.0650	1.0385 ± 0.0257
$\overline{H}_0$	**1.0296** ± 0.1034	1.0353 ± 0.0813	1.0226 ± 0.0398	**1.0286** ± 0.0506	**1.0033** ± 0.0209
$\overline{M}_1$	0.9607 ± 0.1274	**1.0091** ± 0.0824	**1.0136** ± 0.0447	1.0300 ± 0.0558	1.0099 ± 0.0211
$H(0)$	1.0540 ± 0.1302	1.1040 ± 0.0740	1.0626 ± 0.0486	1.0697 ± 0.0480	1.0373 ± 0.0282
$H(0.1)$	1.0714 ± 0.1430	1.0832 ± 0.0683	1.0743 ± 0.0499	1.0770 ± 0.0604	1.0504 ± 0.0155
$H(0.25)$	1.0471 ± 0.1752	1.0863 ± 0.0867	1.1003 ± 0.0662	1.0793 ± 0.0656	1.0495 ± 0.0234
$H(0.5)$	1.0763 ± 0.1877	1.1428 ± 0.1132	1.1070 ± 0.0824	1.0693 ± 0.0606	1.0445 ± 0.0302
$M(0)$	1.0719 ± 0.1262	1.0761 ± 0.0869	1.0687 ± 0.0530	1.0736 ± 0.0677	1.0414 ± 0.0275
$M(.1)$	1.0574 ± 0.1572	1.0836 ± 0.1007	1.0768 ± 0.0573	1.0845 ± 0.0747	1.0467 ± 0.0310
$M(.25)$	1.0811 ± 0.1663	1.0904 ± 0.1134	1.0847 ± 0.0637	1.0889 ± 0.0835	1.0497 ± 0.0335
$M(.5)$	1.1069 ± 0.1832	1.1002 ± 0.1386	1.0978 ± 0.0729	1.0976 ± 0.0957	1.0571 ± 0.0418
Student $\mathbf{t_2}$ $(\gamma = \mathbf{0.5},\ \rho = -1)$					
H	0.5599 ± 0.1079	0.6104 ± 0.0698	0.5548 ± 0.0325	0.5271 ± 0.0328	0.5246 ± 0.0270
M	0.4063 ± 0.1195	0.5123 ± 0.1031	0.4924 ± 0.0285	0.4754 ± 0.0472	0.4969 ± 0.0299
$\overline{H}_0$	0.4714 ± 0.0607	0.5233 ± 0.0454	0.5023 ± 0.0213	**0.5019** ± 0.0215	**0.4988** ± 0.0155
$\overline{M}_1$	0.3427 ± 0.0009	0.4655 ± 0.0005	0.4703 ± 0.0002	0.4483 ± 0.0001	0.4781 ± 0.0001
$H(0)$	0.1270 ± 0.0488	0.2613 ± 0.1874	0.2535 ± 0.1125	0.2457 ± 0.1554	0.1645 ± 0.0793
$H(0.1)$	0.4761 ± 0.0423	0.4849 ± 0.0349	0.4940 ± 0.0225	0.4941 ± 0.0137	0.4962 ± 0.0087
$H(0.25)$	**0.5111** ± 0.0680	0.4891 ± 0.0224	**0.5000** ± 0.0179	0.4968 ± 0.0150	0.4964 ± 0.0093
$H(0.5)$	0.5595 ± 0.1090	0.6074 ± 0.0675	0.5561 ± 0.0316	0.5241 ± 0.0326	0.5245 ± 0.0269
$M(0)$	0.2290 ± 0.0781	0.3323 ± 0.0780	0.3571 ± 0.0454	0.3436 ± 0.0459	0.3681 ± 0.0322
$M(.1)$	0.2979 ± 0.0950	0.4028 ± 0.0797	0.4248 ± 0.0371	0.4121 ± 0.0444	0.4360 ± 0.0388
$M(.25)$	0.3521 ± 0.1068	0.4406 ± 0.0768	0.4497 ± 0.0358	0.4285 ± 0.0394	0.4539 ± 0.0356
$M(.5)$	0.4153 ± 0.1188	**0.5107** ± 0.1015	0.4990 ± 0.0315	0.4760 ± 0.0468	0.4950 ± 0.0289
$\mathbf{EV}_\gamma$ $(\gamma = \mathbf{1})$ $(\rho = -\mathbf{1})$					
H	1.0430 ± 0.1501	1.0668 ± 0.1310	1.1578 ± 0.0625	1.0805 ± 0.0600	1.0478 ± 0.0287
M	0.9902 ± 0.1870	1.0006 ± 0.1000	1.1287 ± 0.0812	1.0931 ± 0.0883	1.0527 ± 0.0373
$\overline{H}_0$	0.8708 ± 0.0823	0.9667 ± 0.0638	**1.0495** ± 0.0579	**1.0377** ± 0.0507	**1.0146** ± 0.0214
$\overline{M}_1$	0.8123 ± 0.1108	0.9220 ± 0.0651	1.0759 ± 0.0703	1.0874 ± 0.0645	1.0518 ± 0.0315
$H(0)$	1.0918 ± 0.1355	**1.0037** ± 0.0717	1.0909 ± 0.0765	1.0697 ± 0.0480	1.0373 ± 0.0282
$H(0.1)$	1.0938 ± 0.1606	1.0267 ± 0.0930	1.0973 ± 0.0646	1.0770 ± 0.0604	1.0504 ± 0.0155
$H(0.25)$	1.0911 ± 0.1505	1.0475 ± 0.1138	1.1369 ± 0.0740	1.0793 ± 0.0656	1.0495 ± 0.0234
$H(0.5)$	1.0854 ± 0.1723	1.0581 ± 0.1348	1.1379 ± 0.0838	1.0693 ± 0.0606	1.0445 ± 0.0302
$M(0)$	1.0082 ± 0.1261	0.9948 ± 0.0659	1.1000 ± 0.0684	1.0736 ± 0.0677	1.0414 ± 0.0275
$M(.1)$	**1.0022** ± 0.1561	0.9904 ± 0.0806	1.1133 ± 0.0763	1.0845 ± 0.0747	1.0467 ± 0.0310
$M(.25)$	1.0162 ± 0.1653	0.9932 ± 0.0916	1.1224 ± 0.0803	1.0889 ± 0.0835	1.0497 ± 0.0335
$M(.5)$	0.9944 ± 0.1805	1.0133 ± 0.1102	1.1399 ± 0.0817	1.0976 ± 0.0957	1.0571 ± 0.0418

- For Fréchet parents and regarding $REFF$ indicators, the reduced-bias estimator $\overline{H}_0$ is the one exhibiting the better behaviour (higher $REFF$). The moment estimator, at the optimal level, slightly overpasses the Hill estimator, also at its optimal level, for all n. Whenever we consider the PORT-estimators, the $REFF$ indicators are always smaller than 1, and they decrease as p increases. For the same p, $M(p)$ and $H(p)$ have $REFF$ indicators close together, with a slightly better performance of the $M(p)$ estimator.

Table 3: Simulated mean values, at optimal levels, of H, $\overline{H}_0$, $\overline{M}_1$, $H(p)$ and $M(p)$, $p = 0, 0.1, 0.25$ and 0.5 for parents with $|\rho| > 1$.

n	200	500	1000	2000	5000
Student t_1 $(\gamma = \mathbf{1},\ \rho = \mathbf{-2})$					
H	1.1198 ± 0.1513	1.1329 ± 0.0991	1.0744 ± 0.0868	1.0401 ± 0.0452	1.0258 ± 0.0490
M	1.0550 ± 0.1661	1.0937 ± 0.1257	1.0349 ± 0.0829	1.0392 ± 0.0509	1.0164 ± 0.0470
$\overline{H}_1$	1.0228 ± 0.1191	1.1255 ± 0.1113	1.0356 ± 0.0766	1.0344 ± 0.0267	1.0071 ± 0.0466
$\overline{M}_1$	1.0767 ± 0.1144	1.0719 ± 0.0661	1.0647 ± 0.0471	1.0739 ± 0.0451	1.0371 ± 0.0269
$H(0)$	0.3105 ± 0.1960	0.3997 ± 0.2939	0.2189 ± 0.1124	0.4788 ± 0.3238	0.4629 ± 0.2564
$H(0.1)$	0.7925 ± 0.1710	0.8584 ± 0.1290	0.8289 ± 0.1213	0.9069 ± 0.0813	0.9355 ± 0.0764
$H(0.25)$	**1.0161** ± 0.0839	**0.9735** ± 0.0395	**1.0138** ± 0.0306	**1.0167** ± 0.0133	**1.0000** ± 0.0134
$H(0.5)$	1.1086 ± 0.1493	1.1348 ± 0.1006	1.0799 ± 0.0842	1.0367 ± 0.0473	1.0256 ± 0.0484
$M(0)$	0.4514 ± 0.0753	0.4716 ± 0.0317	0.4753 ± 0.0197	0.4948 ± 0.0097	0.4967 ± 0.0064
$M(.1)$	0.7012 ± 0.1303	0.8056 ± 0.1272	0.81389 ± 0.0903	0.9015 ± 0.0732	0.9004 ± 0.0702
$M(.25)$	0.8428 ± 0.1220	0.9015 ± 0.0979	0.8959 ± 0.0755	0.9343 ± 0.0545	0.9350 ± 0.0495
$M(.5)$	1.0523 ± 0.1721	1.0964 ± 0.1270	1.0391 ± 0.0818	1.0374 ± 0.0506	1.0165 ± 0.0463
GP$_\gamma$ $(\gamma = \mathbf{2})$ $(\rho = \mathbf{-2})$					
H	2.1099 ± 0.1881	2.0214 ± 0.0990	2.0849 ± 0.0893	2.0389 ± 0.0551	2.0606 ± 0.0674
M	2.0832 ± 0.2040	1.9605 ± 0.0895	2.0861 ± 0.1023	2.0562 ± 0.0661	2.0444 ± 0.0558
$\overline{H}_1$	2.1310 ± 0.1325	**2.0030** ± 0.0954	2.0574 ± 0.1051	2.0307 ± 0.0610	2.0464 ± 0.0540
$\overline{M}_1$	**1.9728** ± 0.1796	1.9280 ± 0.0875	**2.0259** ± 0.0916	**2.0078** ± 0.0629	**2.0042** ± 0.0473
$H(0)$	2.1092 ± 0.1876	2.0216 ± 0.0989	2.0850 ± 0.0893	2.0389 ± 0.0551	2.0606 ± 0.0674
$H(0.1)$	2.1367 ± 0.1969	2.0115 ± 0.1013	2.0861 ± 0.0953	2.0317 ± 0.0722	2.0503 ± 0.0641
$H(0.25)$	2.1531 ± 0.2338	2.0343 ± 0.1223	2.0844 ± 0.1148	2.0433 ± 0.0862	2.0383 ± 0.0594
$H(0.5)$	2.0667 ± 0.2715	2.0213 ± 0.0973	2.1414 ± 0.1075	2.0757 ± 0.0814	2.0623 ± 0.0662
$M(0)$	2.0828 ± 0.2022	1.9605 ± 0.0905	2.0863 ± 0.1023	2.0563 ± 0.0661	2.0444 ± 0.0559
$M(.1)$	2.0689 ± 0.2206	1.9666 ± 0.0975	2.0844 ± 0.1023	2.0537 ± 0.0661	2.0440 ± 0.0559
$M(.25)$	2.0524 ± 0.2302	1.9592 ± 0.1096	2.0949 ± 0.1027	2.0573 ± 0.0684	2.0453 ± 0.0646
$M(.5)$	2.0348 ± 0.2767	1.9249 ± 0.1253	2.1156 ± 0.1204	2.0813 ± 0.0794	2.0322 ± 0.0657

Table 4: Simulated mean squared errors of H (first row) and $REFF$-indicators of M, $\overline{H}_0$, $\overline{M}_1$, $H(p)$ and $M(p)$, $p = 0,\ 0.1,\ 0.25$ and 0.5, for Fréchet parents with $\gamma = 1$ $(\rho = -1)$.

n	200	500	1000	2000	5000
Frechet$(\gamma = \mathbf{1})$ $(\rho = \mathbf{-1})$					
MSE_H	0.02590 ± 0.0004	0.0135 ± 0.0002	0.0083 ± 0.0001	0.0051 ± 0.0000	0.0027 ± 0.0000
$REFF_{M\|H}$	1.0229 ± 0.0036	1.0176 ± 0.0054	1.0131 ± 0.0028	1.0077 ± 0.0039	1.0080 ± 0.0052
$REFF_{\overline{H}_0\|H}$	**1.3290** ± 0.0096	**1.3763** ± 0.0141	**1.4731** ± 0.0071	**1.5752** ± 0.0129	**1.7902** ± 0.0196
$REFF_{\overline{M}_1\|H}$	1.0447 ± 0.0068	1.1372 ± 0.0095	1.2383 ± 0.0088	1.3428 ± 0.0100	1.5352 ± 0.0154
$REFF_{H(0)\|H}$	0.9065 ± 0.0037	0.9172 ± 0.0022	0.9238 ± 0.0020	0.9279 ± 0.0023	0.9370 ± 0.0017
$REFF_{H(.1)\|H}$	0.8144 ± 0.0033	0.8131 ± 0.0037	0.8132 ± 0.0035	0.8101 ± 0.0048	0.8120 ± 0.0037
$REFF_{H(.25)\|H}$	0.7416 ± 0.0042	0.7421 ± 0.0042	0.7405 ± 0.0034	0.7382 ± 0.0044	0.7413 ± 0.0048
$REFF_{H(.5)\|H}$	0.6251 ± 0.0055	0.6284 ± 0.0046	0.6307 ± 0.0040	0.6295 ± 0.0043	0.6316 ± 0.0049
$REFF_{M(0)\|H}$	0.9345 ± 0.0033	0.9381 ± 0.0048	0.9395 ± 0.0027	0.9391 ± 0.0042	0.9449 ± 0.0049
$REFF_{M(.1)\|H}$	0.8413 ± 0.0038	0.8358 ± 0.0051	0.8302 ± 0.0027	0.8239 ± 0.0043	0.8229 ± 0.0044
$REFF_{M(.25)\|H}$	0.7658 ± 0.0045	0.7636 ± 0.0049	0.7588 ± 0.0026	0.7530 ± 0.0042	0.7526 ± 0.0042
$REFF_{M(.5)\|H}$	0.6416 ± 0.0048	0.6488 ± 0.0050	0.6467 ± 0.0029	0.6420 ± 0.0039	0.6433 ± 0.0039

Table 5: Simulated mean squared errors of H (first row) and $REFF$-indicators of M, $\overline{H}_0$, $\overline{M}_1$, $H(p)$ and $M(p)$, $p = 0,\ 0.1,\ 0.25$ and 0.5, for EV_γ parents with $\gamma = 0.25$.

n	200	500	1000	2000	5000
$\mathbf{EV}_\gamma$ (γ = 0.25) (ρ = −0.25)					
MSE_H	0.0402 ± 0.0006	0.0246 ± 0.0004	0.0176 ± 0.0003	0.0127 ± 0.0002	0.0085 ± 0.0001
$REFF_{M\|H}$	1.0929 ± 0.0122	1.2719 ± 0.0109	1.3587 ± 0.0137	1.4152 ± 0.0071	1.5529 ± 0.0143
$REFF_{\overline{H}_0\|H}$	1.2339 ± 0.0043	1.1713 ± 0.0066	1.1332 ± 0.0048	1.1023 ± 0.0029	1.0711 ± 0.0033
$REFF_{\overline{M}_1\|H}$	1.4665 ± 0.0170	1.8416 ± 0.0172	2.1562 ± 0.0226	2.5308 ± 0.0232	3.1365 ± 0.0345
$REFF_{H(0)\|H}$	1.4959 ± 0.0068	1.5169 ± 0.0093	1.5336 ± 0.0095	1.5407 ± 0.0057	1.5597 ± 0.0134
$REFF_{H(.1)\|H}$	1.2293 ± 0.0057	1.2136 ± 0.0056	1.2072 ± 0.0041	1.1994 ± 0.0032	1.1902 ± 0.0047
$REFF_{H(.25)\|H}$	1.0880 ± 0.0026	1.0810 ± 0.0034	1.0779 ± 0.0023	1.0751 ± 0.0008	1.0721 ± 0.0021
$REFF_{H(.5)\|H}$	0.9095 ± 0.0037	0.9147 ± 0.0024	0.9183 ± 0.0025	0.9202 ± 0.0020	0.9256 ± 0.0017
$REFF_{M(0)\|H}$	**1.5073** ± 0.0171	**1.9175** ± 0.0227	**2.3055** ± 0.0208	**2.7677** ± 0.0130	**3.5114** ± 0.0467
$REFF_{M(.1)\|H}$	1.3995 ± 0.0134	1.7507 ± 0.0143	2.0598 ± 0.0143	2.4442 ± 0.0126	3.1323 ± 0.0279
$REFF_{M(.25)\|H}$	1.2298 ± 0.0124	1.4341 ± 0.0119	1.5345 ± 0.0153	1.7109 ± 0.0125	2.1631 ± 0.0177
$REFF_{M(.5)\|H}$	0.9544 ± 0.0209	1.1164 ± 0.0087	1.1985 ± 0.0114	1.2547 ± 0.0058	1.3204 ± 0.0128

Table 6: Simulated mean squared errors of H (first row) and $REFF$-indicators of M, $\overline{H}_0$, $\overline{M}_1$, $H(p)$ and $M(p)$, $p = 0,\ 0.1,\ 0.25$ and 0.5, for EV_γ parents with $\gamma = 1$.

n	200	500	1000	2000	5000
$\mathbf{EV}_\gamma$ (γ = 1) (ρ = −1)					
MSE_H	0.0558 ± 0.0010	0.0286 ± 0.0003	0.0175 ±0.0002	0.0107 ± 0.0001	0.0057 ± 0.0001
$REFF_{M\|H}$	1.0262 ± 0.0052	1.0307 ± 0.0040	1.0242 ± 0.0045	1.0194 ± 0.0055	1.0165 ± 0.0026
$REFF_{\overline{H}_0\|H}$	1.2253 ± 0.0286	**1.4324** ± 0.0316	**1.6822** ± 0.0327	**1.9978** ± 0.0249	**2.5151** ± 0.0221
$REFF_{\overline{M}_1\|H}$	1.0830 ± 0.0222	1.2712 ± 0.0188	1.3845 ± 0.0107	1.3456 ± 0.0112	1.1382 ± 0.0052
$REFF_{H(0)\|H}$	1.3168 ± 0.0054	1.3331 ± 0.0072	1.3390 ± 0.0090	1.3476 ± 0.0096	1.3565 ± 0.0075
$REFF_{H(.1)\|H}$	1.1832 ± 0.0053	1.1813 ± 0.0081	1.1786 ± 0.0058	1.1764 ± 0.0077	1.1755 ± 0.0048
$REFF_{H(.25)\|H}$	1.0773 ± 0.0047	1.0781 ± 0.0032	1.0731 ± 0.0036	1.0719 ± 0.0045	1.0731 ± 0.0027
$REFF_{H(.5)\|H}$	0.9096 ± 0.0036	0.9131 ± 0.0033	0.9141 ± 0.0027	0.9141 ± 0.0040	0.9143 ± 0.0026
$REFF_{M(0)\|H}$	**1.3544** ± 0.0090	1.3613 ± 0.0064	1.3604 ± 0.0076	1.3637 ± 0.0085	1.3678 ± 0.0054
$REFF_{M(.1)\|H}$	1.2187 ± 0.0076	1.2128 ± 0.0052	1.2021 ± 0.0059	1.1964 ± 0.0070	1.1913 ± 0.0039
$REFF_{M(.25)\|H}$	1.1100 ± 0.0065	1.1081 ± 0.0050	1.0988 ± 0.0050	1.0935 ± 0.0062	1.0895 ± 0.0031
$REFF_{M(.5)\|H}$	0.9320 ± 0.0052	0.9414 ± 0.0035	0.9364 ± 0.0037	0.9323 ± 0.0052	0.9312 ± 0.0025

- For the EV_γ, $\gamma = 0.25$, and regarding $REFF$ indicators, only $H(0.5)$ exhibits a $REFF$ measure smaller than one for all n. The reduced-bias estimator $\overline{H}_0$ behaves better than the Hill and quite close to $H(0.25)$, but not so high as for Fréchet parents. Both for $H(p)$ and $M(p)$ the $REFF$ indicators increase as p decreases, with the moment estimator behaving better than the Hill estimator, for the same p. The estimator with the highest $REFF$-indicator, among the ones considered is $M(0)$. However $H(0)$ provides a $REFF$ indicator quite close to 1.5 for all n.

  For the EV_γ with $\gamma = 1$ the main difference lies in the fact that now the reduced-bias indicator $\overline{H}_0$ provides the highest $REFF$ indicators for all $n \geq 500$. The relative behaviour

Table 7: Simulated mean squared errors of H (first row) and $REFF$-indicators of M, $\overline{H}_0$, $\overline{M}_1$, $H(p)$ and $M(p)$, $p = 0,\ 0.1,\ 0.25$ and 0.5, for EV_γ parents with $\gamma = 2$.

n	200	500	1000	2000	5000
EV$_\gamma$ $(\gamma = \mathbf{2})$ $(\rho = \mathbf{-1})$					
MSE_H	0.0129 ± 0.0029	0.0612 ± 0.0013	0.0372 ± 0.0003	0.0226 ± 0.0002	0.0118 ± 0.0002
$REFF_{M\|H}$	1.0637 ± 0.0050	1.0689 ± 0.0042	1.0576 ± 0.0039	1.0411 ± 0.0048	1.0344 ± 0.0028
$REFF_{\overline{H}_0\|H}$	0.5819 ± 0.1552	0.9426 ± 0.0094	1.0074 ± 0.0059	1.1089± 0.0044	**1.3571** ± 0.0028
$REFF_{\overline{M}_1\|H}$	0.7405 ± 0.1205	1.0356 ± 0.0182	1.1190 ± 0.0060	**1.1616** ± 0.0091	1.1334 ± 0.0051
$REFF_{H(0)\|H}$	1.0953 ± 0.0047	1.0722 ± 0.0023	1.0640 ± 0.0027	1.0561 ± 0.0030	1.0412 ± 0.0019
$REFF_{H(.1)\|H}$	1.0747 ± 0.0037	1.0566 ± 0.0017	1.0505 ± 0.0022	1.0451 ± 0.0026	1.0322 ± 0.0021
$REFF_{H(.25)\|H}$	1.0388 ± 0.0029	1.0302 ± 0.0014	1.0278 ± 0.0019	1.0252 ± 0.0016	1.0181 ± 0.0016
$REFF_{H(.5)\|H}$	0.9426 ± 0.0033	0.9509 ± 0.0034	0.9556 ± 0.0026	0.9588 ± 0.0025	0.9664 ± 0.0017
$REFF_{M(0)\|H}$	**1.1839** ± 0.0096	**1.1554** ± 0.0045	**1.1328** ± 0.0040	1.1052 ± 0.0039	1.0836 ± 0.0044
$REFF_{M(.1)\|H}$	1.1563 ± 0.0082	1.1365 ± 0.0044	1.1162 ± 0.0039	1.0918 ± 0.0039	1.0733 ± 0.0044
$REFF_{M(.25)\|H}$	1.1131 ± 0.0060	1.1056 ± 0.0040	1.0896 ± 0.0038	1.0683 ± 0.0040	1.0561 ± 0.0036
$REFF_{M(.5)\|H}$	0.9958 ± 0.0040	1.0134 ± 0.0057	1.0060 ± 0.0036	0.9959 ± 0.0043	0.9947 ± 0.0030

Table 8: Simulated mean squared errors of H (first row) and $REFF$-indicators of M, $\overline{H}_0$, $\overline{M}_1$, $H(p)$ and $M(p)$, $p = 0,\ 0.1,\ 0.25$ and 0.5, for Student parents t_ν, with $\nu = 4$ degrees of freedom.

n	200	500	1000	2000	5000
Student t$_\mathbf{4}$ $(\gamma = \mathbf{0.25},\ \rho = \mathbf{-0.5})$					
MSE_H	0.0204 ± 0.0004	0.0112 ± 0.0002	0.0073 ± 0.0001	0.0048 ± 0.0001	0.0029 ± 0.0000
$REFF_{M\|H}$	0.5277 ± 0.0069	0.6198 ± 0.0068	0.6696 ± 0.0052	0.7078 ± 0.0040	0.7481 ± 0.0082
$REFF_{\overline{H}_0\|H}$	1.3992 ± 0.0171	1.3600 ± 0.0097	1.3249 ± 0.0108	1.2811 ± 0.0105	1.2360 ± 0.0100
$REFF_{\overline{M}_1\|H}$	0.5837 ± 0.0079	0.6227 ± 0.0066	0.6547 ± 0.0042	0.6855 ± 0.0053	0.7280 ± 0.0083
$REFF_{H(0)\|H}$	1.9181 ± 0.0233	1.2850 ± 0.0113	0.8359 ± 0.0047	0.5610 ± 0.0042	0.3620 ± 0.0022
$REFF_{H(.1)\|H}$	**3.0107** ± 0.0310	**3.4637** ± 0.0290	**3.9376** ± 0.0358	**4.4930** ± 0.0502	**5.4485** ± 0.0625
$REFF_{H(.25)\|H}$	1.7002 ± 0.0156	1.7846 ± 0.0127	1.8815 ± 0.0126	1.9872 ± 0.0144	2.1792 ± 0.0143
$REFF_{H(.5)\|H}$	1.0035 ± 0.0011	1.0013 ± 0.0009	1.0007 ± 0.0004	1.0004 ± 0.0000	1.0001 ± 0.0004
$REFF_{M(0)\|H}$	0.5062 ± 0.0059	0.5192 ± 0.0052	0.5198 ± 0.0025	0.5135 ± 0.0040	0.5044 ± 0.0046
$REFF_{M(.1)\|H}$	0.5250 ± 0.0065	0.5603 ± 0.0058	0.5760 ± 0.0024	0.5844 ± 0.0046	0.5942 ± 0.0057
$REFF_{M(.25)\|H}$	0.5336 ± 0.0070	0.5841 ± 0.0062	0.6086 ± 0.0027	0.6252 ± 0.0046	0.6442 ± 0.0065
$REFF_{M(.5)\|H}$	0.5289 ± 0.0069	0.6197 ± 0.0067	0.6692 ± 0.0051	0.7075 ± 0.0040	0.7479 ± 0.0082

of the $REFF$ indicators for $H(p)$ and $M(p)$ follows a pattern similar to the one associated to a $EV_{0.25}$, but both $H(0.5)$ and $M(0.5)$ have $REFF$ indicators smaller than one for all n.

- For all Student models, and as expected due to the symmetry of the model around 0, $H(0.5)$ is almost coincident with H, as well as $M(0.5)$ almost equals M. For the Student model with $\nu = 4$ degrees of freedom, the reduced-bias estimator $\overline{H}_0$ behaves quite well, even for small values of n, but $H(0.25)$ overpasses it, being $H(0.1)$ the best estimator among the ones considered. All $M(p)$ estimators behave worse than the Hill estimator at optimal levels when ρ approaches 0, but for $\nu = 2$ the moment estimator M behaves

Table 9: Simulated mean squared errors of H (first row) and $REFF$-indicators of M, $\overline{H}_0$, $\overline{M}_1$, $H(p)$ and $M(p)$, $p = 0,\ 0.1,\ 0.25$ and 0.5, for Student parents t_ν, with $\nu = 2$ degrees of freedom.

n	200	500	1000	2000	5000
Student $\mathbf{t_2}$ ($\gamma = \mathbf{0.5},\ \rho = -\mathbf{1}$)					
MSE_H	0.0230 ± 0.0004	0.0116 ± 0.0001	0.0070 ± 0.0001	0.0043 ± 0.0001	0.0022 ± 0.0000
$REFF_{M\|H}$	0.6813 ± 0.0041	0.8123 ± 0.0062	0.9086 ± 0.0048	1.0057 ± 0.0065	1.1488 ± 0.0112
$REFF_{\overline{H}_0\|H}$	1.4179 ± 0.0195	1.6942 ± 0.0247	1.9507 ± 0.0214	2.2143 ± 0.0255	2.6311 ± 0.0317
$REFF_{\overline{M}_1\|H}$	0.6258 ± 0.0058	0.7016 ± 0.0084	0.7619 ± 0.0066	0.8207 ± 0.0107	0.8988 ± 0.0076
$REFF_{H(0)\|H}$	0.4506 ± 0.0041	0.3190 ± 0.0022	0.2483 ± 0.0016	0.1947 ± 0.0017	0.1403 ± 0.0010
$REFF_{H(.1)\|H}$	**2.3302** ± 0.0277	**2.5868** ± 0.0176	**2.8415** ± 0.0233	**3.1373** ± 0.0223	**3.5726** ± 0.0243
$REFF_{H(.25)\|H}$	1.9862 ± 0.0166	2.2168 ± 0.0163	2.4293 ± 0.0166	2.6650 ± 0.0189	3.0537 ± 0.0364
$REFF_{H(.5)\|H}$	1.0060 ± 0.0018	1.0017 ± 0.0008	1.0003 ± 0.0007	1.0000 ± 0.0004	0.9998 ± 0.0003
$REFF_{M(0)\|H}$	0.5173 ± 0.0047	0.4765 ± 0.0032	0.4345 ± 0.0033	0.3886 ± 0.0038	0.3243 ± 0.0025
$REFF_{M(.1)\|H}$	0.6043 ± 0.0053	0.6121 ± 0.0047	0.5993 ± 0.0048	0.5756 ± 0.0067	0.5294 ± 0.0049
$REFF_{M(.25)\|H}$	0.6572 ± 0.0056	0.7039 ± 0.0053	0.7161 ± 0.0055	0.7163 ± 0.0077	0.6898 ± 0.0055
$REFF_{M(.5)\|H}$	0.6830 ± 0.0047	0.8109 ± 0.0060	0.9059 ± 0.0045	1.0023 ± 0.0060	1.1458 ± 0.0114

Table 10: Simulated mean squared errors of H (first row) and $REFF$-indicators of M, $\overline{H}_0$, $\overline{M}_1$, $H(p)$ and $M(p)$, $p = 0,\ 0.1,\ 0.25$ and 0.5, for Student parents t_ν, with $\nu = 1$ degrees of freedom.

n	200	500	1000	2000	5000
Student $\mathbf{t_1}$ ($\gamma = \mathbf{1},\ \rho = -\mathbf{2}$)					
MSE_H	0.0370 ± 0.0005	0.0166 ± 0.0003	0.0095 ± 0.0001	0.0053 ± 0.0001	0.0025 ± 0.0000
$REFF_{M\|H}$	0.8668 ± 0.0074	0.9151 ± 0.0068	0.9234 ± 0.0052	0.9232 ± 0.0044	0.9272 ± 0.0044
$REFF_{\overline{H}_1\|H}$	0.7966 ± 0.1693	1.1591 ± 0.0109	1.1584 ± 0.0078	1.1610 ± 0.0075	1.1636 ± 0.0055
$REFF_{\overline{M}_1\|H}$	0.5245 ± 0.0749	0.7026 ± 0.0120	0.7528 ± 0.0119	0.8086 ± 0.0127	0.8816 ± 0.0087
$REFF_{H(0)\|H}$	0.2529 ± 0.0023	0.1712 ± 0.0017	0.1303 ± 0.0011	0.0976 ± 0.0009	0.0671 ± 0.0007
$REFF_{H(.1)\|H}$	0.5569 ± 0.0061	0.4928 ± 0.0047	0.4602 ± 0.0048	0.4290 ± 0.0052	0.3909 ± 0.0044
$REFF_{H(.25)\|H}$	**1.5400** ± 0.0146	**1.6404** ± 0.0098	**1.7504** ± 0.0151	**1.8564** ± 0.0185	**2.0077** ± 0.0242
$REFF_{H(.5)\|H}$	1.0060 ± 0.0023	1.0022 ± 0.0010	0.9998 ± 0.0017	0.9992 ± 0.0011	0.9988 ± 0.0008
$REFF_{M(0)\|H}$	0.3431 ± 0.0029	0.2423 ± 0.0021	0.1879 ± 0.0013	0.1435 ± 0.0012	0.1004 ± 0.0009
$REFF_{M(.1)\|H}$	0.5361 ± 0.0055	0.4734 ± 0.0047	0.4455 ± 0.0045	0.4177 ± 0.0051	0.3826 ± 0.0042
$REFF_{M(.25)\|H}$	0.7588 ± 0.0082	0.6971 ± 0.0074	0.6559 ± 0.0065	0.6139 ± 0.0064	0.5614 ± 0.0067
$REFF_{M(.5)\|H}$	0.8698 ± 0.0078	0.9179 ± 0.0069	0.9234 ± 0.0047	0.9225 ± 0.0050	0.9264 ± 0.0043

slightly better than the Hill for large n. As mentioned before, $H(0)$, possibly not even consistent for the estimation of γ, behaves really very badly, with sample paths quite stable, but around a value a long way from the target.

4 An application to the Nasdaq Composite index

As an empirical example, we place ourselves in a context from finance, analyzing the risk for investors holding short positions in the Nasdaq Composite index, i.e., for investors betting on a fall in the index.

Table 11: Simulated mean squared errors of H (first row) and $REFF$-indicators of M, $\overline{H}_0$, $\overline{M}_1$, $H(p)$ and $M(p)$, $p = 0,\ 0.1,\ 0.25$ and 0.5, for GP parents, with $\gamma = 0.5$.

n	200	500	1000	2000	5000
GP$_\gamma$ ($\gamma = \mathbf{0.5}$) ($\rho = \mathbf{-0.5}$)					
MSE_H	0.0362 ± 0.0006	0.0208 ± 0.0003	0.0139 ± 0.0003	0.0094 ± 0.0001	0.0057 ± 0.0000
$REFF_{M\|H}$	1.0687 ± 0.0071	1.1037 ± 0.0063	1.1145 ± 0.0060	1.1192 ± 0.0041	1.1317 ± 0.0063
$REFF_{\overline{H}_0\|H}$	**1.3803** ± 0.0082	**1.3390** ± 0.0069	**1.3019** ± 0.0105	**1.2648** ± 0.0063	**1.2336** ± 0.0061
$REFF_{\overline{M}_1\|H}$	1.1892 ± 0.0095	1.1682 ± 0.0059	1.1557 ± 0.0059	1.1486 ± 0.0046	1.1493 ± 0.0062
$REFF_{H(0)\|H}$	0.9984 ± 0.0000	0.9994 ± 0.0004	0.9997 ± 0.0003	0.9999 ± 0.0028	0.9999 ± 0.0004
$REFF_{H(.1)\|H}$	0.9667 ± 0.0013	0.9693 ± 0.0019	0.9703 ± 0.0009	0.9692 ± 0.0015	0.9711 ± 0.0014
$REFF_{H(.25)\|H}$	0.9119 ± 0.0031	0.9189 ± 0.0040	0.9206 ± 0.0022	0.9193 ± 0.0024	0.9242 ± 0.0020
$REFF_{H(.5)\|H}$	0.7982 ± 0.0048	0.8131 ± 0.0063	0.8168 ± 0.0045	0.8192 ± 0.0023	0.8270 ± 0.0018
$REFF_{M(0)\|H}$	1.0668 ± 0.0071	1.1029 ± 0.0063	1.1142 ± 0.0060	1.1190 ± 0.0041	1.1317 ± 0.0063
$REFF_{M(.1)\|H}$	1.0280 ± 0.0062	1.0664 ± 0.0061	1.0796 ± 0.0060	1.0849 ± 0.0038	1.0989 ± 0.0059
$REFF_{M(.25)\|H}$	0.9599 ± 0.0058	1.0070 ± 0.0053	1.0205 ± 0.0067	1.0290 ± 0.0036	1.0431 ± 0.0046
$REFF_{M(.5)\|H}$	0.8179 ± 0.0040	0.8790 ± 0.0036	0.8996 ± 0.0062	0.9130 ± 0.0029	0.9297 ± 0.0042

Table 12: Simulated mean squared errors of H (first row) and $REFF$-indicators of M, $\overline{H}_0$, $\overline{M}_1$, $H(p)$ and $M(p)$, $p = 0,\ 0.1,\ 0.25$ and 0.5, for GP parents, with $\gamma = 2$.

n	200	500	1000	2000	5000
GP$_\gamma$ ($\gamma = \mathbf{2}$) ($\rho = \mathbf{-2}$)					
MSE_H	0.0658 ± 0.0012	0.0309 ± 0.0006	0.0176 ± 0.0003	0.0100 ± 0.0001	0.0047 ± 0.0001
$REFF_{M\|H}$	1.0162 ± 0.0040	1.0075 ± 0.0034	1.0040 ± 0.0059	0.9998 ± 0.0043	0.9957 ± 0.0037
$REFF_{\overline{H}_1\|H}$	**1.1632** ± 0.0072	**1.1542** ± 0.0062	**1.1490** ± 0.0060	**1.1431** ± 0.0044	**1.1307** ± 0.0048
$REFF_{\overline{M}_1\|H}$	1.1601 ± 0.0067	1.2215 ± 0.0086	1.2874 ± 0.0109	1.3343 ± 0.0067	1.4110 ± 0.0070
$REFF_{H(0)\|H}$	0.9979 ± 0.0000	0.9991 ± 0.0031	0.9996 ± 0.0043	0.9998 ± 0.0000	0.9999 ± 0.0003
$REFF_{H(.1)\|H}$	0.9563 ± 0.0014	0.9553 ± 0.0029	0.9563 ± 0.0027	0.9583 ± 0.0030	0.9579 ± 0.0028
$REFF_{H(.25)\|H}$	0.8860 ± 0.0035	0.88556 ± 000.39	0.8887 ± 0.0035	0.8900 ± 0.0043	0.8903 ± 0.0040
$REFF_{H(.5)\|H}$	0.7469 ± 0.0070	0.7472 ± 0.0029	0.7511 ± 0.0034	0.7533 ± 0.0055	0.7549 ± 0.0044
$REFF_{M(0)\|H}$	1.0142 ± 0.0039	1.0067 ± 0.0033	1.0036 ± 0.0059	0.9996 ± 0.0043	0.9957 ± 0.0037
$REFF_{M(.1)\|H}$	0.9730 ± 0.0031	0.9654 ± 0.0027	0.9614 ± 0.0054	0.9583 ± 0.0040	0.9543 ± 0.0014
$REFF_{M(.25)\|H}$	0.9033 ± 0.0040	0.8961 ± 0.0026	0.8931 ± 0.0056	0.8907 ± 0.0039	0.8873 ± 0.0039
$REFF_{M(.5)\|H}$	0.7599 ± 0.0062	0.7585 ± 0.0019	0.7572 ± 0.0067	0.7574 ± 0.0047	0.7537 ± 0.0042

Since we are interested in the analysis of the risk of holding short positions, we begin with the positive log-returns, i.e., with $X_i = \ln(S_{i+1}/S_i) = -L_i$, $1 \le i \le n-1$, assumed to be stationary and weakly dependent. With the purpose of comparison with a case study from Drees (2003), we have used the daily log-returns from 1997 to 2000, which corresponds to a sample size $n = 1037$. Although there is some increasing trend in the volatility, stationarity is assumed, under the same considerations as in Drees (2003).

In Figure 4.1 we display the estimates for the tail index associated to $\hat\gamma^H_{n,k}$, $\hat\gamma^M_{n,k}$, $\hat\gamma^{H(p)}_{n,k}$ and $\hat\gamma^{M(p)}_{n,k}$ for some values of p.

It is clear from the analysis of the γ-scatterplots that all estimates are positive for k from

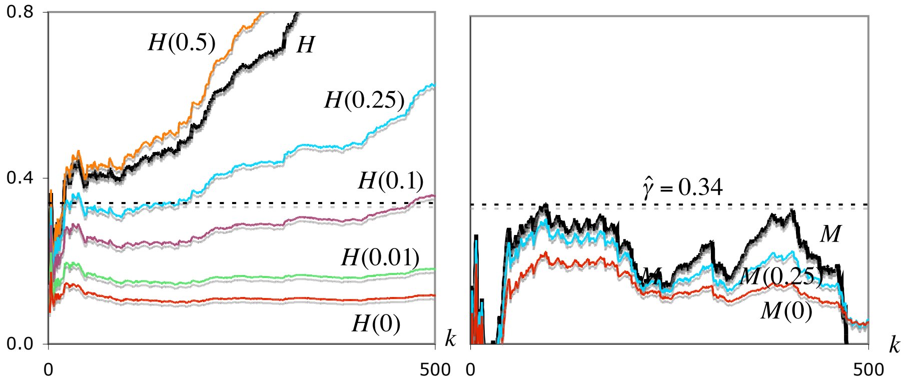

Figure 4.1: Tail index estimates based on $\hat{\gamma}^{H}_{n,k}$ and $\hat{\gamma}^{H(p)}_{n,k}$ (*left*) and on $\hat{\gamma}^{M}_{n,k}$ and $\hat{\gamma}^{M(p)}_{n,k}$ (*right*), for $p = 0.05, 0.25, 0.5$.

about 50 up to 450, i.e., there is a strong evidence for a heavy-tailed underlying distribution. However, the patterns exhibited by the different estimators $\hat{\gamma}^{H(p)}_{n,k}$ are significantly different for different values of the *tuning* parameter p. We have been, at a first sight, particularly puzzled with the sample paths of $\hat{\gamma}^{H(0)}_{n,k}$, and such sample paths immediately suggest a possible non-consistency of $\hat{\gamma}^{H(0)}_{n,k}$ due to an infinite left endpoint of the underlying model. We have thus decided to analyze more deeply both tails of the model underlying the sample $X_i, 1 \le i \le 1036$. For that we have used not only the Hill estimator, but also the MVRB-estimator $\overline{H}_0$ in (1.7), which is, for heavy tails, an alternative to the Hill estimator not only at the optimal levels or for large k, as happens with the "classical" second order reduced bias tail index estimators, but for all k. It was indeed this estimator that led us to the estimate $\hat{\gamma} = 0.34$ pictured in Figure 4.1 and consequently to the choice $p = 0.25$ for the class of estimators $H(p)$ in (1.10).

Right tail analysis of Nasdaq data. In Figure 4.2, and working with the $n_0 = 570$ positive values of the log-returns X_i on NASDAQ data, we picture the sample paths of $\widehat{\rho}_0(k)$ and $\widehat{\rho}_1(k)$. The algorithm in sub-section 3.2 leads us to choose, on the basis of any stability criterion for large values of k, the estimate associated to $\tau = 0$. We have considered $\widehat{\rho} = \widehat{\rho}_0(k_1)$, with $k_1 = n_0^{0.995}$. We have got $\widehat{\rho}_0 = \widehat{\rho}_0(552) = -0.71$. The use of the β-estimate suggested in the above mentioned algorithm, led us to the estimate $\widehat{\beta}_0 = 1.04$. For the estimation of γ through the reduced bias tail index estimators, we have used the heuristic estimate of the level provided in Gomes and Pestana (2007a), i.e., the value $k_{01} \equiv k_{01}(n;\beta,\rho) = \left(1.96(1-\rho)n_0^{-\rho}/|\beta|\right)^{2/(1-2\rho)}$. Levels of this type are still levels such that $\sqrt{k}\ (n/k)^{\rho} \to \lambda$, finite, and are not yet optimal for the tail index estimation through second order reduced-bias tail index estimators. However, do not forget that with a tail index estimator like $\overline{H}$, in (1.7), we are always safe and able to provide a more reliable estimation than through the Hill estimates. We came to $\widehat{k}_{01} = 109$ and to the estimate $\widehat{\gamma} = \overline{H}_0(109) = 0.34$. Note that the estimation of the optimal threshold (Hall and Welsh, 1985) for the estimation through the Hill estimator in (1.4), leads us to

$$\widehat{k}_0 = \left(\frac{(1-\widehat{\rho})\ n_0^{-\widehat{\rho}}}{\widehat{\beta}\ \sqrt{-2\widehat{\rho}}}\right)^{2/(1-2\widehat{\rho})} = 55 \quad\Longrightarrow\quad \widehat{\gamma}^H_{n,k}(\widehat{k}_0) = 0.41.$$

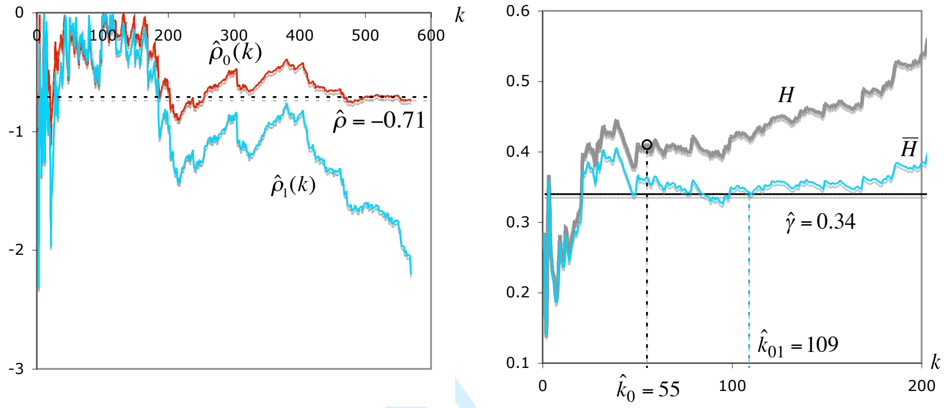

Figure 4.2: Estimates of the second order parameters ρ, through $\widehat{\rho}_0(k)$ and $\widehat{\rho}_1(k)$ (*left*), and the tail index γ (*right*), for the positive log-returns P, on NASDAQ data.

Left tail analysis of Nasdaq data. Figure 4.3 is related to a similar data analysis, carried on the $n_0 = 466$ positive values of L_i. We have now obtained $\widehat{\rho} = -0.71$, $\widehat{\beta} = 1.05$, $\hat{k}_0 = 48$, $\widehat{\gamma}^H_{n,k}(\widehat{k}_0) = 0.35$, $\hat{k}_{01} = 97$ and $\widehat{\gamma} = \overline{H}_0(97) = 0.30$.

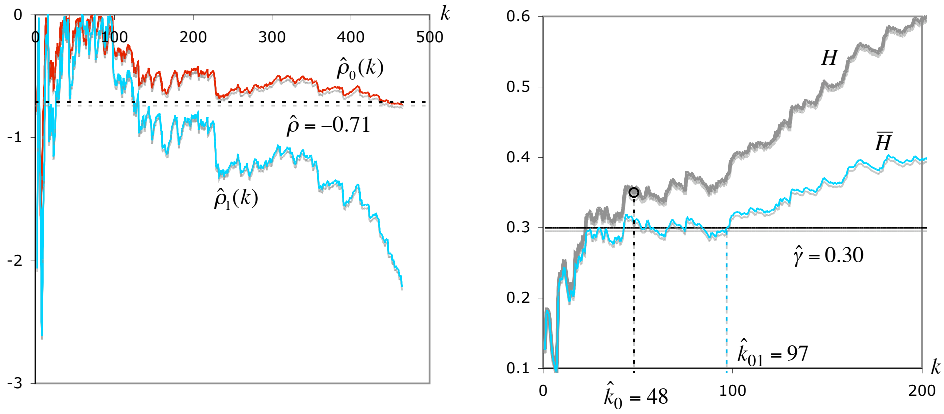

Figure 4.3: Estimates of the second order parameters ρ, through $\widehat{\rho}_0(k)$ and $\widehat{\rho}_1(k)$ (*left*), and the tail index γ (*right*), for the negative log-returns L, on NASDAQ data.

This data analysis leads us to the conclusion that the underlying model detains a location median not far from 0. Indeed, when we induce a shift associated to the tuning parameter

$p = 0.5$, we get a sample path not a long way from that of the Hill estimator (see Figure 4.1, left). Moreover, relying on the observed results for the γ estimates, it is not sensible to discard the possibility that both tails are heavy (with the right tail underlying the X_i slightly heavier than the left tail ($\widehat{\gamma} = 0.34$ for the right tail versus $\widehat{\gamma} = 0.30$ for the left tail). This obviously implies an underlying model with support $(-\infty, +\infty)$. It is then not at all sensible to induce a shift $X_{1:n}$, like it is suggested in Drees (2003). Such a shift is appealing, because it induces for the Hill estimator an almost flat sample path (see again Figure 4.1, left), but as mentioned before, the "flat zone" leads, in this case, to a severe underestimation of the tail index γ. To support this statement, look again at Figures 3.7 and 3.8, with the pattern of mean values (E) and mean squared errors (MSE) of the PORT-Hill and -moment estimators, respectively, for models from a Student-t_ν parent with $\nu = 4$ degrees of freedom ($\gamma = 0.25$).

Although a parametric data analysis of this data is outside the scope of the present paper, the similarities between the behavior of the mean value patterns in Figures 3.7 and 3.8 and the sample paths of the Hill and moment PORT-estimators in Figure 4.2, suggest that the d.f. underlying these returns is not a long way from a Student-t d.f. or its skewed extensions, which are very common models in the area of extremes and finance. For recent references see Jones and Faddy (2003) and McNeil, Frey and Embrechts (2005).

In this application, and taking into account the previous analysis, it seems sensible to consider as a compromise choice in the PORT-Hill estimator, the shift induced by the first empirical quartile, i.e., to pick the value $p = 0.25$, as we have already seen in Figure 4.1, but the possibility of working simultaneously with other estimators, like the $MVRB$ estimator here considered, should not at all be discarded, because this can help us to better estimate the extreme value index, a parameter of primordial importance in all subsequent extreme value analysis needed.

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