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Abstract

We derive several multivariate control charts to monitor the mean vector of multivariate GARCH processes under the presence of changes, by means of maximizing the generalized likelihood ratio. This presentation is rounded up by a comparative performance study based on extensive Monte Carlo simulations. An empirical illustration shows how the obtained results can be applied to real data.

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1 Introduction

In 1982 Engle invented a new method of modeling the volatility of financial time series which is based on the recursive calculation of the conditional variance. The structure of the autoregressive process was applied in order to forecast the future values of the conditional volatility. The model was extended by Bollerslev (1986) by using the autoregressive moving average process in the conditional variance equation. Later on, different univariate GARCH processes were suggested in the literature. Nelson (1991) proposed the exponential GARCH process to model the asymmetry of the returns. The other asymmetric GARCH processes were derived by Engle and Ng (1993) and Zakoian (1994) among others.

The extension of the univariate GARCH model to the multivariate case is not straightforward. As a result several models are suggested in the literature. The first multivariate GARCH model was derived by Bollerslev et al. (1988), which is based on the VEC representation of the conditional variance equation. Although the model is quite general it possesses some difficulties in the practical application. First, the number of unknown parameters is extremely large. Second, the calculated recursively conditional covariance matrices are not obviously positive definite. It leads to further developments of the theory of multivariate GARCH processes. The second widespread model was suggested by Engle and Kroner (1995). The conditional covariance matrices obtained by this model are always positive definite. Moreover, the number of unknown parameters is significantly reduced, especially for the high-dimensional processes.

The second approach of modeling conditional covariance matrix is based on modeling conditional correlations and variances separately. Bollerslev (1990) designed the CCC process which is based on the assumption of the constant conditional correlations. The model seems to be very useful in its practical application (see, e.g., Ling and McAleer (2003)). It was generalized independently by Engle (2002) and Tse and Tsui (2002) by allowing the conditional correlations to vary with time. The third possibility of constructing a multivariate GARCH process is to consider the factor or orthogonal models (Diebold and Nerlove(1989), Engle et al. (1990), Alexander (2000), van der Weide (2002)). A recent survey of multivariate GARCH processes of Bauwens et al. (2006) described these procedures in details.

Usually, the multivariate GARCH processes assume the mean vector to be equal to zero. Alternatively, the VARMA-GARCH models have been discussed in the literature (see, e.g., Ling and McAleer (2003)). These processes are designed to model the conditional mean structure of the data generating process. As a partial case the constant mean vector is assumed.

Detecting structural changes in the parameters of a stochastic process is a well-known decision problem which has been of interest in a huge number of scientific papers, especially in the field of econometrics (see e.g., Broemeling and Tsurumi (1987), Maddala and Kim (2002), Greene (2003)). Tests for structural change are derived to verify the equality of parameters in two separate subsamples. Alternatively, sequential methods are used. The starting point of the sequential procedures is a unique observation of the stochastic process. Based on this realization we make a decision about the parameter constancy of the target process. When we decide that a structural break has occurred the monitoring procedure is stopped. Otherwise, we use the next realization of the process. It continues until the first decision about a change is made.

Control charts are the main tool of the statistical process control that mainly deals with the sequential monitoring. A control chart is obtained by plotting a control statistic that is calculated based on the last observation. The value of the control statistic is compared with the control limit, i.e. with the preselected threshold value. When the control statistic exceeds the control limit the chart signals an alarm about a change in the process. The control charts for multivariate independently and normally distributed observations were proposed by Hotelling (1947), Crosier (1988), Pignatiello and Runger (1990), Lowry et al. (1992), and Ngai and Zhang (2001). Kramer and Schmid (1997) and Bodnar and Schmid (2006, 2007) extended these results by suggesting the control schemes for multivariate time series. We contribute to existing literature by deriving control charts to monitor the mean vector of multivariate GARCH processes. They are obtained by maximizing the generalized likelihood ratio and from the sequential probability ratio test.

The paper is organized as follows. In the next section the change point models are introduced. The main results are presented in Section 3.1. The control schemes of Theorem 1 are obtained by maximizing the generalized likelihood ratio (GLR), while in Theorem 2 we present the control charts derived from the sequential probability ratio test (SPRT). In Section 3.2 the numerical comparison of the control schemes is given. An empirical illustration in Section 4 shows how the obtained results can be applied to real data. The returns of USD/JPY and USD/GBP exchange rates are analyzed. The paper concludes with a short summary in Section 5. The designs of multivariate GARCH processes are given in the appendix (Section 6.1). In Section 6.2 the proof of Theorem 1 is presented.

2 Model

The main goal of the statistical process control is to monitor whether the observed process (the actual process) coincides with the target process. In engineering the target process is equal to the process which fulfills the quality requirements. In economics it is obtained by fitting a model to previous data. In the following $\{\mathbf{Y}_t\}$ stands for the target process. We assume that $\{\mathbf{Y}_t\}$ follows a multivariate GARCH process, i.e.

$$\mathbf{Y}_t = \mathbf{H}_t^{1/2}\boldsymbol{\varepsilon}_t\,, \tag{1}$$

where $\{\boldsymbol{\varepsilon}_t\}$ are identical independent normally distributed with $\boldsymbol{\varepsilon}_t \sim \mathcal{N}_p(\mathbf{0}, \mathbf{I})$. $\mathbf{H}_t$ is a conditional covariance matrix given the sigma field $\mathcal{F}_{t-1}$ generated by all information till time $t-1$. Consequently, it holds that $\mathbf{Y}_t|\mathcal{F}_{t-1} \sim \mathcal{N}_p(\mathbf{0}, \mathbf{H}_t)$. $\mathbf{H}_t$ is calculated recursively depending on the specification of a selected GARCH model. Definitions of different multivariate GARCH processes are given in the appendix (Section 6.1).

Without loss of generality we assume that all parameters of the target process $\{\mathbf{Y}_t\}$ are known. In case there are some unknown parameters they have to be replaced with the estimated counterparts based on previous data. Consequently, the estimated parameters are measurable by all the sigma fields $\mathcal{F}_0$, ..., $\mathcal{F}_{t-1}$ used in deriving control statistics. Under the additional assumption of consistency they can be treated as true values (see Section 6.2). Jeantheau (1998), Engle and Sheppard (2001), and Comte and Lieberman (2003) showed that the quasi-maximum likelihood estimators for the parameters of the considered in the paper multivariate GARCH processes are consistent under middle conditions.

The observed process is denoted by $\{\mathbf{X}_t\}$. Our aim is to detect changes in the mean vector of the observed process. In order to describe the relationship between the observed and the target processes we consider two change point models. The first one incorporates time-varying changes and it is given by

$$\mathbf{X}_t = \mathbf{Y}_t + \mathbf{a}_t\mathbf{1}_{q,q+1,\ldots}(t)\,, \tag{2}$$

where $\mathbf{a}_t \in I\!R^p$ and $q \in I\!N$ are unknown quantities. $\mathbf{1}_A(t)$ denotes the indicator function of the set A at point t. In case $\mathbf{a}_t \neq \mathbf{0}$ for $t \geq q$ we say that a change at the time point q is present. $\mathbf{a}_t$ describes the size and the direction of the shift. In case of no change the target process coincides with the observed process. We say that the observed process is in control. Else, it is denoted to be out of control. The in-control mean $\boldsymbol{\mu}_0$ is called the target value. Note that $\mathbf{X}_t = \mathbf{Y}_t$ for $t \leq 0$, i.e. both processes are the same up to time point 0.

The second change point model with a constant change is a partial case of (2) when $\mathbf{a}_t = \mathbf{a}$. It is given by

$$\mathbf{X}_t = \mathbf{Y}_t + \mathbf{a}\mathbf{1}_{q,q+1,\ldots}(t)\,. \tag{3}$$

Although, there is a strong relationship between (2) and (3) in the next section we show that different types of the control charts correspond to each model.

3 Control Charts for Multivariate GARCH Processes

In the present section we derive control charts for detecting changes in the mean vector of multivariate GARCH processes. In the derivation of the control schemes, the classical methods of the sequential analysis are used. The statistics are obtained by maximizing the generalized likelihood ratio and by using the sequential probability ratio test (see, e.g. Siegmund (1985), Nikiforov (1986)).

3.1 Design of the Control Schemes

Let $\|\mathbf{a}\|_{\mathbf{\Sigma}}$ denote the norm of the $\mathbf{a}$ with respect to the positive definite matrix $\mathbf{\Sigma}$, i.e. $\|\mathbf{a}\|^2_{\mathbf{\Sigma}} = \mathbf{a}'\mathbf{\Sigma}^{-1}\mathbf{a}$. Let $\mathbf{a}_{q:n} = vec(\mathbf{a}_q, \mathbf{a}_{q+1}, ..., \mathbf{a}_n)$ and $\mathbf{G}_2 = diag(\mathbf{H}_q^{-1}, ..., \mathbf{H}_n^{-1})$ where $vec(.)$ denotes the vec operator (see Harville (1997), ch. 16.2). We define

$$k_{q,n} = \|\mathbf{a}_{q:n}\|_{\mathbf{G}_2^{-1}} = \sum_{i=q}^{n} \mathbf{a}_i'\mathbf{H}_i^{-1}\mathbf{a}_i = \sum_{i=q}^{n} (\mathrm{E}_q(\mathbf{X}_i))'\mathbf{H}_i^{-1}\mathrm{E}_q(\mathbf{X}_i)\,, \tag{4}$$

where $\mathrm{E}_q(.)$ is the expectation calculated with respect to one of the change point models of Section 2 under the assumption that a change occurs at time q. The results presented in Theorem 1 are obtained by maximizing the generalized likelihood ratio. Note that they are general and can be applied to an arbitrary multivariate GARCH process.

Theorem 1. *Let $\{\mathbf{Y}_t\}$ be a multivariate GARCH process with the conditional covariance matrix $\mathbf{H}_t$ at time t.*

a) The control statistic obtained by the GLR approach for model (3) is given by

$$GLR1_n = \max\left\{0, \max_{1\le q\le n}\{\sqrt{k_{q,n}}(\|\mathbf{S}_{q,n}\|_{\sum_{i=q}^{n}\mathbf{H}_i^{-1}} - \frac{\sqrt{k_{q,n}}}{2})\}\right\} \tag{5}$$

with

$$\mathbf{S}_{q,n} = \sum_{i=q}^{n} \mathbf{H}_i^{-1} \mathbf{X}_i \,. \tag{6}$$

b) The control statistic obtained by the GLR approach for model (2) is given by

$$GLR2_n = \max\left\{0, \max_{1\le q\le n}\{\sqrt{k_{q,n}}(\sqrt{D_n - D_{q-1}} - \frac{\sqrt{k_{q,n}}}{2})\}\right\} \tag{7}$$

with

$$D_n = \sum_{i=1}^{n} \mathbf{X}_i' \mathbf{H}_i^{-1} \mathbf{X}_i \,, \quad D_0 = 0 \,. \tag{8}$$

Next, we choose another approach to derive control schemes. It is based on the sequential probability ratio test (SPRT) of Wald (see, e.g. Nikiforov (1986), Knoth and Schmid (2004)). For the change point model (3), Nikiforov (1986) suggested several modifications of the CUSUM control chart applied to multivariate time series. The design of each chart depends on the assumptions imposed on the change vector $\mathbf{a}$ or on $\mathbf{a}_t$, $t \ge q$. When neither the direction of the change nor the size of the change are known, Nikiforov (1986) suggested a modification of the SPRT test. In order to synthesize the cumulative sum algorithm, the Wald's weight function method is used. In the present section, we consider an alternative approach. Instead of using a weighting of all possible directions, we take the maximum with respect to the direction following the idea of the likelihood ratio approach. The corresponding charts are presented in Theorem 2.

Theorem 2. *Let $\{\mathbf{Y}_t\}$ be a multivariate GARCH process with the conditional covariance matrix $\mathbf{H}_t$ at time t.*

a) The control statistic obtained by the SPRT approach for model (3) is given by

$$SPR1_n = \max\left\{0, \sqrt{k_{n-\tau_n^{(1)}+1:n}} \left(\|\mathbf{S}_{n-\tau_n^{(1)}+1:n}\|_{\sum_{i=n-\tau_n^{(1)}+1}^{n} \mathbf{H}_i^{-1}} - \frac{\sqrt{k_{n-\tau_n^{(1)}+1:n}}}{2} \right) \right\} \tag{9}$$

for $n \ge 1$ with $\mathbf{S}_{v:n}$ as defined in (6) and

$$\tau_n^{(1)} = \begin{cases} 1 & \text{for} \quad SPR1_{n-1} = 0 \\ \tau_{n-1}^{(1)} + 1 & \text{for} \quad SPR1_{n-1} > 0 \end{cases} \quad \text{for} \quad n \ge 1. \tag{10}$$

b) The control statistic obtained by the SPRT approach for model (2) is given by

$$SPR2_n = \max\left\{0, \sqrt{k_{n-\tau_n^{(2)}+1:n}}\left(\sqrt{D_n - D_{n-\tau_n^{(2)}}} - \frac{\sqrt{k_{n-\tau_n^{(2)}+1:n}}}{2}\right)\right\} \qquad (11)$$

for $n \geq 1$ with D_n as defined in (8), $SPR2_0 = 0$, and

$$\tau_n^{(2)} = \begin{cases} 1 & for \quad SPR2_{n-1} = 0 \\ \tau_{n-1}^{(2)} + 1 & for \quad SPR2_{n-1} > 0 \end{cases} \quad for \quad n \geq 1. \qquad (12)$$

Remark: The derived control schemes can be easily extended to the case when the coefficients of the recursion for the conditional covariance matrix are unknown and they have to be estimated using previous data. In this case the control charts are obtained conditionally on the estimated values of the unknown parameters. Because data from the previous period is used, when the process is in control, the designs of the control schemes do not change. The only difference is that in the recursion for the conditional covariance matrix the estimated quantities are used. The only requirement is that the parameters are consistently estimated, which is fulfilled for the considered in the next section multivariate GARCH processes according to Jeantheau (1998), Engle and Sheppard (2001), and Comte and Lieberman (2003). The proof of this fact is given in the appendix (Section 6.2).

3.2 Comparison of the Control Schemes

The goal of this section is to compare the performance of the derived in the previous section control charts with each other. We use the maximum expected delay (MED) for measuring the performance of a control chart. The expected delay is defined as the average number of observations of a control chart from the change point in the process until the chart gives a signal provided that this signal is not a false alarm. Consequently, the maximum expected delay is calculated by maximizing the expected delay of a stopping time with respect to all possible positions of the change point. Mathematically, the expected delay of a stopping time N with a changed occurred at time $q \leq N$ is expressed as

$$ED_{\mathbf{a},q}(N) = \mathrm{E}_{\mathbf{a},q}(N - q + 1 | N \geq q) \qquad (13)$$

provided $\mathrm{E}_{\mathbf{a},q}(N) < \infty$. Pollak and Siegmund (1975) proposed the use of the maximum expected delay

$$MED = \sup_{q \geq 1} ED_{\mathbf{a},q}(N) \qquad (14)$$

which can be considered as a worst-case criterion.

All charts are calibrated in the same way. The control limit for each chart is determined such that the in-control average run length (ARL) is equal to a prespecified value. The ARL is defined as the average number of observations taken until a signal is given. **In the comparison study we chose $ARL = 200$.** Using the calculated control limits the MEDs are compared with each other. Note that the control limits depend neither on the process parameters of the target process nor on the type of the multivariate GARCH process. It constitutes a great advantage of the approach. It makes its application to real data much easier.

Because no explicit formulas for the ARLs and MEDs are available a Monte Carlo study is used to estimate these quantities. The estimators are obtained by averaging the corresponding sample values. In our simulation study 10^5 independent realizations of the target process are generated to estimate the in-control ARLs. The control limits of all charts are determined by applying the regula falsi (see, e.g., Conte and de Boor (1981)) to the estimated in-control ARLs. For the estimation of the MEDs 10^6 realizations are taken.

As a target process four two dimensional GARCH processes are considered. the control schemes of Section 3.1 are applied to each process. The shift in the mean vector are generated according to the model (3) with $\mathbf{a} = (a_1, a_2)'$. The performance of the control charts is given in Figures 1-4. The figure shows the MEDs of the GLR1, SPR1, and SPR2 control schemes for $a_2 = 0.3$ and $a_1 \in \{-0.9, -0.6, -0.3, 0, 0.3, 0.6, 0.9\}$. The GLR2 control is not included in the figures because it always has the largest MEDs.

The BEKK model of Engle and Kroner (1995) is used as a first target process. The results for this model are given in Figure 1. In Figure 2, the results for the CCC process of Bollerslev (1990) are shown, while Figure 3 presents the results for the DCC model of Engle (2002). The partial case of the CCC process with the constant correlation coefficient equals to 0 is given in Figure 4. Here, the two independent univariate GARCH(1,1) processes are generated.

Figure 1. *MEDs of the GLR1, SPR1, and SPR2 control charts (c.f. Section 3.1) as a function of the shifts a_1 ($a_2 = 0.3$) for the two dimensional BEKK process. The in-control ARL is* 200.

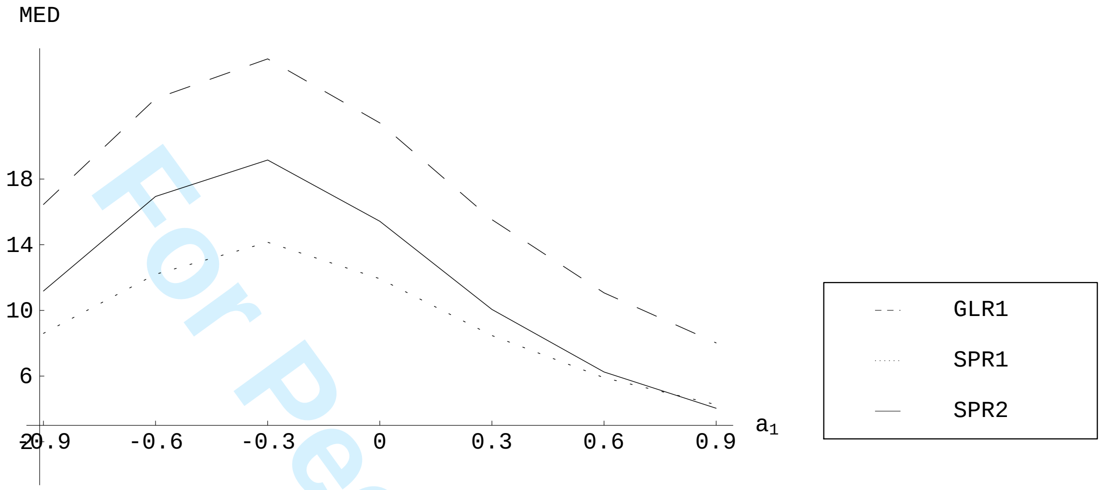


Figure 2. *MEDs of the GLR1, SPR1, and SPR2 control charts (c.f. Section 3.1) as a function of the shifts a_1 ($a_2 = 0.3$) for the two dimensional CCC process. The in-control ARL is* 200.

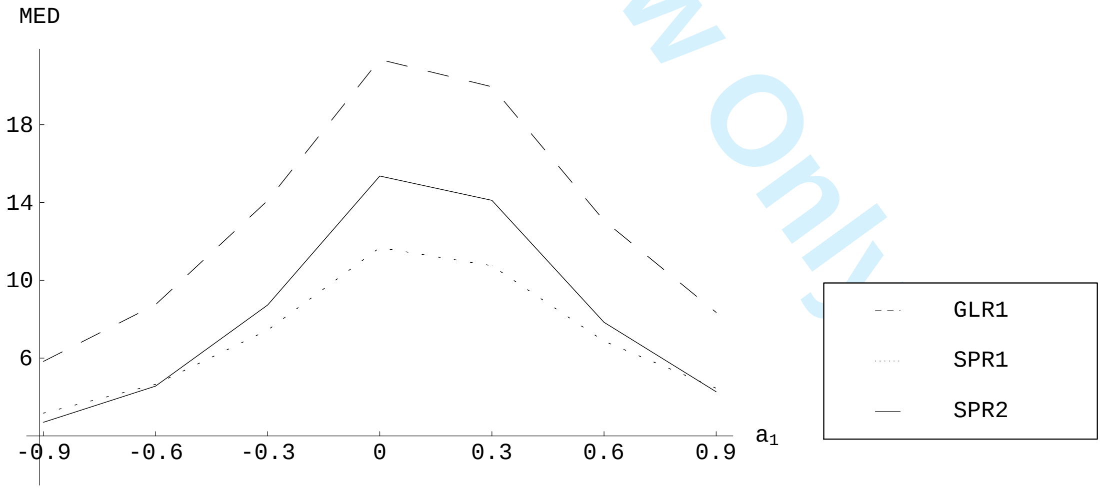


Figure 3. *MEDs of the GLR1, SPR1, and SPR2 control charts (c.f. Section 3.1) as a function of the shifts a_1 ($a_2 = 0.3$) for the two dimensional DCC process. The in-control ARL is* 200.

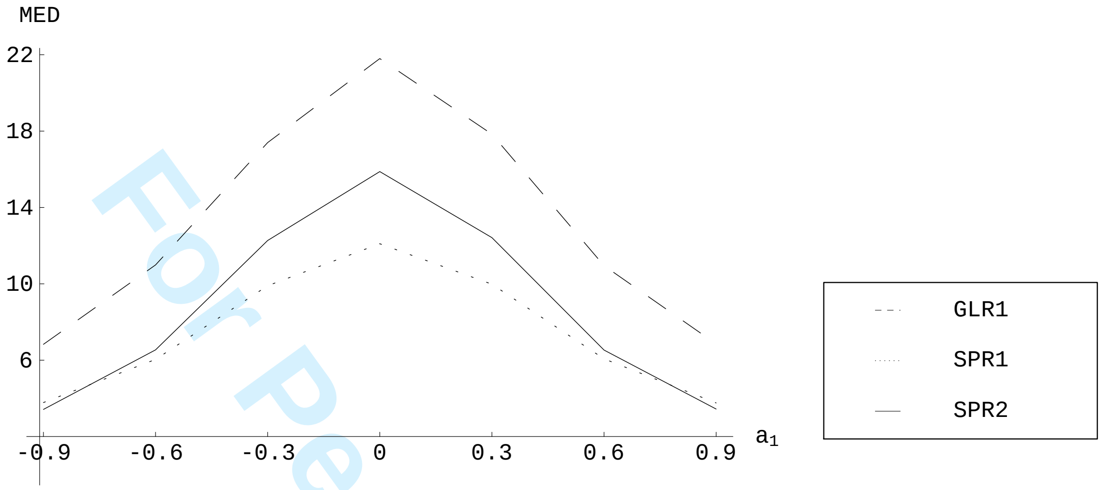
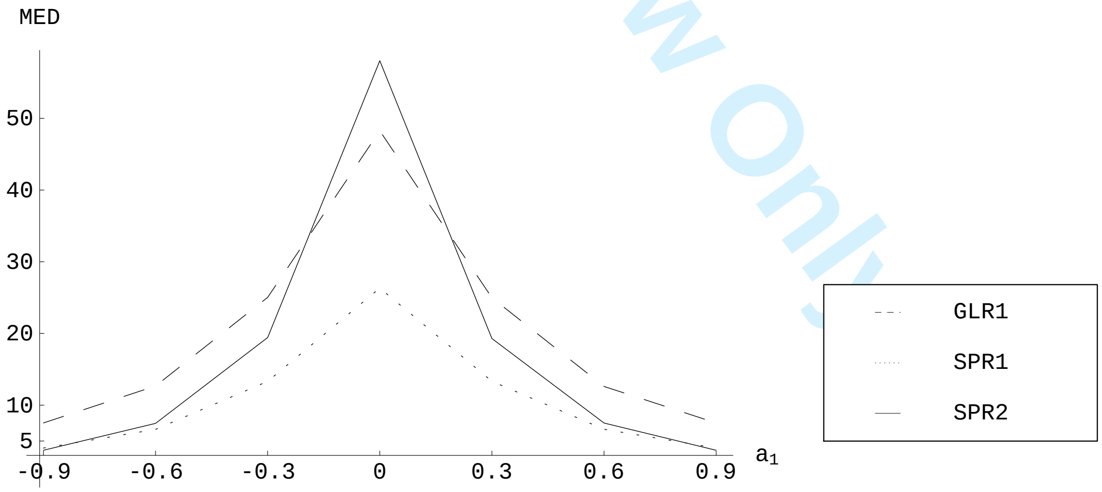


Figure 4. *MEDs of the GLR1, SPR1, and SPR2 control charts (c.f. Section 3.1) as a function of the shifts a_1 ($a_2 = 0.3$) for the two independent univariate GARCH processes. The in-control ARL is* 200.



The comparison of the control schemes leads to the interesting results. For shifts of the small and moderate size the SPR1 chart shows the best results. For the larger shifts SPR2 control scheme has to be selected. These two schemes are ranked on the first and second places correspondingly. On the third place we put the GLR1 control chart. It always performs worst. Only in one case out of seven for the two independent univariate GARCH processes it has a smaller MED as the SPR2 scheme. The poor performance of the control schemes based on maximizing the generalized likelihood ratio is explained by the additional uncertainty about the time of a change that is maintained in the design of these schemes. From the other side ignoring the uncertainty about the time of the change leads to the SPR1 and the SPR2 control charts that are able to detect changes in the mean vector much faster.

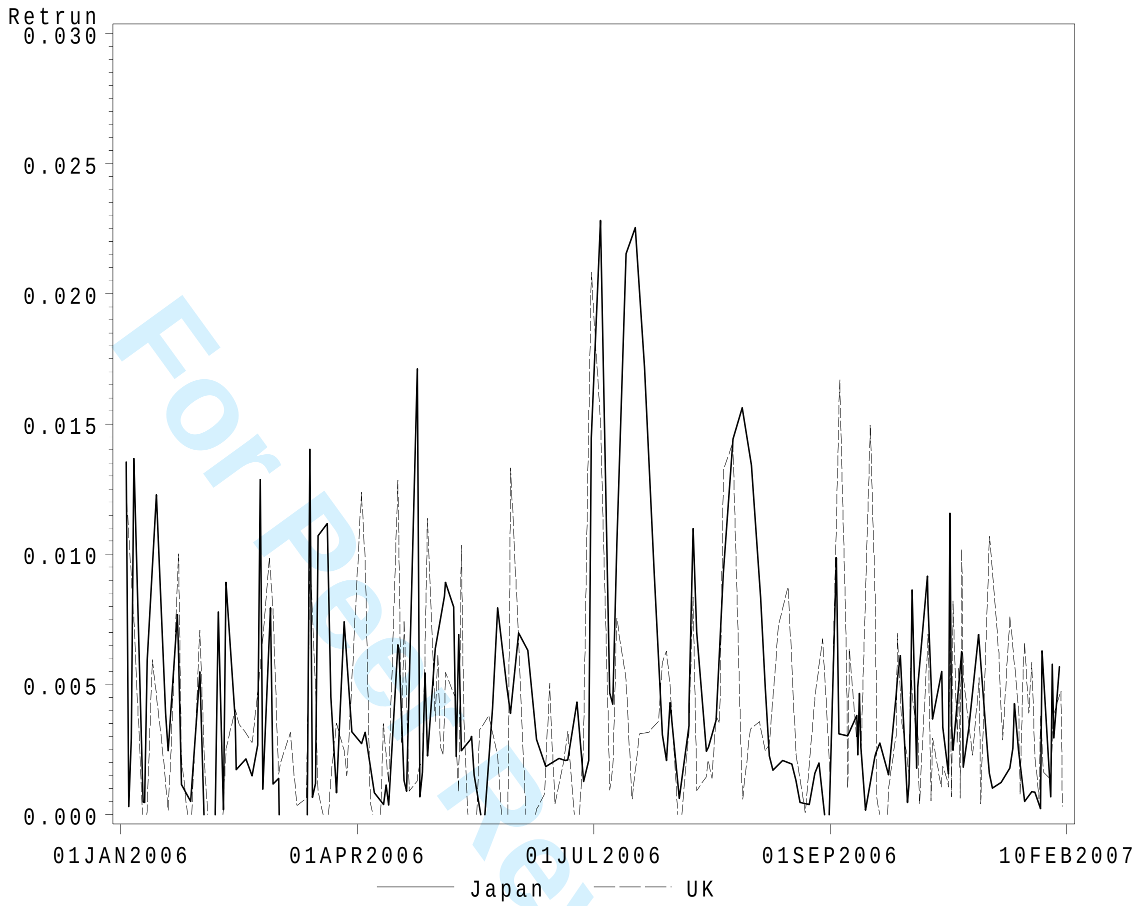
4 Empirical Illustration

In this section, we present an empirical example about the returns of two exchange rates. We show how the results of the previous sections can be applied to monitor the mean vector of the returns. We consider the daily returns of USD/JPY and USD/GBP exchange rates from January, 2nd 1996 to February, 9th 2007. This sample we partition into two subsamples. The first one, that consists of data from January, 2nd 1996 to December, 30th 2006 with 2511 observations is used to estimate the parameters of multivariate GARCH processes. Based on these data the two dimensional BEKK process, the two dimensional CCC process, the two dimensional DCC process, and the two univariate independent GARCH processes are fitted. The rest of data is collected into the second subsample and it is used to monitor changes in the mean vector of the fitted models.

For the illustration purposes we plot the daily returns of USD/JPY and USD/GBP exchange rates from January, 3nd 2006 to February, 9th 2007 in Figure 5. Here, it is observed significant deviations from the means of USD/JPY and USD/GBP returns at the end of June, and in the mean of USD/JPY return at the beginning of May. There is also a change in the mean of USD/GBP return at the beginning of September.

The control limits of the control charts do not depend on the process parameters and the type of the target process. Having once calculated the control limits they can be used for the whole period of interest, even after a restart. This is a great advantage of the control charts. It makes these charts very attractive in practice. For the SPR1 scheme we use $h = 5.79$ with $k = 0.4$, while for the SPR2 scheme $h = 2.00$ and $k = 1.4$ are selected. The control limits correspond to the in-control ARL equals to 200. The choice of k is based on the numerical study of Section 3.2.

Figure 5. *Returns of the USD/JPY and USD/GBP exchange rates for the period from 3 January 2006 to 9 February 2007.*



In Figures 6 to 9 the control statistics of the SPR1 and SPR2 schemes are plotted over time. It is interesting that for both charts the time points of the signals are different. While the larger shifts are detected by the SPR2 scheme, the smaller shifts are detected by the SPR1 scheme. This results is in-line with the Monte Carlo study of Section 3.2.

In principle, the charts detect four periods of significant changes in the mean vector of the considered returns. The first one is detected by the SPR1 scheme at the beginning of May and it corresponds to the change in the mean of USD/JPY return. The second change occurs at the end of June, which is detected by the SPR2 chart. The SPR2 scheme also signals about a change at the beginning of September, when a change in the mean of USD/GBP return occurs. The SPR1 scheme detect drifts at the end of November. At this time period we observe a volatile behavior in both means.

Figure 6. *SPR1 and SPR2 control charts (c.f. Section 3.1) applied to the data of the returns of the USD/JPY and USD/GBP exchange rates for the period from 3 January 2006 to 9 February 2007. The in-control process is modeled as a two dimensional BEKK process. The in-control ARL is* 200.

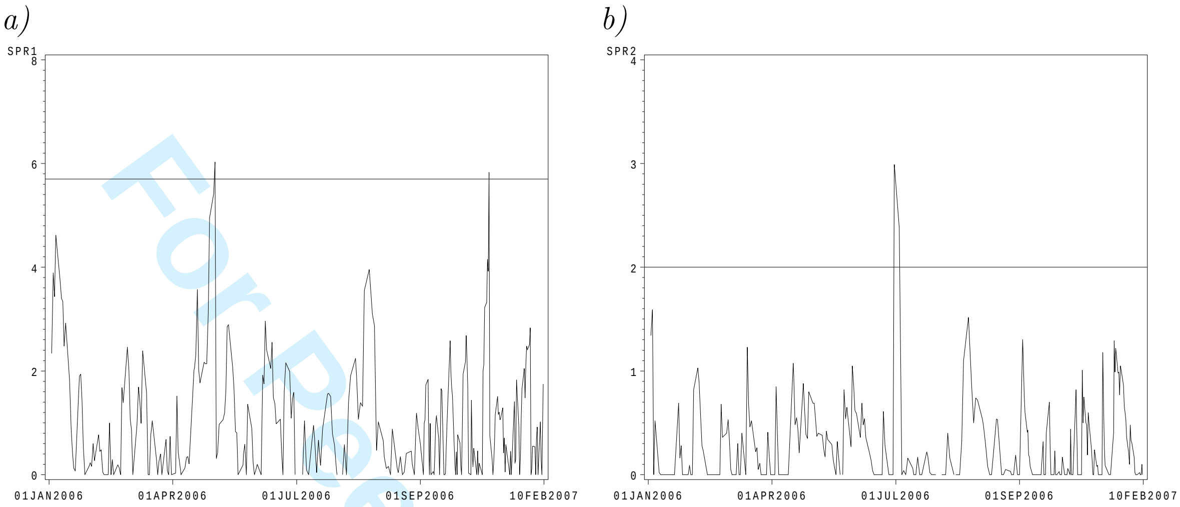


Figure 7. *SPR1 and SPR2 control charts (c.f. Section 3.1) applied to the data of the returns of the USD/JPY and USD/GBP exchange rates for the period from 3 January 2006 to 9 February 2007. The in-control process is modeled as a two dimensional CCC process. The in-control ARL is* 200.

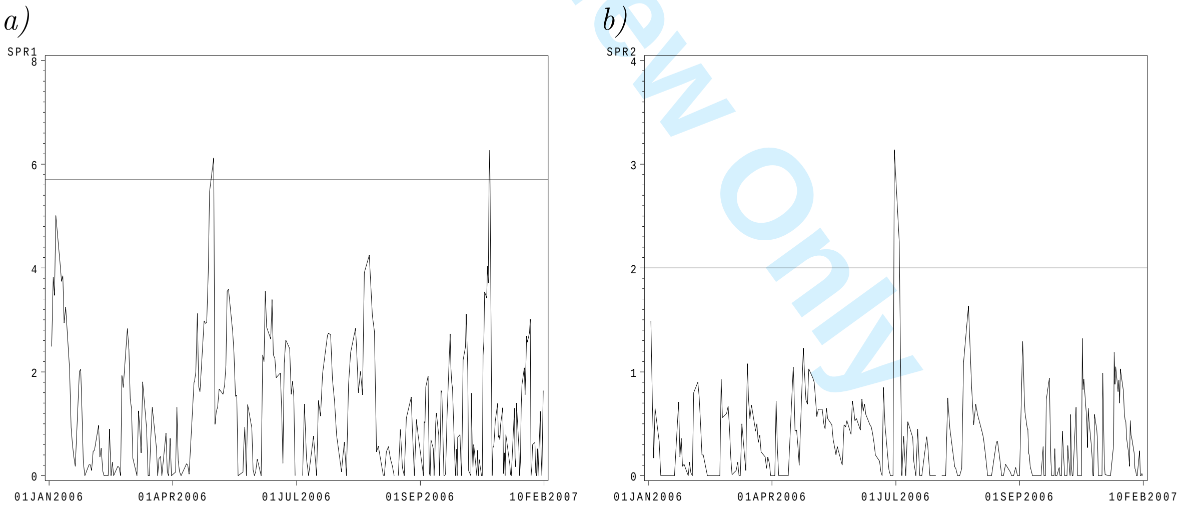


Figure 8. *SPR1 and SPR2 control charts (c.f. Section 3.1) applied to the data of the returns of the USD/JPY and USD/GBP exchange rates for the period from 3 January 2006 to 9 February 2007. The in-control process is modeled as a two dimensional DCC process. The in-control ARL is* 200.

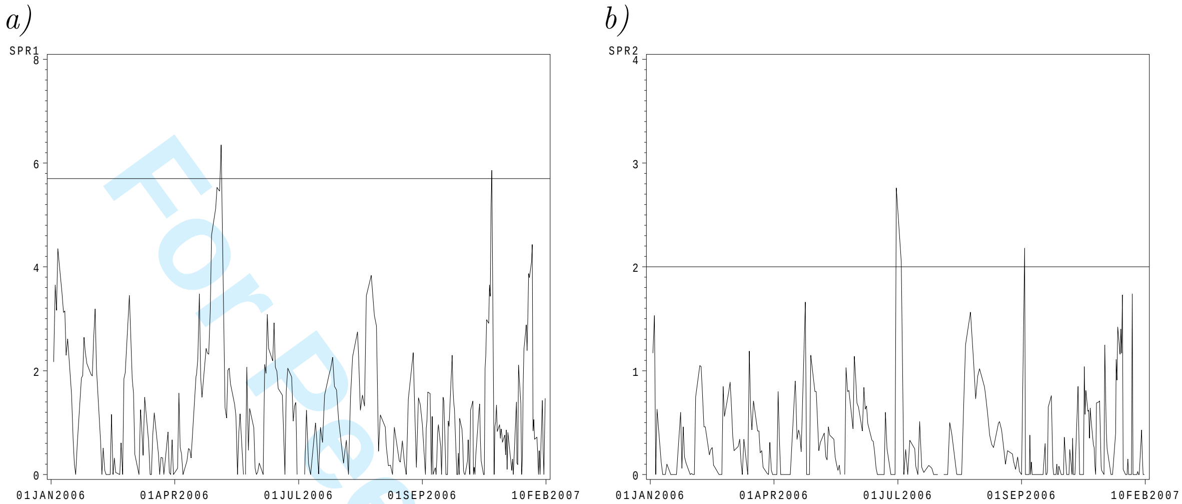
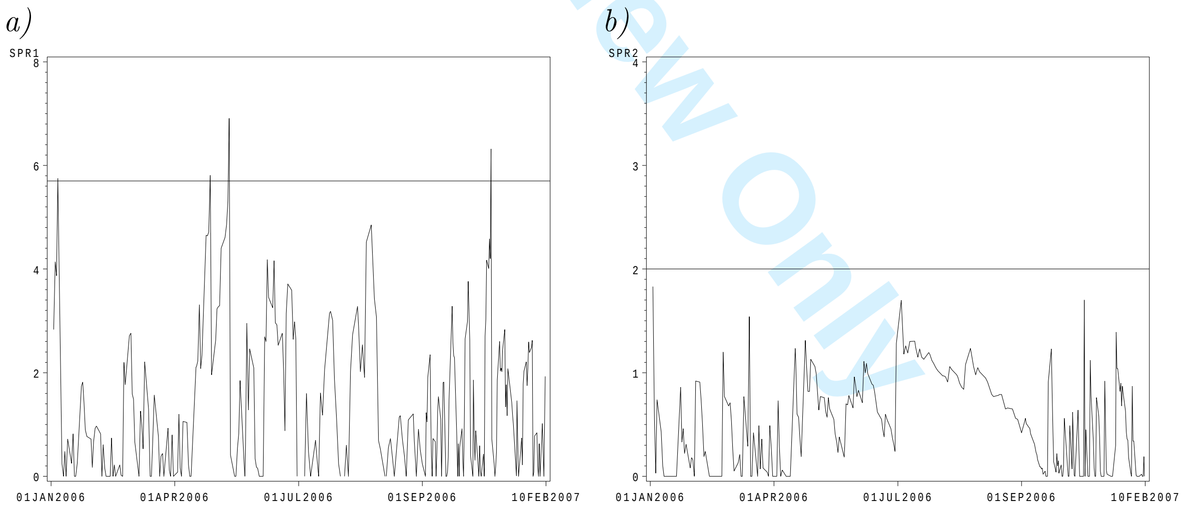


Figure 9. *SPR1 and SPR2 control charts (c.f. Section 3.1) applied to the data of the returns of the USD/JPY and USD/GBP exchange rates for the period from 3 January 2006 to 9 February 2007. The in-control process is modeled as two independent univariate GARCH processes. The in-control ARL is* 200.



5 Summary

In this paper we introduce several control charts for detecting changes in the mean vector of a multivariate GARCH process. The control designs are obtained by maximizing the generalized likelihood ratio. They are general and can be applied to different classes of multivariate GARCH processes. Moreover, the problem of parameter uncertainty is treated as well. The control charts are compared with each other via the extensive Monte Carlo study. We conclude that for larger shifts the SPR2 scheme is the best one, while for shifts of the smaller size the SPR1 scheme has to be selected. These results are also confirmed in the empirical study, where the returns of the two exchange rates are considered.

6 Appendix

6.1 Multivariate GARCH Processes

Since the seminal work of Engle (1982) several multivariate generalizations of the univariate GARCH process are suggested. All these models impose a time varying conditional covariance matrix which is calculated using a recursive procedure. The difference between the model is that they are based on different recursions.

In general, we assume that the vector of the returns is given by

$$\mathbf{Y}_t = \mathbf{H}_t^{\frac{1}{2}} \boldsymbol{\varepsilon}_t\,, \tag{15}$$

where $\mathbf{H}_t = \mathrm{Cov}(\mathbf{Y}_t|\mathcal{F}_{t-1})$ is the conditional covariance matrix of $\mathbf{Y}_t$ given the sigma field $\mathcal{F}_{t-1}$. $\mathcal{F}_{t-1}$ is generated by all information till time $t-1$. Next we review different specifications of multivariate GARCH models that are used for modeling $\mathbf{H}_t$. For a detailed survey we refer to Bauwens et al. (2006).

The first multivariate GARCH process was derived by Bollerslev et al. (1988). It is known as a VEC-parametrization of the multivariate GARCH process. Let $\mathbf{h}_t = vech(\mathbf{H}_t)$ and $\boldsymbol{\xi}_t = vech(\mathbf{Y}_t\mathbf{Y}_t')$, where $vech(.)$ denotes the vech operator (see Harville (1997), ch. 16.4). The conditional covariance matrix is given by

$$\mathbf{h}_t = \mathbf{C} + \sum_{i=1}^{p} \mathbf{A}_i \mathbf{h}_{t-i} + \sum_{j=1}^{q} \mathbf{B}_j \boldsymbol{\xi}_{t-j}\,, \tag{16}$$

where $\mathbf{C}$ is a $(k+1)k/2$ dimensional parameter vector, $\mathbf{A}_i$ and $\mathbf{B}_j$ are $((k+1)k/2) \times ((k+1)k/2)$ parameter matrices. Each component of the vector $\mathbf{h}_t$ is presented as a function of lagged squared errors, cross products of errors and lagged values of $\mathbf{h}_t$. The considered framework is rather general. However, even for a small dimension of the returns vector the number of parameters to be estimated is large. E.g., the number of parameters for the VEC(1,1) process is equal to $k(k+1)(k(k+1)+1)/2$. The second problem related to this process is the positive definiteness of the matrix $\mathbf{H}_t$. The sufficient conditions of positive definiteness are given in Gourieroux (1997, ch. 6.1).

In order to avoid the problem of positive definiteness of $\mathbf{H}_t$, Engle and Kroner (1995) proposed a new parametrization for $\mathbf{H}_t$, the so-called BEKK(p,q,K) process

$$\mathbf{H}_t = \mathbf{C}'\mathbf{C} + \sum_{k=1}^{K}\sum_{i=1}^{p}\mathbf{A}'_{ik}\mathbf{H}_{t-i}\mathbf{A}_{ik} + \sum_{k=1}^{K}\sum_{j=1}^{q}\mathbf{B}'_{jk}\mathbf{Y}_{t-j}\mathbf{Y}'_{t-j}\mathbf{B}_{jk}\,. \tag{17}$$

K determines the generality of the process. The BEKK specification of $\mathbf{H}_t$ allows a reduction of the number of unknown parameters to $k(5k+1)/2$ for the first order case. Under certain conditions, the VEC and the BEKK specifications are equivalent (see Proposition 2.4 of Engle and Kroner (1995)).

The number of parameter to be estimated can be reduced by using the scalar and diagonal versions of the VEC and the BEKK models. In this case the matrices $\mathbf{C}$, $\mathbf{A}_i$, $\mathbf{B}_j$, $\mathbf{A}_{ki}$, and $\mathbf{B}_{kj}$ are replaced by scalars or diagonal matrices. The second possibility is to consider the factor or orthogonal models (Diebold and Nerlove(1989), Engle et al. (1990), Alexander (2000), van der Weide (2002)). The procedure of Alexander (2000) is based on constructing unconditionally uncorrelated linear combinations of the process $\mathbf{Y}_t$. I.e. $\tilde{\mathbf{Y}}_t = \mathbf{F}\mathbf{Y}_t$, $\mathrm{E}(\tilde{\mathbf{Y}}_t\tilde{\mathbf{Y}}'_t) = \mathbf{V}$ with $\mathbf{V}$ to be a diagonal matrix. Then univariate GARCH models are fitted for all elements of the vector $\tilde{\mathbf{Y}}_t$ or to some of them and the whole covariance matrix is estimated under the assumption of zero conditional correlations. The conditional covariance equation is given by

$$\mathbf{H}_t = \mathbf{F}'\mathbf{V}_{t-1}\mathbf{F} \quad \text{with} \quad \mathbf{V}_{t-1} = \mathrm{E}(\tilde{\mathbf{Y}}_t\tilde{\mathbf{Y}}'_t|\mathcal{F}_{t-1})\,. \tag{18}$$

In this formulation the number of the unknown parameters is equal to $2k$.

Bollerslev (1990) proposed a class of multivariate GARCH processes with constant conditional correlations (CCC process). The conditional covariances are modeled as the product of the corresponding conditional standard deviations

$$\mathbf{H}_t = \mathbf{D}_t\mathbf{R}\mathbf{D}_t \quad \text{with} \quad \mathbf{D}_t = diag(h_{11;t}^{1/2}, ..., h_{kk;t}^{1/2})\,. \tag{19}$$

$\mathbf{R} = (\rho_{ij})$ is a symmetric positive definite matrix with $\rho_{ii} = 1$ for each i. The diagonal elements of $\mathbf{D}_t$ are modeled by fitting univariate GARCH processes.

The results of Bollerslev (1990) were extended by Engle (2002) and Tse and Tsui (2002) who allowed the correlation matrix $\mathbf{R}$ to be time varying. The corresponding approaches are known as a dynamic conditional correlation (DCC) model. The DCC process of Engle (2002) (see also Engle and Sheppard (2001)) is given by

$$\mathbf{H}_t = \mathbf{D}_t\mathbf{R}_t\mathbf{D}_t\,, \tag{20}$$

where $\mathbf{D}_t$ is given in (19), $h_{ii;t}$ are modeled by univariate GARCH processes, and

$$\mathbf{R}_t = (diag(\mathbf{Q}_t))^{-\frac{1}{2}}\mathbf{Q}_t(diag(\mathbf{Q}_t))^{-\frac{1}{2}} \tag{21}$$

with the symmetric positive definite matrix $\mathbf{Q}_t$ given by

$$\mathbf{Q}_t = (1 - \sum_{i=1}^{P} \alpha_i - \sum_{j=1}^{Q} \beta_j)\overline{\mathbf{Q}} + \sum_{i=1}^{P} \alpha_i \mathbf{u}_{t-i}\mathbf{u}'_{t-i} + \sum_{j=1}^{Q} \beta_j \mathbf{Q}_{t-j} \,. \tag{22}$$

$\overline{\mathbf{Q}}$ is the unconditional covariance matrix of $\mathbf{u}_t = \mathbf{D}_t^{-1}\mathbf{Y}_t/$. $\alpha_i > 0$, $\beta_j \geq 0$ are scaler parameters satisfying $\sum_{i=1}^{P} \alpha_i + \sum_{j=1}^{Q} \beta_j < 1$. The elements of $\overline{\mathbf{Q}}$ can be alternatively set to the empirical counterparts.

6.2 Derivation of the Control Schemes

In this section all the proofs are given.

Proof of Theorem 1

Let L_n denotes the logarithm of the likelihood ratio to test the hypothesis of shift's appearance at $q \in 1, ..., n$ against the null hypothesis "no shift". Then

$$-2\ln L_n = -2\ln\left(\frac{f_0(\mathbf{X})}{\max_{0\leq q\leq n}\, f_{a_t,q}(\mathbf{X})}\right),$$

where $f_0(\mathbf{X})$ presents the joint density function of the sample $\mathbf{X}_1, ..., \mathbf{X}_n$ and the initial vector $\mathbf{X}_0$. In the case of no change it is given by

$$f_0(\mathbf{X}) \;\; = \;\; f(\mathbf{X}_n|\mathbf{X}_{n-1}, ..., \mathbf{X}_0) ... f(\mathbf{X}_1|\mathbf{X}_0) f(\mathbf{X}_0) \,.$$

Because the unconditional density of a multivariate GARCH process is unknown we use the density of the random vector $\mathbf{X}_0$ in the definition of the likelihood function. The density function of the initial vector $\mathbf{X}_0$ we denote by $f(\mathbf{X}_0)$, which is unspecified unconditional density of a multivariate GARCH process. **Let** $C_{np} = (2\pi)^{-np/2}\left(\prod_{i=1}^{n}|\mathbf{H}_i|\right)^{-1/2}$**.** Because $\mathbf{X}_i|\mathbf{X}_{i-1}, ..., \mathbf{X}_0 \sim \mathcal{N}_p(\mathbf{0}, \mathbf{H}_i)$ it holds that

$$f_0(\mathbf{X}) \;\; = \;\; C_{np} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} \mathbf{X}'_i \mathbf{H}_i^{-1} \mathbf{X}_i\right) f(\mathbf{X}_0) \,. \tag{23}$$

The same idea leads to the definition of $f_{a_t,q}(\mathbf{X})$, i.e. the joint density in the out of control state with respect to the model (2). It is given by

$$f_{a_t,q}(\mathbf{X}) = C_{np} \exp\left(-\frac{1}{2}\sum_{i=1}^{q-1} \mathbf{X}'_i \mathbf{H}_i^{-1} \mathbf{X}_i - \frac{1}{2}\sum_{i=q}^{n} (\mathbf{X}_i - \mathbf{a}_i)' \mathbf{H}_i^{-1} (\mathbf{X}_i - \mathbf{a}_i)\right) f(\mathbf{X}_0) \,. \tag{24}$$

Note that $f(\mathbf{X}_0)$ is the same in (23) and (24). It follows from the fact that in both cases $\mathbf{X}_0$ is taken from the in-control state. Putting everything together we get

$$\frac{f_0(\mathbf{X})}{f_{a_t,q}(\mathbf{X})} \;\; = \;\; \exp\left(-\frac{1}{2} S_{q,n}\right)$$

with

$$S_{q,n} = \sum_{i=q}^{n}\Big(\mathbf{X}_i'\mathbf{H}_i^{-1}\mathbf{X}_i - (\mathbf{X}_i-\mathbf{a}_i)'\mathbf{H}_i^{-1}(\mathbf{X}_i-\mathbf{a}_i)\Big)$$

$$= 2\sum_{i=q}^{n}\mathbf{X}_i'\mathbf{H}_i^{-1}\mathbf{a}_i - \sum_{i=q}^{n}\mathbf{a}_i'\mathbf{H}_i^{-1}\mathbf{a}_i = 2\mathbf{G}_1'\mathbf{a}_{q:n} - \mathbf{a}_{q:n}'\mathbf{G}_2\mathbf{a}_{q:n}\,,$$

where $\mathbf{a}_{q:n} = vec(\mathbf{a}_q, \mathbf{a}_{q+1}, ..., \mathbf{a}_n)$, $\mathbf{G}_1 = vec(\mathbf{X}_q'\mathbf{H}_q^{-1}, ..., \mathbf{X}_n'\mathbf{H}_n^{-1})$, and $\mathbf{G}_2 = diag(\mathbf{H}_q^{-1}, ..., \mathbf{H}_n^{-1})$. Hence,

$$\frac{f_0(\mathbf{X})}{f_{a_t,q}(\mathbf{X})} = \exp\left(-\Big(\mathbf{G}_1'\mathbf{a}_{q:n} - \frac{1}{2}\mathbf{a}_{q:n}'\mathbf{G}_2\mathbf{a}_{q:n}\Big)\right)$$

and

$$\frac{f_0(\mathbf{X})}{\max_{\mathbf{a}_{q:n}} f_{a_t,q}(\mathbf{X})} = \exp\Big(-\max_{\mathbf{a}_{q:n}}\Big(\mathbf{G}_1'\mathbf{a}_{q:n} - \frac{1}{2}k_{q:n}\Big)\Big)\,, \tag{25}$$

where

$$k_{q:n} = \mathbf{a}_{q:n}'\mathbf{G}_2\mathbf{a}_{q:n} = \sum_{i=q}^{n}(\mathrm{E}_q(\mathbf{X}_i))'\mathbf{H}_i^{-1}\mathrm{E}_q(\mathbf{X}_i)\,. \tag{26}$$

Let $g(\mathbf{a}_{q:n}) = \mathbf{G}_1'\mathbf{a}_{q:n}$. Our task is to maximize the function $g(\mathbf{a}_{q:n})$ with respect to (w.r.t.) $\mathbf{a}_{q:n}$ given the condition (26). The rest of the proof is separately done for both types of the shifts, i.e. constant change and time-varying change.

a) Here, the model (3) is maintained, i.e. $\mathbf{a}_t = 0$ for $t < q$ and $\mathbf{a}_t = \mathbf{a}$ for $t \geq q$. In this case the maximization problem transforms to

$$g(\mathbf{a}_{q:n}) = g(\mathbf{a}) = \mathbf{S}_{q,n}'\mathbf{a} \quad \rightarrow \quad max \tag{27}$$

with respect to $\mathbf{a}$ under the constraint $k_{q:n} = \mathbf{a}'\sum_{i=q}^{n}\mathbf{H}_i^{-1}\mathbf{a}$. Applying the Lagrangian method to (27) we get

$$\mathbf{a} = \frac{\sqrt{k_{q:n}}(\sum_{i=q}^{n}\mathbf{H}_i^{-1})^{-1}\mathbf{S}_{q:n}}{\sqrt{\mathbf{S}_{q:n}'(\sum_{i=q}^{n}\mathbf{H}_i^{-1})^{-1}\mathbf{S}_{q:n}}}\,. \tag{28}$$

Hence,

$$\frac{f_0(\mathbf{X})}{\max_{\mathbf{a}} f_{a_t,q}(\mathbf{X})} = \exp\left(-\sqrt{k_{q:n}}\left(\sqrt{\mathbf{S}_{q:n}'(\sum_{i=q}^{n}\mathbf{H}_{i-1}^{-1})^{-1}\mathbf{S}_{q:n}} - \frac{\sqrt{k_{q:n}}}{2}\right)\right),$$

what completes the proof of the part 'a'.

b) Here, it is allowed the drift vector $\mathbf{a}_t$ to vary with the time. The control chart is derived by maximizing $g(\mathbf{a}_{q:n})$ given (26). From the Lagrangian method we obtain

$$\mathbf{a}_{q:n} = \frac{\sqrt{k_{q:n}}}{\sqrt{\mathbf{G}_1'\mathbf{G}_2^{-1}\mathbf{G}_1}}\mathbf{G}_2^{-1}\mathbf{G}_1\,, \tag{29}$$

where

$$\mathbf{G}_1'\mathbf{G}_2^{-1}\mathbf{G}_1 = \sum_{i=q}^{n}\mathbf{X}_i'\mathbf{H}_i^{-1}(\mathbf{H}_i^{-1})^{-1}\mathbf{H}_i^{-1}\mathbf{X}_i = \sum_{i=q}^{n}\mathbf{X}_i'\mathbf{H}_i^{-1}\mathbf{X}_i = D_n - D_{q-1}\,.$$

Hence,

$$\frac{f_0(\mathbf{X})}{\max_{\mathbf{a}} f_{a_t,q}(\mathbf{X})} = \exp\left(-\sqrt{k_{q:n}}\left(\sqrt{D_n - D_{q-1}} - \frac{\sqrt{k_{q:n}}}{2}\right)\right). \tag{30}$$

The theorem is proved.

Proof of Remark

Let $\tilde{f}_0(\mathbf{X})$ denote the joint density function of $\mathbf{X}_0, \mathbf{X}_1, ..., \mathbf{X}_n$ when the unknown parameters are replaced with the corresponding estimated counterparts. It holds that

$$\tilde{f}_0(\mathbf{X}) = f_0(\mathbf{X}|\hat{\mathbf{\Theta}})f(\hat{\mathbf{\Theta}})\,,$$

where $\hat{\mathbf{\Theta}}$ is the vector with the estimated parameters. The last equality holds because the unknown parameters are estimated using previous data and, thus, they are measurable for all the sigma fields $\mathcal{F}_i$, $i = 0, ..., n-1$. Analogically, it holds that

$$\tilde{f}_{a_t,q}(\mathbf{X}) = f_{a_t,q}(\mathbf{X}|\hat{\mathbf{\Theta}})f(\hat{\mathbf{\Theta}})$$

for each q and $\mathbf{a}_t$. Hence,

$$-2\ln\tilde{L}_n = -2\ln\left(\frac{\tilde{f}_0(\mathbf{X})}{\max_{1\le q\le n}\ \tilde{f}_{a_t,q}(\mathbf{X})}\right) = -2\ln\left(\frac{f_0(\mathbf{X})}{\max_{1\le q\le n}\ f_{a_t,q}(\mathbf{X})}\right) = -2\ln L_n\,.$$

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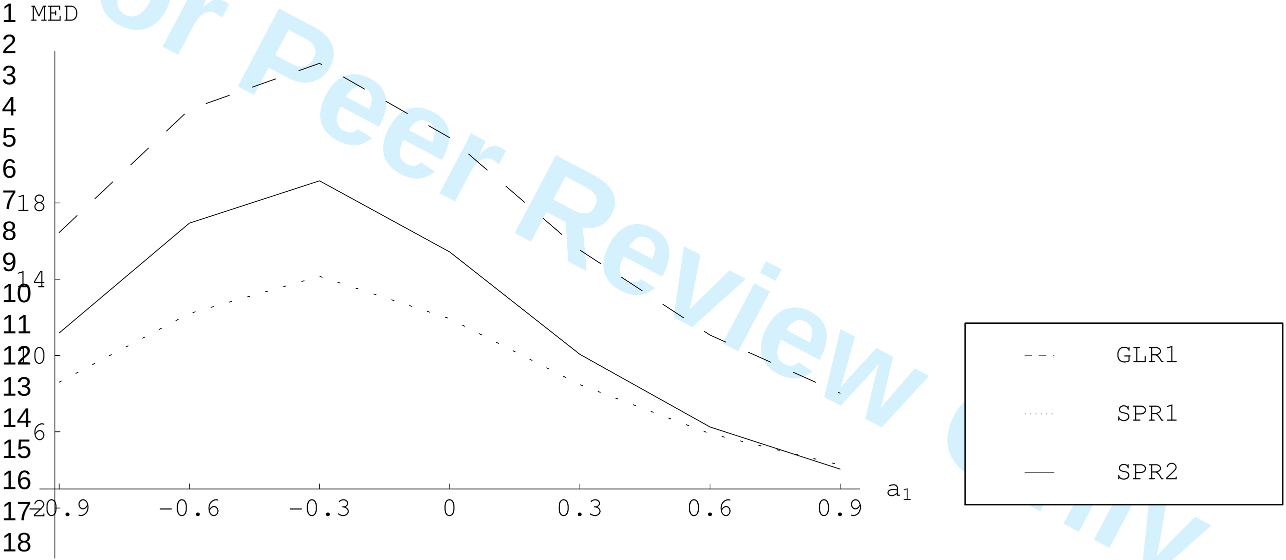
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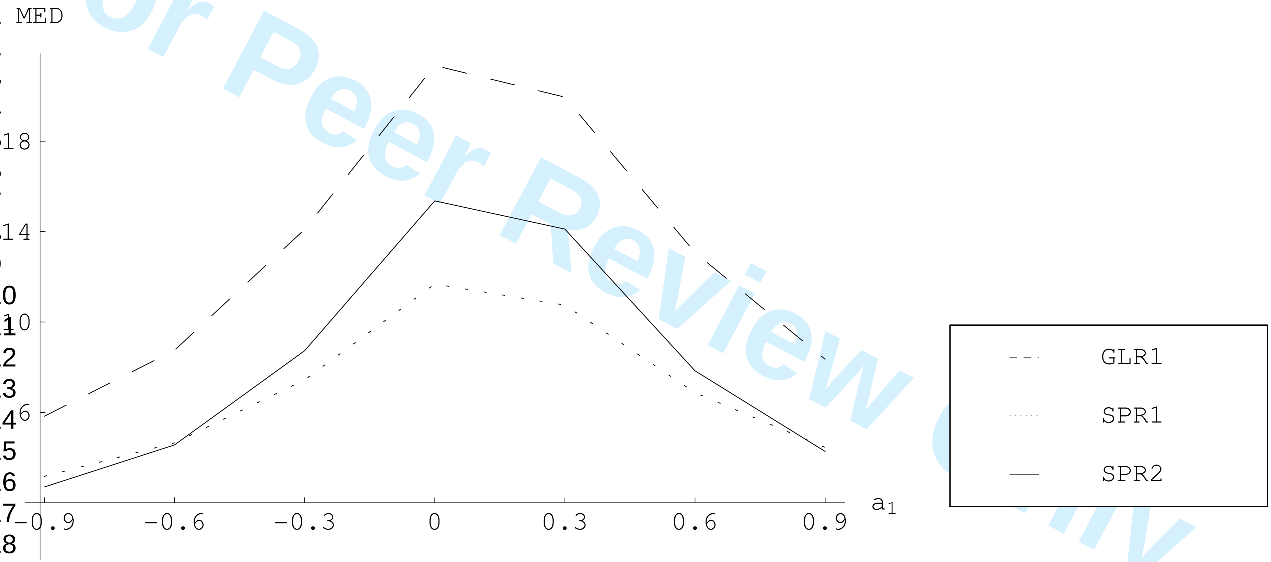
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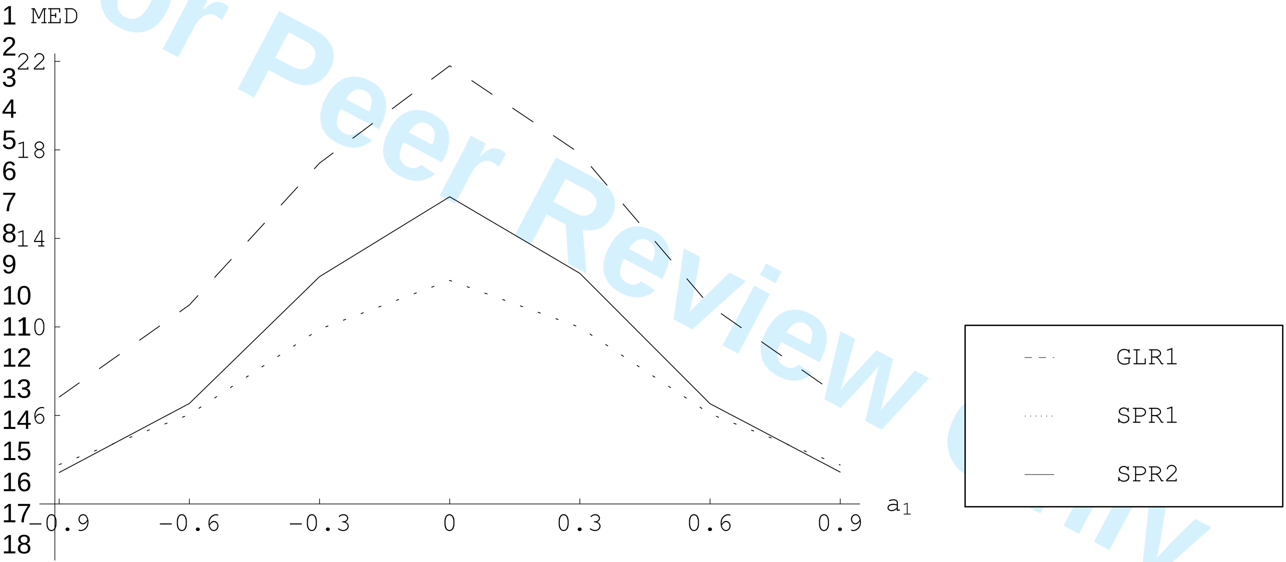
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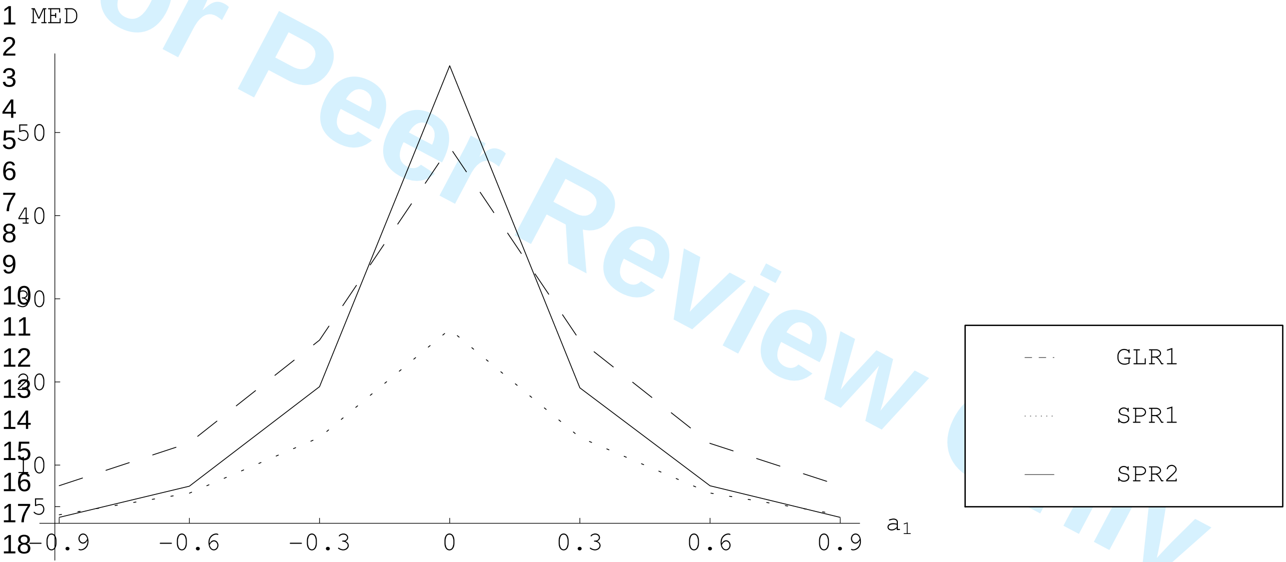
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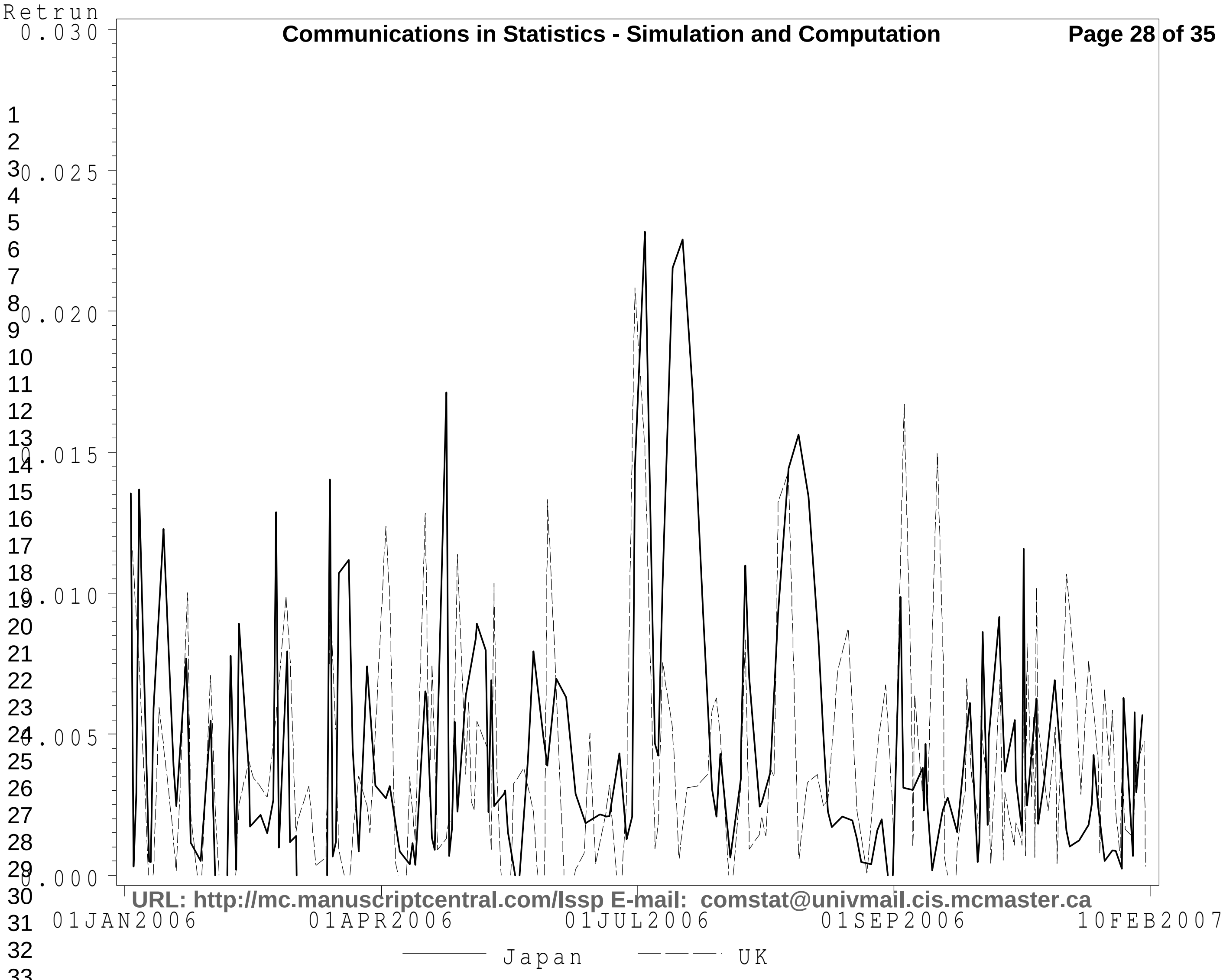
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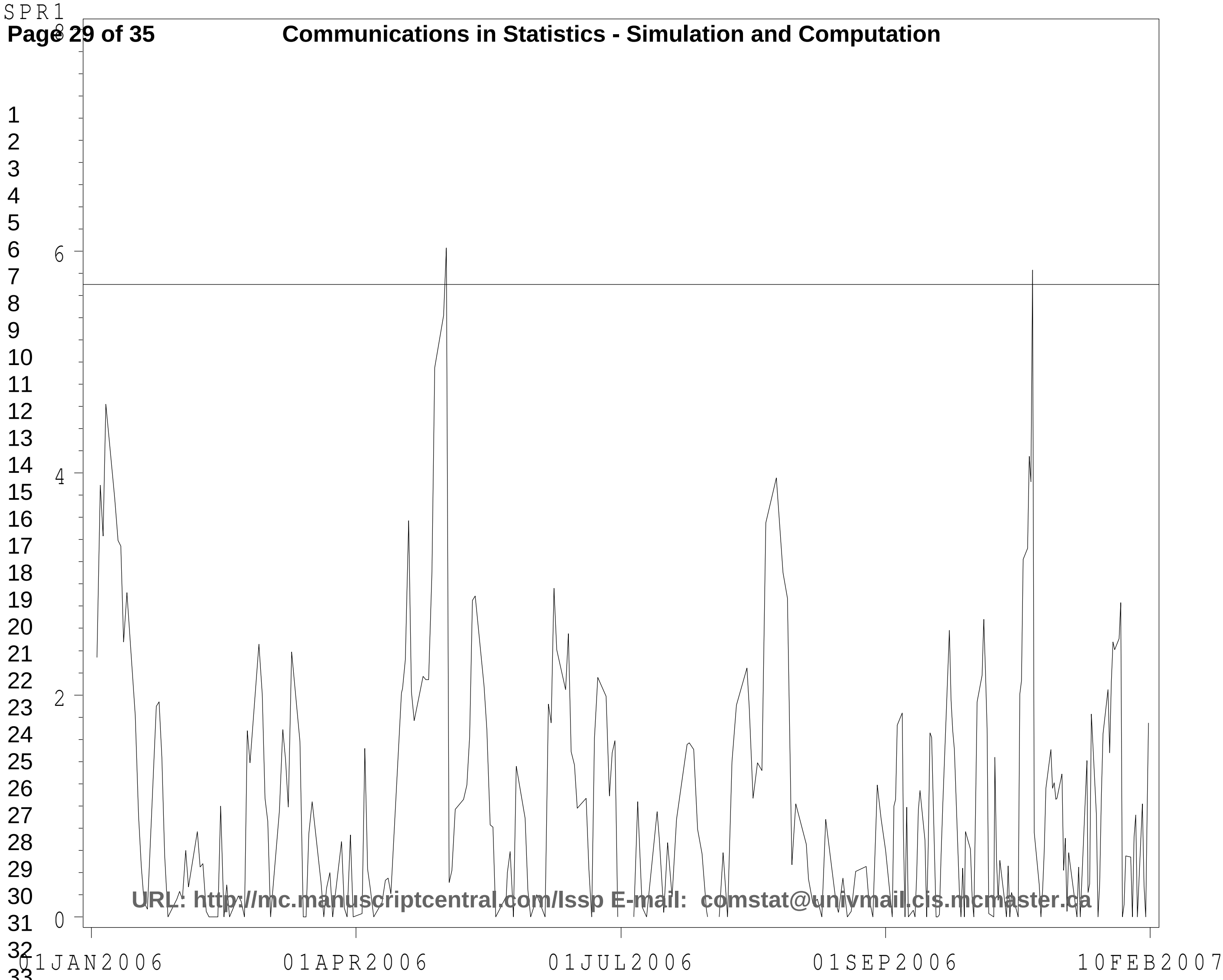
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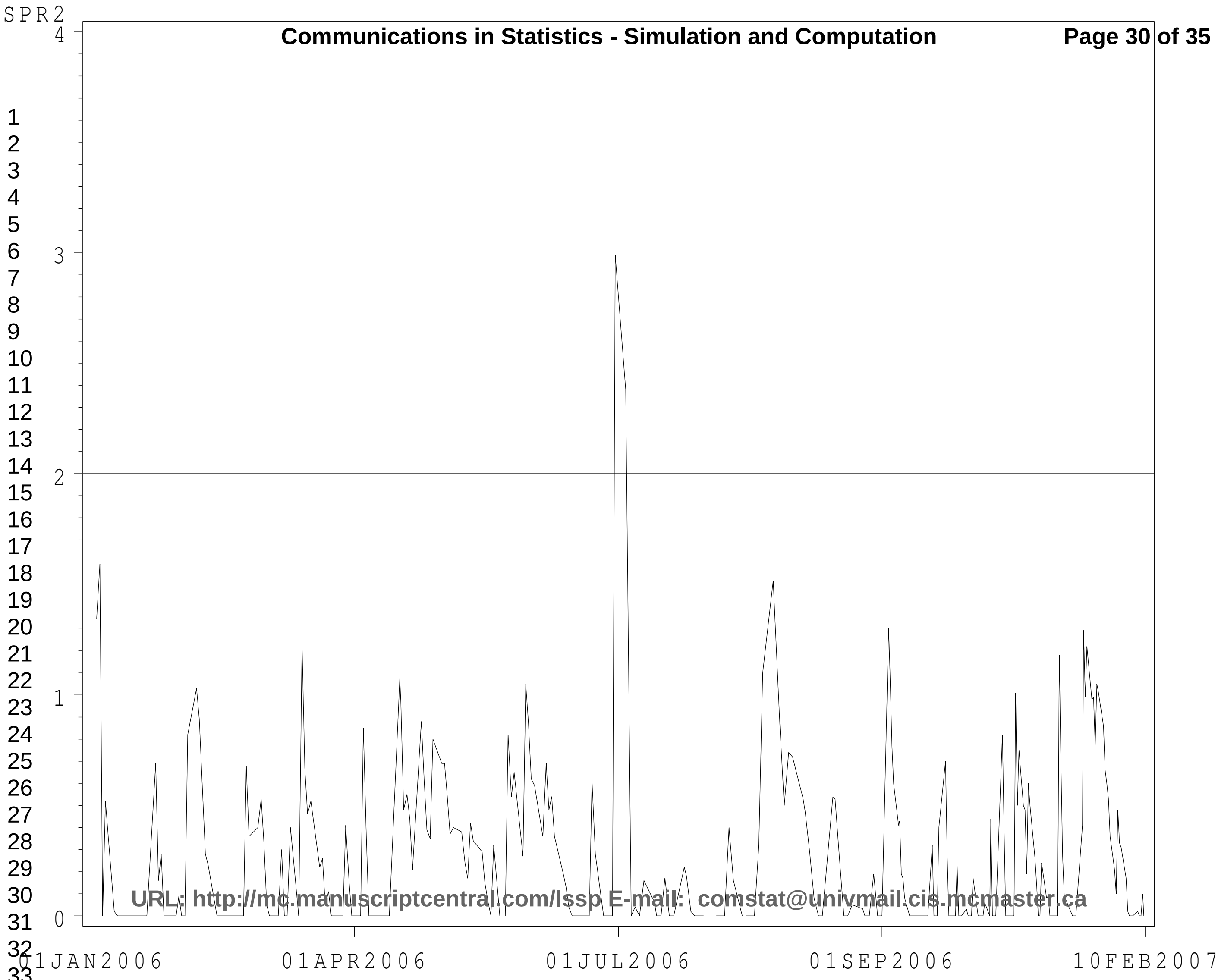
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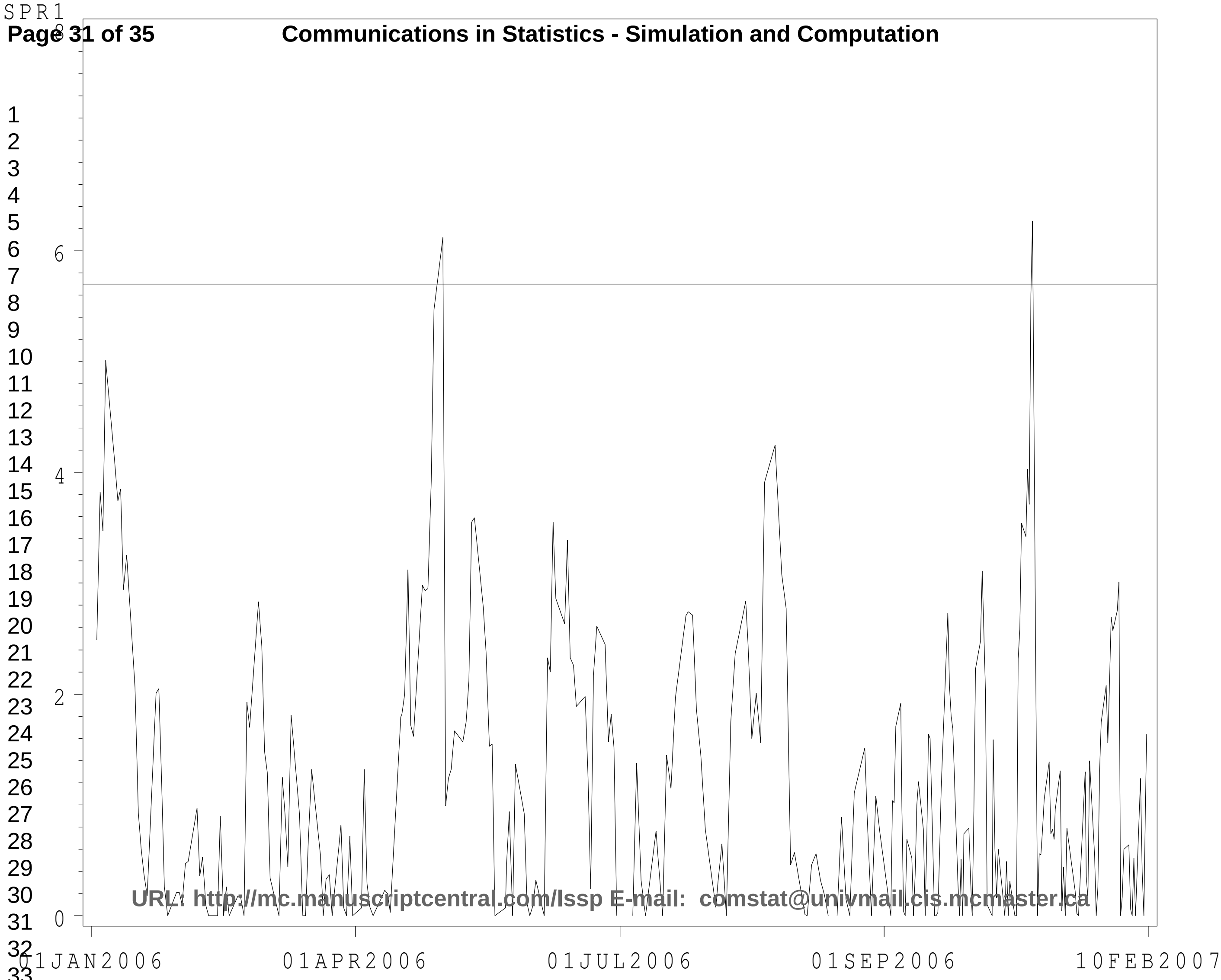
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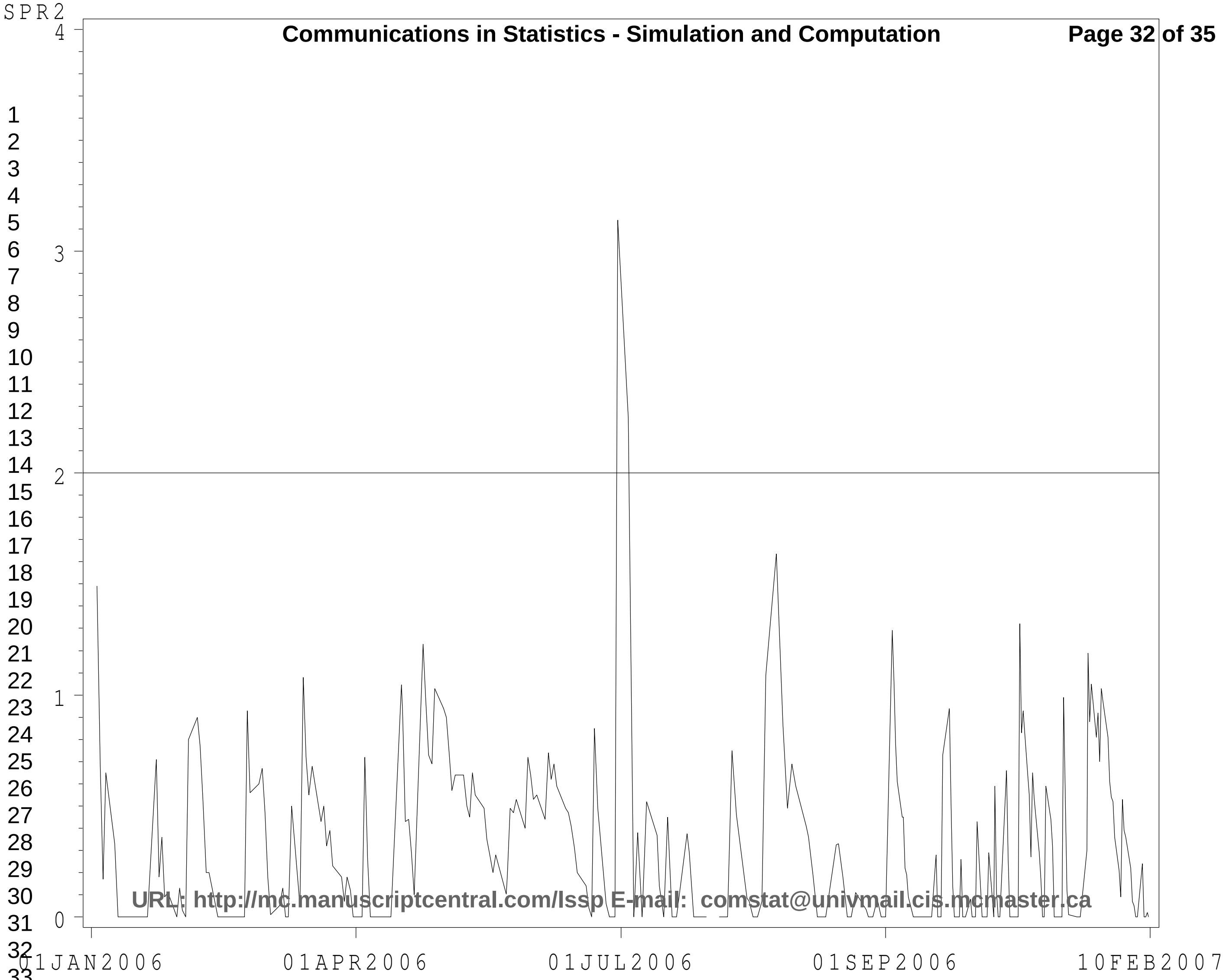
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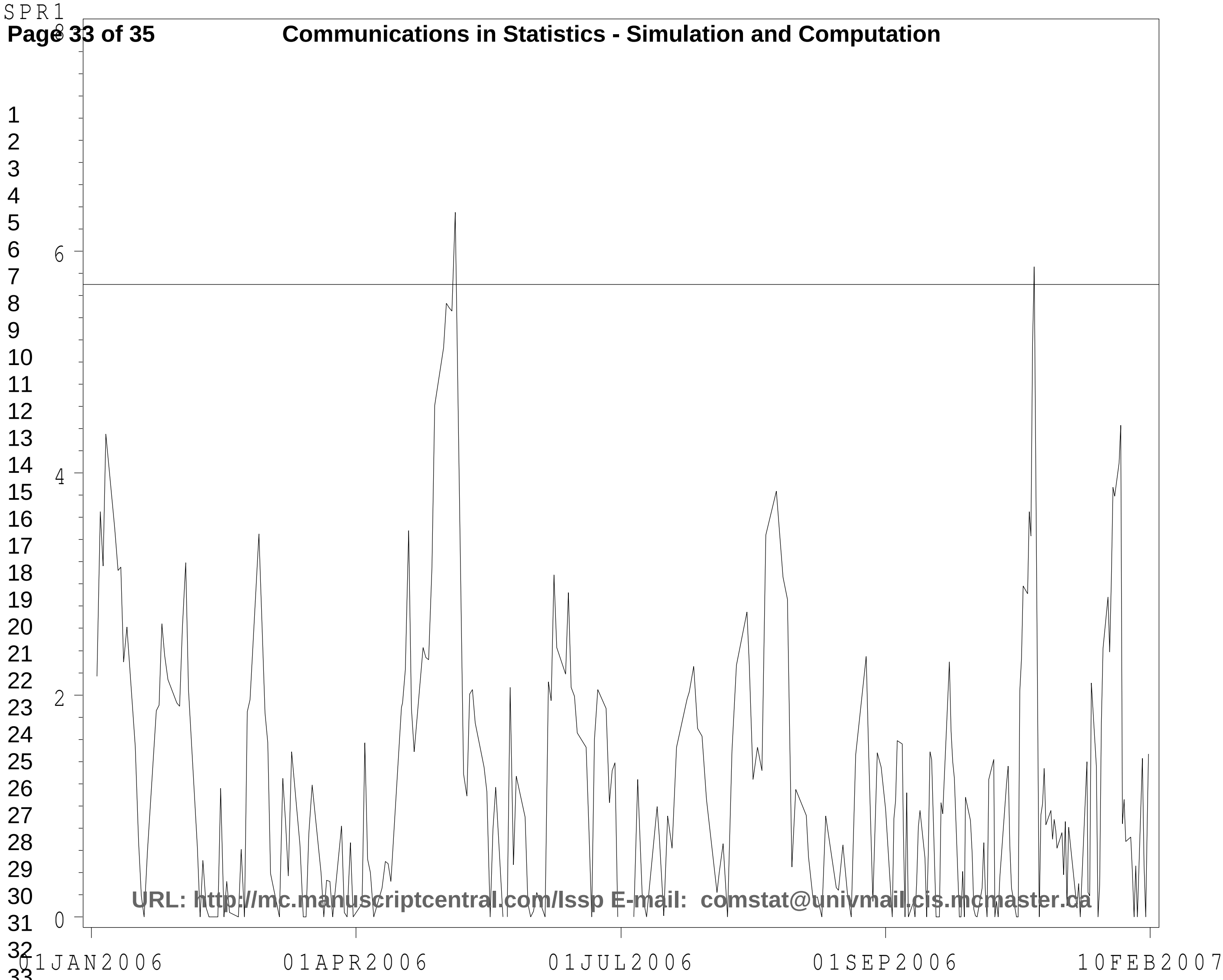
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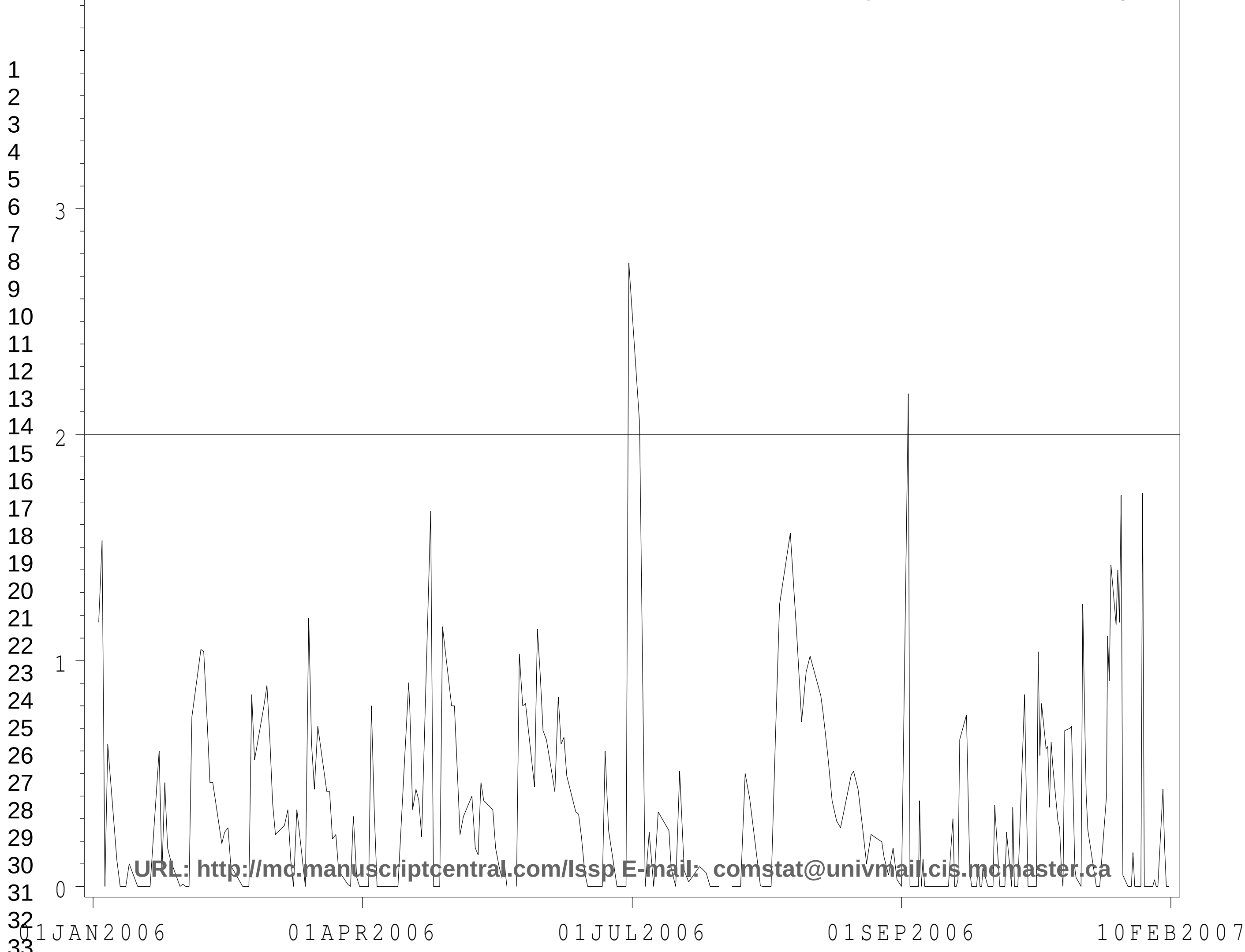
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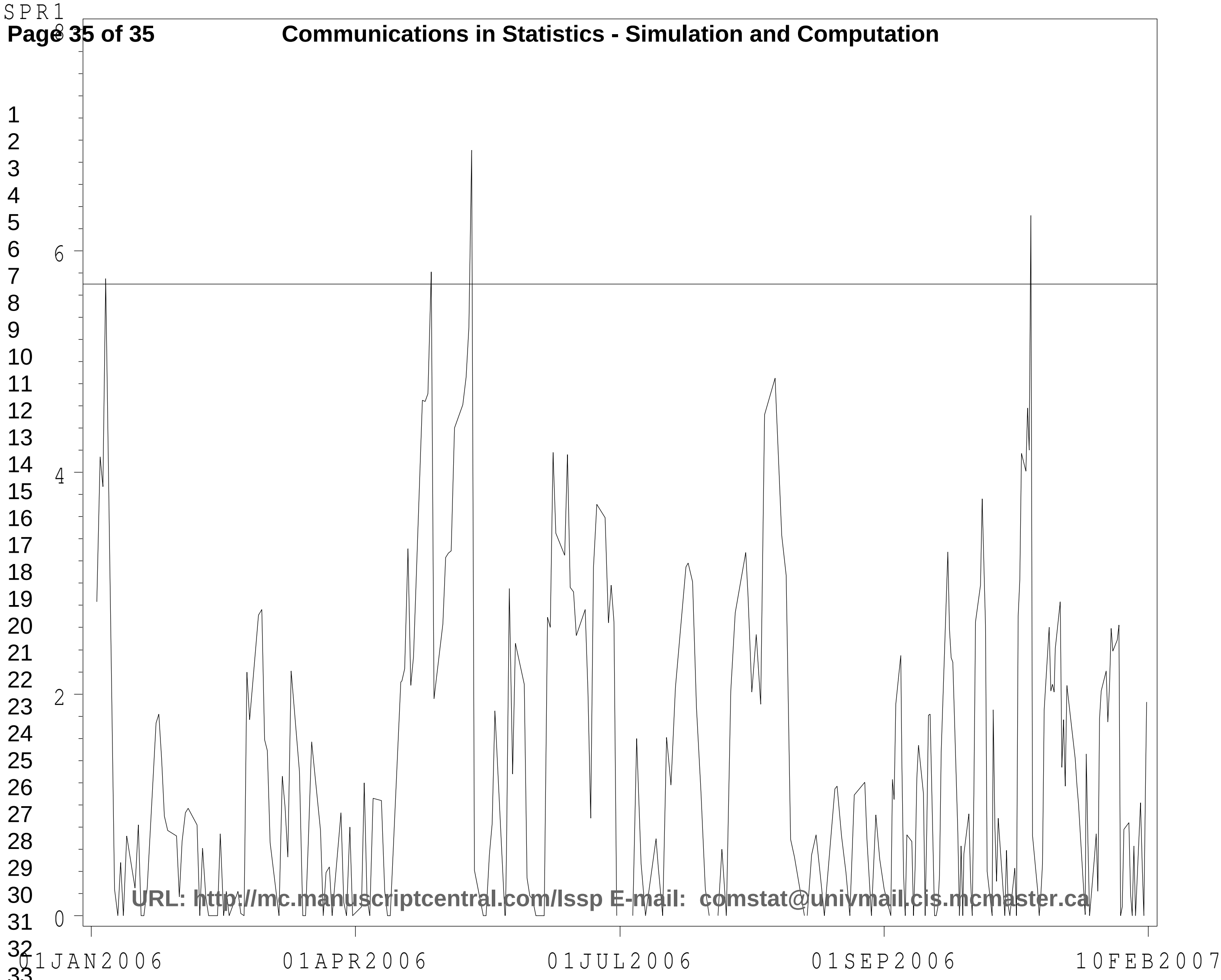
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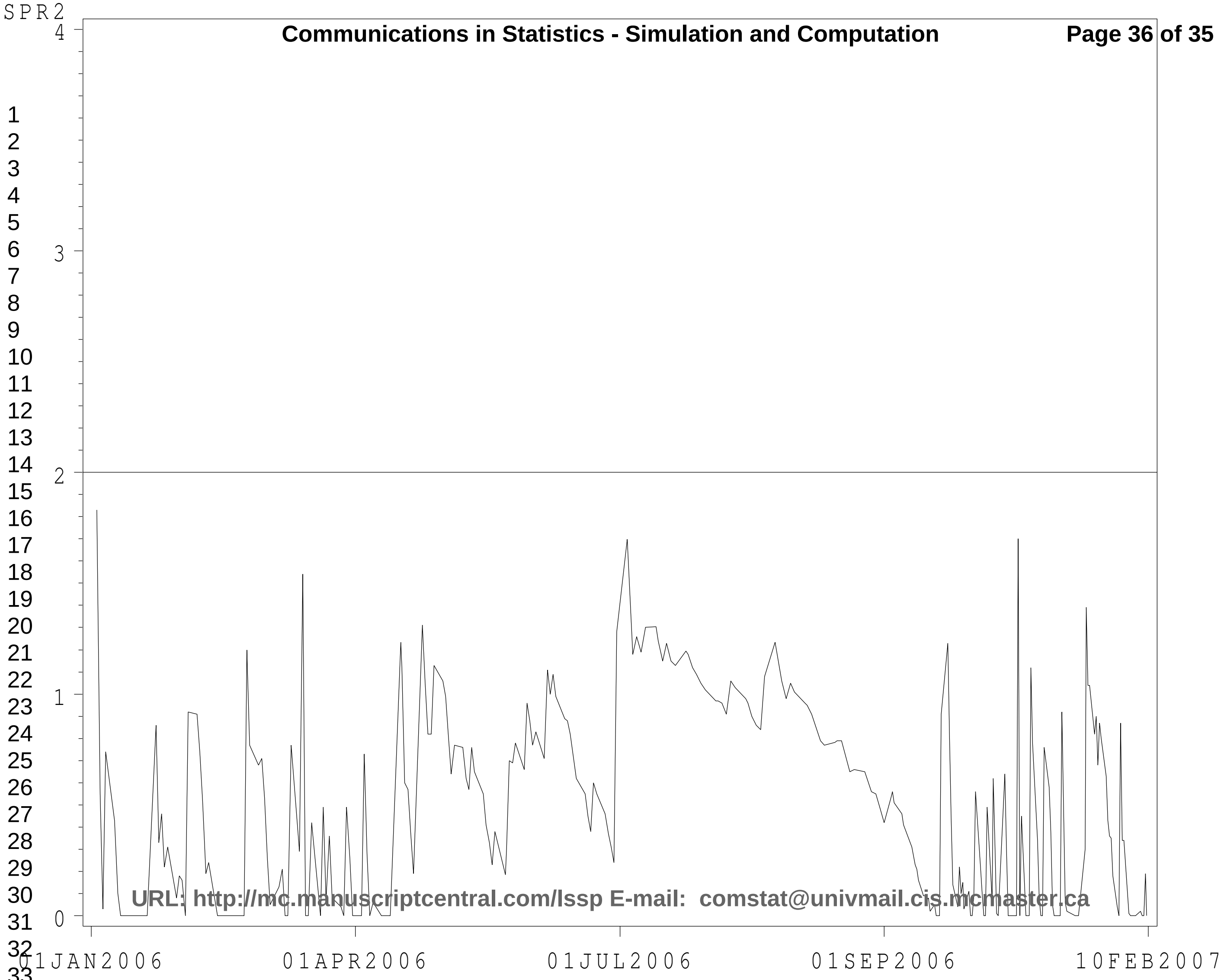
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