# Monte Carlo for estimating exponential convolution 

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#### Abstract

In this note we study the numerical stability problem that may take place when calculating the cumulative distribution function of the Hypoexponential random variable. This computation is extensively used during the execution of Monte Carlo network reliability estimation algorithms. In spite of the fact that analytical formulas are available, they can be unstable in practice. This instability occurs frequently when estimating very small failure probabilities $\left(10^{-30}-10^{-40}\right)$ that can happen for example while estimating the unreliability of telecommunication systems. In order to address this problem, we propose a simple unbiased estimation algorithm that is capable of handling a large number of variables. We show that the proposed estimator has a bounded relative error and that it compares favorably with other existing methods.


Keywords. Hypoexponential distribution, Monte Carlo, Rare Events, Network Reliability.

## 1 Introduction

Network Reliability problem appears in many real life applications such as transportation, social and computer networks, communication, and many more. One approach to handle this problem is by using a Monte Carlo (MC) technique. Some MC methods require computation of the Cumulative Distribution Function (CDF) of the Hypoexponential random variable.

We can state the reliability problem as follows. Suppose we are given an undirected graph $G(V, E, T)$ where $V$ and $E$ are the vertex and edge sets respectively and let $T \subseteq V$ be some terminal set of nodes. Suppose also that edges are subject to failure and for all $e \in E$ there is a corresponding failure probability $q_{e}$. Under this setting we can ask for the probability that the terminal set $T$ be connected. We call the latter an $U P$ state.

One of the well-studied approaches to handle this problem is called an Evolution Monte Carlo (EMC) method [4. The main idea is that at time zero
no edges are present in the graph. Next, we assign each edge a corresponding exponential random variable that represents the time this edge is "born". Naturally, there is a time when the network enters the UP state. The Evolution Monte Carlo method studies those times and delivers the corresponding network reliability (for details see [4). While executing the EMC algorithm, we need to perform many calculations of the form $\mathbb{P}\left(\sum_{i=0}^{n} X_{i} \leq t\right)$. Note that $X_{i} \sim \exp \left(\lambda_{i}\right)$, so this sum is distributed Hypoexponentially and the corresponding complementary CDF can be computed using a matrix exponential:

$$
\begin{equation*}
\mathbb{P}\left(\sum_{i=0}^{n} X_{i} \geq t\right)=e_{1} e^{D t} \mathbf{1}=e_{1} \sum_{k=0}^{\infty} \frac{D^{k} t^{k}}{k!} \mathbf{1} \tag{1}
\end{equation*}
$$

where $e_{1}=(1,0, \cdots, 0)$ is a $1 \times n$ vector, $\mathbf{1}$ is a $n \times 1$ column vector of ones, and

$$
D=\left(\begin{array}{ccccc}
-\lambda_{1} & \lambda_{1} & 0 & \cdots & 0 \\
0 & -\lambda_{2} & \lambda_{2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & -\lambda_{n-1} & \lambda_{n-1} \\
0 & \cdots & 0 & 0 & \lambda_{n}
\end{array}\right)
$$

is a $n \times n$ matrix 2. For the rest of this section we concentrate on the methods used to perform this computation.

First, we examine the exact algorithms available.

- If $\lambda_{1}>\lambda_{2}>\cdots>\lambda_{n}$ is satisfied, formula (1) can be written as

$$
\begin{equation*}
\mathbb{P}\left(\sum_{i=0}^{n} X_{i} \leq t\right)=1-\sum_{i=0}^{n} e^{-\lambda_{i} t} \prod_{j \neq i} \frac{\lambda_{j}}{\lambda_{j}-\lambda_{i}} \tag{2}
\end{equation*}
$$

and computed in $O\left(n^{2}\right)$ time following Ross in [8]. Unfortunately, it was noted that this formula suffers from numerical instability. For example, consider the following $\lambda$ values.

$$
\begin{aligned}
& \lambda_{1}=10.00, \lambda_{2}=9.99, \lambda_{3}=9.98, \lambda_{4}=9.97, \lambda_{5}=9.96, \lambda_{6}=9.95, \\
& \lambda_{7}=9.94 \lambda_{8}=9.93, \lambda_{9}=9.92, \lambda_{10}=9.91, \lambda_{11}=9.9, \lambda_{12}=9.89 .
\end{aligned}
$$

Using a MatLab code proposed in [3] to calculate $\mathbb{P}\left(\sum_{i=0}^{12} X_{i} \leq 1\right)$ we observe that this probability is equal to $-134,217,727$. The result can be verified using the convolution1 code in Appendix B

- A much better approach was tested by Botev et al. [2] and exploited a new matrix exponential algorithm called scaling and squaring that was introduced by Higham in [5. The convolution2 MatLab implementation is attached in Appendix . This method is very stable but more expensive in the sense of CPU time when compared to convolution1.

Next, we introduce randomized methods that output the estimation of the desired value.

- The Cross Entropy (CE) method is a powerful technique for solving difficult estimation and optimization problems, based on Kullback-Leibler (or cross-entropy) minimization [1]. This method was pioneered by Rubinstein in 1999 [9] and is based on an adaptive importance sampling procedure for the estimation of rare-event probabilities.
- The Splitting method is another common technique to deal with counting, combinatorial optimization and rare-event estimation, but unlike the $C E$ method that is based on Importance Sampling, the Splitting procedure relies on the Markov Chain Monte Carlo (MCMC) approach. Splitting dates back to Kahn and Harris [6] and Rosenbluth and Rosenbluth [7]. The main idea is to partition the state-space of a system into a series of nested subsets and to consider the rare event as the intersection of a nested sequence of events.
- The Conditional Monte Carlo Algorithm ( $G-S$ ) proposed by Gertsbakh and Shpungin in [4], Section 7.3, p. 91. The main idea of this approach is to sample the exponential random variables recursively while avoiding rare-event settings. This technique was especially designed to handle the numerical problems that may occur during the exponential convolution calculation.

The rest of the note is organized as follows. In section 2 we introduce our algorithm and prove that it is unbiased and has a bounded relative error. In section 3 we present numerical results and show that our approach can be compared with other methods. Finally, section 4 presents some concluding remarks.

## 2 IS Algorithm

Given independent exponential random variables $X_{1}, \ldots, X_{n}$ such that $X_{i} \sim$ $\exp \left(\lambda_{i}\right)$, we propose to sample from different densities and use likelihood ratios respectively. The details are presented in the following algorithm.

```
Algorithm 2.1 IS Algorithm
Input: \(\lambda_{1}, \cdots, \lambda_{n}\)
Output: \(\widehat{\mathbb{P}}\left(\sum_{i=1}^{n} X_{i} \leq 1\right)\)
    res \(\leftarrow 0\)
    for \(i=1 \rightarrow N\) do
        Sample \(y_{1}, \cdots, y_{n}\), such that \(y_{i} \sim \exp (n)\)
        if \(\sum_{i=1}^{n} y_{i} \leq 1\) then
        \(r e s \leftarrow r e s+\frac{\prod_{i=1}^{n} \lambda_{i} e^{-\lambda_{i} y_{i}}}{\prod_{i=1}^{n} n e^{-n y_{i}}}\)
    end if
    end for
    return \(\frac{\text { res }}{N}\)
```

Let us define

$$
\begin{equation*}
\ell=\mathbb{P}\left(\sum_{i=1}^{n} X_{i} \leq 1\right) \tag{3}
\end{equation*}
$$

Note that the algorithm outputs an estimator to $\mathbb{E}(Z)$, where

$$
\begin{equation*}
Z=1_{\left\{\sum_{i=1}^{n} Y_{i} \leq 1\right\}} \frac{\prod_{i=1}^{n} \lambda_{i} e^{-\lambda_{i} Y_{i}}}{\prod_{i=1}^{n} n e^{-n Y_{i}}} \tag{4}
\end{equation*}
$$

For a formal proof that $\mathbb{E}[Z]$ is an unbiased estimator of $\mathbb{P}\left(\sum_{i=1}^{n} X_{i} \leq 1\right)$ see Lemma A. 1 .

The following corollary immediately follows from the definition of a relative error and from Theorem A.2.

Corollary 2.1 The relative error of the IS Algorithm satisfies

$$
\begin{equation*}
R E \leq \sqrt{\frac{\sqrt{n} e^{2(\bar{\lambda}-\underline{\lambda})+1}}{N}} \tag{5}
\end{equation*}
$$

where $n$ is a number of exponential random variables in the sum, $\bar{\lambda}=\max _{i=1, \ldots, n}\left\{\lambda_{i}\right\}$, $\underline{\lambda}=\min _{i=1, \ldots, n}\left\{\lambda_{i}\right\}$, and $N$ is the sample size.

## 3 Numerical Results

We conducted many numerical experiments using all the algorithms mentioned earlier. In general, we came to the conclusion that for most practical purposes, the exact algorithm convolution2 should be preferred. Unfortunately, when rare event settings are involved the latter may fail. In this section we consider the performance of the proposed algorithms on 3 models. We performed all computations on an Intel Core i5 laptop with 4 GB RAM. We use the same algorithm parameters for all models.

- IS: $N=100 n$ sample size
- Cross Entropy: $\rho=0.3, \alpha=0.5$ and $N=100 n$ sample size both for parameter estimation and final sampling
- Splitting: $\rho=0.3$ and $N=1,000$ sample size
- $G-S: N=100,000$ sample size
- The relative error $(\widehat{R E})$ calculation is based on $K=10$ independent runs. The $\widehat{R E}$ was calculated as

$$
\begin{equation*}
\widehat{R E}=\frac{S}{\tilde{\ell}} \tag{6}
\end{equation*}
$$

where

$$
\widehat{\ell}=\widehat{\mathbb{P}}\left(\sum_{i=1}^{n} X_{i} \leq 1\right), S^{2}=\frac{1}{K-1} \sum_{i=1}^{K}\left(\widehat{\ell}_{i}-\widetilde{\ell}\right)^{2} \text { and } \tilde{\ell}=\frac{1}{\mathrm{~K}} \sum_{\mathrm{i}=1}^{\mathrm{K}} \widehat{\ell}_{\mathrm{i}} .
$$

- $\widehat{R T V}$ - relative time variance is used to compare different algorithms; it is defined as the simulation time in seconds multiplied by the squared relative error.

We consider the following models.

- Model 1: $\sum_{i=1}^{10} X_{i}$ where $X_{i} \sim \exp (\lambda)$ are i.i.d exponential random variables with $\lambda=0.03$.
- Model 2: $\sum_{i=1}^{10} X_{i}$ where $X_{i} \sim \exp (\lambda)$ are i.i.d exponential random variables with $\lambda=0.01$.
- Model 3: $\sum_{i=1}^{10} X_{i}$ where $X_{i} \sim \exp \left(\lambda_{i}\right)$. The corresponding $\lambda$ values are given below.

$$
\begin{aligned}
& \lambda_{1}=0.01, \lambda_{2}=0.011, \lambda_{3}=0.009, \lambda_{4}=0.01, \lambda_{5}=0.011, \\
& \lambda_{6}=0.009, \lambda_{7}=0.01 \lambda_{8}=0.011, \lambda_{9}=0.009, \lambda_{10}=0.01 .
\end{aligned}
$$

The following tables summarize our results.
Table 1: Average performance of 10 runs of the algorithms for Model 1

| Algorithm | $\widehat{\mathbb{P}}\left(\sum X_{i} \leq 1\right)$ | $\widehat{R E}$ | $\widehat{R T V}$ | CPU |
| :---: | :---: | :---: | :---: | :---: |
| IS | $1.61 \times 10^{-22}$ | $5.98 \times 10^{-2}$ | $3.38 \times 10^{-4}$ | 0.094 |
| Cross Entropy | $1.58 \times 10^{-22}$ | $9.24 \times 10^{-2}$ | $1.90 \times 10^{-3}$ | 0.222 |
| Splitting | $1.32 \times 10^{-22}$ | $5.53 \times 10^{-1}$ | 1.37 | 4.501 |
| $G-S$ | $1.44 \times 10^{-22}$ | $3.94 \times 10^{-1}$ | $2.85 \times 10^{-2}$ | 0.184 |

The exact convolution2 algorithm delivers $\mathbb{P}\left(\sum X_{i} \leq 1\right)=1.1102 \times 10^{-16}$ as an output. Unfortunately, Algorithm convolution1 cannot be used for equal $\lambda$ values.

Table 2: Average performance of 10 runs of the algorithms for Model 2

| Algorithm | $\widehat{\mathbb{P}}\left(\sum X_{i} \leq 1\right)$ | $\widehat{R E}$ | $\widehat{R T V}$ | CPU |
| :---: | :---: | :---: | :---: | :---: |
| $I S$ | $2.75 \times 10^{-27}$ | $6.05 \times 10^{-2}$ | $3.50 \times 10^{-4}$ | 0.096 |
| Cross Entropy | $2.74 \times 10^{-27}$ | $6.73 \times 10^{-2}$ | $5.82 \times 10^{-4}$ | 0.128 |
| Splitting | $2.97 \times 10^{-27}$ | $5.55 \times 10^{-1}$ | 1.74 | 5.667 |
| $G$-S | $2.35 \times 10^{-27}$ | $3.77 \times 10^{-1}$ | $2.57 \times 10^{-2}$ | 0.181 |

The exact convolution2 algorithm delivers $\mathbb{P}\left(\sum X_{i} \leq 1\right)=1.1102 \times 10^{-16}$ as an output. Note that the algorithm outputs the same value for both Model 1 and Model 2.

Table 3: Average performance of 10 runs of the algorithms for Model 3

| Algorithm | $\widehat{\mathbb{P}}\left(\sum X_{i} \leq 1\right)$ | $\widehat{R E}$ | $\widehat{R T V}$ | CPU |
| :---: | :---: | :---: | :---: | :---: |
| IS | $2.56 \times 10^{-27}$ | $3.19 \times 10^{-2}$ | $9.75 \times 10^{-5}$ | 0.096 |
| Cross Entropy | $2.60 \times 10^{-27}$ | $3.42 \times 10^{-2}$ | $1.61 \times 10^{-4}$ | 0.138 |
| Splitting | $2.35 \times 10^{-27}$ | $4.18 \times 10^{-1}$ | $9.75 \times 10^{-1}$ | 5.588 |
| $G$-S | $2.14 \times 10^{-27}$ | $2.47 \times 10^{-1}$ | $1.10 \times 10^{-2}$ | 0.180 |

Algorithm convolution1 cannot deliver a meaningful answer and convolution2 algorithm delivers $\mathbb{P}\left(\sum X_{i} \leq 1\right)=-2.2204 \times 10^{-16}$ as an output. Note that in this case the stability is lost and the algorithm outputs $\mathbb{P}\left(\sum X_{i} \geq 1\right)>1$.

## 4 Conclusions

In this note, we developed a new importance sampling algorithm for computing the CDF of the Hypoexponential random variable. We proved that the proposed estimator is efficient and its performance compares favorably with other existing methods. Based on our numerical results we conclude that in situations with no rare events involved, one should prefer to use the exact convolution2 method that is still relatively fast and very stable. Naturally, when the exact method fails, which may happen in case of very small probabilities, one should apply some Monte Carlo approximation. MCMC based Splitting is too slow to be used in reliability applications. The $G$ - $S$ has a good performance and very easy to implement but it seems that its relative error is inferior when compared to $I S$. The Cross Entropy and the proposed $I S$ algorithm are comparable, but $I S$ is much simpler to implement.

## ACKNOWLEDGMENT

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## Appendix A Proofs

Lemma A. 1 The output of IS Algorithm 2.1 is unbiased.
Proof. Let $Y_{i} \sim \exp (n), i=1,2, \ldots, n$ be independent exponentially distributed random variables. Recall that we are looking for an unbiased estimator of $P\left(X_{1}+X_{2}+\ldots+X_{n} \leq 1\right)$, where $X_{i}$ are independent and $X_{i} \sim \exp \left(\lambda_{i}\right)$. Our estimator is

$$
\begin{equation*}
Z=1_{\left\{\sum_{i=1}^{n} Y_{i} \leq 1\right\}} \frac{\prod_{i=1}^{n} \lambda_{i} e^{-\lambda_{i} Y_{i}}}{\prod_{i=1}^{n} n e^{-n Y_{i}}} \tag{7}
\end{equation*}
$$

Note that the joint density function of $\mathbf{Y}=\left(Y_{1}, Y_{2}, \ldots, Y_{n}\right)$ is $\Psi(\mathbf{v})=\prod_{i=1}^{n} n e^{-n v_{i}}$. Now

$$
\begin{gathered}
E[Z]=\int(n) \int_{v_{i} \geq 0, v_{1}+\ldots+v_{n} \leq 1} \frac{\prod_{i=1}^{n} \lambda_{i} e^{-\lambda_{i} v_{i}}}{\prod_{i=1}^{n} n e^{-n v_{i}}} \prod_{i=1}^{n} n e^{-n v_{i}} d v_{1} d v_{2} \ldots d v_{n}= \\
\int(n) \int_{v_{i} \geq 0, v_{1}+\ldots+v_{n} \leq 1} \prod_{i=1}^{n} \lambda_{i} e^{-\lambda_{i} v_{i}} d v_{1} d v_{2} \ldots d v_{n}=P\left(X_{1}+X_{2}+\ldots+X_{n} \leq 1\right) . Q . E . D .
\end{gathered}
$$

Theorem A. 2 Let $Z$ be defined as in (4). Then we have,

$$
\begin{equation*}
\frac{\mathbb{E}\left(Z^{2}\right)}{(\mathbb{E}(Z))^{2}} \leq \sqrt{n} e^{2(\bar{\lambda}-\underline{\lambda})+1} \tag{8}
\end{equation*}
$$

where $\bar{\lambda}=\max _{i=1, \ldots, n}\left\{\lambda_{i}\right\}$ and $\underline{\lambda}=\min _{i=1, \ldots, n}\left\{\lambda_{i}\right\}$.
Proof. Denote by

$$
\begin{equation*}
Y:=\sum_{i=1}^{n} Y_{i} \tag{9}
\end{equation*}
$$

By the definition of the random variables $y_{i}, i=1, \ldots, n$, we have that $Y$ is distributed $\operatorname{Erlang}(n, n)$ and therefore it has the following probability density

$$
\begin{equation*}
f_{Y}(y)=\frac{n^{n}}{(n-1)!} y^{n-1} e^{-n y}, \quad y>0 \tag{10}
\end{equation*}
$$

Define

$$
\begin{equation*}
I(n, x):=\int_{0}^{x} t^{n-1} e^{t} d t \tag{11}
\end{equation*}
$$

From (11), (41), (21) and (10) we have,

$$
\begin{align*}
\mathbb{E}\left(Z^{2}\right) & =\left(\frac{\prod_{i=1}^{n} \lambda_{i}}{\prod_{i=1}^{n} n}\right)^{2} \mathbb{E}\left(1_{\left\{\sum_{i=1}^{n} Y_{i} \leq 1\right\}} \frac{\prod_{i=1}^{n} e^{-2 \lambda_{i} Y_{i}}}{\prod_{i=1}^{n} e^{-2 n Y_{i}}}\right)  \tag{12}\\
& \leq\left(\frac{\prod_{i=1}^{n} \lambda_{i}}{\prod_{i=1}^{n} n}\right)^{2} \mathbb{E}\left(1_{\left\{\sum_{i=1}^{n} Y_{i} \leq 1\right\}} \frac{e^{-2 \underline{2} \sum_{i=1}^{n} Y_{i}}}{e^{-2 n \sum_{i=1}^{n} Y_{i}}}\right) \\
& =\left(\frac{\prod_{i=1}^{n} \lambda_{i}}{\prod_{i=1}^{n} n}\right)^{2} \mathbb{E}\left(1_{\{Y \leq 1\}} e^{2(n-\underline{\lambda}) Y}\right) \\
& =\left(\frac{\prod_{i=1}^{n} \lambda_{i}}{\prod_{i=1}^{n} n}\right)^{2} \frac{n^{n}}{(n-1)!} \int_{0}^{1} x^{n-1} e^{(n-2 \underline{\lambda}) x} d x \\
& =\left(\frac{\prod_{i=1}^{n} \lambda_{i}}{\prod_{i=1}^{n} n}\right)^{2} \frac{n^{n}}{(n-1)!} \frac{1}{(n-2 \underline{\lambda})^{n}} \int_{0}^{n-2 \underline{\lambda}} x^{n-1} e^{x} d x \\
& =\left(\frac{\prod_{i=1}^{n} \lambda_{i}}{\prod_{i=1}^{n} n}\right)^{2} \frac{n^{n}}{(n-1)!} \frac{1}{(n-2 \underline{\lambda})^{n}} I(n, n-2 \underline{\lambda}) .
\end{align*}
$$

Recall the definition of the lower incomplete gamma function,

$$
\begin{equation*}
\gamma(n, x):=\int_{0}^{x} t^{n-1} e^{-t} d t \tag{13}
\end{equation*}
$$

Use (13), (4), (91) and (10) to get,

$$
\begin{align*}
\mathbb{E}(Z) & =\frac{\prod_{i=1}^{n} \lambda_{i}}{\prod_{i=1}^{n} n} \mathbb{E}\left(1_{\left\{\sum_{i=1}^{n} Y_{i} \leq 1\right\}} \frac{e^{-\sum_{i=1}^{n} \lambda_{i} Y_{i}}}{e^{-n \sum_{i=1}^{n} Y_{i}}}\right)  \tag{14}\\
& \geq \frac{\prod_{i=1}^{n} \lambda_{i}}{\prod_{i=1}^{n} n} \mathbb{E}\left(1_{\left\{\sum_{i=1}^{n} Y_{i} \leq 1\right\}} \frac{e^{-\bar{\lambda} \sum_{i=1}^{n} Y_{i}}}{e^{-n \sum_{i=1}^{n} Y_{i}}}\right) \\
& =\frac{\prod_{i=1}^{n} \lambda_{i}}{\prod_{i=1}^{n} n} \mathbb{E}\left(1_{\{Y \leq 1\}} e^{(n-\bar{\lambda}) Y}\right) \\
& =\frac{\prod_{i=1}^{n} \lambda_{i}}{\prod_{i=1}^{n} n} \cdot \frac{n^{n}}{(n-1)!} \int_{0}^{1} x^{n-1} e^{-\bar{\lambda} x} d x \\
& =\frac{\prod_{i=1}^{n} \lambda_{i}}{\prod_{i=1}^{n} n} \cdot \frac{n^{n}}{(n-1)!} \frac{1}{\bar{\lambda}^{n}} \int_{0}^{\bar{\lambda}} x^{n-1} e^{-x} d x \\
& =\frac{\prod_{i=1}^{n} \lambda_{i}}{\prod_{i=1}^{n} n} \cdot \frac{n^{n}}{(n-1)!} \frac{1}{\bar{\lambda}^{n}} \gamma(n, \bar{\lambda}) .
\end{align*}
$$

From (12) and (14) we get

$$
\begin{equation*}
\frac{\mathbb{E}\left(Z^{2}\right)}{(\mathbb{E}(Z))^{2}} \leq \frac{(n-1)!}{n^{n}} \frac{\bar{\lambda}^{2 n}}{(n-2 \underline{\lambda})^{n}} \frac{I(n, n-2 \underline{\lambda})}{(\gamma(n, \bar{\lambda}))^{2}} . \tag{15}
\end{equation*}
$$

By a simple calculation we obtain the following bounds on the functions $I$ and $\gamma$,

$$
\begin{align*}
& I(n, x) \leq \frac{x^{n} e^{n}}{n}, \forall x \in[0, \infty), n \in \mathbb{N},  \tag{16}\\
& \gamma(n, x) \geq \frac{x^{n}}{n e^{x}}, \forall x \in[0, \infty), n \in \mathbb{N} . \tag{17}
\end{align*}
$$

Recall Stirling's formula

$$
\begin{equation*}
n!\leq n^{n+1 / 2} e^{-n+1}, \forall n \in \mathbb{N} \tag{18}
\end{equation*}
$$

Apply (16) (18) on (15) to get

$$
\begin{align*}
\frac{\mathbb{E}\left(Z^{2}\right)}{(\mathbb{E}(Z))^{2}} & \leq \frac{e^{-n+1}}{\sqrt{n}} \frac{\bar{\lambda}^{2 n}}{(n-2 \underline{\lambda})^{n}} \frac{I(n, n-2 \underline{\lambda})}{(\gamma(n, \bar{\lambda}))^{2}} \\
& \leq \frac{e^{-n+1}}{\sqrt{n}} \frac{\bar{\lambda}^{2 n}}{(n-2 \underline{\lambda})^{n}} \frac{\frac{1}{n}(n-2 \underline{\lambda})^{n} e^{n-2 \underline{\lambda}}}{\frac{\bar{\lambda}^{2 n}}{n^{2} e^{2 \lambda}}} \\
& =\sqrt{n} e^{2(\bar{\lambda}-\underline{\lambda})+1}, \tag{19}
\end{align*}
$$

and we get (8).

## Appendix B MatLab code for exact computation

```
function ell=convolution1(t,nu)
    % computes P(A_1+...+A_b>t) exactly,
    % where A_i distributed Exp(nu(i)) independently;
    % nu has to be decreasing (sorted) sequence
    b=length(nu); % parameters of the waiting times
    w=zeros(b,b); % b is critical number
    w (l,l)=l;
    for k=l:b-l
            for j=l:k
                w (k+l,j)=w (k,j) *nu(b-k)/(nu(b-k)-nu(b-j+l));
                w(k+l,k+l)=l-sum(w(k+l,1 :k));
            end
    end
    ell=w(b,:) *exp(-nu(end:-l: 1) *t); % probability
end
```

```
function ell=convolution2(t, nu)
    % computes P(A_1+...+A_b>t) exactly,
    % where A_i }\neg\operatorname{Exp}(nu(i)) independently
    % nu has to be decreasing (sorted) sequence
    b=length(nu); % parameters of the waiting times
    A=diag(-nu)+diag(nu(1:b-1),1);
    A=expm(A*t);
    ell=sum(A(1,:));
end
```

