## Robust Parameter Estimation of Regression Model with AR(p) Error Terms

Y. Tuaç, Y. Güney B. Şenoğlu and O. Arslan

Ankara University, Faculty of Science, Department of Statistics, 06100 Ankara/Turkey

ytuac@ankara.edu.tr, ydone@ankara.edu.tr, senoglu@science.ankara.edu.tr, oarslan@ankara.edu.tr

#### Abstract

In this paper, we consider a linear regression model with AR(p) error terms with the assumption that the error terms have a *t* distribution as a heavy tailed alternative to the normal distribution. We obtain the estimators for the model parameters by using the conditional maximum likelihood (CML) method. We conduct an iteratively reweighting algorithm (IRA) to find the estimates for the parameters of interest. We provide a simulation study and three real data examples to illustrate the performance of the proposed robust estimators based on *t* distribution.

**Keywords:** autoregressive stationary process; conditional maximum likelihood; linear regression; non normal distributions; robust estimation.

## **1. Introduction**

Consider the following linear regression model

$$y_t = \sum_{i=1}^{M} x_{t,i} \beta_i + e_t \quad , \qquad t = 1, 2, \dots, N$$
(1)

where,  $y_t$  is the response variable,  $x_{t,i}$  are the explanatory variables,  $\beta_i$  are the unknown regression parameters and  $e_t$  is the error term. In classical regression analysis, the general assumptions on the error term are zero mean, constant variance and not correlated with each other. It is well known that under these assumptions the ordinary least squares (OLS) estimator is the best. However, one of the problems in application is that the error term may be correlated with each other. In this case, although, the OLS estimators are unbiased and consistence, they may be no longer efficient even in large sample cases, and hence this may cause large estimated standard errors for the estimators of the regression parameters (see Olaomi and Ifederu [13]). There are many ways to deal with autocorrelated structures in the disturbances; the most common way is to assume autoregressive error terms in regression model.

We assume that  $e_t$  is a stationary autoregressive error process of order p (AR(p)) given as

$$e_t = \phi_1 e_{t-1} + \dots + \phi_p e_{t-p} + a_t, \tag{2}$$

where  $\phi_j$ , for j = 1, 2, ..., p, are unknown autoregressive parameters.

For simplicity we use  $a_t = e_t - \phi_1 e_{t-1} - \dots - \phi_p e_{t-p} = \Phi(B)e_t$ , where  $E(a_t) = 0$ ,  $Var(a_t) = \sigma^2$  and  $a_t$ 's are uncorrelated random variables with constant variance. Here *B* is called the backshift operator and where  $\Phi(\cdot)$  is the function defining the autoregression. Then using the backshift operator the regression model given in (1) can be rewritten as

(3) 
$$\Phi(B)y_t = \sum_{i=1}^{M} \beta_i \Phi(B) x_{t,i} + a_t, \quad t = p + 1, 2, \dots, N,$$

...

where

$$\Phi(B)y_t = y_t - \phi_1 y_{t-1} - \dots - \phi_p y_{t-p},$$
(4)

$$\Phi(B)x_{t,i} = x_{t,i} - \phi_1 x_{t-1,i} - \dots - \phi_p x_{t-p,i}$$
(5)

In general, it is assumed that  $a_t$  is normally distributed. For instance, Alpuim and El-Shaarawi [2] estimated the parameters of the regression model with AR(p) error term using the OLS estimation method. They also used the maximum likelihood (ML) estimation and CML estimation method under the assumption of normality and studied the asymptotic properties of the resulting estimators. Beach and Mackinnon [5] used ML estimation method to estimate the parameters of AR(1) error term regression models. Tiku [17] estimated the parameters by using the modified maximum likelihood (MML) method for the regression model with AR(1) error terms under the assumption that the error term has the Long Tailed Symmetric (LTS) distribution. There are some other studies used heavy tailed distributions in time series. For instance, Hill [7] used tail-trimming and/or weighting to show how robust to any type of light or heavy tailed distribution in infinite variance autoregressions case. Also, heavy tailed asymmetrically distributed errors in GARCH model were discussed with tail-trimmed QML estimator in Hill [8,9].

Another challenging problem in a regression analysis is the presence of outliers in data. Since the parameter estimators based on normal distribution are very sensitive to the outliers, the corresponding estimators will be no longer efficient. One way to combat with the outliers is to use heavy tailed distributions as alternatives to the normal distribution. Thus, the *t* distribution provides a useful alternative to the normal distribution for statistical modelling of data sets that have heavier tailed empirical distribution. The motivation of this paper is to propose conditional maximum likelihood estimators for unknown parameters of a linear regression model with autoregressive errors under the assumption that the independent identically distributed (iid) error term  $a_t$  given in equation (2) has a *t* distribution with known degrees of freedom. The estimators for the parameters of interest obtained under this assumption will be robust in terms of the influence function. It is known that if the degrees of freedom is estimated along with the other parameters the influence function of the resulting estimators will be unbounded and hence they are not going to be robust (Lucas [10]). Therefore, the degrees of freedom is usually taken as fixed and treated as a robustness tuning parameter in robustness studies, for example see Lange et al. [11].

The rest of the paper is organized as follows. In section 2, we first summarize the CML estimation method. Then, we move on the CML estimation for the parameters of regression model with AR(p) error terms under the assumption that  $a_t$ 's have t distribution. We also give the observed Fisher information matrix for the estimators. Note that the observed Fisher information matrix will be used in section 4 to form confidence intervals and to compute the standard errors of the estimators. In section 3, we give an

IRA to compute the estimates. A simulation study and three real data examples are given in section 4 to illustrate the performance of the proposed estimators. Finally we conclude the paper with a discussion section.

## 2. Parameters estimation of the AR(p) error term regression model

In this section, since the exact likelihood function could be well approximated by the conditional likelihood function (Ansley [1]) we will first give the CML estimation method. CML estimators are used mainly in cases where ML estimators are difficult to compute. We will briefly give the conditional likelihood estimators under the normality assumption and move on the t distribution case. We also provide observed Fisher matrices for both cases.

# 2.1 Conditional likelihood under normality

Let  $a_t$ 's have the probability density function  $f(a_t, \theta)$ . If we condition on  $a_1, a_2, ..., a_p$ , the conditional log-likelihood function will be

$$lnL = \sum_{t=p+1}^{N} lnf(a_t | a_1, a_2, \dots, a_{t-p}, \boldsymbol{\theta}).$$
(6)

Consider the regression model given in (3). If it is assumed that  $a_t$ 's are normally distributed the conditional log-likelihood function will be as follows (Alpuim and El-Shaarawi [2]).

$$lnL = c - \frac{N-p}{2} ln\sigma^2 - \frac{1}{2\sigma^2} \sum_{t=p+1}^{N} \left( \Phi(B)y_t - \sum_{i=1}^{M} \beta_i \Phi(B)x_{t,i} \right)^2$$
(7)

Taking the derivatives of the conditional log-likelihood function with respect to unknown parameters and setting to zero yield the following estimating equations.

$$\frac{\partial lnL}{\partial \beta_k} = \frac{1}{\sigma^2} \sum_{t=p+1}^N \left( \Phi(B) y_t - \sum_{i=1}^M \beta_i \, \Phi(B) x_{t,i} \right) \Phi(B) x_{t,k} = 0 \tag{8}$$

$$\frac{\partial lnL}{\partial \phi_l} = \frac{1}{\sigma^2} \sum_{t=p+1}^N \left( \Phi(B) y_t - \sum_{i=1}^M \beta_i \, \Phi(B) x_{t,i} \right) \left( y_{t-l} - \sum_{i=1}^M \beta_i \, x_{t-l,i} \right) = 0 \tag{9}$$

$$\frac{\partial lnL}{\partial \sigma^2} = -\frac{N-p}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{t=p+1}^N \left( \Phi(B) y_t - \sum_{i=1}^M \beta_i \, \Phi(B) x_{t,i} \right)^2 = 0 \tag{10}$$

Rearranging these equations we get the following estimators.

$$\underline{\hat{\beta}} = \left[\sum_{t=p+1}^{N} \widehat{\Phi}(B) x_t \widehat{\Phi}(B) x_t^T\right]^{-1} \left[\sum_{t=p+1}^{N} \widehat{\Phi}(B) y_t \widehat{\Phi}(B) x_t\right]$$
(11)

$$\underline{\hat{\phi}} = R^{-1}(\hat{\beta})R_0(\hat{\beta}) \tag{12}$$

$$\hat{\sigma}^2 = \frac{1}{N-p} \sum_{t=p+1}^{N} \left( \widehat{\Phi}(B) y_t - \underline{\hat{\beta}} \widehat{\Phi}(B) x_t \right)^2$$
(13)

where

$$R_{0}(\beta) = \begin{bmatrix} \sum_{t=p+1}^{N} e_{t}e_{t-1} \\ \sum_{t=p+1}^{N} e_{t}e_{t-2} \\ \vdots \\ \sum_{t=p+1}^{N} e_{t}e_{t-p} \end{bmatrix}, R(\beta) = \begin{bmatrix} \sum_{t=p+1}^{N} e_{t-1}^{2} & \sum_{t=p+1}^{N} e_{t-1}e_{t-2} & \cdots & \sum_{t=p+1}^{N} e_{t-1}e_{t-p} \\ \sum_{t=p+1}^{N} e_{t-2}e_{t-1} & \sum_{t=p+1}^{N} e_{t-2}^{2} & \cdots & \sum_{t=p+1}^{N} e_{t-2}e_{t-p} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{t=p+1}^{N} e_{t}e_{t-p} \end{bmatrix}$$
(14)

and  $\widehat{\Phi}(B)$  is the backshift operator with the estimates of  $\phi_j$ .

Using these equations we can rewrite  $\hat{\beta}$  and  $\hat{\sigma}^2$ ,

$$\underline{\hat{\beta}} = \left[\widehat{\Phi}(B)X^T\widehat{\Phi}(B)X\right]^{-1} \left[\widehat{\Phi}(B)X^T\widehat{\Phi}(B)\underline{Y}\right]$$
(15)

$$\hat{\sigma}^{2} = \frac{1}{N-p} [\hat{\Phi}(B)\underline{Y} - \hat{\Phi}(B)\underline{X}\underline{\hat{\beta}}]^{T} [\hat{\Phi}(B)\underline{Y} - \hat{\Phi}(B)\underline{X}\underline{\hat{\beta}}]$$
(16)

where

$$\widehat{\Phi}(B)X = [\widehat{\Phi}(B)x_{t,i}],$$
$$\widehat{\Phi}(B)\underline{Y} = [\widehat{\Phi}(B)y_t].$$

These estimators depend on the estimators of the other parameters. Therefore, the values of the estimators should be computed using numerical methods. We use IRA to compute these estimators to guarantee the convergence (see Lange et al. [11], Arslan and Genç [4]). These estimators correspond also to the OLS estimators obtained through the minimization of the sum of squares of the  $a_t$ . These estimators are sensitive to the outliers in the data. Therefore, an alternative error distribution should be considered to deal

with this problem. In the following section we will assume that  $a_t$ 's have a t distribution with known degrees of freedom and carry out the estimation under this assumption.

Further, we also give the observed Fisher information matrix for the unknown parameters of the regression model defined in equation (3) with normally distributed error terms. Note that the observed Fisher information matrices will be used to compute the standard errors and the confidence intervals in simulation study and the real data examples.

After some straightforward algebra the observed Fisher information matrix for the normal distribution case can be obtained as follows.

$$F(\hat{\beta}, \hat{\phi}, \hat{\sigma}) = \begin{bmatrix} \frac{1}{\hat{\sigma}^{2}} (\hat{\Phi}(B) X^{T} \hat{\Phi}(B) X) & 0 & 0 \\ 0 & \frac{1}{(\hat{\sigma}^{2}) R(\hat{\beta})} & 0 \\ 0 & 0 & \frac{N-p}{\hat{\sigma}^{4}} \end{bmatrix}$$
(17)

# 2.2 Parameters estimation under t distribution

Consider the regression model given in equation (3) and assume that  $a_t$ 's have *t* distribution with the density function

$$f(a_t) = \frac{c_v}{\sigma} \left( \nu + \frac{a_t^2}{\sigma^2} \right)^{-\frac{\nu+1}{2}},\tag{18}$$

where  $c_{\nu} = \frac{\Gamma(\frac{\nu+1}{2})\nu^{\nu/2}}{\sqrt{\pi}\Gamma(\frac{\nu}{2})}$ ,  $\nu > 0$  degrees of freedom and  $\sigma > 0$  scale parameter. Under this assumption the conditional log-likelihood function will be obtained as

$$lnL = lnc_{v} - (N - p)ln\sigma - \frac{v + 1}{2} \sum_{t=p+1}^{N} \ln\left[v + \frac{\left(\Phi(B)y_{t} - \sum_{i=1}^{M}\beta_{i} \Phi(B)x_{t,i}\right)^{2}}{\sigma^{2}}\right].$$
 (19)

Taking the derivatives of log-likelihood function with respect to the unknown parameters and setting them to zero yield the following estimating equations.

$$\frac{\partial lnL}{\partial \beta_k} = \frac{(v+1)}{\sigma^2} \sum_{t=p+1}^{N} \frac{\left(\Phi(B)y_t - \sum_{i=1}^{M} \beta_i \Phi(B) x_{t,i}\right) \Phi(B) x_{t,k}}{\left[v + \frac{\left(\Phi(B)y_t - \sum_{i=1}^{M} \beta_i \Phi(B) x_{t,i}\right)^2}{\sigma^2}\right]} = 0,$$
(20)

$$\frac{\partial lnL}{\partial \phi_l} = \frac{(\nu+1)}{\sigma^2} \sum_{t=p+1}^{N} \frac{\left(\Phi(B)y_t - \sum_{i=1}^{M} \beta_i \,\Phi(B)x_{t,i}\right) \left(y_{t-l} - \sum_{i=1}^{M} \beta_i \,x_{t-l,i}\right)}{\left[\nu + \frac{\left(\Phi(B)y_t - \sum_{i=1}^{M} \beta_i \,\Phi(B)x_{t,i}\right)^2}{\sigma^2}\right]} = 0, \tag{21}$$

$$\frac{\partial lnL}{\partial \sigma} = -\frac{(N-p)}{\sigma} + \left(\frac{\nu+1}{\sigma^3}\right) \sum_{t=p+1}^{N} \frac{\left(\Phi(B)y_t - \sum_{i=1}^{M} \beta_i \Phi(B)x_{t,i}\right)^2}{\left[\nu + \frac{\left(\Phi(B)y_t - \sum_{i=1}^{M} \beta_i \Phi(B)x_{t,i}\right)^2}{\sigma^2}\right]} = 0.$$
(22)

Rearranging these equations we get

$$\frac{\partial lnL}{\partial \beta_k} = \frac{1}{\sigma^2} \sum_{t=p+1}^N w_t(v) \left( \Phi(B) y_t - \sum_{i=1}^M \beta_i \Phi(B) x_{t,i} \right) \Phi(B) x_{t,k} = 0,$$
(23)

$$\frac{\partial lnL}{\partial \phi_l} = \frac{1}{\sigma^2} \sum_{t=p+1}^N w_t(v) \left( \Phi(B) y_t - \sum_{i=1}^M \beta_i \, \Phi(B) x_{t,i} \right) \left( y_{t-l} - \sum_{i=1}^M \beta_i \, x_{t-l,i} \right) = 0, \tag{24}$$

$$\frac{\partial lnL}{\partial \sigma} = -\frac{(N-p)}{\sigma} + \frac{1}{\sigma^3} \sum_{t=p+1}^N w_t(v) \left(\Phi(B)y_t - \sum_{i=1}^M \beta_i \Phi(B)x_{t,i}\right)^2 = 0$$
(25)

where

$$w_{t}(v) = \frac{v+1}{\left[v + \frac{\left(\Phi(B)y_{t} - \sum_{i=1}^{M}\beta_{i} \Phi(B)x_{t,i}\right)^{2}}{\sigma^{2}}\right]}.$$
(26)

These equations yield the following estimators provided that  $\left[\sum_{t=p+1}^{N} w_t(\nu)\widehat{\Phi}(B)x_t\widehat{\Phi}(B)x_t^T\right]^{-1}$ and  $R_w^{-1}(\widehat{\beta})$  exist.

$$\underline{\hat{\beta}} = \left[\sum_{t=p+1}^{N} w_t(\nu)\widehat{\Phi}(B)x_t\widehat{\Phi}(B)x_t^T\right]^{-1} \left[\sum_{t=p+1}^{N} w_t(\nu)\widehat{\Phi}(B)y_t\widehat{\Phi}(B)x_t\right],\tag{27}$$

$$\underline{\hat{\phi}} = R_w^{-1}(\hat{\beta})R_{w0}(\hat{\beta}),\tag{28}$$

$$\hat{\sigma}^2 = \frac{1}{N-p} \sum_{t=p+1}^{N} w_t(v) \left(\widehat{\Phi}(B)y_t - \widehat{\Phi}(B)x_t^T \underline{\hat{\beta}}\right)^2$$
(29)

where

$$R_{w0}(\beta) = \begin{bmatrix} \sum_{t=p+1}^{N} w_t e_t e_{t-1} \\ \sum_{t=p+1}^{N} w_t e_t e_{t-2} \\ \vdots \\ \sum_{t=p+1}^{N} w_t e_t e_{t-p} \end{bmatrix}, \qquad R_w(\beta) = \begin{bmatrix} \sum_{t=p+1}^{N} w_t e_{t-1}^2 & \sum_{t=p+1}^{N} w_t e_{t-1} e_{t-2} & \cdots & \sum_{t=p+1}^{N} w_t e_{t-1} e_{t-p} \\ \sum_{t=p+1}^{N} w_t e_{t-2} e_{t-1} & \sum_{t=p+1}^{N} w_t e_{t-2}^2 & \cdots & \sum_{t=p+1}^{N} w_t e_{t-2} e_{t-p} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{t=p+1}^{N} w_t e_t e_{t-p} e_{t-1} & \sum_{t=p+1}^{N} w_t e_{t-p} e_{t-2} & \cdots & \sum_{t=p+1}^{N} w_t e_{t-p}^2 \\ \end{bmatrix}.$$
(30)

Further, these equations can be rewritten as

$$\underline{\hat{\beta}} = \left[\widehat{\Phi}(B)X^T W \widehat{\Phi}(B)X\right]^{-1} \left[\widehat{\Phi}(B)X^T W \widehat{\Phi}(B)\underline{Y}\right],\tag{31}$$

$$\hat{\sigma}^{2} = \frac{1}{N-p} [\hat{\Phi}(B)\underline{Y} - \hat{\Phi}(B)\underline{X}\underline{\hat{\beta}}]^{T} \boldsymbol{W} \Big[ \hat{\Phi}(B)\underline{Y} - \hat{\Phi}(B)\underline{X}\underline{\hat{\beta}} \Big],$$
(32)

by using vector notation. Here

$$\widehat{\Phi}(B)X = \left[\widehat{\Phi}(B)x_{t,i}\right]_{t=p+1,\dots,N},\\ \widehat{\Phi}(B)\underline{Y} = \left[\widehat{\Phi}(B)y_t\right]_{t=p+1,\dots,N},\\ W = diag\{w_t\}_{t=p+1,\dots,N}.$$

Since the weight function  $w_t$  is a decreasing function of  $\frac{(\Phi(B)y_t - \sum_{i=1}^M \beta_i \Phi(B)x_{t,i})^2}{\sigma^2}$  the observations with larger residuals receive small weights. Therefore, the effect of the corresponding point on the estimator will be downweighted. This behavior of the *t* distribution guarantees the robustness of the resulting estimators (Lucas [10], Arslan and Genç [3, 4]). Note that as  $\nu$  tends to infinity  $w_t(\nu) \rightarrow 1$  and this case gives the estimators given in equations (11)-(13).

Similarly the observed Fisher information matrix for the unknown parameters of the regression model defined in equation (3) with the t distributed error terms can be obtained as follows.

$$F(\hat{\beta}, \hat{\phi}, \hat{\sigma}) = \begin{bmatrix} \frac{1}{\hat{\sigma}^{2}} (\hat{\Phi}(B) X^{T} W \hat{\Phi}(B) X) & 0 & 0 \\ 0 & \frac{1}{(\hat{\sigma}^{2}) R_{w}(\hat{\beta})} & 0 \\ 0 & 0 & \frac{N-p}{\hat{\sigma}^{4}} \end{bmatrix}$$
(33)

We should also note that since the estimators given in (27)-(29) are dependent on the weights and since the weights are also functions of the estimators these equations cannot be solved explicitly. Therefore, the numerical methods are also needed to solve these equations to get the estimates. Because of the form of the equations the IRA can be easily implemented to get the estimates as it is done for all the procedures based on *t* distribution. Note that in the *t* distribution case the IRA is an expectation-maximization (EM) algorithm so that its convergence is guaranteed (see Lange et al. [11], McLachlan and Krishnan, [12] Arslan and Genç [4]). The following section is devoted to the IRA.

## 3. Iteratively reweighted algorithm

Using the updating equations (28, 31, 32) and the weight function given in equation (26) the following iteratively reweighted algorithm can be formed to calculate the estimates for,  $\beta$ ,  $\phi$  and  $\sigma^2$ . Note that the degrees of freedom of the *t* distribution will be taken as known and fixed.

- (i) Set the initial values  $\beta^{(0)}$ ,  $\phi^{(0)}$  and  $\sigma^{2^{(0)}}$  and fix a stopping rule  $\delta$ .
- (ii) Calculate the following weight function for m = 0,1,2...

$$w_t^{(m)} = \frac{\nu + 1}{\left[\nu + \frac{\left(\Phi^{(m)}(B)y_t - \sum_{i=1}^M \beta_i^{(m)} \Phi^{(m)}(B)x_{t,i}\right)^2}{\sigma^{2^{(m)}}}\right]}.$$
(34)

(iii) Using these values calculate

$$\underline{\phi}^{(m+1)} = R_w^{-1(m)} (\hat{\beta}^{(m)}) R_{w0}^{(m)} (\hat{\beta}^{(m)}).$$
(35)

(iv) Using  $w_t^{(m)}$  and  $\phi^{(m+1)}$  calculate

$$\underline{\beta}^{(m+1)} = \left[ \Phi^{(m+1)}(\boldsymbol{B}) \boldsymbol{X}^{T} \boldsymbol{W}^{(m)} \Phi^{(m+1)}(\boldsymbol{B}) \boldsymbol{X} \right]^{-1} \left[ \Phi^{(m+1)}(\boldsymbol{B}) \boldsymbol{X}^{T} \boldsymbol{W}^{(m)} \Phi^{(m+1)}(\boldsymbol{B}) \underline{Y} \right].$$
(36)

(v) Using  $w_t^{(m)}$ ,  $\phi^{(m+1)}$  and  $\beta^{(m+1)}$  calculate

$$(\sigma^2)^{(m+1)} = \frac{1}{N-p} \Big[ \Phi^{(m+1)}(B)\underline{Y} - \Phi^{(m+1)}(B)\underline{X}\underline{\beta}^{(m+1)} \Big]^T W^{(m)} \Big[ \Phi^{(m+1)}(B)\underline{Y} - \Phi^{(m+1)}(B)\underline{X}\underline{\beta}^{(m+1)} \Big]$$
(37)

(vi) Repeat the steps (ii)-(v) until the convergence condition  $max(\|\beta^{(m+1)} - \beta^{(m)}\|, \|\phi^{(m+1)} - \phi^{(m)}\|, \|(\sigma^2)^{(m+1)} - (\sigma^2)^{(m)}\|) < \delta$  is satisfied.

In section 4, we will use this algorithm to compute the CML estimates in simulation and the real data examples.

## 4. Numerical study

In this section, we give a small simulation study and three real data examples to illustrate the performance of the regression estimators obtained from the *t* distribution (with AR (2) error terms for finite sample case) with and without outliers in the data. The CML estimates are computed using the IRA given in section 3. Note that throughout the simulation study and the real data examples the degrees of freedom ( $\nu$ ) of the *t* distribution is taken as 3 since the small values such as 3 are suggested for the sake of robustness in literature (e.g see Lange et.al, 1989).

#### 4.1 A simulation study

Simulation design. Firstly we generate three independent variables  $x_t$  from standard normal distribution  $(x_{t,i} \sim N(0,1))$ . The values of the parameters are  $\underline{\beta} = (\beta_1, \beta_2, \beta_3)' = (0.1, 0.5, 0.9)'$  and  $\underline{\phi} = (\phi_1, \phi_2)' = (-0.7, 0.12)'$ . Note that the AR (2) model values of  $\underline{\phi}$  are taken to guarantee the stationarity assumption for the model of the error terms. Then the values of the response variable are generated using  $\Phi(B)y_t = \sum_{i=1}^M \beta_i \Phi(B) x_{t,i} + a_t$ .

Simulation cases. In first case the  $a_t$ 's in (3) are generated from standard normal distribution  $(a_t \sim N(0,1))$  and parameters are estimated by using normal distribution's conditional likelihood and t distribution's conditional likelihood. In the second case the  $a_t$ 's are generated from t distribution with v = 3 degrees of freedom  $(a_t \sim t_v(0,1))$  and parameters are estimated by using normal and t assumptions again. The last case is for the symmetric Pareto distributed error terms, where we use the Pareto distribution symmetric by zero as given in [6] with the tail index (shape) parameter  $\kappa = 1.25$ . For the Pareto error case we compute the values of the estimators obtained from the normal and the t distributions.

*Outlier case.* To add some outliers to the data 10 percent of  $\underline{Y}$  is replaced by the points generated from N(0, 100).

*Performance measures.* Mean squared error (MSE) and bias values are calculated to compare the estimators. These values are calculated by using R = 100 replications for the sample sizes n = 25, 50 and 100. Also, using the observed Fisher information matrix given in section 2, the standard errors (SE) and the confidence intervals (CIL - CIU) are calculated.

The MSE values and the biases are calculated using

$$MSE(\hat{\beta}) = \frac{1}{R} \sum_{i=1}^{R} (\hat{\beta}_{i} - \beta)^{2}, bias(\hat{\beta}) = \bar{\beta} - \beta,$$
  

$$MSE(\hat{\phi}) = \frac{1}{R} \sum_{i=1}^{R} (\hat{\phi}_{i} - \phi)^{2}, bias(\hat{\phi}) = \bar{\phi} - \phi,$$
  

$$MSE(\hat{\sigma}^{2}) = \frac{1}{R} \sum_{i=1}^{R} (\hat{\sigma}_{i}^{2} - \sigma^{2})^{2}, bias(\hat{\sigma}^{2}) = \overline{\sigma^{2}} - \sigma^{2},$$
  

$$\bar{\alpha} = \frac{1}{R} \sum_{i=1}^{R} (\hat{\sigma}_{i}^{2} - \sigma^{2})^{2}, bias(\hat{\sigma}^{2}) = \overline{\sigma^{2}} - \sigma^{2},$$

where  $\bar{\beta} = \frac{1}{R} \sum_{i=1}^{R} \hat{\beta}_i$ ,  $\bar{\phi} = \frac{1}{R} \sum_{i=1}^{R} \hat{\phi}_i$ ,  $\bar{\sigma}^2 = \frac{1}{R} \sum_{i=1}^{R} \hat{\sigma}_i^2$ .

Simulation results. Simulation results are given in Tables 1 and 2.

## Simulation results without outliers

Table 1. Bias, MSE, SE, CIL and CIU values of the estimates without outliers True Values  $(\beta_1, \beta_2, \beta_3)' = (0.1, 0.5, 0.9)', (\phi_1, \phi_2)' = (-0.7, 0.12)$  and  $\sigma = 1$ 

			Normal E	Error	t Erro	or	Paretian Error	$(\kappa = 1.25)$
n			Norm	St- <i>t</i>	Norm	St- <i>t</i>	Norm	St- <i>t</i>
25		$\hat{\beta}_1$	0.0854	0.0749	0.1135	0.1360	25.3813	-0.2909
		Bias	0.0230	0.0302	-0.0136	-0.0075	25.2813	-0.3909
	ß	MSE	0.0453	0.0578	0.0789	0.0708	26940.5	20.0117
	$\rho_1$	SE	0.1598	0.3965	0.2678	0.5131	5.0918	0.7380
		CIL	0.0501	-0.0366	0.0082	0.0272	23.4241	-0.5746
		CIU	0.1754	0.2484	0.2189	0.2449	27.3385	-0.0073
		$\hat{\beta}_2$	0.4817	0.4717	0.4945	0.4600	24.2895	-0.0754
		Bias	0.0182	-0.0293	-0.0263	0.0086	23.6061	-0.5754
	ß	MSE	0.0372	0.0656	0.0945	0.1150	61730.1	56.5089
	$\rho_2$	SE	0.2119	0.2997	0.2235	0.4930	5.4498	1.8644
		CIL	0.4044	0.3513	0.4045	0.3677	22.1946	-0.7921
		CIU	0.6008	0.6417	0.5846	0.5522	26.3843	0.6412
		$\hat{\beta}_3$	0.8959	0.9039	0.8800	0.9008	-20.515	0.6152
		Bias	0.0128	0.0130	0.0112	-0.0125	-21.415	-0.2848
	ß	MSE	0.0468	0.0537	0.0901	0.1110	17290.3	10.3356
	$\rho_3$	SE	0.1813	0.3596	0.3182	0.4130	2.9110	0.4780
		CIL	0.8266	0.7264	0.7574	0.7370	-21.634	0.4315
		CIU	0.9687	1.0499	1.0025	1.0645	-19.396	0.7989
		$\widehat{\phi}_1$	-0.7034	-0.5273	-0.7484	-0.5611	-0.5226	-0.5219
		Bias	-0.0326	0.0923	-0.0240	0.0818	0.1774	0.2995
	ሐ.	MSE	0.0715	0.0810	0.0699	0.0618	0.0973	0.1307
	$\Psi_1$	SE	0.0261	0.1414	0.0826	0.2476	0.1617	0.1971
		CIL	-0.7136	-0.6208	-0.7833	-0.6007	-0.5845	-0.4763
		CIU	-0.6931	-0.4338	-0.7136	-0.5214	-0.4606	-0.3247
		$\widehat{\phi}_2$	0.0276	0.0626	0.0283	0.0413	0.2662	0.2888
		Bias	-0.1222	-0.1412	-0.1013	-0.1295	0.1462	0.1251
	<i>ф</i> .	MSE	0.0832	0.0854	0.0825	0.0669	0.0857	0.0404
	$\Psi_2$	SE	0.1403	0.1397	0.1227	0.1835	0.9441	0.1710
		CIL	-0.0644	-0.0111	-0.0213	-0.0027	-0.0987	0.1794
		CIU	0.1196	0.1364	0.0778	0.0852	0.6310	0.3108

		$\hat{\sigma}$	0.8711	0.6923	1.3911	0.8149	169.849	0.9941
		Bias	-0.1101	0.3134	0.4137	0.2423	168.849	-0.0059
		MSE	0.0349	0.1105	0.3948	0.1188	652360	0.2501
	σ	SE	0.1241	0.0963	0.2948	0.1381	34.675	0.2017
		CIL	0.7894	0.6490	1.2774	0.7606	156.522	0.9166
		CIU	0.9300	0.7264	1.5048	0.8692	183.176	1.0717
			Normal E	Error	t Erro	r	Paretian Error	$\kappa = 1.25$ )
			Norm	St- <i>t</i>	Norm	St-t	Norm	St-t
		Â1	0.0863	0.0792	0.0677	0.0883	-2.9119	0.3695
50		Bias	-0.0137	-0.0208	-0.0323	-0.0117	-3.0119	-0.2695
	0	MSE	0.0198	0.0258	0.0557	0.0399	499.264	4.5549
	$\beta_1$	SE	0.3551	0.2667	0.2473	0.3190	1.8167	0.3675
		CIL	-0.0121	0.0229	0.0069	0.0252	-3.4155	0.2693
		CIU	0.1847	0.1555	0.1346	0.1513	-2.4083	0.4697
		β <sub>2</sub>	0.5004	0.4949	0.5349	0.5310	-1.0789	0.3599
		Bias	0.0004	-0.0051	-0.0349	0.0310	-1.5789	-0.1401
	0	MSE	0.0118	0.0163	0.0401	0.0307	353.475	2.4658
	$p_2$	SE	0.1746	0.2404	0.2325	0.2985	0.9547	0.9719
		CIL	0.4520	0.4787	0.4687	0.4448	-1.3435	0.0898
		CIU	0.5488	0.5112	0.6010	0.6172	-0.8142	0.6300
		$\hat{\beta}_3$	0.8986	0.9083	0.9122	0.9088	1.7127	1.1110
		Bias	-0.0014	0.0083	-0.0122	0.0088	-2.6127	0.2110
	ß	MSE	0.0186	0.0196	0.0496	0.0310	792.217	1.4357
	$P_3$	SE	0.2209	0.1215	0.1092	0.2660	2.0044	1.3471
		CIL	0.8363	0.8831	0.8832	0.8802	-2.2683	0.7256
		CIU	0.9610	0.9531	0.9411	0.9374	-1.1572	1.4965
		$\widehat{\phi}_1$	-0.7449	-0.6906	-0.6860	-0.5060	-0.5743	-0.4243
		Bias	-0.0449	0.1968	0.0140	0.1932	0.1257	0.2757
	ሐ.	MSE	0.0264	0.0431	0.0273	0.0567	0.0426	0.1184
	$\Psi_1$	SE	0.2443	0.0962	0.0974	0.1196	0.0725	0.3794
		CIL	-0.8122	-0.8290	-0.7127	-0.5345	-0.6029	-0.5284
		CIU	-0.6775	-0.5575	-0.6593	-0.4747	-0.5627	-0.3202
		$\widehat{\phi}_2$	0.0543	0.0803	0.0788	0.0945	0.1740	0.1980
		Bias	-0.0657	-0.0441	-0.0412	-0.0896	0.1078	0.0780
	<i>φ</i> 2	MSE	0.0309	0.0206	0.0279	0.0199	0.0347	0.0210
	42	SE	0.1643	0.1237	0.1146	0.1161	0.0825	0.3210
		CIL	0.0076	0.0390	0.0464	0.0559	0.1512	0.1090
		CIU	0.1010	0.1127	0.1112	0.1250	0.1969	0.2871
		σ	0.9334	0.7397	1.5421	0.9104	60.540	1.1750
		Bias	-0.0666	-0.2603	0.5421	-0.0896	59.540	0.1750
	σ	MSE	0.0170	0.0742	0.4703	0.0199	25203	0.3459
		SE	0.1347	0.0836	0.2265	0.1157	8.7382	0.1975
		CIL	0.8960	0.71/8	1.4804	0.8772	58.1179	1.1198
		CIU	0.9707	0.7616	1.6038	0.9435	65.9622	1.2303
			Normal F	rror	t Erro	r	Paretian Error	$\kappa = 1.25$
			Norm	St-t	Norm	St-t	Norm	St-t
100		$\hat{\beta}_1$	0.1001	0.0964	0.1288	0.1085	4.9025	0.1551
	$\beta_1$	Bias	0.0001	-0.0036	0.0288	0.0085	4.8025	0.0551
		MSE	0.0072	0.0106	0.0186	0.0103	2028.2	0.7551

	SE	0.1271	0.1666	0.2197	0.1995	3.0280	0.3802
	CIL	0.0805	0.0769	0.0929	0.0653	4.2516	0.0596
	CIU	0.1197	0.1159	0.1647	0.1516	5.5535	0.3095
	$\hat{\beta}_2$	0.5123	0.5129	0.5054	0.4990	19.0216	0.3903
	Bias	0.0123	0.0129	0.0054	-0.0010	18.5216	-0.1097
ß	MSE	0.0078	0.0101	0.0186	0.0135	9080.24	1.5062
$P_2$	SE	0.1415	0.1827	0.2314	0.2146	3.0746	0.4928
	CIL	0.4969	0.4885	0.4890	0.4632	18.3705	0.1974
	CIU	0.5277	0.5372	0.5217	0.5348	19.6726	0.5832
	$\hat{\beta}_3$	0.8996	0.9047	0.9211	0.9197	20.3116	0.7991
	Bias	-0.0004	0.0047	0.0211	0.0197	19.4116	-0.1009
ß	MSE	0.0062	0.0083	0.0174	0.0146	3555.75	0.6635
$P_3$	SE	0.1340	0.1736	0.2144	0.1940	3.1505	0.2957
	CIL	0.8745	0.8754	0.8888	0.8853	19.8952	0.7643
	CIU	0.9248	0.9341	0.9534	0.9541	20.7280	0.8339
	$\widehat{\phi}_1$	-0.7217	-0.5309	-0.6996	-0.5160	-0.6227	-0.3619
	Bias	-0.0217	0.1691	0.0004	0.1840	0.0773	0.3381
<i>ф</i> .	MSE	0.0148	0.0389	0.0091	0.0425	0.0188	0.1393
$\Psi_1$	SE	0.0881	0.0795	0.1357	0.0807	0.6589	0.7619
	CIL	-07408	-0.5467	-0.7180	-0.5330	-0.6344	-0.3768
	CIU	-07026	-0.5150	-0.6812	-0.4990	-0.6110	-0.3469
	$\widehat{\phi}_2$	0.0912	0.0980	0.1072	0.0973	0.1961	0.2485
	Bias	-0.0288	-0.0220	-0.0128	-0.0227	0.0761	0.1285
ሰ-	MSE	0.0161	0.0101	0.0110	0.0070	0.0161	0.0276
$\Psi_2$	SE	0.0882	0.0772	0.1359	0.0724	1.0997	0.7410
	CIL	0.0693	0.0797	0.0868	0.0807	0.1826	0.2337
	CIU	0.1131	0.1162	0.1276	0.1139	0.2096	0.2632
	$\hat{\sigma}$	0.9748	0.7721	1.5719	0.9178	175.337	1.2211
	Bias	-0.0252	-0.2279	0.5719	-0.0822	174.332	0.2211
σ	MSE	0.0064	0.0550	0.4113	0.0126	444540	0.3115
	SE	0.0891	0.0612	0.1660	0.0843	16.5886	0.1521
	CIL	0.9555	0.7603	1.5407	0.9012	191.887	1.1916
	CIU	0.9941	0.7839	1.6030	0.9345	159.144	1.2507

Table 1 displays the simulation results for the case without outlier in the data with different sample sizes. From the table we can say that error terms based on normal distribution and t distribution cases have similar performance. When the error distribution is normal the CML estimators based on the normal distribution perform the best and the performance of the CML estimator based on the t distribution is comparable with the estimators based on the normal distribution. On the other hand, if the error distribution is the t distribution the estimators based on t distribution are the best, and it is followed by the normal distribution. Finally, for the Pareto distributed error case the estimators based on the normal distribution drastically affected and give the worst results with the larger MSE and the bias values. But, for the Pareto distributed error case the estimators behave much better than the normal case.

# Simulation results with 10% outliers

			Normal Error		t Error	
n	Parameter		Normal	t	Normal	t
		$\hat{\beta}_1$	-0.1805	0.1277	-0.7210	-0.0001
25	0	Bias	-0.2805	0.0277	-0.8210	-0.1001
	$\rho_1$	MSE	14.395	0.0666	24.5021	0.1529
		SE	0.7206	0.6763	0.6216	0.6371
		CIL	-0.3517	-0.2601	-0.8925	-0.2184
		CIU	-0.0094	0.5154	-0.5494	0.2182
		$\hat{\beta}_2$	1.1081	0.5161	0.4523	0.5571
	0	Bias	0.6081	0.0161	-0.0477	0.0571
	$\rho_2$	MSE	19.419	0.1170	14.6355	0.2807
		SE	2.2292	0.7016	0.6484	0.5687
		CIL	0.9673	0.1159	0.2394	0.2805
		CIU	1.2490	0.9163	0.6652	0.8336
		β <sub>3</sub>	0.2261	0.9344	0.6559	0.9139
	0	Bias	-0.6739	0.0344	-0.2441	0.0139
	$\beta_3$	MSE	15.333	0.0984	18.1168	0.1785
		SE	2.2848	0.8055	0.9610	0.9656
		CIL	0.0442	0.7948	0.5443	0.7959
		CIU	0.4079	1.0740	0.7674	1.0320
		$\hat{\phi}_1$	-0.0335	-0.2303	0.0294	-0.2678
	,	Bias	0.6665	0.4697	0.7294	0.4322
	$\phi_1$	MSE	3.0020	0.3129	2.1621	0.3226
		SE	5.2471	0.0421	0.3258	0.0030
		CIL	-0.2352	-0.3086	-0.1113	-0.3572
		CIU	0.1683	-0.1521	0.1701	-0.1784
		$\hat{\phi}_2$	-2.0472	0.1866	-1.7184	0.2123
	,	Bias	-2.1672	0.0666	-1.8384	0.1005
	$\phi_2$	MSE	19.1661	0.0970	13.2772	0.0923
		SE	2.2688	0.0995	0.3081	0.1152
		CIL	-2.1668	0.0904	-2.2067	0.1214
		CIU	-1.9275	0.2829	-1.2301	0.3031
		$\hat{\sigma}$	17.8423	0.9916	19.3621	1.1540
	σ	Bias	16.8423	-0.0084	18.3621	0.1540
		MSE	417.615	0.0277	452.471	0.0759
		SE	87.2684	0.1956	4.2045	0.2637
		CIL	16.3839	0.9112	17.7795	1.0451
		CIU	19.3007	1.0720	20.9447	1.2628
			Normal E	error	t Erro	r
			Normal	t	Normal	t
		$\hat{\beta}_1$	-0.2296	0.1139	-0.2276	0.0913
50	D	Bias	-0.3296	0.0139	-0.3276	-0.0087
	$p_1$	MSE	11.890	0.0363	9.6600	0.0531
		SE	30.410	0.4740	2.9891	0.7061
		CIL	-0.8091	-0.0874	-0.6726	-0.0615
		CIU	0.3500	0.3152	0.2173	0.2440

Table 2. Bias, MSE, SE, CIL and CIU values of the estimates with outliers True Values  $(\beta_1, \beta_2, \beta_3)' = (0.1, 0.5, 0.9)', (\phi_1, \phi_2)' = (-0.7, 0.12)$  and  $\sigma = 1$ 

		$\hat{\beta}_2$	0.9483	0.4933	0.3304	0.5128
	0	Bias	0.4483	-0.0067	-0.1696	0.0128
	$\beta_2$	MSE	14.957	0.0228	13.637	0.0557
		SE	25.740	0.3973	2.8076	0.4466
		CIL	0.3299	0.3797	-0.2893	0.3497
		CIU	1.5667	0.6069	0.9501	0.6758
		β <sub>3</sub>	0.4665	0.8961	0.9969	0.9010
	0	Bias	-0.4335	-0.0039	0.0969	0.0010
	$\beta_3$	MSE	16.155	0.0307	10.6562	0.0540
		SE	31.805	0.2983	1.4603	0.7257
		CIL	0.1967	0.7335	0.5512	0.8402
		CIU	0.7363	1.0587	1.4426	0.9619
		$\widehat{\phi}_1$	0.0013	-0.2284	0.0789	-0.2730
	L	Bias	0.7013	0.4716	0.7789	0.4270
	$\phi_1$	MSE	0.9785	0.2699	1.0454	0.2211
		SE	3.8399	0.1041	0.3321	0.1633
		CIL	-0.0391	-0.2299	0.0054	-0.2899
		CIU	0.0417	-0.2269	0.1524	-0.2560
		$\hat{\phi}_2$	-0.3809	0.2138	-0.3230	0.1971
	,	Bias	-0.7013	0.0938	-0.4430	0.0771
	$\phi_2$	MSE	0.9788	0.0449	1.4744	0.0450
		SE	0.3777	0.1455	0.9165	0.1320
		CIL	-0.5184	0.1850	-0.5298	0.1826
		CIU	-0.2434	0.2427	-0.1163	0.2116
		$\hat{\sigma}$	23.1810	1.0546	22.1079	1.2178
	σ	Bias	22.1810	0.0546	21.1079	0.2178
		MSE	557.262	0.0189	592.552	0.0812
		SE	3.3555	0.1666	3.3084	0.2140
		CIL	22.2536	1.0101	21.2234	1.1584
		CIU	24.1085	1.0991	22.9924	1.2771
		-	Normal E	Error	t Erro	r
			Normal	t	Normal	t
		$eta_1$	0.3444	0.0977	0.5861	0.1064
100	ß.	Bias	0.2444	-0.0023	0.4861	0.0064
	Ρ1	MSE	6.6409	0.0164	7.8438	0.0199
		SE	5.4338	0.3981	4.7214	0.4410
		CIL	-0.2259	0.0287	0.3555	0.0258
		CIU	0.9146	0.1668	0.8168	0.1869
		$\beta_2$	0.5080	0.5049	0.9318	0.4921
	ße	Bias	0.0080	0.0049	0.4318	-0.0079
	P2	MSE	8.6819	0.0164	8.2327	0.0310
		SE	8.2289	0.4066	4.1940	0.3476
		CIL	-0.2487	0.4306	-0.6039	0.4320
		CIU	1.2647	0.5793	2.4675	0.5521
		$\hat{eta}_3$	0.9028	0.9085	0.6209	0.9054
	R	Bias	0.0028	0.0085	-0.2791	0.0054
	$\mu_3$	MSE	5.9380	0.0162	11.6861	0.0278
		SE	9.1633	0.2197	5.5664	0.1939
		CIL	-0.0449	0.8551	-1.4063	0.8350
		CIU	1.8505	0.9619	2.6481	0.9759

	$\widehat{\phi}_1$	0.0245	-0.1722	-0.0047	-0.2039
đ	Bias	0.7245	0.5278	0.6953	0.4961
$arphi_1$	MSE	0.6194	0.3110	0.5554	0.2652
	SE	0.7856	0.0835	0.0976	0.0734
	CIL	0.0101	-0.1792	-0.0167	-0.2183
	CIU	0.0389	-0.1653	0.0073	-0.1894
	$\widehat{\phi}_2$	-0.1095	0.2172	-0.0639	0.2186
đ	Bias	-0.2295	0.5278	-0.1839	0.0986
$arphi_2$	MSE	0.1745	0.0259	0.1300	0.0281
	SE	0.3773	0.0513	0.0244	0.0555
	CIL	-0.1226	0.2068	-0.0763	0.1989
	CIU	-0.0965	0.2275	-0.0515	0.2383
	$\hat{\sigma}$	27.4424	1.1533	28.8648	1.3182
Ţ.	Bias	26.4424	0.1533	27.8648	0.3182
8	MSE	770.938	0.0422	755.250	0.1214
	SE	2.9298	0.1340	2.7141	0.1875
	CIL	26.8991	1.1269	28.2933	1.2838
	CIU	27.9858	1.1796	29.4363	1.3526

Table 2 shows the simulation results with 10 percent outlier in the data. When outliers are introduced in the data the estimators based on normal distribution are drastically worsen which is reflected to the higher MSE values. However, the estimators based on t distribution still have excellent performance with outliers. The estimators based on t distribution superior to the estimators of the normal distribution in terms of MSE and bias values.

# 4.2 Real data examples

*Example 1.* In this example we will analyze the data set given by Sheather [16]. The data shows that Australian Film Commission's (ACF) yearly gross box office receipts from movies screened in Australia. Table 3 shows the data set. In that book two different scenarios have been applied to this data set. The first one is the ordinary regression model and the second one is the regression model with autoregressive error terms. In this paper we also consider two models and estimate the parameters of interest using the normal and *t* distributions. Table 4 shows the summary of the estimates, standard errors and the 95% confidence intervals for  $\hat{\beta}$  and  $\hat{\phi}$ . We calculated the standard errors and the 95% confidence intervals using the observed Fisher information.

Gross box office		Gross box office				
(\$M)	Year	(\$M)	Year			
95.3	1976	334.3	1992			
86.4	1977	388.7	1993			
119.4	1978	476.4	1994			
124.4	1979	501.4	1995			
154.2	1980	536.8	1996			
174.3	1981	583.9	1997			

Table 3. Australian Gross Box Office Results from 1976 to 2007

210	1982	629.3	1998
208	1983	704.1	1999
156	1984	689.5	2000
160.6	1985	812.4	2001
188.6	1986	844.8	2002
182.1	1987	865.8	2003
223.8	1988	907.2	2004
257.6	1989	817.5	2005
284.6	1990	866.6	2006
325	1991	895.4	2007

We consider the following model

 $GrossBoxOffice = \beta_0 + \beta_1 Years + e_t$ 

and we assume that the error terms have AR(1) model. The LS estimates show that  $\beta_0$  is insignificant so that  $\beta_0$  is not included in the model. The LS estimates are used as the initial values for the algorithms.

Table 4. Parameter estimates for Australian Gross Box Office Re	esults
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		Normal	t
	$\hat{\beta}_1$	27.1927	26.7505
$\beta_1$	SE	0.4263	0.7030
	95% CI	(27.0449) - (27.3404)	(26.5069) – (26.9941)
	$\widehat{\phi}$	0.8816	0.2558
$\phi$	SE	2.9578	0.0066
	95% CI	(-0.1436) - (1.9067)	(0.2358) - (0.2817)
AIC		344.2041	344.2835
BIC	2	347.1356	347.2149

Table 4 gives a summary of estimation with both cases. The table shows the estimates standard errors and 95% confidence intervals for two different estimation methods.



Figure 1. Australian Gross Box Office Results data set estimated by t and normal distributions methods

Figure 1 shows the fitted regression lines on the data. The solid line shows the fitted regression obtained from the t distribution and the dashed line corresponds to the normal case. When data has no outlier the fitted lines obtained from the normal and the t distributions have similar behavior.

*Example 2.* In this example we will analyze the data set given by Rousseeuw and Leroy [15]. The data shows the proportion of the number of ten million international phone calls from Belgium in the years 1950-1973. From Figure 2 we can observe that there are outliers in the data. Rousseeuw and Leroy [15] modeled this data set with a linear regression model to illustrate the performance of the robust regression method the least median of squares (LMS). The following table displays the data.

Number of Calls <sup>a</sup>	Year	Number of Calls <sup>a</sup>	Year
$(y_i)$	$(x_i)$	$(y_i)$	$(x_i)$
0.44	50	1.61	62
0.47	51	2.12	63
0.47	52	11.90	64
0.59	53	12.40	65
0.66	54	14.20	66
0.73	55	15.90	67
0.81	56	18.20	68
0.88	57	21.20	69
1.06	58	4.30	70
1.20	59	2.40	71
1.35	60	2.70	72
1.49	61	2.90	73

Table 5. Number of International Calls from Belgium

<sup>a</sup> In tens of millions.

We observe that the OLS residuals show an autocorrelated structure with type AR(1). This observation based on autocorrelation function and the partial autocorrelation function graphs of the OLS residuals. Therefore, we use a regression model with autoregressive error term with AR(1) to model this data set and use normal and the *t* distribution to obtain estimates for the parameters. The following table gives the summary of the estimates along with the standard errors and the 95% confidence intervals. We also provide the values of the AIC and BIC criteria. The values of AIC and BIC show that the *t* distribution gives the better fit then the normal distribution. It should be noticed that the estimates obtain from *t* distribution are much closed to the values obtained from the LMS.

		Normal	t
	$\hat{\beta}_0$	-13.8142	-5.3724
$\beta_0$	SĚ	10.9242	11.6686
	95% CI	(-18.1848) – (-9.4436)	(-10.0408) - (-0.7040)
	$\hat{\beta}_1$	0.2980	0.1131
$\beta_1$	SĒ	0.1796	0.1918
	95% CI	(0.2262) - (0.3699)	(0.0363) - (0.1898)
	$\widehat{\phi}$	0.7366	0.1627
$\phi$	SE	50.0647	0.2422
	95% CI	(-19.2934) – (20.7667)	(0.0658) - (0.2596)
AIC		61.3656	37.4305
BIC	2	120.2654	72.3951

Table 6. Parameter Estimates for International Calls From Belgium



Figure 2. Number of International Phone Calls from Belgium Data Set Estimated by *t* and Normal Distributions

Figure 2 depicts the scatter plot of the data with the fitted regression lines obtain from normal and the t distributions. From this figure we observe that unlike the fitted line obtain from the normal distribution, the fitted line from the t distribution is not affected from the outliers.

*Example 3.* We use the data set has been previously analyzed by Ramanathan [14] to show the performance of autoregressive error terms regression model. The data provides consumption of electricity by residential customers served by San Diego Gas and Electric Company. This data set consists of 87 quarterly observations for each 4 covariates from the second quarter of 1972 through fourth quarter of 1993. The response variable is electricity consumption as measured by the logarithm of the kwh (LKWH) sales per residential customer. The explanatory variables are the per-capita income (LY), the price of electricity (LPRICE), cooling degree days (CCD) and heating degree days (HDD). The linear regression model and the expected signs of the  $\beta$ 's considered in Ramanathan [14] are as follows:

 $LKWH = \beta_0 + \beta_1 LY + \beta_2 LPRICE + \beta_3 CDD + \beta_4 HDD + \varepsilon_t$  $\beta_1 > 0, \ \beta_2 < 0, \beta_3 > 0, \beta_4 > 0.$ 

It is pointed out by Ramanathan [14] that when the OLS method is used to obtain the estimates the signs of LPRICE, CDD and HDD are consistent with the expected ones, but estimation of LY has the reverse sign. They note that this unexpected result may happen due to ignoring the autocorrelation structure of the error term, hence they suggest using autoregressive error term regression model to model the data. They select the AR order 4 which minimizes the BIC. Then, they used the OLS method to find the estimators for the parameters, and they observed that the sign of the LY is changed towards the expectations.

Here we use the normal and the *t* distributions as the error distribution and obtain the estimators for the parameters of interest. In Table 7 we give the estimates and the AIC values obtained from the normal and the *t* cases. Note that the estimates obtained from the normal distribution are the same with the OLS estimates for the AR(4) model given in Ramanathan [14].

		OLS	LS	LASSO	Bridge	MMLASSO	MMBridge				
LY	$\hat{\boldsymbol{\beta}}_1$	-0.00234	0.18625	0.05879	-	0.14879	-				
LPRICE	$\hat{\beta}_2$	-0.01856	-0.09354	-0.08455	-0.08563	-0.06455	-0.08563				
CDD	$\hat{\beta}_3$	0.06365	0.00029	0.00028	0.00028	0.00028	0.00028				
HDD	$\hat{\beta}_4$	0.08564	0.00022	0.00022	0.00023	0.00022	0.00023				
AR order		-	4	4	4	4	4				

Table 7. The estimated coefficients without outlier

Table 7. The estimated coefficients with outlier

		OLS	LS	LASSO	Bridge	MMLASSO	MMBridge
LY	$\hat{\boldsymbol{\beta}}_1$	-2.69756	1.69845	3.65769	-	0.25654	-
LPRICE	$\hat{\beta}_2$	-0.96123	-0.00154	-0.98555	-0.64786	-0.00547	-0.02645
CDD	$\hat{\beta}_3$	0.57743	0.07264	0.64135	0.91231	0.00072	0.00036
HDD	$\hat{\beta}_4$	0.96874	0.04622	0.95344	0.84521	0.00095	0.00041
AR order		-	4	4	4	4	4

We notice from this table that unlike the OLS estimate without AR(4) structure the sign of  $\beta_1$  is positive when the autoregressive errors are assumed as Ramanathan [14] reported. Further, when the estimation is carried out using the *t* distribution, the AIC value is smaller than the AIC obtained from the normal distribution. Thus, it can be concluded that the *t* distribution may provide better fit than the normal distribution for this data.

## 5. Discussion

In this paper, we have proposed to use the t distribution as an alternative to the normal distribution as the error distribution in linear regression model with autoregressive error terms. The simulation results and the real data examples have shown that the t and the normal distributions give similar results when there is no outlier in the data. On the other hand, when the data have some outliers the t distribution has better performance than the normal distribution for all the settings. Further, Example 3 has shown that the OLS method may fail to accurately estimate the unknown parameters when the error terms have autocorrelation structure. However, when the autoregressive error form is introduced into the model the estimates are correctly obtained in terms of sign from the OLS method, the normal and the t distributions. For the same example we have also noticed that the result obtained from the t distribution can be better model than the normal distribution. To sum up, all of these results show that the t distribution can be used as an alternative to the normal distribution for parameter estimation in a linear regression model with autoregressive error terms when the data sets have outliers and/or heavy tailed error distributions.

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