Supplementary Material for

"Finite mixture modeling using shape mixtures of the skew scale mixtures of normal distributions"

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This supporting information is a longer version of the printed paper. It contains more details on finite sample properties of ML estimates and some more simulation studies to demonstrate the adaptability and flexibility when using the finite mixture of SMSSMN distributions.

A. More details on simulation 1

In section 4.1 of the paper, we have just presented some figures to show how the biases and MSEs of the estimates of shape parameters in the mixture models are changed when sample size tend to increase. Here, more details of this experiment are given in Tables S.1-S.3 for all of the parameters. We see that the absolute biases of ML estimates is decreasing in n and tends to zero, showing empirically the asymptotic unbiasedness of the estimates obtained via the ECM algorithm. Moreover, MSEs are decreasing in n and vanishes as n goes to infinity, showing empirically the consistency of the estimates.

Table S.1: SMSTN-MIX: Bias and MSE for EM estimates of simulated data and CPU time (CT; in second). True parameter values are $\pi = 0.6,\ \xi_1 = -3,\ \sigma_1 = 1, \alpha_1 = 5;\ \xi_2 = 3,\ \sigma_2 = 1,\ \alpha_2 = 10$ and $\nu = 3$.

			n=50		n=100		n=250		n=500		n=1000		n=2000	
λ_1	λ_2	Estimates	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
-1	4	π_1	0.0004	0.0729	0.0072	0.0522	0.0006	0.0322	0.0015	0.0226	0.0004	0.0168	0.00007	0.0112
		ξ_1	0.0649	0.3634	0.0352	0.3165	0.0072	0.2472	0.0152	0.1822	0.0029	0.1421	0.0123	0.1064
		ξ_2	0.0524	0.2838	0.0123	0.2055	0.0015	0.1113	0.00008	0.0839	0.0052	0.0574	0.0018	0.0400
		σ_1	0.1312	0.3351	0.0798	0.2295	0.0389	0.1388	0.0244	0.1066	0.0159	0.0714	0.0122	0.0563
		σ_2	0.0534	0.2903	0.0686	0.2414	0.0318	0.1439	0.0111	0.0962	0.0016	0.0697	0.0049	0.0466
		λ_1	4.6367	9.6647	2.0250	5.0684	0.5770	1.5839	0.4006	1.0629	0.1827	0.6283	0.0901	0.3919
		λ_2	17.6583	28.7120	7.9823	14.7892	2.8199	5.8102	1.1524	3.0357	0.4821	1.6318	0.2087	0.9350
		α_1	158.844	408.459	57.2912	174.131	11.1017	26.6222	6.0560	14.4625	2.9119	6.4745	0.9628	3.2817
		α_2	724.556	1302.9	240.104	578.297	42.9686	109.561	14.4496	40.2985	4.7046	14.5248	1.9028	6.7099
		ν	1381.61	5446.44	1546.70	1567.55	0.5548	1.3410	0.2574	0.9185	0.0767	0.4281	0.9006	0.3101
		CT	0.3258		0.4183		0.7060		1.1800		2.1120		3.7708	
-1	-4	π_1	0.0003	0.0765	0.0004	0.0583	0.00003	0.0350	0.0003	0.0273	0.00006	0.0171	0.0008	0.0125
		ξ_1	0.0103	0.4184	0.0524	0.3289	0.0324	0.2517	0.0174	0.1880	0.0047	0.1470	0.0003	0.1107
		ξ_2	0.0058	0.2692	0.0145	0.2096	0.0135	0.1337	0.0009	0.0851	0.0025	0.0556	0.0017	0.0396
		σ_1	0.1302	0.3241	0.1135	0.2650	0.0475	0.1621	0.0312	0.1179	0.0109	0.0781	0.0105	0.0577
		σ_2	0.0836	0.3250	0.0604	0.2500	0.0138	0.1582	0.0092	0.1102	0.0048	0.0719	0.0012	0.0546
		λ_1	3.9018	10.244	1.9823	4.2483	0.7230	1.6703	0.3377	0.9028	0.1710	0.6803	0.0938	0.4176
		λ_2	18.936	29.542	7.5627	15.261	2.6537	5.8531	1.2321	3.0816	0.5365	1.7561	0.1988	1.0280
		α_1	168.67	486.67	46.094	111.96	12.597	32.797	5.1903	11.093	2.9941	7.8573	1.5042	3.7832
		α_2	743.30	1372.01	213.21	542.34	42.830	121.02	14.536	40.692	5.6426	16.841	1.7317	7.2897
		ν	26109.2	6919.9	3426.5	1873.6	286.68	486.96	0.2948	0.8856	0.1588	0.5289	0.0958	0.3605
		CT	0.3069		0.4226		0.7114		1.1766		2.1027		3.9152	

Table S.2: SMSCN-MIX: Bias and MSE for EM estimates of simulated data and CPU time (CT; in second). True parameter values are $\pi = 0.6,\ \xi_1 = -3,\ \sigma_1 = 1, \alpha_1 = 5,\ \nu_{11} = 0.3,\ \nu_{12} = 0.7;\ \xi_2 = 3,\ \sigma_2 = 1,\ \alpha_2 = 10,\ \nu_{21} = 0.6,$ and $\nu_{12} = 0.4$.

			n=50		n=100		n=250		n=500		n=1000		n=2000	
λ_1	λ_2	Estimates	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
-1	4	π_1	0.0009	0.0687	0.0001	0.0503	0.0013	0.0295	0.0011	0.0244	0.0016	0.0161	0.0001	0.0111
		ξ_1	0.0109	0.3277	0.0191	0.2661	0.0258	0.2002	0.0126	0.1581	0.0260	0.1188	0.0339	0.0933
		ξ_2	0.0983	0.3344	0.0383	0.2412	0.0007	0.1308	0.0023	0.0844	0.0172	0.0620	0.0106	0.0437
		σ_1	0.1646	0.2431	0.1476	0.2063	0.1769	0.2078	0.1719	0.1892	0.1793	0.1885	0.1871	0.1918
		σ_2	0.0256	0.1754	0.0371	0.1493	0.0636	0.1255	0.0633	0.0949	0.0769	0.0940	0.0706	0.0800
		λ_1	3.7059	9.2739	1.1832	3.1196	0.4270	1.5971	0.1007	0.7483	0.0239	0.4685	0.1088	0.3343
		λ_2	13.5294	22.2106	6.6903	13.3240	2.5546	5.2886	1.2085	3.0088	0.8035	1.9487	0.5338	1.1675
		α_1	141.026	332.951	33.4381	77.0129	12.2883	38.7877	3.5810	9.2820	1.7703	4.8671	0.5750	2.8571
		α_2	524.848	956.779	175.686	443.771	34.8846	81.8113	11.8364	31.0755	6.8154	18.1967	3.5874	8.0247
		ν_{11}	0.5026	0.5952	0.4553	0.5628	0.4608	0.5433	0.4675	0.5251	0.4413	0.4779	0.4256	0.4464
		ν_{12}	0.1026	0.3349	0.0553	0.3353	0.0608	0.2941	0.0675	0.2484	0.0413	0.1880	0.0256	0.1370
		ν_{21}	0.1078	0.2365	0.1126	0.2282	0.1254	0.1878	0.1179	0.1558	0.1202	0.1368	0.1272	0.1366
		ν_{22}	0.0921	0.2298	0.0873	0.2168	0.0745	0.1584	0.0820	0.1307	0.0797	0.1030	0.0727	0.0882
		CT	0.5186		0.7431		1.3338		2.6495		5.3628		11.7930	
-1	-4	π_1	0.00010	0.0758	0.00025	0.0498	0.00008	0.0335	0.00024	0.0234	0.00021	0.0161	0.00013	0.0106
		ξ_1	0.0124	0.3044	0.0100	0.2688	0.0152	0.2107	0.0051	0.1796	0.0170	0.1323	0.0110	0.0976
		ξ_2	0.0785	0.2883	0.0192	0.2212	0.0001	0.1341	0.0115	0.0895	0.0107	0.0585	0.0085	0.0429
		σ_1	0.1957	0.2845	0.1859	0.2416	0.1644	0.2023	0.1620	0.1889	0.1614	0.1775	0.1640	0.1707
		σ_2	0.0066	0.1971	0.0412	0.1701	0.0719	0.1450	0.0708	0.1169	0.0777	0.1025	0.0722	0.0852
		λ_1	3.3126	8.1829	1.6589	4.2568	0.3677	1.3305	0.1153	0.7163	0.0355	0.4557	0.0548	0.3454
		λ_2	15.6813	25.0013	8.1793	14.7124	2.9745	6.0169	1.5777	3.2581	0.8883	1.9091	0.5058	1.2333
		α_1	122.876	310.894	44.7643	131.154	8.3678	20.7597	2.8936	8.2578	1.3675	4.2660	0.6640	2.6405
		α_2	600.322	1090.19	213.516	498.802	44.4523	108.735	16.5432	40.3804	7.4913	17.5366	3.6584	8.6838
		ν_{11}	0.5174	0.5951	0.5345	0.5956	0.4734	0.5522	0.4991	0.5519	0.4503	0.5046	0.4568	0.4873
		ν_{12}	0.1174	0.3165	0.1345	0.2951	0.0734	0.2935	0.0991	0.2556	0.0503	0.2332	0.0568	0.1787
		ν_{21}	0.1089	0.2536	0.1095	0.2140	0.1167	0.1865	0.1047	0.1554	0.1053	0.1412	0.1006	0.1212
		ν_{22}	0.0910	0.2465	0.0904	0.2049	0.0832	0.1676	0.0952	0.1492	0.0946	0.1334	0.0993	0.1201
		CT	0.4387		0.6418		1.1843		2.2826		4.6534		10.7073	

Table S.3: SMSSL-MIX: Bias and MSE for EM estimates of simulated data and CPU time (CT; in second). True parameter values are $\pi = 0.6,\ \xi_1 = -3,\ \sigma_1 = 1, \alpha_1 = 5;\ \xi_2 = 3,\ \sigma_2 = 1,\ \alpha_2 = 10$ and $\nu = 3$.

			n=50		n=100		n=250		n=500		n=1000		n=2000	
λ_1	λ_2	Estimates	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
-1	4	π_1	0.0062	0.0707	0.0032	0.0530	0.0006	0.0312	0.0023	0.0224	0.0014	0.0161	0.0003	0.0114
		ξ_1	0.0121	0.3586	0.0068	0.3146	0.0152	0.2303	0.0068	0.1896	0.0077	0.1506	0.0016	0.0885
		ξ_2	0.0711	0.2764	0.0312	0.2239	0.0147	0.1266	0.0010	0.0794	0.0018	0.0498	0.0017	0.0369
		σ_1	0.0852	0.2470	0.0642	0.2008	0.0362	0.1307	0.0298	0.1140	0.0227	0.0856	0.0077	0.0556
		σ_2	0.0244	0.2548	0.0236	0.1887	0.0257	0.1395	0.0130	0.1103	0.0189	0.0839	0.0044	0.0571
		λ_1	4.8966	10.415	1.9674	4.5295	0.5778	1.5170	0.1948	0.8652	0.0807	0.5755	0.0717	0.3388
		λ_2	15.866	25.822	9.9999	17.901	3.0354	5.7621	1.4964	3.2634	0.5045	1.6978	0.1831	0.9968
		α_1	199.66	469.88	50.875	127.39	11.943	26.426	3.9971	10.125	1.8651	5.2772	1.2782	3.2021
		α_2	601.67	1145.9	291.86	653.48	42.525	94.495	16.564	37.254	4.8684	14.880	1.7882	6.7305
		ν	8.3564	17.697	7.4095	16.740	3.7151	9.0491	1.6451	4.6076	0.9838	3.2499	0.2557	1.0972
		CT	0.8485		1.3569		2.8016		5.2011		9.8023		17.969	
-1	-4	π_1	0.0072	0.0786	0.00094	0.0473	0.0007	0.0304	0.0004	0.0221	0.0010	0.0166	0.0012	0.0118
		ξ_1	0.0191	0.3335	0.0021	0.2872	0.0134	0.2215	0.0002	0.1943	0.0062	0.1500	0.0041	0.1038
		ξ_2	0.0381	0.2594	0.0205	0.1908	0.0020	0.1318	0.0077	0.0804	0.0044	0.0563	0.0028	0.0370
		σ_1	0.0594	0.2409	0.0527	0.1914	0.0371	0.1478	0.0477	0.1291	0.0267	0.1091	0.0173	0.0767
		σ_2	0.0659	0.2637	0.0488	0.2110	0.0386	0.1616	0.0338	0.1261	0.0175	0.0849	0.0088	0.0573
		λ_1	4.7443	10.263	2.2097	4.8652	0.6630	1.6566	0.2650	0.9043	0.1237	0.5812	0.1004	0.3624
		λ_2	17.761	26.904	9.1433	15.960	2.7473	5.9888	0.9753	3.1010	0.5103	1.5792	0.1698	0.9615
		α_1	202.54	454.56	62.655	157.46	13.952	32.924	4.7303	11.982	2.5006	5.7446	1.3082	3.1939
		α_2	764.56	1389.1	253.32	534.57	40.368	114.92	12.284	46.407	4.9915	13.192	1.6941	6.8619
		ν	7.3109	14.324	6.1324	12.874	4.2939	10.042	4.1661	9.9181	1.6227	4.6779	0.6706	2.3055
		CT	0.8389		1.3550		2.8273		5.1834		9.8480		18.360	

Tables S.1-S.3 also present the CPU time (in seconds) for various sample sizes in each model. As seen in these tables, the average CPU time for the SMSTN-MIX is less than the other models for all sample sizes, while looking on the other tables, we can see that SMSSL-MIX has a low speed process. Also from all tables, the elapsed time depends heavily on the size of generated samples. All computations were carried out by R package 3.5.1 in win64 environment of a desktop PC machine with 3.60-GHz/Intel Core(TM) i3-4160 CPU Processor and 8.0 GB RAM.

B. Another simulation

In this experiment, 300 samples of size $n = 1000$ are generated from the SGN distribution with two components. The SGN distribution can be obtained from representation (7), when $\tau_1 = 1$ and $\tau_2 \sim N(\lambda, \alpha)$. We generate samples from SGN distribution due to that all studied models reduce to this model with particular parameter settings, which is recommended in section 2. Moreover, we set the true parameters as $\pi_1 = 0.6,\ \xi_1 = -3,\ \sigma_1 = 1,\ \lambda_1 = 3,\ \alpha_1 = 5$, and $\xi_2 = 3,\ \sigma_2 = 1,\ \lambda_2 = 4,\ \alpha_2 = 10$. This values are chosen to have a well separated mixture SGN distribution. This fact depicts from Fig. S.1. Then, we fit these Artificial data with the STN-MIX, SMSTN-MIX, SMSCN-MIX, and SMSSL-MIX models.

Fig. S.2 displays 300 empirical fitted density curves. It is clearly seen that SMSTN-MIX, SMSCN-MIX and SMSSL-MIX models adapt the true underlying distribution and behave better at the tails than STN-MIX. This gives further evidence that the shape mixture of skew scale mixture of normal distributions can be taken a prominent alternative to several other skew distributions as it is more capable of capturing distinct non-normal features.

All numerical results indicate that the finite mixtures of SMSSMN distributions provide similar modeling strength for data generated from other skew distributions as it include two shape parameters to regulate the body and tail skewness. Further, the fitted SMSCN-MIX density curves can enclose the true densities quite compactly in most cases.

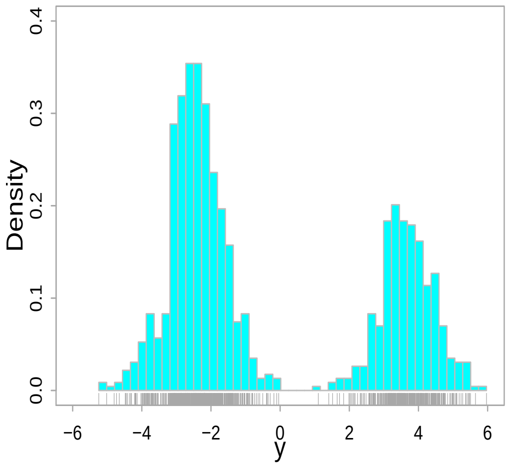

Figure S.1: Artificial data with two components from SGN distribution with parameters $\pi_1 = 0.6,\ \xi_1 = -3,\ \sigma_1 = 1,\ \lambda_1 = 3,\ \alpha_1 = 5$, and $\xi_2 = 3,\ \sigma_2 = 1,\ \lambda_2 = 4,\ \alpha_2 = 10$.

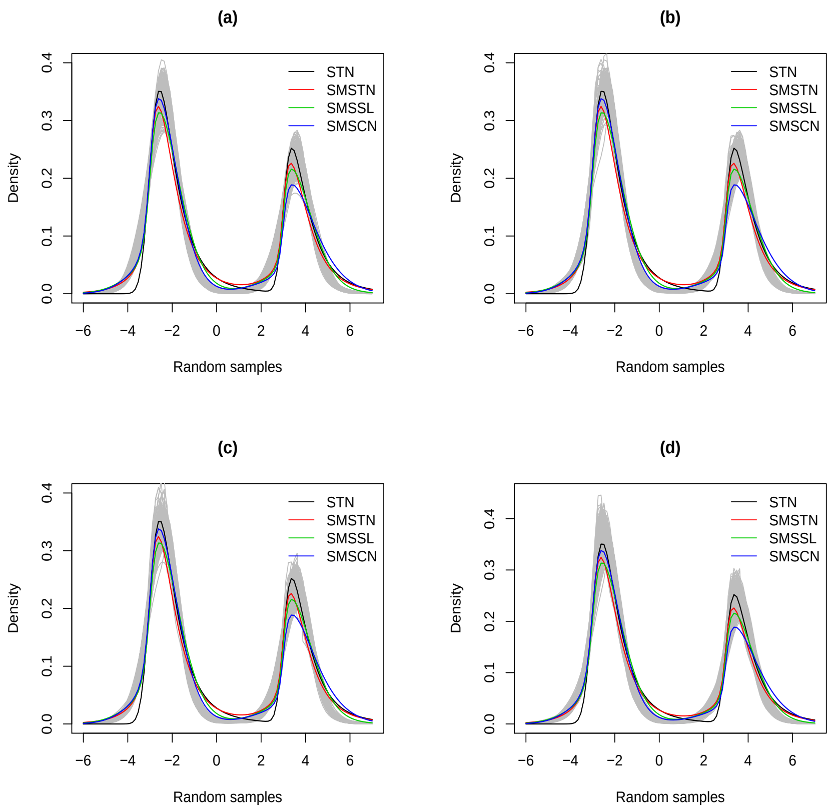

Figure S.2: The true density of the SMSSMN distributions (solid line) and 300 estimated densities (grey lines): (a) STN-MIX , (b) SMSTN-MIX, (c) SMSSL-MIX, and (d) SMSCN-MIX.