SUPPLEMENTAL MATERIALS

A FULL CONDITIONAL DISTRIBUTION OF PARAMETERS IN MSOP MODEL

A.1 THE FULL CONDITIONAL DISTRIBUTION OF $\boldsymbol{\beta}$

Let $\star$ denotes all the other parameters except the given parameters, and then the full conditional distribution (FCD) of $\boldsymbol{\beta}$ is expressed as follows:

$$
\begin{aligned}
p(\boldsymbol{\beta}|\star) &\propto p(\boldsymbol{z}|\boldsymbol{\beta},\boldsymbol{\delta},\boldsymbol{V})\pi(\boldsymbol{\beta}) \\
&\propto \exp\left\{-\frac{1}{2}(\boldsymbol{z}-\boldsymbol{X}\boldsymbol{\beta}-(\boldsymbol{\Delta}\otimes\boldsymbol{I}_S)\boldsymbol{\delta})'(\boldsymbol{I}_n\otimes\boldsymbol{V}^{-1})(\boldsymbol{z}-\boldsymbol{X}\boldsymbol{\beta}-(\boldsymbol{\Delta}\otimes\boldsymbol{I}_S)\boldsymbol{\delta})\right\}\times \\
&\quad\ \exp\left[-\frac{1}{2}(\boldsymbol{\beta}-\boldsymbol{c})'\boldsymbol{T}^{-1}(\boldsymbol{\beta}-\boldsymbol{c})\right] \\
&\propto \exp\left\{-\frac{1}{2}\left[\boldsymbol{\beta}'\boldsymbol{X}'(\boldsymbol{I}_n\otimes\boldsymbol{V}^{-1})\boldsymbol{X}\boldsymbol{\beta}-2\boldsymbol{\beta}'\boldsymbol{X}'(\boldsymbol{I}_n\otimes\boldsymbol{V}^{-1})(\boldsymbol{z}-(\boldsymbol{\Delta}\otimes\boldsymbol{I}_S)\boldsymbol{\delta})+\boldsymbol{\beta}'\boldsymbol{T}^{-1}\boldsymbol{\beta}-2\boldsymbol{\beta}'\boldsymbol{T}^{-1}\boldsymbol{c}\right]\right\} \\
&= \exp\left\{-\frac{1}{2}\boldsymbol{\beta}'[\boldsymbol{X}'(\boldsymbol{I}_n\otimes\boldsymbol{V}^{-1})\boldsymbol{X}+\boldsymbol{T}^{-1}]\boldsymbol{\beta}+\boldsymbol{\beta}'\left[\boldsymbol{X}'(\boldsymbol{I}_n\otimes\boldsymbol{V}^{-1})(\boldsymbol{z}-(\boldsymbol{\Delta}\otimes\boldsymbol{I}_S)\boldsymbol{\delta})+\boldsymbol{T}^{-1}\boldsymbol{c}\right]\right\}.
\end{aligned}
$$

Defining $\boldsymbol{B}=\boldsymbol{X}'(\boldsymbol{I}_n\otimes\boldsymbol{V}^{-1})\boldsymbol{X}+\boldsymbol{T}^{-1}$, and $\boldsymbol{b}=\boldsymbol{X}'(\boldsymbol{I}_n\otimes\boldsymbol{V}^{-1})(\boldsymbol{z}-(\boldsymbol{\Delta}\otimes\boldsymbol{I}_S)\boldsymbol{\delta})+\boldsymbol{T}^{-1}\boldsymbol{c}$, the FCD of $\boldsymbol{\beta}$ can be expressed as:

$$
\boldsymbol{\beta}|\boldsymbol{\delta},\rho,\boldsymbol{\Sigma}_\epsilon,\boldsymbol{V},\boldsymbol{\gamma},\boldsymbol{z},\boldsymbol{y}\sim \mathrm{N}(\boldsymbol{B}^{-1}\boldsymbol{b},\boldsymbol{B}^{-1}).
$$

A.2 THE FULL CONDITIONAL DISTRIBUTION OF $\boldsymbol{\delta}$

The FCD of $\boldsymbol{\delta}$ is expressed as follows:

$$
\begin{aligned}
p(\boldsymbol{\delta}|\star) &\propto p(\boldsymbol{z}|\boldsymbol{\beta},\boldsymbol{\delta},\boldsymbol{V})p(\boldsymbol{\delta}|\rho,\boldsymbol{\Sigma}_\epsilon) \\
&\propto \exp\left\{-\frac{1}{2}(\boldsymbol{z}-\boldsymbol{X}\boldsymbol{\beta}-(\boldsymbol{\Delta}\otimes\boldsymbol{I}_S)\boldsymbol{\delta})'(\boldsymbol{I}_n\otimes\boldsymbol{V}^{-1})(\boldsymbol{z}-\boldsymbol{X}\boldsymbol{\beta}-(\boldsymbol{\Delta}\otimes\boldsymbol{I}_S)\boldsymbol{\delta})\right\}\times \\
&\quad\ \exp\left(-\frac{1}{2}\boldsymbol{\delta}'\boldsymbol{B}_\rho'(\boldsymbol{I}_m\otimes\boldsymbol{\Sigma}_\epsilon^{-1})\boldsymbol{B}_\rho\boldsymbol{\delta}\right) \\
&\propto \exp\left\{-\frac{1}{2}\left[\boldsymbol{\delta}'(\boldsymbol{\Delta}\otimes\boldsymbol{I}_S)'(\boldsymbol{I}_n\otimes\boldsymbol{V}^{-1})(\boldsymbol{\Delta}\otimes\boldsymbol{I}_S)\boldsymbol{\delta}-2\boldsymbol{\delta}'(\boldsymbol{\Delta}\otimes\boldsymbol{I}_S)'(\boldsymbol{I}_n\otimes\boldsymbol{V}^{-1})(\boldsymbol{z}-\boldsymbol{X}\boldsymbol{\beta})+\right.\right. \\
&\qquad\qquad \left.\left.\boldsymbol{\delta}'\boldsymbol{B}_\rho'(\boldsymbol{I}_m\otimes\boldsymbol{\Sigma}_\epsilon^{-1})\boldsymbol{B}_\rho\boldsymbol{\delta}\right]\right\} \\
&= \exp\left\{-\frac{1}{2}\boldsymbol{\delta}'\left[(\boldsymbol{\Delta}\otimes\boldsymbol{I}_S)'(\boldsymbol{I}_n\otimes\boldsymbol{V}^{-1})(\boldsymbol{\Delta}\otimes\boldsymbol{I}_S)+\boldsymbol{B}_\rho'(\boldsymbol{I}_m\otimes\boldsymbol{\Sigma}_\epsilon^{-1})\boldsymbol{B}_\rho\right]\boldsymbol{\delta}+\right. \\
&\qquad\qquad \left.\boldsymbol{\delta}'(\boldsymbol{\Delta}\otimes\boldsymbol{I}_S)'(\boldsymbol{I}_n\otimes\boldsymbol{V}^{-1})(\boldsymbol{z}-\boldsymbol{X}\boldsymbol{\beta})\right\}.
\end{aligned}
$$

Defining $\boldsymbol{\Omega} = (\boldsymbol{\Delta} \otimes \boldsymbol{I}_S)'(\boldsymbol{I}_n \otimes \boldsymbol{V}^{-1})(\boldsymbol{\Delta} \otimes \boldsymbol{I}_S) + \boldsymbol{B}'_\rho(\boldsymbol{I}_m \otimes \boldsymbol{\Sigma}_\epsilon^{-1})\boldsymbol{B}_\rho$, and $\boldsymbol{\eta} = (\boldsymbol{\Delta} \otimes \boldsymbol{I}_S)'(\boldsymbol{I}_n \otimes \boldsymbol{V}^{-1})(\boldsymbol{z} - \boldsymbol{X}\boldsymbol{\beta})$, the FCD of $\boldsymbol{\delta}$ can be expressed as:

$$\boldsymbol{\delta}|\boldsymbol{\beta}, \rho, \boldsymbol{\Sigma}_\epsilon, \boldsymbol{V}, \gamma, \boldsymbol{z}, \boldsymbol{y} \sim \mathrm{N}(\boldsymbol{\Omega}^{-1}\boldsymbol{\eta}, \boldsymbol{\Omega}^{-1}).$$

A.3 THE FULL CONDITIONAL DISTRIBUTION OF ρ

The FCD of ρ is expressed as follows:

$$\begin{aligned} p(\rho|\star) &\propto p(\boldsymbol{\delta}|\rho, \boldsymbol{\Sigma}_\epsilon)\pi(\rho) \\ &\propto |\boldsymbol{B}_\rho| \exp\left(-\frac{1}{2}\boldsymbol{\delta}'\boldsymbol{B}'_\rho(\boldsymbol{I}_m \otimes \boldsymbol{\Sigma}_\epsilon^{-1})\boldsymbol{B}_\rho\boldsymbol{\delta}\right) 1(\lambda_{min}^{-1} \le \rho \le \lambda_{max}^{-1}), \end{aligned}$$

Since $p(\rho|\star)$ is not a standard distribution, we can use a M-H algorithm to obtain the samples.

1) Sample ρ^{new} from the candidate density: $\rho^{new} = \rho^{old} + aG$, where a is a known constant, and G can be a standard normal distribution or t distribution. In this study, we set $G \sim \mathrm{N}(0, 1)$.

2) Since $\rho \in [\lambda_{min}^{-1}, \lambda_{max}^{-1}]$, if proposed ρ^{new} is not within the interval, it is rejected with probability 1. Otherwise, we calculate the acceptance probability $\alpha(\rho^{new}, \rho^{old}) = \min(\frac{p(\rho^{new}|\star)}{p(\rho^{old}|\star)}, 1)$.

3) Draw a uniform random deviate U. If $U < \alpha$, we set $\rho = \rho^{new}$, otherwise, $\rho = \rho^{old}$.

4) Calculate the updated ratio of ρ. Then, a is adjusted based on the updated ratio. If the updated ratio is less than 0.4, and then $a = a/1.1$; and if the updated ratio is less than 0.6, and then $a = a * 1.1$. Thus, the updated ratio is always close to 0.5.

A.4 THE FULL CONDITIONAL DISTRIBUTION OF $\boldsymbol{\Sigma}_\epsilon$

The FCD of $\boldsymbol{\Sigma}_\epsilon$ is expressed as follows:

$$\begin{aligned}
p(\boldsymbol{\Sigma}_\epsilon|\star) &\propto p(\boldsymbol{\delta}|\rho,\boldsymbol{\Sigma}_\epsilon)\pi(\boldsymbol{\Sigma}_\epsilon) \\
&\propto |\boldsymbol{\Sigma}_\epsilon|^{-m/2}\exp\left[-\frac{1}{2}\boldsymbol{\delta}'\boldsymbol{B}_\rho'(\boldsymbol{I}_m\otimes\boldsymbol{\Sigma}_\epsilon^{-1})\boldsymbol{B}_\rho\boldsymbol{\delta}\right]\times|\boldsymbol{\Sigma}_\epsilon|^{-\frac{n_\epsilon+S+1}{2}}\exp\left[-\frac{1}{2}\operatorname{tr}(\boldsymbol{Q}_\epsilon\boldsymbol{\Sigma}_\epsilon^{-1})\right] \\
&= |\boldsymbol{\Sigma}_\epsilon|^{-\frac{m+n_\epsilon+S+1}{2}}\exp\left\{-\frac{1}{2}\left[\boldsymbol{\delta}'\boldsymbol{B}_\rho'(\boldsymbol{I}_m\otimes\boldsymbol{\Sigma}_\epsilon^{-1})\boldsymbol{B}_\rho\boldsymbol{\delta}+\operatorname{tr}(\boldsymbol{Q}_\epsilon\boldsymbol{\Sigma}_\epsilon^{-1})\right]\right\} \\
&= |\boldsymbol{\Sigma}_\epsilon|^{-\frac{m+n_\epsilon+S+1}{2}}\exp\left\{-\frac{1}{2}\left[\operatorname{tr}\left((\boldsymbol{B}_\rho\boldsymbol{\delta})(\boldsymbol{B}_\rho\boldsymbol{\delta})'(\boldsymbol{I}_m\otimes\boldsymbol{\Sigma}_\epsilon^{-1})\right)+\operatorname{tr}\left(\boldsymbol{Q}_\epsilon\boldsymbol{\Sigma}_\epsilon^{-1}\right)\right]\right\}.
\end{aligned}$$

Let $\boldsymbol{B}_\rho\boldsymbol{\delta}=\boldsymbol{\varpi}=\begin{pmatrix}\boldsymbol{\varpi}_1\\ \vdots\\ \boldsymbol{\varpi}_i\\ \vdots\\ \boldsymbol{\varpi}_m\end{pmatrix}_{mS\times 1}$, with $\boldsymbol{\varpi}_i=\begin{pmatrix}\boldsymbol{\varpi}_{i1}\\ \vdots\\ \boldsymbol{\varpi}_{is}\\ \vdots\\ \boldsymbol{\varpi}_{iS}\end{pmatrix}_{S\times 1}$. Then,

$$\operatorname{tr}\left((\boldsymbol{B}_\rho\boldsymbol{\delta})(\boldsymbol{B}_\rho\boldsymbol{\delta})'(\boldsymbol{I}_m\otimes\boldsymbol{\Sigma}_\epsilon^{-1})\right)=\operatorname{tr}\left(\boldsymbol{\varpi}\boldsymbol{\varpi}'(\boldsymbol{I}_m\otimes\boldsymbol{\Sigma}_\epsilon^{-1})\right)=\operatorname{tr}\left(\sum_{i=1}^m\boldsymbol{\varpi}_i\boldsymbol{\varpi}_i'\boldsymbol{\Sigma}_\epsilon^{-1}\right).$$

Therefore,

$$\begin{aligned}
p(\boldsymbol{\Sigma}_\epsilon|\star) &\propto |\boldsymbol{\Sigma}_\epsilon|^{-\frac{m+n_\epsilon+S+1}{2}}\exp\left\{-\frac{1}{2}\left[\operatorname{tr}\left(\sum_{i=1}^m\boldsymbol{\varpi}_i\boldsymbol{\varpi}_i'\boldsymbol{\Sigma}_\epsilon^{-1}\right)+\operatorname{tr}\left(\boldsymbol{Q}_\epsilon\boldsymbol{\Sigma}_\epsilon^{-1}\right)\right]\right\} \\
&= |\boldsymbol{\Sigma}_\epsilon|^{-\frac{m+n_\epsilon+S+1}{2}}\exp\left\{-\frac{1}{2}\operatorname{tr}\left[\left(\sum_{i=1}^m\boldsymbol{\varpi}_i\boldsymbol{\varpi}_i'+\boldsymbol{Q}_\epsilon\right)\boldsymbol{\Sigma}_\epsilon^{-1}\right]\right\}.
\end{aligned}$$

Defining $\tilde{n}_\epsilon=m+n_\epsilon$, and $\tilde{\boldsymbol{Q}}_\epsilon=\sum_{i=1}^m\boldsymbol{\varpi}_i\boldsymbol{\varpi}_i'+\boldsymbol{Q}_\epsilon$, the FCD of $\boldsymbol{\Sigma}_\epsilon$ can be expressed as:

$$\boldsymbol{\Sigma}_\epsilon|\boldsymbol{\beta},\boldsymbol{\delta},\rho,\boldsymbol{V},\boldsymbol{\gamma},\boldsymbol{z},\boldsymbol{y}\sim\mathrm{IW}(\tilde{n}_\epsilon,\tilde{\boldsymbol{Q}}_\epsilon).$$

A.5 THE FULL CONDITIONAL DISTRIBUTION OF $\boldsymbol{V}$

The FCD of $\boldsymbol{V}$ is expressed as follows:

$$
\begin{aligned}
p(\boldsymbol{V}|\star) &= p(\boldsymbol{z}|\boldsymbol{\beta},\boldsymbol{\delta},\boldsymbol{V})\pi(\boldsymbol{V}) \\
&\propto \prod_{i=1}^{m}\prod_{k=1}^{n_i}\left\{|\boldsymbol{V}|^{-1/2}\exp\left[-\frac{1}{2}(\boldsymbol{z}_{ik}-\boldsymbol{X}_{ik}\boldsymbol{\beta}-\boldsymbol{\delta}_i)'\boldsymbol{V}^{-1}(\boldsymbol{z}_{ik}-\boldsymbol{X}_{ik}\boldsymbol{\beta}-\boldsymbol{\delta}_i)\right]\right\}\times \\
&\quad |\boldsymbol{V}|^{-\frac{n_0+S+1}{2}}\exp\left[-\frac{1}{2}\operatorname{tr}(\boldsymbol{Q}_0\boldsymbol{V}^{-1})\right] \\
&= |\boldsymbol{V}|^{-\frac{n+n_0+S+1}{2}}\exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{m}\sum_{k=1}^{n_i}(\boldsymbol{z}_{ik}-\boldsymbol{X}_{ik}\boldsymbol{\beta}-\boldsymbol{\delta}_i)'\boldsymbol{V}^{-1}(\boldsymbol{z}_{ik}-\boldsymbol{X}_{ik}\boldsymbol{\beta}-\boldsymbol{\delta}_i)+\operatorname{tr}(\boldsymbol{Q}_0\boldsymbol{V}^{-1})\right]\right\} \\
&= |\boldsymbol{V}|^{-\frac{n+n_0+S+1}{2}}\exp\left\{-\frac{1}{2}\operatorname{tr}\left[\left(\sum_{i=1}^{m}\sum_{k=1}^{n_i}(\boldsymbol{z}_{ik}-\boldsymbol{X}_{ik}\boldsymbol{\beta}-\boldsymbol{\delta}_i)(\boldsymbol{z}_{ik}-\boldsymbol{X}_{ik}\boldsymbol{\beta}-\boldsymbol{\delta}_i)'+\boldsymbol{Q}_0\right)\boldsymbol{V}^{-1}\right]\right\}.
\end{aligned}
$$

Defining $\tilde{n}=n+n_0$, and $\tilde{\boldsymbol{Q}}=\sum_{i=1}^{m}\sum_{k=1}^{n_i}(\boldsymbol{z}_{ik}-\boldsymbol{X}_{ik}\boldsymbol{\beta}-\boldsymbol{\delta}_i)(\boldsymbol{z}_{ik}-\boldsymbol{X}_{ik}\boldsymbol{\beta}-\boldsymbol{\delta}_i)'+\boldsymbol{Q}_0$, the FCD of $\boldsymbol{V}$ can be expressed as:

$$
\boldsymbol{V}|\boldsymbol{\beta},\boldsymbol{\delta},\rho,\boldsymbol{\Sigma}_\epsilon,\boldsymbol{\gamma},\boldsymbol{z},\boldsymbol{y}\sim \mathrm{IW}(\tilde{n},\tilde{\boldsymbol{Q}}).
$$

A.6 THE FULL CONDITIONAL DISTRIBUTION OF $\boldsymbol{z}$

Let $\boldsymbol{z}_s=(z_{11s},\ldots,z_{1n_is},\ldots,z_{m1s},\ldots,z_{mn_ms})'$ denotes the vector of the sth element z_{iks} from $\boldsymbol{z}_{ik}(i=1,\ldots,m;k=1,\ldots,n_i)$, and let $\boldsymbol{z}_{-s}$ denotes the vector removing $\boldsymbol{z}_s$ from $\boldsymbol{z}$. Similarly, let $z_{ik,-s}$ denotes the vector removing z_{iks} from $\boldsymbol{z}_{ik}$. Furthermore, let $\boldsymbol{\gamma}=(\boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_s,\ldots,\boldsymbol{\gamma}_S)'$ and $\boldsymbol{\gamma}_s=(\gamma_{s,2},\ldots,\gamma_{s,Ls-2})'$, and then $\boldsymbol{\gamma}_{-s}$ denotes the vector removing $\boldsymbol{\gamma}_s$ from $\boldsymbol{\gamma}$. We can get the joint conditional distribution of $\boldsymbol{\gamma}_s$ and $\boldsymbol{z}_s$:

$$
\begin{aligned}
&p(\boldsymbol{\gamma}_s,\boldsymbol{z}_s|\boldsymbol{\beta},\boldsymbol{\delta},\rho,\boldsymbol{\Sigma}_\epsilon,\boldsymbol{V},\boldsymbol{\gamma}_{-s},\boldsymbol{z}_{-s},\boldsymbol{y}) \\
&\quad = p(\boldsymbol{z}_s|\boldsymbol{\gamma}_s,\boldsymbol{\beta},\boldsymbol{\delta},\rho,\boldsymbol{\Sigma}_\epsilon,\boldsymbol{V},\boldsymbol{\gamma}_{-s},\boldsymbol{z}_{-s},\boldsymbol{y})\times p(\boldsymbol{\gamma}_s|\boldsymbol{\beta},\boldsymbol{\delta},\rho,\boldsymbol{\Sigma}_\epsilon,\boldsymbol{V},\boldsymbol{\gamma}_{-s},\boldsymbol{z}_{-s},\boldsymbol{y}).
\end{aligned}
$$

Furthermore, given $\boldsymbol{\beta},\boldsymbol{\delta}$ and $\boldsymbol{V}$, $\boldsymbol{z}_{ik}$ is independent of $\boldsymbol{z}_{-ik}$, and thus

$$
\begin{aligned}
p(\boldsymbol{z}_s|\boldsymbol{\gamma}_s,\boldsymbol{\beta},\boldsymbol{\delta},\rho,\boldsymbol{\Sigma}_\epsilon,\boldsymbol{V},\boldsymbol{\gamma}_{-s},\boldsymbol{z}_{-s},\boldsymbol{y}) &= \prod_{i=1}^{m}\prod_{k=1}^{n_i}p(z_{iks}|\boldsymbol{\gamma}_s,\boldsymbol{\beta},\boldsymbol{\delta},\rho,\boldsymbol{\Sigma}_\epsilon,\boldsymbol{V},\boldsymbol{\gamma}_{-s},\boldsymbol{z}_{-s},\boldsymbol{y}) \\
=\prod_{i=1}^{m}\prod_{k=1}^{n_i}p(\boldsymbol{z}_{iks}|\boldsymbol{\beta},\boldsymbol{\delta},\boldsymbol{V},\boldsymbol{\gamma},\boldsymbol{z}_{-s},\boldsymbol{y}) &= \prod_{i=1}^{m}\prod_{k=1}^{n_i}p(\boldsymbol{y}_{ik}|\gamma,\boldsymbol{z}_{ik})p(\boldsymbol{z}_{iks}|\boldsymbol{z}_{-s},\boldsymbol{\beta},\boldsymbol{\delta},\boldsymbol{V}).
\end{aligned}
$$

Defining $\boldsymbol{h}_{ik} = \boldsymbol{X}_{ik}\boldsymbol{\beta} + \boldsymbol{\delta}_i$, because $\boldsymbol{z}_{ik} \sim \mathrm{N}(\boldsymbol{X}_{ik}\boldsymbol{\beta} + \boldsymbol{\delta}_i, \boldsymbol{V})$, and then $\boldsymbol{z}_{ik} \sim \mathrm{N}(\boldsymbol{h}_{ik}, \boldsymbol{V})$. Furthermore, for the sake of convenience of expression, we reorganize $\boldsymbol{z}_{ik}$, $\boldsymbol{h}_{ik}$ and $\boldsymbol{V}$ as follows (the sth element become the first element):

$$\boldsymbol{z}_{ik} = \begin{pmatrix} z_{iks} \\ \boldsymbol{z}_{ik,-s} \end{pmatrix}, \ \boldsymbol{h}_{ik} = \begin{pmatrix} h_{iks} \\ \boldsymbol{h}_{ik,-s} \end{pmatrix}, \text{ and } \boldsymbol{V} = \begin{pmatrix} V_{s,s} & \boldsymbol{v}_{s,-s} \\ \boldsymbol{v}_{-s,s} & \boldsymbol{V}_{-s,-s} \end{pmatrix},$$

where $\boldsymbol{v}_{s,-s} = (V_{s,1}, \ldots, V_{s,s-1}, V_{s,s+1}, \ldots, V_{s,S})$ is a $1 \times (S-1)$ vector, $\boldsymbol{v}_{-s,s} = (V_{1,s}, \ldots, V_{s-1,s}, V_{s+1,s}, \ldots, V_{S}$ is an $(S-1) \times 1$ vector, and $\boldsymbol{V}_{-s,-s}$ denotes an $(S-1) \times (S-1)$ matrix removing the elements $V_{s,s}$, $\boldsymbol{v}_{-s,s}$ and $\boldsymbol{v}_{s,-s}$ from $\boldsymbol{V}$.

Based on the property of multivariate normal distribution, we can obtain the conditional probability distribution of z_{iks} given $\boldsymbol{z}_{-s}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{V}$; that is,

$$z_{iks} | \boldsymbol{z}_{-s}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{V} \sim \mathrm{N}(\tilde{h}_{iks}, \tilde{v}_{iks}), \ \text{for } i = 1, \ldots, m, \text{and } k = 1, \ldots, n_i,$$

where

$$\begin{aligned} \tilde{h}_{iks} &= h_{iks} + \boldsymbol{v}_{s,-s}\boldsymbol{V}_{-s,-s}^{-1}(\boldsymbol{z}_{ik,-s} - \boldsymbol{h}_{ik,-s}), \\ \tilde{v}_{iks} &= V_{s,s} - \boldsymbol{v}_{s,-s}\boldsymbol{V}_{-s,-s}^{-1}\boldsymbol{v}_{-s,s}. \end{aligned}$$

Thus,

$$\begin{aligned} p(z_{iks} | \boldsymbol{\gamma}_s, \boldsymbol{\beta}, \boldsymbol{\delta}, \rho, \boldsymbol{\Sigma}_\epsilon, \boldsymbol{V}, \boldsymbol{\gamma}_{-s}, \boldsymbol{z}_{-s}, \boldsymbol{y}) &\propto p(\boldsymbol{y}_{ik} | \boldsymbol{\gamma}, \boldsymbol{z}_{ik}) p(\boldsymbol{z}_{iks} | \boldsymbol{z}_{ik,-s}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{V}) \\ &\propto 1(\boldsymbol{z}_{ik} \in G_{ik}) \exp\left[-\frac{1}{2\tilde{v}_{iks}}(z_{iks} - \tilde{h}_{iks})^2\right]. \end{aligned}$$

Therefore, the FCD of z_{iks} can be expressed as:

$$z_{iks} | \boldsymbol{\beta}, \boldsymbol{\delta}, \rho, \boldsymbol{\Sigma}_\epsilon, \boldsymbol{V}, \boldsymbol{\gamma}, \boldsymbol{z}_{-s}, \boldsymbol{y} \sim \mathrm{N}(\tilde{h}_{iks}, \tilde{v}_{iks}) 1(\boldsymbol{z}_{ik} \in G_{ik}).$$

A.7 THE FULL CONDITIONAL DISTRIBUTION OF $\boldsymbol{\gamma}$

Given $\boldsymbol{\beta}, \boldsymbol{\delta}, \rho, \boldsymbol{\Sigma}_\epsilon, \boldsymbol{V}$ and $\boldsymbol{\gamma}_s$, $z_{11s}, z_{12s}, \ldots, z_{mn_ms}$ are independent, and thus

$$\begin{aligned}
&p(\boldsymbol{\gamma}_s|\boldsymbol{\beta}, \boldsymbol{\delta}, \rho, \boldsymbol{\Sigma}_\epsilon, \boldsymbol{V}, \boldsymbol{\gamma}_{-s}, \boldsymbol{z}_{-s}, \boldsymbol{y}) \propto p(\boldsymbol{y}|\boldsymbol{\beta}, \boldsymbol{\delta}, \rho, \boldsymbol{\Sigma}_\epsilon, \boldsymbol{V}, \boldsymbol{\gamma}, \boldsymbol{z}_{-s})\pi(\boldsymbol{\gamma}_s) \\
&= \prod_{i=1}^{m}\prod_{k=1}^{n_i}\sum_{l_s=2}^{L_s-1} p(y_{iks} = l_s|\boldsymbol{\beta}, \boldsymbol{\delta}, \rho, \boldsymbol{\Sigma}_\epsilon, \boldsymbol{V}, \boldsymbol{\gamma}_s, \boldsymbol{z}_{-s})1(\gamma_{s,2} < \cdots < \gamma_{s,L_s-1}) \\
&= \prod_{i=1}^{m}\prod_{k=1}^{n_i}\sum_{l_s=2}^{L_s-1} 1(y_{iks} = l_s)p(\gamma_{s,l_s-1} < z_{iks} \leq \gamma_{s,l_s}|\boldsymbol{z}_{-s}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{V})1(\gamma_{s,2} < \cdots < \gamma_{s,L_s-1}).
\end{aligned}$$

As explained before, $z_{iks}|\boldsymbol{z}_{-s}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{V} \sim \mathrm{N}(\tilde{h}_{iks}, \tilde{v}_{iks})$. Therefore,

$$p(\gamma_{s,l_s-1} < z_{iks} \leq \gamma_{s,l_s}|\boldsymbol{z}_{-s}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{V}) = \Phi\left(\frac{\gamma_{s,l_s} - \tilde{h}_{iks}}{\tilde{v}_{iks}^{1/2}}\right) - \Phi\left(\frac{\gamma_{s,l_s-1} - \tilde{h}_{iks}}{\tilde{v}_{iks}^{1/2}}\right).$$

As a result,

$$\begin{aligned}
&p(\boldsymbol{\gamma}_s|\boldsymbol{\beta}, \boldsymbol{\delta}, \rho, \boldsymbol{\Sigma}_\epsilon, \boldsymbol{V}, \boldsymbol{\gamma}_{-s}, \boldsymbol{z}_{-s}, \boldsymbol{y}) \\
&\propto \prod_{i=1}^{m}\prod_{k=1}^{n_i}\sum_{l_s=2}^{L_s-1} 1(y_{iks} = l_s)\left[\Phi\left(\frac{\gamma_{s,l_s} - \tilde{h}_{iks}}{\tilde{v}_{iks}^{1/2}}\right) - \Phi\left(\frac{\gamma_{s,l_s-1} - \tilde{h}_{iks}}{\tilde{v}_{iks}^{1/2}}\right)\right]1(\gamma_{s,2} < \cdots < \gamma_{s,L_s-1}).
\end{aligned}$$

Based on this form of $p(\boldsymbol{\gamma}|\star)$, Chen and Dey (2000) offered a better way to get the parameters $\boldsymbol{\gamma}$. This new method removes the ordering constraint by using the one-to-one map:

$$\gamma_{s,l_s} = \frac{\gamma_{s,l_s-1} + \exp(\kappa_{s,l_s})}{1 + \exp(\kappa_{s,l_s})}, 2 < l_s < L_s - 2.$$

Defining $\boldsymbol{\kappa}_s = (\kappa_{s,2}, \ldots, \kappa_{s,Ls-2})'$, and we assume the prior for $\boldsymbol{\kappa}_s$ follows a normal distribution: $\boldsymbol{\kappa}_s \sim \mathrm{N}(\boldsymbol{\kappa}_{s0}, \boldsymbol{D}_{s0})$. That is,

$$\pi(\boldsymbol{\kappa}_s) \propto \exp\left[-\frac{1}{2}(\boldsymbol{\kappa}_s - \boldsymbol{\kappa}_{s0})'\boldsymbol{D}_{s0}^{-1}(\boldsymbol{\kappa}_s - \boldsymbol{\kappa}_{s0})\right].$$

The Jacobian of transformation from $\boldsymbol{\gamma}_s$ to $\boldsymbol{\kappa}_s$ is as follows:

$$\left|\frac{\partial\boldsymbol{\gamma}_s}{\partial\boldsymbol{\kappa}_s'}\right| = \prod_{l_s=2}^{L_s-2}\left|\frac{\partial\gamma_{s,l_s}}{\partial\kappa_{s,l_s}}\right| = \prod_{l_s=2}^{L_s-2}\frac{(1 - \gamma_{s,l_s-1})\exp(\kappa_{s,l_s})}{(1 + \exp(\kappa_{s,l_s}))^2}.$$

Therefore,

$$
\begin{aligned}
p(\boldsymbol{\kappa}_s|\star) &= p(\boldsymbol{\gamma}_s|\star)\left|\frac{\partial \boldsymbol{\gamma}_s}{\partial \boldsymbol{\kappa}_s'}\right|\pi(\boldsymbol{\kappa}_s) \\
&\propto \left\{\prod_{i=1}^{m}\prod_{k=1}^{n_i}\sum_{l_s=2}^{L_s-1} 1(y_{iks}=l_s)\left[\Phi\left(\frac{\gamma_{s,l_s}-\tilde{h}_{iks}}{\tilde{v}_{iks}^{1/2}}\right)-\Phi\left(\frac{\gamma_{s,l_s-1}-\tilde{h}_{iks}}{\tilde{v}_{iks}^{1/2}}\right)\right]\right\} \\
&\times \left\{\prod_{l_s=2}^{L_s-2}\frac{(1-\gamma_{s,l_s-1})\exp(\kappa_{s,l_s})}{(1+\exp(\kappa_{s,l_s}))^2}\right\}\exp\left[-\frac{1}{2}(\boldsymbol{\kappa}_s-\boldsymbol{\kappa}_{s0})'\boldsymbol{D}_{s0}^{-1}(\boldsymbol{\kappa}_s-\boldsymbol{\kappa}_{s0})\right].
\end{aligned}
$$

Since $p(\boldsymbol{\kappa}_s|\star)$ is not a standard distribution, we can use the a M-H algorithm to generate $\boldsymbol{\gamma}_s$ as follows:

1) Sample $\boldsymbol{\kappa}_s|\star$ marginally of $\boldsymbol{z}_s$ by drawing $\boldsymbol{\kappa}_s^{new}$ from the candidate student-t distribution $f_T(\boldsymbol{\kappa}_s|\hat{\boldsymbol{\kappa}}_s, \hat{\boldsymbol{D}}_s, \hat{v})$, where $\hat{\boldsymbol{\kappa}}_s = \text{argmax} p(\boldsymbol{\kappa}_s|\star)$, $\hat{\boldsymbol{D}}_s = \left\{\left[-\frac{\partial^2 \log p(\boldsymbol{\kappa}_s|\star)}{\partial \boldsymbol{\kappa}_s \partial \boldsymbol{\kappa}_s'}\right]_{\boldsymbol{\kappa}_s=\hat{\boldsymbol{\kappa}}_s}\right\}^{-1}$, and $\hat{v}$ is the freedom.

2) Given the current $\boldsymbol{\kappa}_s^{old}$ and the proposed $\boldsymbol{\kappa}_s^{new}$, accept $\boldsymbol{\kappa}_s^{new}$ with the probability:

$$
\alpha(\boldsymbol{\kappa}_s^{old}, \boldsymbol{\kappa}_s^{new}) = \min\left(\frac{p(\boldsymbol{\kappa}_s^{new}|\star) f_T(\boldsymbol{\kappa}_s^{old}|\hat{\boldsymbol{\kappa}}_s, \hat{\boldsymbol{D}}_s, \hat{v})}{p(\boldsymbol{\kappa}_s^{old}|\star) f_T(\boldsymbol{\kappa}_s^{new}|\hat{\boldsymbol{\kappa}}_s, \hat{\boldsymbol{D}}_s, \hat{v})}, 1\right).
$$

3) Use the one-o-one map: $\gamma_{s,l_s} = \frac{\gamma_{s,l_s-1}+\exp(\kappa_{s,l_s})}{1+\exp(\kappa_{s,l_s})}, 2 < l_s < L_s - 2$ to obtain the cutpoints $\boldsymbol{\gamma}_s = (\gamma_{s,2}, \ldots, \gamma_{s,L_s-2})'$.

B BAYESIAN INFERENCE FOR USOP MODEL

B.1 MODEL SPECIFICATION FOR USOP MODEL

Following Smith and Lesage (2004), the univariate spatial ordered probit (USOP) model is expressed in the form:

$$y_{ik} = l \text{ if } \gamma_{l-1} < z_{ik} \leq \gamma_l, \;\; l = 1, \ldots, L,$$

with $\gamma_0 = -\infty < \gamma_1 < \gamma_2 < \cdots < \gamma_{L-1} < \gamma_L = \infty$.

The observed response variable y_{ik} for the latend response variable z_{ik} is expressed as:

$$\begin{aligned}
z_{ik} &= \boldsymbol{x}_{ik}'\boldsymbol{\beta} + \delta_i + \xi_{ik}, \;\; \text{for}\, i = 1, \ldots, m, \text{and } k = 1, \ldots, n_i, \\
\delta_i &= \rho \sum_{j=1}^{m} w_{ij}\delta_j + \epsilon_i, i = 1, \ldots, m, \\
\xi_{ik} &\sim \mathrm{N}(0, v), \\
\epsilon_i &\sim \mathrm{N}(0, \sigma^2).
\end{aligned}$$

Stacking all regions, then above USOP model can be summarized as:

$$\begin{aligned}
&\boldsymbol{z} = \boldsymbol{X\beta} + \boldsymbol{\Delta\delta} + \boldsymbol{\xi}, \\
&\boldsymbol{\delta} = \rho \boldsymbol{W\delta} + \boldsymbol{\epsilon}, \\
&\boldsymbol{\xi} \sim \mathrm{N}(0, \boldsymbol{V}), \;\text{with } \boldsymbol{V} = v\boldsymbol{I}_n, \\
&\boldsymbol{\epsilon} \sim \mathrm{N}(0, \sigma^2\boldsymbol{I}_m).
\end{aligned}$$

Following Albert and Chib (1993), the likelihood function of $\boldsymbol{y}$ can be expressed as follows:

$$f(\boldsymbol{y}|\boldsymbol{\beta}, \boldsymbol{\gamma}, \delta, \boldsymbol{V}) = \prod_{i=1}^{m}\prod_{k=1}^{n_i}\sum_{l=1}^{L} 1(y_{ik} = l)\left[\Phi\left(\frac{\gamma_l - \boldsymbol{x}_{ik}'\boldsymbol{\beta} - \delta_i}{{v_i}^{1/2}}\right) - \Phi\left(\frac{\gamma_{l-1} - \boldsymbol{x}_{ik}'\boldsymbol{\beta} - \delta_i}{{v_i}^{1/2}}\right)\right].$$

B.2 BAYESIAN INFERENCE

B.2.1 PRIORS OF PARAMETERS The following prior distributions of the parameters are applied in the USOP model:

$$\begin{aligned}
\boldsymbol{\beta} &\sim \mathrm{N}(\boldsymbol{c}, \boldsymbol{T}), \\
\frac{\iota}{v} &\sim \chi^2(\iota) \Rightarrow v \sim \text{Inv-}\Gamma\left(\frac{\iota}{2}, \frac{\iota}{2}\right) \textit{(be proved later)}, \\
\frac{1}{\sigma^2} &\sim \Gamma(e, g) \Rightarrow \sigma^2 \sim \text{Inv-}\Gamma(e, g), \\
\rho &\sim \mathrm{U}(\lambda_{min}^{-1}, \lambda_{max}^{-1}),
\end{aligned}$$

where λ_{min} and λ_{max} are the minimum and maximum eigenvalues of $\boldsymbol{W}$.

In addition, $\boldsymbol{\gamma}$ has the limitation of $\gamma_1 < \gamma_2 < \cdots < \gamma_{L-1}$ with $\gamma_0 = -\infty$ and $\gamma_L = \infty$, therefore,

$$\pi(\boldsymbol{\gamma}) = 1(\gamma_1 < \gamma_2 < \cdots < \gamma_{L-1}).$$

where $1(\cdot)$ denotes an indicator function.

B.2.2 POSTERIORS OF PARAMETERS

The joint posterior density function of the parameters is as follows:

$$\begin{aligned}
p(\boldsymbol{\beta}, \boldsymbol{\delta}, \rho, \sigma^2, \boldsymbol{V}, \boldsymbol{\gamma}, \boldsymbol{z}|\boldsymbol{y}) &\propto p(\boldsymbol{y}|\boldsymbol{\beta}, \boldsymbol{\delta}, \rho, \sigma^2, \boldsymbol{V}, \boldsymbol{\gamma}, \boldsymbol{z})p(\boldsymbol{\beta}, \boldsymbol{\delta}, \rho, \sigma^2, \boldsymbol{V}, \boldsymbol{\gamma}, \boldsymbol{z}) \\
&= p(\boldsymbol{y}|\boldsymbol{z}, \boldsymbol{\gamma})p(\boldsymbol{z}|\boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{V})p(\boldsymbol{\delta}|\rho, \sigma^2)\pi(\boldsymbol{\beta})\pi(\rho)\pi(\sigma^2)\pi(\boldsymbol{V})\pi(\boldsymbol{\gamma}).
\end{aligned}$$

where $p(\boldsymbol{y}|\boldsymbol{\beta}, \boldsymbol{\delta}, \rho, \mu, \sigma^2, \boldsymbol{V}, \boldsymbol{\gamma}, \boldsymbol{z}) = p(\boldsymbol{y}|\boldsymbol{z}, \boldsymbol{\gamma})$, because, given z_{ik} and $\boldsymbol{\gamma}$, the probability of y_{ik} equals 1 when $\gamma_{l-1} < z_{ik} \leq \gamma_l$ and 0 otherwise, which has no relationship with other parameters.

Thus, the density function of $\boldsymbol{y}$ given $\boldsymbol{z}$ and $\boldsymbol{\gamma}$ is:

$$p(\boldsymbol{y}|\boldsymbol{z}, \boldsymbol{\gamma}) = \prod_{i=1}^{m}\prod_{k=1}^{n_i}\left[\sum_{l=1}^{L} 1(y_{ik} = l)1(\gamma_{l-1} < z_{ik} \leq \gamma_l)\right], \tag{1}$$

where $1(\cdot)$ is an indicator function.

B.2.2.1 THE FULL CONDITIONAL DISTRIBUTION OF $\boldsymbol{\beta}$

Let $\star$ denotes all the other parameters except the given parameters, then the full conditional distribution (FCD) of $\boldsymbol{\beta}$ is expressed as follows:

$$
\begin{aligned}
p(\boldsymbol{\beta}|\star) &\propto p(\boldsymbol{z}|\boldsymbol{\beta},\boldsymbol{\delta},\boldsymbol{V})\pi(\boldsymbol{\beta}) \\
&\propto \exp\left\{-\frac{1}{2}(\boldsymbol{z}-\boldsymbol{X\beta}-\boldsymbol{\Delta\delta})'\boldsymbol{V}^{-1}(\boldsymbol{z}-\boldsymbol{X\beta}-\boldsymbol{\Delta\delta})\right\}\exp\left[-\frac{1}{2}(\boldsymbol{\beta}-\boldsymbol{c})'\boldsymbol{T}^{-1}(\boldsymbol{\beta}-\boldsymbol{c})\right] \\
&\propto \exp\left\{-\frac{1}{2}\left[\boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X\beta}-2\boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{V}^{-1}(\boldsymbol{z}-\boldsymbol{\Delta\delta})+\boldsymbol{\beta}'\boldsymbol{T}^{-1}\boldsymbol{\beta}-2\boldsymbol{\beta}'\boldsymbol{T}^{-1}\boldsymbol{c}\right]\right\} \\
&= \exp\left\{-\frac{1}{2}\boldsymbol{\beta}'(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X}+\boldsymbol{T}^{-1})\boldsymbol{\beta}+\boldsymbol{\beta}'\left[\boldsymbol{X}'\boldsymbol{V}^{-1}(\boldsymbol{z}-\boldsymbol{\Delta\delta})+\boldsymbol{T}^{-1}\boldsymbol{c}\right]\right\}.
\end{aligned}
$$

Let $\boldsymbol{B}=\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X}+\boldsymbol{T}^{-1}$, and $\boldsymbol{b}=\boldsymbol{X}'\boldsymbol{V}^{-1}(\boldsymbol{z}-\boldsymbol{\Delta\delta})+\boldsymbol{T}^{-1}\boldsymbol{c}$, then, the FCD of $\boldsymbol{\beta}$ can be expressed as:

$$\boldsymbol{\beta}|\boldsymbol{\delta},\rho,\sigma^2,\boldsymbol{V},\boldsymbol{\gamma},\boldsymbol{z},\boldsymbol{y}\sim \mathrm{N}(\boldsymbol{B}^{-1}\boldsymbol{b},\boldsymbol{B}^{-1}).$$

B.2.2.2 THE FULL CONDITIONAL DISTRIBUTION OF $\boldsymbol{\delta}$

The FCD of $\boldsymbol{\delta}$ is expressed as follows:

$$
\begin{aligned}
p(\boldsymbol{\delta}|\star) &\propto p(\boldsymbol{z}|\boldsymbol{\beta},\boldsymbol{\delta},\boldsymbol{V})p(\boldsymbol{\delta}|\rho,\sigma^2) \\
&\propto \exp\left\{-\frac{1}{2}(\boldsymbol{z}-\boldsymbol{X\beta}-\boldsymbol{\Delta\delta})'\boldsymbol{V}^{-1}(\boldsymbol{z}-\boldsymbol{X\beta}-\boldsymbol{\Delta\delta})\right\}\exp\left(-\frac{1}{2\sigma^2}\boldsymbol{\delta}'\boldsymbol{B}_\rho'\boldsymbol{B}_\rho\boldsymbol{\delta}\right) \\
&\propto \exp\left\{-\frac{1}{2}\left[\boldsymbol{\delta}'\boldsymbol{\Delta}'\boldsymbol{V}^{-1}\boldsymbol{\Delta\delta}-2\boldsymbol{\delta}'\boldsymbol{\Delta}'\boldsymbol{V}^{-1}(\boldsymbol{z}-\boldsymbol{X\beta})+\frac{1}{\sigma^2}\boldsymbol{\delta}'\boldsymbol{B}_\rho'\boldsymbol{B}_\rho\boldsymbol{\delta}\right]\right\} \\
&= \exp\left\{-\frac{1}{2}\boldsymbol{\delta}'(\boldsymbol{\Delta}'\boldsymbol{V}^{-1}\boldsymbol{\Delta}+\sigma^{-2}\boldsymbol{B}_\rho'\boldsymbol{B}_\rho)\boldsymbol{\delta}+\boldsymbol{\delta}'\boldsymbol{\Delta}'\boldsymbol{V}^{-1}(\boldsymbol{z}-\boldsymbol{X\beta})\right\}.
\end{aligned}
$$

Let $\boldsymbol{\Omega}=\boldsymbol{\Delta}'\boldsymbol{V}^{-1}\boldsymbol{\Delta}+\sigma^{-2}\boldsymbol{B}_\rho'\boldsymbol{B}_\rho$, and $\boldsymbol{\eta}=\boldsymbol{\Delta}'\boldsymbol{V}^{-1}(\boldsymbol{z}-\boldsymbol{X\beta})$, then, the FCD of $\boldsymbol{\delta}$ can be expressed as:

$$\boldsymbol{\delta}|\boldsymbol{\beta},\rho,\sigma^2,\boldsymbol{V},\boldsymbol{\gamma},\boldsymbol{z},\boldsymbol{y}\sim \mathrm{N}(\boldsymbol{\Omega}^{-1}\boldsymbol{\eta},\boldsymbol{\Omega}^{-1}).$$

B.2.2.3 THE FULL CONDITIONAL DISTRIBUTION OF ρ

The FCD of ρ is expressed as follows:

$$
\begin{aligned}
p(\rho|\star) &\propto p(\boldsymbol{\delta}|\rho,\sigma^2)\pi(\rho) \\
&\propto |\boldsymbol{I}-\rho\boldsymbol{W}|\exp\left[-\frac{1}{2\sigma^2}\boldsymbol{\delta}'(\boldsymbol{I}-\rho\boldsymbol{W})'(\boldsymbol{I}-\rho\boldsymbol{W})\boldsymbol{\delta}\right]1(\lambda_{min}^{-1}\le\rho\le\lambda_{max}^{-1}),
\end{aligned}
$$

Since $p(\rho|\star)$ is not a standard distribution, we can use the M-H algorithm to get the sample as in A.3.

B.2.2.4 THE FULL CONDITIONAL DISTRIBUTION OF σ^2

The FCD of σ^2 is expressed as follows:

$$\begin{aligned} p(\sigma^2|\star) &\propto p(\boldsymbol{\delta}|\rho,\sigma^2)\pi(\sigma^2) \\ &\propto (\sigma^2)^{-m/2}\exp\left(-\frac{1}{2\sigma^2}\boldsymbol{\delta}'\boldsymbol{B}_\rho'\boldsymbol{B}_\rho\boldsymbol{\delta}\right)(\sigma^2)^{-(e+1)}\exp\left(-\frac{g}{\sigma^2}\right) \\ &\propto (\sigma^2)^{-(m/2+e+1)}\exp\left(-\frac{\boldsymbol{\delta}'\boldsymbol{B}_\rho'\boldsymbol{B}_\rho\boldsymbol{\delta}+2g}{2\sigma^2}\right). \end{aligned}$$

Let $\theta = \frac{\boldsymbol{\delta}'\boldsymbol{B}_\rho'\boldsymbol{B}_\rho\boldsymbol{\delta}+2g}{\sigma^2}$, that is, $\sigma^2 = \frac{\boldsymbol{\delta}'\boldsymbol{B}_\rho'\boldsymbol{B}_\rho\boldsymbol{\delta}+2g}{\theta}$. Therefore, $\left|\frac{\mathrm{d}\,\sigma^2}{\mathrm{d}\,\theta}\right| = \frac{\boldsymbol{\delta}'\boldsymbol{B}_\rho'\boldsymbol{B}_\rho\boldsymbol{\delta}+2g}{\theta^2}$.

$$\begin{aligned} p(\theta|\star) &= p(\sigma^2(\theta)|\star)\left|\frac{\mathrm{d}\,\sigma^2}{\mathrm{d}\,\theta}\right| \\ &\propto (\frac{\boldsymbol{\delta}'\boldsymbol{B}_\rho'\boldsymbol{B}_\rho\boldsymbol{\delta}+2g}{\theta})^{-(m/2+e+1)}\exp\left(-\frac{\theta}{2}\right)\frac{\boldsymbol{\delta}'\boldsymbol{B}_\rho'\boldsymbol{B}_\rho\boldsymbol{\delta}+2g}{\theta^2} \\ &\propto \theta^{\frac{m+2e}{2}-1}\exp\left(-\frac{\theta}{2}\right). \end{aligned}$$

Therefore, the FCD of σ^2 can be expressed as:

$$\frac{\boldsymbol{\delta}'\boldsymbol{B}_\rho'\boldsymbol{B}_\rho\boldsymbol{\delta}+2g}{\sigma^2}|\boldsymbol{\beta},\boldsymbol{\delta},\rho,\boldsymbol{V},\boldsymbol{\gamma},\boldsymbol{z},\boldsymbol{y} \sim \chi^2(m+2e).$$

B.2.2.5 THE FULL CONDITIONAL DISTRIBUTION OF v

The FCD of v is expressed as follows:

$$\begin{aligned} p(v|\star) = p(\boldsymbol{z}|\boldsymbol{\beta},\boldsymbol{\delta},\boldsymbol{V})\pi(v) &= \left\{\prod_{i=1}^{m}\prod_{k=1}^{n_i} v^{-1/2}\exp\left[-\frac{1}{2v}(z_{ik}-\boldsymbol{x}_{ik}'\boldsymbol{\beta}-\delta_i)^2\right]\right\}\pi(v) \\ &= v^{-n/2}\exp\left[-\frac{\sum_{i=1}^{m}\sum_{k=1}^{n_i}(z_{ik}-\boldsymbol{x}_{ik}'\boldsymbol{\beta}-\delta_i)^2}{2v}\right]v^{-(\frac{\iota}{2}+1)}\exp\left(-\frac{\iota}{2v}\right) \\ &= v^{-(\frac{n+\iota}{2}+1)}\exp\left[-\frac{\sum_{i=1}^{m}\sum_{k=1}^{n_i}(z_{ik}-\boldsymbol{x}_{ik}'\boldsymbol{\beta}-\delta_i)^2+\iota}{2v}\right]. \end{aligned}$$

Let $\theta = \frac{\sum_{i=1}^{m}\sum_{k=1}^{n_i}(z_{ik}-\boldsymbol{x}'_{ik}\boldsymbol{\beta}-\delta_i)^2+\iota}{v}$, then $v = \frac{\sum_{i=1}^{m}\sum_{k=1}^{n_i}(z_{ik}-\boldsymbol{x}'_{ik}\boldsymbol{\beta}-\delta_i)^2+\iota}{\theta}$, and, $\left|\frac{\mathrm{d}\,v}{\mathrm{d}\,\theta}\right| = \frac{\sum_{i=1}^{m}\sum_{k=1}^{n_i}(z_{ik}-\boldsymbol{x}'_{ik}\boldsymbol{\beta}-\delta_i)^2+\iota}{\theta^2}$, therefore,

$$
\begin{aligned}
p(\theta|\star) &= p(v(\theta)|\star)\left|\frac{\mathrm{d}\,v}{\mathrm{d}\,\theta}\right| \\
&\propto \left(\frac{\sum_{i=1}^{m}\sum_{k=1}^{n_i}(z_{ik}-\boldsymbol{x}'_{ik}\boldsymbol{\beta}-\delta_i)^2+\iota}{\theta}\right)^{-\left(\frac{n+\iota}{2}+1\right)} \exp\left(-\frac{\theta}{2}\right)\frac{\sum_{i=1}^{m}\sum_{k=1}^{n_i}(z_{ik}-\boldsymbol{x}'_{ik}\boldsymbol{\beta}-\delta_i)^2+\iota}{\theta^2} \\
&\propto \theta^{\left(\frac{n+\iota}{2}-1\right)}\exp\left(-\frac{\theta}{2}\right).
\end{aligned}
$$

Thus, the FCD of v can be expressed as:

$$
\frac{\sum_{i=1}^{m}\sum_{k=1}^{n_i}(z_{ik}-\boldsymbol{x}'_{ik}\boldsymbol{\beta}-\delta_i)^2+\iota}{v}|\boldsymbol{\beta},\boldsymbol{\delta},\rho,\sigma^2,\boldsymbol{\gamma},\boldsymbol{z},\boldsymbol{y} \sim \chi^2(n+\iota).
$$

Here, ι is a hyperparameter. Lesage (1999) points out that for most applications, $2 \le \iota \le 7$ is appropriate. In this study, we refer to many literatures to set $\iota = 4$.

B.2.2.6 THE FULL CONDITIONAL DISTRIBUTION OF $\boldsymbol{z}$

The FCD of $\boldsymbol{z}$ is expressed as follows:

$$
\begin{aligned}
p(\boldsymbol{z}|\star) &\propto p(\boldsymbol{y}|\boldsymbol{\gamma},\boldsymbol{z})p(\boldsymbol{z}|(\boldsymbol{\beta},\boldsymbol{\delta},\boldsymbol{V})) \\
&\propto \prod_{i=1}^{m}\prod_{k=1}^{n_i}\left\{\left[\sum_{l=1}^{L}1(y_{ik}=l)1(\gamma_{l-1}<z_{ik}\le\gamma_l)\right]v^{-1/2}\exp\left[-\frac{1}{2v}(z_{ik}-\boldsymbol{x}'_{ik}\boldsymbol{\beta}-\delta_i)^2\right]\right\},
\end{aligned}
$$

that is to say,

$$
p(z_{ik}|\star) \propto \left[\sum_{l=1}^{L}1(y_{ik}=l)1(\gamma_{l-1}<z_{ik}\le\gamma_l)\right]v^{-1/2}\exp\left[-\frac{1}{2v}(z_{ik}-\boldsymbol{x}'_{ik}\boldsymbol{\beta}-\delta_i)^2\right].
$$

Therefore, the FCD of z_{ik} can be expressed as:

$$
z_{ik}|\boldsymbol{\beta},\boldsymbol{\delta},\rho,\sigma^2,\boldsymbol{V},\boldsymbol{\gamma},z_{-ik},y_{ik}=l \sim TN_{(\gamma_{l-1},\gamma_l)}(\boldsymbol{x}'_{ik}\boldsymbol{\beta}+\delta_i,v),
$$

where $TN_{(\gamma_{l-1},\gamma_l)}$ means a truncated normal distribution on the left by γ_{l-1} and on the right by γ_l.

B.2.2.7 THE FULL CONDITIONAL DISTRIBUTION OF $\boldsymbol{\gamma}$

The FCD of $\boldsymbol{\gamma}$ is expressed as follows:

$$p(\boldsymbol{\gamma}|\star) \propto f(\boldsymbol{y}|\boldsymbol{\beta}, \boldsymbol{\gamma}, \delta, \boldsymbol{V})\pi(\boldsymbol{\gamma}) = \prod_{i=1}^{m}\prod_{k=1}^{n_i}\sum_{l=2}^{L-1} 1(y_{ik}=l)\left[\Phi\left(\frac{\gamma_l - \boldsymbol{x}'_{ik}\boldsymbol{\beta} - \delta_i}{v^{1/2}}\right)\right.$$
$$\left. -\Phi\left(\frac{\gamma_{l-1} - \boldsymbol{x}'_{ik}\boldsymbol{\beta} - \delta_i}{v^{1/2}}\right)\right] 1(\gamma_2 < \cdots < \gamma_{L-2}).$$

Removes the ordering constraint by the following the one-to-one map:

$$\gamma_l = \frac{\gamma_{l-1} + \exp(\kappa_l)}{1 + \exp(\kappa_l)}, 2 < l < L-2.$$

The prior for κ_l is assumed to follow a normal distribution. Let $\boldsymbol{\kappa} = (\kappa_2, \cdots, \kappa_{L-2})'$, then, $\boldsymbol{\kappa} \sim \mathrm{N}(\boldsymbol{\kappa}_0, \boldsymbol{D}_0)$, that is,

$$\pi(\boldsymbol{\kappa}) \propto \exp\left[-\frac{1}{2}(\boldsymbol{\kappa} - \boldsymbol{\kappa}_0)'\boldsymbol{D}_0^{-1}(\boldsymbol{\kappa} - \boldsymbol{\kappa}_0)\right].$$

The Jacobian of transformation from $\boldsymbol{\gamma}$ to $\boldsymbol{\kappa}$ is as follows:

$$\left|\frac{\partial \boldsymbol{\gamma}}{\partial \boldsymbol{\kappa}'}\right| = \prod_{l=2}^{L-2}\left|\frac{\partial \gamma_l}{\partial \kappa_l}\right| = \prod_{l=2}^{L-2}\frac{(1-\gamma_{l-1})\exp(\kappa_l)}{(1+\exp(\kappa_l))^2}.$$

Therefore,

$$\begin{aligned} p(\boldsymbol{\kappa}|\star) &= p(\boldsymbol{\gamma}|\star)\left|\frac{\partial \boldsymbol{\gamma}}{\partial \boldsymbol{\kappa}'}\right|\pi(\boldsymbol{\kappa}) \\ &\propto \left\{\prod_{i=1}^{m}\prod_{k=1}^{n_i}\sum_{l=2}^{L-1} 1(y_{ik}=l)\left[\Phi\left(\frac{\gamma_l - \boldsymbol{x}'_{ik}\boldsymbol{\beta} - \delta_i}{v_i{}^{1/2}}\right) - \Phi\left(\frac{\gamma_{l-1} - \boldsymbol{x}'_{ik}\boldsymbol{\beta} - \delta_i}{v_i{}^{1/2}}\right)\right]\right\} \\ &\times \left\{\prod_{l=2}^{L-2}\frac{(1-\gamma_{l-1})\exp(\kappa_l)}{(1+\exp(\kappa_l))^2}\right\}\exp\left[-\frac{1}{2}(\boldsymbol{\kappa} - \boldsymbol{\kappa}_0)'\boldsymbol{D}_0^{-1}(\boldsymbol{\kappa} - \boldsymbol{\kappa}_0)\right]. \end{aligned}$$

Since $p(\boldsymbol{\kappa}|\star)$ is not a standard distribution, we can use the M-H algorithm to generate $\boldsymbol{\gamma}$ as in A.7.

C MARGINAL EFFECTS OF MSOP MODEL

Learning from A.7, the probability of $y_{iks} = l_s$ can be written as:

$$\Pr(y_{iks} = l_s | \boldsymbol{\beta}, \boldsymbol{\delta}, \rho, \sigma^2, \boldsymbol{V}, \boldsymbol{\gamma}_s, \boldsymbol{z}_{-s}) = \Phi\left(\frac{\gamma_{s,l_s} - \tilde{h}_{iks}}{\tilde{v}_{iks}^{1/2}}\right) - \Phi\left(\frac{\gamma_{s,l_s-1} - \tilde{h}_{iks}}{\tilde{v}_{iks}^{1/2}}\right),$$

where

$$\begin{aligned}
\tilde{h}_{iks} &= h_{iks} + \boldsymbol{v}_{s,-s}\boldsymbol{V}_{-s,-s}^{-1}(\boldsymbol{z}_{ik,-s} - \boldsymbol{h}_{ik,-s}) = \boldsymbol{x}_{iks}'\boldsymbol{\beta}_s + \delta_{is} + \boldsymbol{v}_{s,-s}\boldsymbol{V}_{-s,-s}^{-1}(\boldsymbol{z}_{ik,-s} - \boldsymbol{h}_{ik,-s}) \\
&= x_{iks1}\boldsymbol{\beta}_{s1} + \cdots + x_{iksc}\boldsymbol{\beta}_{sc} + \cdots + x_{iksq_s}\boldsymbol{\beta}_{sq_s} + \delta_{is} + \boldsymbol{v}_{s,-s}\boldsymbol{V}_{-s,-s}^{-1}(\boldsymbol{z}_{ik,-s} - \boldsymbol{h}_{ik,-s}), \\
\tilde{v}_{iks} &= V_{s,s} - \boldsymbol{v}_{s,-s}\boldsymbol{V}_{-s,-s}^{-1}\boldsymbol{v}_{-s,s}.
\end{aligned}$$

If $x_{iksc}(c = 1, \ldots, q_s)$ is a continuous variable, the marginal effect can be obtained by the derivative as follows:

$$\frac{\partial \Pr(y_{iks} = l_s | \boldsymbol{\beta}, \boldsymbol{\delta}, \rho, \sigma^2, \boldsymbol{V}, \boldsymbol{\gamma}_s, \boldsymbol{z}_{-s})}{\partial x_{iksc}} = \left[\phi\left(\frac{\gamma_{s,l_s} - \tilde{h}_{iks}}{\tilde{v}_{iks}^{1/2}}\right) - \phi\left(\frac{\gamma_{s,l_s-1} - \tilde{h}_{iks}}{\tilde{v}_{iks}^{1/2}}\right)\right] \times (-\beta_{sc}).$$

If $x_{iksc}(c = 1, \ldots, q_s)$ is a discrete variable, the marginal effect of x_{iksc} going from l_c to $l_c + 1$ can be calculated by:

$$\left[\Phi\left(\frac{\gamma_{s,l_s} - \tilde{h}_{iks,l_c+1}}{\tilde{v}_{iks}^{1/2}}\right) - \Phi\left(\frac{\gamma_{s,l_s-1} - \tilde{h}_{iks,l_c+1}}{\tilde{v}_{iks}^{1/2}}\right)\right] - \left[\Phi\left(\frac{\gamma_{s,l_s} - \tilde{h}_{iks,l_c}}{\tilde{v}_{iks}^{1/2}}\right) - \Phi\left(\frac{\gamma_{s,l_s-1} - \tilde{h}_{iks,l_c}}{\tilde{v}_{iks}^{1/2}}\right)\right],$$

where

$$\begin{aligned}
\tilde{h}_{iks,l_c+1} &= x_{iks1}\boldsymbol{\beta}_{s1} + \cdots + (l_c + 1)\boldsymbol{\beta}_{sc} + \cdots + x_{iksq_s}\boldsymbol{\beta}_{sq_s} + \delta_{is} + \boldsymbol{v}_{s,-s}\boldsymbol{V}_{-s,-s}^{-1}(\boldsymbol{z}_{ik,-s} - \boldsymbol{h}_{ik,-s}), \\
\tilde{h}_{iks,l_c} &= x_{iks1}\boldsymbol{\beta}_{s1} + \cdots + l_c\boldsymbol{\beta}_{sc} + \cdots + x_{iksq_s}\boldsymbol{\beta}_{sq_s} + \delta_{is} + \boldsymbol{v}_{s,-s}\boldsymbol{V}_{-s,-s}^{-1}(\boldsymbol{z}_{ik,-s} - \boldsymbol{h}_{ik,-s}).
\end{aligned}$$

D APPENDIX TABLES

Table A. 1: Frequency table of the balanced (bell-shaped) cases

m	n_i	$\boldsymbol{\beta}_{true}$	ρ_{true}	y1 1	y1 2	y1 3	y1 4	y1 5	y2 1	y2 2	y2 3	y2 4	y2 5
6	100	(0.55,0.35,0.41,0.56)’	0.3	40	118	212	158	72	30	111	196	163	100
			0.5	30	107	215	163	85	23	91	174	197	115
			0.7	21	67	213	168	131	12	49	166	190	183
		(0.40,0.49,0.45,0.52)’	0.3	33	124	220	150	73	30	109	201	161	99
			0.5	32	103	227	150	88	22	93	182	185	118
			0.7	22	70	202	180	126	12	50	167	187	184
	200	(0.55,0.35,0.41,0.56)’	0.3	82	248	405	321	144	69	201	407	318	205
			0.5	71	227	378	354	170	46	160	409	327	258
			0.7	42	156	355	403	244	25	97	313	384	381
		(0.40,0.49,0.45,0.52)’	0.3	77	264	407	310	142	70	202	409	311	208
			0.5	63	235	391	343	168	47	157	405	341	250
			0.7	40	181	344	389	246	24	106	305	384	381
	300	(0.55,0.35,0.41,0.56)’	0.3	122	355	649	469	205	105	306	606	498	285
			0.5	100	309	635	518	238	66	263	575	546	350
			0.7	57	241	555	586	361	34	180	455	579	552
		(0.40,0.49,0.45,0.52)’	0.3	138	349	650	463	200	102	305	615	497	281
			0.5	111	317	639	496	237	66	266	584	536	348
			0.7	65	239	567	563	366	36	175	462	582	545
10	100	(0.55,0.35,0.41,0.56)’	0.3	86	200	363	246	105	91	222	339	243	105
			0.5	107	231	361	227	74	44	178	323	296	159
			0.7	104	227	363	229	77	25	123	304	316	232
		(0.40,0.49,0.45,0.52)’	0.3	87	203	363	240	107	93	218	343	241	105
			0.5	115	224	367	222	72	47	172	328	299	154
			0.7	112	217	366	231	74	24	129	298	318	231
	200	(0.55,0.35,0.41,0.56)’	0.3	202	419	689	492	198	188	449	694	474	195
			0.5	191	401	680	504	224	179	431	698	487	205
			0.7	149	373	650	539	289	151	400	698	523	228
		(0.40,0.49,0.45,0.52)’	0.3	188	444	701	474	193	192	433	703	486	186
			0.5	175	425	690	494	216	182	421	699	499	199
			0.7	147	379	675	520	279	157	392	696	527	228
	300	(0.55,0.35,0.41,0.56)’	0.3	361	702	1026	683	228	195	599	1031	757	418
			0.5	356	693	1033	688	230	163	535	986	834	482
			0.7	339	681	1036	705	239	96	412	913	904	675
		(0.40,0.49,0.45,0.52)’	0.3	365	708	1026	707	194	197	597	1038	754	414
			0.5	357	705	1032	707	199	160	538	990	832	480
			0.7	350	685	1042	714	209	97	409	917	903	674

(to be continued)

(continued)

m	n_i	$\boldsymbol{\beta}_{true}$	ρ_{true}	y1 1	2	3	4	5	y2 1	2	3	4	5
15	100	(0.55,0.35,0.41,0.56)'	0.3	159	369	531	331	110	122	302	525	370	181
			0.5	166	378	519	327	110	110	292	516	375	207
			0.7	181	370	519	316	114	85	250	498	420	247
		(0.40,0.49,0.45,0.52)'	0.3	171	368	526	337	98	117	309	519	378	177
			0.5	179	373	514	333	101	111	286	519	387	197
			0.7	181	389	517	311	102	83	259	484	424	250
	200	(0.55,0.35,0.41,0.56)'	0.3	361	712	1035	662	230	253	636	1008	740	363
			0.5	362	725	1038	641	234	226	603	1003	766	402
			0.7	384	729	1022	628	237	170	540	946	845	499
		(0.40,0.49,0.45,0.52)'	0.3	384	711	1030	671	204	246	647	1012	738	357
			0.5	387	719	1031	653	210	226	604	999	774	397
			0.7	408	708	1038	628	218	172	537	952	842	497
	300	(0.55,0.35,0.41,0.56)'	0.3	573	1016	1556	1053	302	348	980	1538	1114	520
			0.5	588	1022	1541	1040	309	313	918	1528	1160	581
			0.7	618	1033	1528	1003	318	241	792	1462	1261	744
		(0.40,0.49,0.45,0.52)'	0.3	561	1060	1577	1001	301	358	959	1551	1117	515
			0.5	579	1073	1555	989	304	314	915	1526	1165	580
			0.7	616	1077	1532	971	304	236	798	1455	1262	749
20	100	(0.55,0.35,0.41,0.56)'	0.3	235	473	715	427	150	171	471	688	467	203
			0.5	241	481	716	415	147	167	451	684	478	220
			0.7	252	498	714	399	137	157	406	685	501	251
		(0.40,0.49,0.45,0.52)'	0.3	229	499	715	416	141	169	469	688	466	208
			0.5	235	508	712	408	137	162	449	688	483	218
			0.7	136	382	661	573	248	213	446	729	438	174
	200	(0.55,0.35,0.41,0.56)'	0.3	485	940	1404	883	288	355	930	1355	934	426
			0.5	492	956	1395	877	280	340	893	1370	940	457
			0.7	534	966	1396	842	262	306	849	1307	1025	513
		(0.40,0.49,0.45,0.52)'	0.3	493	970	1380	884	273	352	933	1374	911	430
			0.5	505	976	1381	872	266	344	899	1360	942	455
			0.7	531	991	1381	847	250	310	834	1323	1012	521
	300	(0.55,0.35,0.41,0.56)'	0.3	723	1466	2111	1268	432	562	1431	1962	1393	652
			0.5	737	1488	2096	1253	426	537	1397	1948	1432	686
			0.7	773	1548	2071	1210	398	497	1282	1955	1484	782
		(0.40,0.49,0.45,0.52)'	0.3	744	1454	2132	1265	405	568	1414	1966	1400	652
			0.5	755	1475	2111	1268	391	546	1382	1953	1428	691
			0.7	801	1506	2093	1222	378	504	1285	1946	1484	781

Table A. 2: Frequency table of the unbalanced cases

m	n_i	$\boldsymbol{\beta}_{true}$	ρ_{true}	y1 1	y1 2	y1 3	y1 4	y1 5	y2 1	y2 2	y2 3	y2 4	y2 5
6	100	(0.55,0.35,0.41,0.56)'	0.3	202	77	84	63	174	177	77	78	80	188
			0.5	202	84	79	60	175	176	68	81	82	193
			0.7	212	82	75	58	173	164	60	90	81	205
		(0.40,0.49,0.45,0.52)'	0.3	202	81	82	66	169	180	75	74	82	189
			0.5	203	82	81	64	170	178	66	81	80	195
			0.7	209	83	80	61	167	166	55	92	83	204
	200	(0.55,0.35,0.41,0.56)'	0.3	405	129	153	151	362	351	125	177	135	412
			0.5	409	128	155	149	359	343	125	169	145	418
			0.7	420	131	158	144	347	315	121	176	142	446
		(0.40,0.49,0.45,0.52)'	0.3	402	132	154	152	360	351	128	172	139	410
			0.5	404	135	154	154	353	338	132	166	144	420
			0.7	418	135	157	148	342	315	119	177	140	449
	300	(0.55,0.35,0.41,0.56)'	0.3	621	189	268	229	493	533	207	261	207	592
			0.5	631	187	265	227	490	516	199	271	202	612
			0.7	646	192	254	228	480	477	188	274	208	653
		(0.40,0.49,0.45,0.52)'	0.3	619	191	281	213	496	536	206	258	211	589
			0.5	629	187	274	221	489	514	196	274	203	613
			0.7	639	194	258	228	481	479	189	275	201	656
10	100	(0.55,0.35,0.41,0.56)'	0.3	344	109	124	120	303	283	115	122	130	350
			0.5	357	104	129	120	290	274	117	116	136	357
			0.7	373	104	131	122	270	261	105	120	123	391
		(0.40,0.49,0.45,0.52)'	0.3	348	105	125	119	303	285	114	118	134	349
			0.5	352	111	125	120	292	273	118	116	132	361
			0.7	285	101	123	133	358	293	116	117	126	348
	200	(0.55,0.35,0.41,0.56)'	0.3	685	236	268	268	543	578	239	263	229	691
			0.5	711	238	263	257	531	562	232	263	231	712
			0.7	759	242	268	250	481	531	213	260	238	758
		(0.40,0.49,0.45,0.52)'	0.3	683	250	258	271	538	575	245	261	228	691
			0.5	712	237	265	261	525	564	227	268	234	707
			0.7	753	249	265	248	485	530	217	252	239	762
	300	(0.55,0.35,0.41,0.56)'	0.3	1007	354	430	369	840	837	354	446	332	1031
			0.5	1027	361	434	361	817	809	350	438	339	1064
			0.7	1100	383	408	353	756	750	338	415	364	1133
		(0.40,0.49,0.45,0.52)'	0.3	1011	360	421	372	836	836	348	451	336	1029
			0.5	1049	343	431	365	812	812	346	435	347	1060
			0.7	1116	363	418	354	749	750	334	428	353	1135

(to be continued)

(continued)

m	n_i	$\boldsymbol{\beta}_{true}$	ρ_{true}	y1 1	y1 2	y1 3	y1 4	y1 5	y2 1	y2 2	y2 3	y2 4	y2 5
15	100	(0.55,0.35,0.41,0.56)’	0.3	535	179	206	170	410	460	163	217	186	474
			0.5	550	178	203	165	404	449	168	208	193	482
			0.7	579	185	193	158	385	428	172	200	201	499
		(0.40,0.49,0.45,0.52)’	0.3	544	173	195	182	406	453	168	216	185	478
			0.5	551	184	191	174	400	441	172	212	193	482
			0.7	594	168	201	157	380	428	169	202	202	499
	200	(0.55,0.35,0.41,0.56)’	0.3	986	375	430	365	844	865	370	437	336	992
			0.5	1029	357	434	358	822	850	370	424	345	1011
			0.7	1096	362	429	337	776	821	360	424	342	1053
		(0.40,0.49,0.45,0.52)’	0.3	1003	354	448	357	838	864	369	440	333	994
			0.5	1034	346	441	352	827	853	363	430	341	1013
			0.7	1100	357	426	340	777	824	359	424	344	1049
	300	(0.55,0.35,0.41,0.56)’	0.3	1574	526	643	496	1261	1339	572	621	551	1417
			0.5	1623	527	629	491	1230	1317	561	627	554	1441
			0.7	1712	555	615	451	1167	1271	538	639	545	1507
		(0.40,0.49,0.45,0.52)’	0.3	1598	507	635	515	1245	1339	564	628	545	1424
			0.5	1633	522	628	491	1226	1320	560	626	546	1448
			0.7	1718	559	595	472	1156	1272	545	635	540	1508
20	100	(0.55,0.35,0.41,0.56)’	0.3	679	228	304	236	553	604	241	253	242	660
			0.5	697	231	296	233	543	599	229	257	246	669
			0.7	734	230	294	238	504	582	231	252	242	693
		(0.40,0.49,0.45,0.52)’	0.3	682	229	292	252	545	599	239	257	245	660
			0.5	699	226	298	242	535	595	231	262	243	669
			0.7	736	233	288	235	508	579	230	259	238	694
	200	(0.55,0.35,0.41,0.56)’	0.3	1373	494	535	485	1113	1199	475	538	510	1278
			0.5	1406	496	533	469	1096	1177	476	547	500	1300
			0.7	1492	490	518	459	1041	1146	464	542	512	1336
		(0.40,0.49,0.45,0.52)’	0.3	1382	493	524	490	1111	1207	463	543	509	1278
			0.5	1410	510	511	481	1088	1174	479	541	511	1295
			0.7	1498	499	516	449	1038	1142	468	543	509	1338
	300	(0.55,0.35,0.41,0.56)’	0.3	2083	694	789	781	1653	1810	750	813	709	1918
			0.5	2128	696	788	780	1608	1780	743	818	715	1944
			0.7	2245	692	793	737	1533	1708	726	842	708	2016
		(0.40,0.49,0.45,0.52)’	0.3	2094	674	816	757	1659	1807	750	814	714	1915
			0.5	2148	680	804	758	1610	1780	735	827	716	1942
			0.7	2256	697	795	737	1515	1710	726	840	712	2012

Table A. 3: Simulation results of balanced cases

ρ_{true}	m	n_i	simulation results					
			β_{11}	β_{12}	β_{21}	β_{22}	ρ	MSE
			$\boldsymbol{\beta}_{true} = (0.55, 0.35, 0.41, 0.56)'$					
0.3	6	100	0.5466**	0.3848**	0.4071**	0.5611**	0.2941	0.0694
		200	0.5596**	0.3511**	0.4455**	0.5792**	0.3950	0.0798
		300	0.5499**	0.3641**	0.4103**	0.5687**	0.3234	0.1135
	10	100	0.5153**	0.3553**	0.3981**	0.5724**	0.3189	0.0932
		200	0.5332**	0.3509**	0.4218**	0.5555**	0.2962	0.1033
		300	0.5680**	0.3650**	0.4155**	0.5623**	0.3509	0.0908
	15	100	0.5146**	0.3515**	0.4190**	0.5797**	0.4141	0.0785
		200	0.5499**	0.3561**	0.4129**	0.5691**	0.4147	0.0673
		300	0.5597**	0.3404**	0.4010**	0.5563**	0.4936*	0.0954
	20	100	0.5657**	0.3277**	0.4180**	0.5631**	0.2943	0.0669
		200	0.5397**	0.3485**	0.4203**	0.5553**	0.3307	0.0492
		300	0.5554**	0.3577**	0.4284**	0.5594**	0.3111	0.0489
			$\boldsymbol{\beta}_{true} = (0.55, 0.35, 0.41, 0.56)'$					
0.5	6	100	0.5527**	0.3669**	0.3765**	0.5377**	0.4649*	0.0573
		200	0.5733**	0.3868**	0.4370**	0.5765**	0.5371**	0.0525
		300	0.5345**	0.3644**	0.4016**	0.5486**	0.5118*	0.0659
	10	100	0.5343**	0.3413**	0.4239**	0.5522**	0.4757*	0.0572
		200	0.5418**	0.3600**	0.4193**	0.5565**	0.4790	0.0809
		300	0.5745**	0.3690**	0.4178**	0.5656**	0.5302**	0.0410
	15	100	0.5189**	0.3436**	0.4332**	0.5734**	0.5545**	0.0410
		200	0.5473**	0.3659**	0.4137**	0.5664**	0.5699**	0.0370
		300	0.5557**	0.3433**	0.4000**	0.5606**	0.6273**	0.0407
	20	100	0.5561**	0.3318**	0.4229**	0.5613**	0.4713**	0.0418
		200	0.5445**	0.3551**	0.4242**	0.5535**	0.5027**	0.0332
		300	0.5549**	0.3585**	0.4272**	0.5578**	0.4195*	0.0583
			$\boldsymbol{\beta}_{true} = (0.55, 0.35, 0.41, 0.56)'$					
0.7	6	100	0.5776**	0.3796**	0.4241**	0.5158**	0.6267**	0.0625
		200	0.5630**	0.3472**	0.4386**	0.6025**	0.6693**	0.0402
		300	0.5246**	0.3492**	0.4345**	0.5566**	0.6437**	0.0589
	10	100	0.5406**	0.3540**	0.4434**	0.5394**	0.7009**	0.0261
		200	0.5461**	0.3392**	0.4065**	0.5280**	0.6636**	0.0473
		300	0.5697**	0.3548**	0.4038**	0.5541**	0.7158**	0.0276
	15	100	0.5225**	0.3271**	0.4316**	0.5734**	0.6998**	0.0216
		200	0.5443**	0.3657**	0.4026**	0.5529**	0.7350**	0.0152
		300	0.5535**	0.3373**	0.4131**	0.5627**	0.7568**	0.0150
	20	100	0.5597**	0.3293**	0.4332**	0.5729**	0.6025**	0.0417
		200	0.5418**	0.3641**	0.4232**	0.5552**	0.6362**	0.0301
		300	0.5461**	0.3490**	0.4319**	0.5593**	0.5899**	0.0385

(to be continued)

(continued)

ρ_{true}	m	n_i	simulation results					
			β_{11}	β_{12}	β_{21}	β_{22}	ρ	MSE
			$\boldsymbol{\beta}_{true} = (0.40, 0.49, 0.45, 0.52)'$					
0.3	6	100	0.3899**	0.5306**	0.4358**	0.5102**	0.3740	0.0718
		200	0.4237**	0.4859**	0.4886**	0.5458**	0.3378	0.0857
		300	0.4008**	0.5215**	0.4536**	0.5250**	0.3402	0.0739
	10	100	0.3571**	0.5097**	0.4488**	0.5346**	0.2540	0.1010
		200	0.3800**	0.4896**	0.4602**	0.5180**	0.3214	0.0941
		300	0.4094**	0.4964**	0.4444**	0.5315**	0.3817	0.0691
	15	100	0.3708**	0.4792**	0.4565**	0.5514**	0.4056	0.0674
		200	0.4122**	0.4998**	0.4517**	0.5343**	0.4491**	0.0655
		300	0.4109**	0.4815**	0.4434**	0.5180**	0.4646**	0.0680
	20	100	0.4101**	0.4717**	0.4554**	0.5128**	0.3620	0.0760
		200	0.3917**	0.4848**	0.4706**	0.5092**	0.3577	0.0579
		300	0.4001**	0.5006**	0.4780**	0.5191**	0.3109	0.0488
			$\boldsymbol{\beta}_{true} = (0.40, 0.49, 0.45, 0.52)'$					
0.5	6	100	0.4139**	0.5314**	0.4312**	0.4873**	0.4860*	0.0610
		200	0.4101**	0.5158**	0.4810**	0.5340**	0.5730**	0.0515
		300	0.4101**	0.4942**	0.4475**	0.5154**	0.5167**	0.0424
	10	100	0.3989**	0.4830**	0.4399**	0.5228**	0.5090**	0.0518
		200	0.3885**	0.5012**	0.4622**	0.5140**	0.4838	0.0970
		300	0.4096**	0.4988**	0.4391**	0.5336**	0.5253**	0.0544
	15	100	0.3674**	0.4960**	0.4655**	0.5370**	0.5435**	0.0466
		200	0.4061**	0.4999**	0.4522**	0.5286**	0.5954**	0.0342
		300	0.4000**	0.4872**	0.4371**	0.5183**	0.6268**	0.0374
	20	100	0.3981**	0.4741**	0.4474**	0.5080**	0.5231**	0.0404
		200	0.3935**	0.4864**	0.4682**	0.5216**	0.4664**	0.0436
		300	0.3962**	0.4961**	0.4758**	0.5171**	0.4461**	0.0359
			$\boldsymbol{\beta}_{true} = (0.40, 0.49, 0.45, 0.52)'$					
0.7	6	100	0.4036**	0.5387**	0.4565**	0.5019**	0.6111**	0.0594
		200	0.4015**	0.4994**	0.4768**	0.5503**	0.6664**	0.0538
		300	0.3910**	0.5035**	0.4802**	0.5134**	0.6845**	0.0321
	10	100	0.3906**	0.4969**	0.4690**	0.5042**	0.7370**	0.0218
		200	0.3988**	0.4974**	0.4434**	0.4874**	0.6970**	0.0331
		300	0.4139**	0.4823**	0.4327**	0.5205**	0.7109**	0.0217
	15	100	0.3460**	0.4740**	0.4696**	0.5475**	0.6374**	0.0639
		200	0.4045**	0.4949**	0.4461**	0.5222**	0.7449**	0.0151
		300	0.4095**	0.4751**	0.4448**	0.5269**	0.7404**	0.0210
	20	100	0.3790**	0.4757**	0.4442**	0.5141**	0.6950**	0.0186
		200	0.3910**	0.4894**	0.4669**	0.5183**	0.6345**	0.0264
		300	0.4018**	0.4911**	0.4743**	0.5229**	0.5930**	0.0361

Table A. 4: Simulation results of unbalanced cases

ρ_{true}	m	n_i	simulation results					
			β_{11}	β_{12}	β_{21}	β_{22}	ρ	MSE
			$\boldsymbol{\beta}_{true} = (0.55, 0.35, 0.41, 0.56)'$					
0.3	6	100	0.5549**	0.4049**	0.4539**	0.6121**	0.4137	0.1731
		200	0.5096**	0.3894**	0.4221**	0.6175**	0.3580	0.1527
		300	0.5131**	0.4172**	0.4864**	0.5710**	0.4098	0.1501
	10	100	0.5380**	0.2814**	0.4092**	0.5901**	0.4014	0.1938
		200	0.5262**	0.3637**	0.4302**	0.7042**	0.2770	0.1646
		300	0.5550**	0.3325**	0.4360**	0.6778**	0.2941	0.2207
	15	100	0.5307**	0.3869**	0.3671**	0.6478**	0.5110	0.1337
		200	0.5564**	0.3338**	0.4328**	0.6410**	0.2481	0.1276
		300	0.4868**	0.3783**	0.3955**	0.5330**	0.5667*	0.1418
	20	100	0.5420**	0.3702**	0.4073**	0.6758**	0.3679	0.1032
		200	0.5351**	0.3167**	0.4648**	0.5636**	0.1830	0.1614
		300	0.5718**	0.3416**	0.3857**	0.5521**	0.4408	0.0994
			$\boldsymbol{\beta}_{true} = (0.55, 0.35, 0.41, 0.56)'$					
0.5	6	100	0.5658**	0.3946**	0.4986**	0.6042**	0.4758	0.1436
		200	0.5297**	0.3770**	0.4331**	0.5976**	0.4964	0.1276
		300	0.5295**	0.4127**	0.5008**	0.5906**	0.5711*	0.0969
	10	100	0.4903**	0.2821**	0.3961**	0.5505**	0.5823*	0.1222
		200	0.5246**	0.3894**	0.4303**	0.7216**	0.3847	0.1889
		300	0.5464**	0.3230**	0.4537**	0.6989**	0.4696	0.1112
	15	100	0.5302**	0.3968**	0.3443**	0.6653**	0.5969**	0.0923
		200	0.5704**	0.3430**	0.4419**	0.6400**	0.4336	0.0927
		300	0.4891**	0.3706**	0.3813**	0.5345**	0.6848**	0.0690
	20	100	0.5331**	0.3934**	0.4089**	0.6807**	0.5004	0.1028
		200	0.5301**	0.3113**	0.4634**	0.5638**	0.3822	0.0967
		300	0.5679**	0.3500**	0.3920**	0.5470**	0.6143**	0.0483
			$\boldsymbol{\beta}_{true} = (0.55, 0.35, 0.41, 0.56)'$					
0.7	6	100	0.5893**	0.4442**	0.5287**	0.6011**	0.5904	0.1534
		200	0.5487**	0.3623**	0.4644**	0.5856**	0.6342	0.1207
		300	0.4961**	0.4233**	0.4784**	0.6212**	0.5614	0.2072
	10	100	0.4615**	0.2448**	0.3833**	0.5628**	0.6874*	0.1201
		200	0.5010**	0.3574**	0.4412**	0.7442**	0.5893*	0.1220
		300	0.5565**	0.3378**	0.4359**	0.7075**	0.6983**	0.0670
	15	100	0.5882**	0.4150**	0.3278**	0.6290**	0.7922**	0.0524
		200	0.5828**	0.3434**	0.4183**	0.6514**	0.6554**	0.0492
		300	0.4990**	0.3637**	0.3989**	0.5527**	0.7683**	0.0237
	20	100	0.5512**	0.3954**	0.4130**	0.7016**	0.6819**	0.0581
		200	0.5324**	0.3221**	0.4684**	0.5462**	0.6462**	0.0446
		300	0.5746**	0.3609**	0.3968**	0.5423**	0.7315**	0.0214

(to be continued)

(continued)

ρ_{true}	m	n_i	simulation results					
			β_{11}	β_{12}	β_{21}	β_{22}	ρ	MSE
			$\boldsymbol{\beta}_{true} = (0.40, 0.49, 0.45, 0.52)'$					
0.3	6	100	0.3816**	0.5114**	0.5127**	0.5236**	0.3503	0.1596
		200	0.3749**	0.5353**	0.4702**	0.5523**	0.4739	0.1176
		300	0.4001**	0.5371**	0.5392**	0.5262**	0.4227	0.1504
	10	100	0.3764**	0.3878**	0.4375**	0.5510**	0.3612	0.2062
		200	0.3933**	0.4915**	0.4782**	0.6491**	0.3494	0.1493
		300	0.4023**	0.4494**	0.4824**	0.6328**	0.2965	0.1371
	15	100	0.3987**	0.4817**	0.3857**	0.6240**	0.4905	0.1367
		200	0.4049**	0.4698**	0.4634**	0.6071**	0.2365	0.1220
		300	0.3499**	0.5073**	0.4373**	0.4961**	0.6428**	0.1559
	20	100	0.3706**	0.5191**	0.4424**	0.6385**	0.3080	0.1636
		200	0.4026**	0.4495**	0.4958**	0.5181**	0.1471	0.1770
		300	0.4355**	0.4998**	0.4271**	0.5083**	0.4686	0.0989
			$\boldsymbol{\beta}_{true} = (0.40, 0.49, 0.45, 0.52)'$					
0.5	6	100	0.3754**	0.5361**	0.5279**	0.5511**	0.3648	0.1610
		200	0.3670**	0.5187**	0.4662**	0.5334**	0.5538	0.1148
		300	0.3753**	0.5134**	0.5390**	0.5505**	0.4938	0.1178
	10	100	0.3780**	0.4122**	0.4358**	0.5342**	0.5939	0.1458
		200	0.3694**	0.5141**	0.4758**	0.6767**	0.4173	0.1385
		300	0.4098**	0.4701**	0.4828**	0.6364**	0.5074*	0.0811
	15	100	0.4240**	0.4960**	0.3745**	0.6120**	0.6726**	0.0892
		200	0.4193**	0.4804**	0.4689**	0.6179**	0.4172	0.0881
		300	0.3482**	0.5060**	0.4332**	0.4970**	0.6939**	0.0707
	20	100	0.3790**	0.5308**	0.4500**	0.6348**	0.5065*	0.0829
		200	0.3966**	0.4618**	0.4995**	0.5014**	0.4743*	0.0591
		300	0.4335**	0.5002**	0.4286**	0.5060**	0.6286**	0.0558
			$\boldsymbol{\beta}_{true} = (0.40, 0.49, 0.45, 0.52)'$					
0.7	6	100	0.3843**	0.5759**	0.5775**	0.5484**	0.5504	0.1729
		200	0.3977**	0.5084**	0.4941**	0.5320**	0.7807**	0.0619
		300	0.3637**	0.5492**	0.5074**	0.5657**	0.7428**	0.0693
	10	100	0.5398**	0.5316**	0.4725**	0.4128**	0.7509**	0.0935
		200	0.3582**	0.4949**	0.4847**	0.6977**	0.6271**	0.0918
		300	0.3994**	0.5046**	0.4504**	0.6408**	0.7418**	0.0530
	15	100	0.4328**	0.5636**	0.3728**	0.5997**	0.8032**	0.0540
		200	0.4198**	0.4812**	0.4580**	0.6455**	0.6281**	0.0695
		300	0.3303**	0.5009**	0.4345**	0.4991**	0.7840**	0.0268
	20	100	0.4182**	0.5266**	0.4541**	0.6537**	0.6662**	0.0611
		200	0.3802**	0.4675**	0.5198**	0.4985**	0.6478**	0.0497
		300	0.4289**	0.4952**	0.4273**	0.4924**	0.7266**	0.0240

Table A. 5: Marginal effects of the explanatory variables

		Gender	Age	Education	Marriage	CR	CD	NC
SRH=1	mean	-0.0159	0.0198	-0.0410	-0.0037	0.0445	0.1178	0.1857
	sd	0.0064	0.0045	0.0071	0.0070	0.0068	0.0058	0.0107
	median	-0.0159**	0.0197**	-0.0412**	-0.0038	0.0447**	0.1178**	0.1853**
SRH=2	mean	-0.0126	0.0155	-0.0357	-0.0029	0.0358	0.1392	0.1915
	sd	0.0051	0.0036	0.0067	0.0055	0.0056	0.0075	0.0082
	median	-0.0125**	0.0156**	-0.0357**	-0.0030	0.0358**	0.1391**	0.1913**
SRH=3	mean	-0.0008	0.0010	-0.0047	-0.0001	0.0018	0.0707	-0.1410
	sd	0.0008	0.0007	0.0023	0.0005	0.0019	0.0082	0.0122
	median	-0.0006	0.0009	-0.0046**	-0.0001	0.0017	0.0706**	-0.1408**
SRH=4	mean	0.0160	-0.0197	0.0439	0.0037	-0.0451	-0.1740	-0.1956
	sd	0.0064	0.0044	0.0079	0.0070	0.0069	0.0087	0.0081
	median	0.0159**	-0.0198**	0.0441**	0.0039	-0.0452**	-0.1737**	-0.1957**
SRH=5	mean	0.0124	-0.0156	0.0346	0.0028	-0.0357	-0.1161	-0.0922
	sd	0.0050	0.0037	0.0065	0.0055	0.0056	0.0075	0.0051
	median	0.0124**	-0.0156**	0.0346**	0.0030	-0.0358**	-0.1159**	-0.0923**
LS=1	mean	0.0022	-0.0233	-0.0258	-0.0203	0.0191	0.0087	0.0363
	sd	0.0035	0.0034	0.0039	0.0042	0.0038	0.0039	0.0060
	median	0.0022	-0.0233**	-0.0258**	-0.0202**	0.0191**	0.0088**	0.0361**
LS=2	mean	0.0021	-0.0219	-0.0254	-0.0185	0.0179	0.0083	0.0313
	sd	0.0033	0.0030	0.0039	0.0037	0.0036	0.0038	0.0046
	median	0.0021	-0.0220**	-0.0253**	-0.0184**	0.0179**	0.0084**	0.0311**
LS=3	mean	0.0030	-0.0320	-0.0418	-0.0264	0.0270	0.0127	0.0385
	sd	0.0048	0.0042	0.0070	0.0052	0.0053	0.0059	0.0048
	median	0.0031	-0.0320**	-0.0416**	-0.0263**	0.0270**	0.0128**	0.0383**
LS=4	mean	-0.0040	0.0423	0.0492	0.0373	-0.0355	-0.0158	-0.0662
	sd	0.0064	0.0057	0.0074	0.0076	0.0070	0.0071	0.0101
	median	-0.0040	0.0424**	0.0492**	0.0371**	-0.0356**	-0.0160**	-0.0660**
LS=5	mean	-0.0037	0.0390	0.0469	0.0319	-0.0312	-0.0153	-0.0492
	sd	0.0059	0.0052	0.0075	0.0062	0.0060	0.0071	0.0065
	median	-0.0037	0.0391**	0.0468**	0.0319**	-0.0314**	-0.0155**	-0.0493**

[1] SRH, LS, CR, CD and NC are the abbreviation of self-rated health, life satisfaction, Census Register, Chronic Disease and Need for Care, respectively.

[2] "$**$" and "$*$" denote that zero is not included in the 95% and 90% credible intervals, respectively. Mean, Sd and Median are posterior mean, posterior standard deviation and posterior median, respectively.