# Transformed Fay-Herriot Model with Measurement Error in Covariates

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#### Abstract

Statistical agencies are often asked to produce small area estimates (SAEs) for positively skewed variables. When domain sample sizes are too small to support direct estimators, effects of skewness of the response variable can be large. As such, it is important to appropriately account for the distribution of the response variable given available auxiliary information. Motivated by this issue and in order to stabilize the skewness and achieve normality in the response variable, we propose an area-level log-measurement error model on the response variable. Then, under our proposed modeling framework, we derive an empirical Bayes (EB) predictor of positive small area quantities subject to the covariates containing measurement error. We propose a corresponding mean squared prediction error (MSPE) of EB predictor using both a jackknife and a bootstrap method. We show that the order of the bias is  $O(m^{-1})$ , where m is the number of small areas. Finally, we investigate the performance of our methodology using both design-based and model-based simulation studies.

KEYWORDS: Small area estimation; official statistics; Bayesian methods; jackknife; parametric bootstrap; applied statistics; simulation studies.

## 1. Introduction

Typically, in small area measurement error models, both the response variable and covariate can be any real number (see Ybarra and Lohr (2008), Arima et al. (2017)). However, statistical agencies are often asked to produce small area estimates (SAEs) for skewed variables, which are also positive in  $\mathbb{R}^+$ . For instance, the Census of the Governments (CoG) provides information on roads, tolls, airports, and other similar information at the local-government level as defined by the United States Census Bureau (USCB). Another example includes the United States National Agricultural Statistics Service (NASS), which provides estimates regarding crop harvests (see Bellow and Lahiri (2011)). The United States Natural Resources

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Conservation Service (NRCS) provides estimates regarding roads at the county-level (e.g., Wang and Fuller (2003)), and the Australian Agricultural and Grazing Industries Survey provides estimates of the total expenditures of Australian farms (e.g., Chandra and Chambers (2011)).

When domain sample sizes are too small to support direct estimators, the effect of skewness can be quite large, and it is critical to account for the distribution of the response variable given auxiliary information at hand. For a review of the SAE literature, we refer to recent work by Rao and Molina (2015) and Pfefferman (2013). The case of positively skewed response variables is one such that the governing parameter in the Box-Cox transformation is zero. Due to the fact that the covariate in the model may be positively skewed and contains measurement error, this has received less attention in the literature. Throughout this paper, we explain the problem which is beyond a simple substitution and address some of its difficulties.

## 1.1. Census of the Governments

As mentioned in Sec. 1, our proposed framework is motivated by data that is positively skewed. One such data set is the Census of Governments (CoG), which is a survey data collected by the United States Census Bureau (USCB) periodically that provides comprehensive statistics about governments and governmental activities. Data is reported on government organizations, finances, and employment. For example, data from organizations refer to location, type, and characteristics of local governments and officials. Data from finances/employment refer to revenue, expenditure, debt, assets, employees, payroll, and benefits.

We utilize data from the CoG from 2007 and 2012 (https://www.census.gov/econ/overview/go0100.html). In the CoG, the small areas consist of the 48 states of the contiguous United States. These 48 areas contain 86,152 local governments defined by the USCB, such as airports, toll roads, bridges, and other federal government corporations. The parameter of interest is the average number of full-time employees per government at the state level from the 2012 data set, which can be defined as the total number of full-time employees from all local governments divided by the total number of local governments per state. The covariate of interest is the average number of full-time employees per government at the state level from the 2007 data set. After studying residual plots and histograms, we observe skewed patterns in the average number of full-time employees in both the 2007 and 2012 data sets, which partially motivate our proposed framework.

#### 1.2. Our Contribution

Motivated by issues that statistical agencies face with skewed response variables we make several contributions to the literature. In order to stabilize the skewness and achieve normality in the response variable, we propose an area-level log-measurement error model on the response variable (Eq. (1)). In addition, we propose a log-measurement error model on the covariates (Eq. (2)). Next, under our proposed modeling framework, we derive an EB predictor of positive small area quantities subject to the covariates containing measurement error. In addition, we propose a corresponding estimate of the MSPE using a jackknife

and a parametric bootstrap, where we illustrate that the order of the bias is  $O(m^{-1})$  under standard regularity conditions. We illustrate the performance of our methodology in both model-based simulation and design-based simulation studies. We summarize our conclusions and provide directions for future work.

The article is organized as follows. Sec. 1.3 details the prior work related to our proposed methodology. In Sec. 2, we propose a log-measurement error model for the response variable. In addition, we consider a measurement error model of the covariates with a log transformation. Further, we derive the EB predictor under our framework. Sec. 2.2 provides the MSPE for our EB predictor. We provide a decomposition of the MSPE to include the uncertainty of the EB predictor through unknown parameters. Sec. 3 provides two estimators of the MSPE, namely a jackknife and a parametric bootstrap, where we prove that the order of the bias is  $O(m^{-1})$  under standard regularity conditions. Sec. 4 provides both design-based and model-based simulation studies. Sec. 5 provides a discussion and directions for future work.

#### 1.3. Prior Work

In this section, we review the prior literature most relevant to our proposed work. There is a rich literature on the area-level Fay-Herriot model, where various additive measurement error models have been proposed on the covariates. Ybarra and Lohr (2008) proposed the first additive measurement error model on the covariates. More specifically, the authors considered covariate information from another survey that was independent of the response variable. More recently, Berg and Chandra (2014) have proposed an EB predictor and an approximately unbiased MSE estimator under a unit-level log-normal model, where no measurement error is assumed present in the covariates. Turning to the Bayesian literature, Arima et al. (2017), Arima et al. (2015a), and Arima et al. (2015b) have provided fully Bayesian solutions to the measurement error problem for both unit-level and area-level small area estimation problems.

Next, we discuss related literature regarding the proposed jackknife and parametric bootstrap estimator of the MSPE of the Bayes estimators, where the order of the bias is  $O(m^{-1})$ , under standard regularity conditions. Our proposed jackknife estimator of the MSPE contrasts that of Jiang et al. (2002), who proposed an MSE using an orthogonal decomposition, where the leading term in the MSE does not depend on the area-specific response and is nearly unbiased. Given that the authors can make an orthogonal decomposition, they can show that the order of the bias of the MSE is  $o(m^{-1})$ , which contrasts our proposed approach. Under our approach, the leading term depends on the area-specific response, and thus, the bias is of order  $O(m^{-1})$ . Turning to the bootstrap, we utilize methods similar to Butar and Lahiri (2003). Using this approach, we propose a parametric bootstrap estimator of the MSPE of our estimator. In a similar manner to the jackknife, the order of the bias for the parametric bootstrap estimator of the MSPE is  $O(m^{-1})$ .

## 2. Area-Level Logarithmic Model with Measurement Error

Consider m small areas and let  $Y_i$  (i = 1, ..., m) denote the population characteristic of interest in area i, where often the information of interest is a population mean or proportion. A primary survey provides a direct estimator  $y_i$  of  $Y_i$  for some or all of the m small areas. In this section, we propose a measurement error model suitable for the inference of positively skewed response variable  $y_i$ . To achieve normality in the response variable, we therefore propose an area-level log-measurement error model on  $Y_i$ . In the rest of this section, we explain our model and the desirable predictor.

Consider the following model:

$$z_i = \theta_i + e_i, \tag{1}$$

where  $z_i := \log y_i$ ,  $\theta_i := \log Y_i$ , and  $e_i$  is the sampling error distributed as  $e_i \sim N(0, \psi_i)$ . Assume

$$\theta_i = \sum_{k=1}^p \beta_k \log X_{ik} + \nu_i,$$

where  $X_{ik}$  is the k-th covariate of the i-th small area, which is unknown but is observed by  $x_{ik}$ . The regression coefficient  $\beta_k$  is unknown and must be estimated, and  $\nu_i$  is the random effect distributed as  $\nu_i \sim N(0, \sigma_{\nu}^2)$ , where  $\sigma_{\nu}^2$  is unknown.

Our measurement error model for the case of positively skewed  $X_{ik}$ 's is proposed as

$$w_{ik} := \log x_{ik} = \log X_{ik} + \eta_{ik}, \qquad k = 1, ..., p,$$

or in a vector form

$$\mathbf{w}_i = \mathbf{W}_i + \mathbf{\eta}_i, \qquad \mathbf{\eta}_i \sim N_p(\mathbf{0}, \Sigma_i),$$
 (2)

where  $\boldsymbol{w}_i = (w_{i1}, ..., w_{ip})^{\top}$  and  $\boldsymbol{W}_i = (W_{i1}, ..., W_{ip})^{\top}$  for  $W_{ik} = \log X_{ik}$ . Note that in Eq. (2),  $\boldsymbol{W}_i$  is non-stochastic within the class of functional measurement error models (c.f. Fuller (2006)). We assume  $\Sigma_i$  is known, and if it is unknown, it can be estimated using microdata or from another independent survey. We refer to Arima et al. (2017) for further details of estimating  $\Sigma_i$ .

Now, one can write

$$\begin{cases} z_i = \mathbf{W}_i^{\top} \boldsymbol{\beta} + \nu_i + \boldsymbol{\beta}^{\top} \boldsymbol{\eta}_i + e_i \\ \theta_i = \mathbf{W}_i^{\top} \boldsymbol{\beta} + \nu_i + \boldsymbol{\beta}^{\top} \boldsymbol{\eta}_i \end{cases}$$

where  $\boldsymbol{\beta} = (\beta_1, ..., \beta_p)^{\top}$ . Thus, for the pair  $(z_i, \theta_i)$ , we have the following joint normal distribution

$$\begin{pmatrix} z_i \\ \theta_i \end{pmatrix} \sim N_2 \begin{bmatrix} \begin{pmatrix} \boldsymbol{W}_i^{\top} \boldsymbol{\beta} \\ \boldsymbol{W}_i^{\top} \boldsymbol{\beta} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\beta}^{\top} \boldsymbol{\Sigma}_i \boldsymbol{\beta} + \sigma_{\nu}^2 + \psi_i & \boldsymbol{\beta}^{\top} \boldsymbol{\Sigma}_i \boldsymbol{\beta} + \sigma_{\nu}^2 \\ \boldsymbol{\beta}^{\top} \boldsymbol{\Sigma}_i \boldsymbol{\beta} + \sigma_{\nu}^2 & \boldsymbol{\beta}^{\top} \boldsymbol{\Sigma}_i \boldsymbol{\beta} + \sigma_{\nu}^2 \end{pmatrix} \right].$$

We assume all the sources of errors  $(e_i, \nu_i, \eta_i)$  for i = 1, ..., m are mutually independent throughout the rest of the paper.

**Remark 1.** Eq. (1) is a Fay-Herriot model for  $z_i$ , however, the parameter of interest is  $Y_i := \exp(\theta_i)$  rather than  $\theta_i$ . Slud and Maiti (2006) and Ghosh et al. (2015) used a similar model in the absence of measurement errors in the covariates.

Next, we give the following conditional distribution  $[\theta_i|z_i]$  to later justify our Bayesian interpretation of the unknown interested parameter  $Y_i$ :

$$\theta_i|z_i \sim N\Big[\boldsymbol{W}_i^{\top}\boldsymbol{\beta} + \frac{\boldsymbol{\beta}^{\top}\boldsymbol{\Sigma}_i\boldsymbol{\beta} + \sigma_{\nu}^2}{\boldsymbol{\beta}^{\top}\boldsymbol{\Sigma}_i\boldsymbol{\beta} + \sigma_{\nu}^2 + \psi_i}(z_i - \boldsymbol{W}_i^{\top}\boldsymbol{\beta}), \boldsymbol{\beta}^{\top}\boldsymbol{\Sigma}_i\boldsymbol{\beta} + \sigma_{\nu}^2 - \frac{(\boldsymbol{\beta}^{\top}\boldsymbol{\Sigma}_i\boldsymbol{\beta} + \sigma_{\nu}^2)^2}{\boldsymbol{\beta}^{\top}\boldsymbol{\Sigma}_i\boldsymbol{\beta} + \sigma_{\nu}^2 + \psi_i}\Big],$$

i.e.

$$\theta_i | z_i \sim N \Big( \gamma_i z_i + (1 - \gamma_i) \boldsymbol{W}_i^{\top} \boldsymbol{\beta}, \gamma_i \psi_i \Big),$$

where  $\gamma_i = (\boldsymbol{\beta}^{\top} \Sigma_i \boldsymbol{\beta} + \sigma_{\nu}^2) / (\boldsymbol{\beta}^{\top} \Sigma_i \boldsymbol{\beta} + \sigma_{\nu}^2 + \psi_i).$ 

Recall that the parameter of interest is  $Y_i := \exp(\theta_i)$  after transforming from the logarithmic scale back to the original scale. Therefore, the corresponding Bayes predictor is given by  $\hat{Y}_i := E(Y_i|z_i)$ . By using the moment generating function of the normal distribution of  $\theta_i|z_i$ , the Bayes predictor has the form of  $\hat{Y}_i = \exp\{\gamma_i z_i + (1 - \gamma_i) \boldsymbol{W}_i^{\top} \boldsymbol{\beta} + \gamma_i \psi_i/2\}$ . In practice,  $\boldsymbol{W}_i$  is unobserved, and since  $E(\boldsymbol{w}_i) = \boldsymbol{W}_i$ , we can replace it with the observed  $\boldsymbol{w}_i$ . Also,  $\boldsymbol{\beta}$  and  $\sigma_{\nu}^2$  are unknown, and we need to replace them with their consistent estimators. Therefore, the EB predictor of  $Y_i$  is

$$\hat{Y}_i^{\text{EB}} = \exp\left\{\hat{\gamma}_i z_i + (1 - \hat{\gamma}_i) \boldsymbol{w}_i^{\top} \hat{\boldsymbol{\beta}} + \frac{\hat{\gamma}_i \psi_i}{2}\right\}.$$
(3)

#### 2.1. Estimation of Unknown Parameters

In this section, we discuss estimation of the unknown parameters  $\boldsymbol{\beta}$  and  $\sigma_{\nu}^2$ . First, an estimator of  $\boldsymbol{\beta}$  is obtained by solving the equation

$$\sum_{i=1}^{m} \left[ D_i \left( \boldsymbol{w}_i \boldsymbol{w}_i^{\top} - \Sigma_i \right) \right] \boldsymbol{\beta} = \sum_{i=1}^{m} D_i \boldsymbol{w}_i z_i.$$
 (4)

The justification for Eq. (4) is as follows. Let  $\mathbf{z} = (z_1, ..., z_m)^{\top}$  and  $\mathbf{W}^{\top} = (\mathbf{W}_1, ..., \mathbf{W}_m)$ . Then,  $\mathbf{z} \sim N_m(\mathbf{W}\boldsymbol{\beta}, D^{-1})$  where  $D^{-1} = \operatorname{diag}(D_1^{-1}, ..., D_m^{-1})$  and  $D_i^{-1} = \boldsymbol{\beta}^{\top} \Sigma_i \boldsymbol{\beta} + \sigma_{\nu}^2 + \psi_i$ . Hence, an estimator of  $\boldsymbol{\beta}$  is obtained by solving

$$\boldsymbol{\beta} = \left(\boldsymbol{W}^{\top} D \boldsymbol{W}\right)^{-1} \boldsymbol{W}^{\top} D \boldsymbol{z} = \left(\sum_{i=1}^{m} D_{i} \boldsymbol{W}_{i} \boldsymbol{W}_{i}^{\top}\right)^{-1} \sum_{i=1}^{m} D_{i} \boldsymbol{W}_{i} z_{i}.$$

Now, notice that  $E(\boldsymbol{w}_i \boldsymbol{w}_i^{\top}) = \boldsymbol{W}_i \boldsymbol{W}_i^{\top} + \Sigma_i$  and  $E(\boldsymbol{w}_i) = \boldsymbol{W}_i$ . Hence, we estimate  $\boldsymbol{\beta}$  from

$$\sum_{i=1}^{m} \left[ D_i \left( \boldsymbol{w}_i \boldsymbol{w}_i^{\top} - \Sigma_i \right) \right] \boldsymbol{\beta} = \sum_{i=1}^{m} D_i \boldsymbol{w}_i z_i.$$

However,  $D_i$  is not known as both  $\boldsymbol{\beta}$  and  $\sigma_{\nu}^2$  are unknown. Take  $E(z_i - \boldsymbol{w}_i^{\top} \boldsymbol{\beta})^2 = \sigma_{\nu}^2 + \psi_i$ . Then  $\sigma_{\nu}^2$  can be estimated from

$$m^{-1} \sum_{i=1}^{m} \left( z_i - \boldsymbol{w}_i^{\top} \boldsymbol{\beta} \right)^2 - m^{-1} \sum_{i=1}^{m} \psi_i.$$
 (5)

If the above is less than zero, estimate  $\sigma_{\nu}^2$  as zero. One can estimate  $\beta$  and  $\sigma_{\nu}^2$  by iteratively solving the Eqs. (4) and (5).

## 2.2. Mean Squared Prediction Error of the EB Predictor

In this section, we first define the MSPE of the EB predictor  $\hat{Y}_i^{\text{EB}}$ . Second, we show that the cross-product term of the MSPE of the EB predictor  $\hat{Y}_i^{\text{EB}}$  is exactly zero. Now, we introduce notation that will be used throughout the rest of the paper. Let

$$M_{1i} := E[(\hat{Y}_i - Y_i)^2 | z_i]$$

$$= \exp\left\{\psi_i \gamma_i\right\} \left[\exp\left\{\psi_i \gamma_i\right\} - 1\right] \exp\left\{2\left[\gamma_i z_i + (1 - \gamma_i) \mathbf{W}_i^{\top} \boldsymbol{\beta}\right]\right\}$$

$$M_{2i} := E[(\hat{Y}_i^{\text{EB}} - \hat{Y}_i)^2 | z_i], \quad M_{3i} := E[(\hat{Y}_i^{\text{EB}} - \hat{Y}_i)(\hat{Y}_i - Y_i) | z_i].$$

Note that we estimate  $W_i$  with  $w_i$ , and the term  $M_{1i}$  depends on the area-specific response variable  $z_i$  unlike Jiang et al. (2002), and its estimator has bias of order  $O(m^{-1})$ . Since we wish to include the uncertainty of the EB predictor  $\hat{Y}_i^{\text{EB}}$  with respect to the unknown parameters  $\beta$  and  $\sigma_{\nu}^2$ , we decompose the MSPE into three terms using Definition 1.

**Definition 1.** The MSPE of the EB predictor  $\hat{Y}_i^{EB}$  is

$$MSPE(\hat{Y}_i^{EB}) = E[(\hat{Y}_i^{EB} - Y_i)^2 | z_i]$$

$$\equiv E[(\hat{Y}_i - Y_i)^2 | z_i] + E[(\hat{Y}_i^{EB} - \hat{Y}_i)^2 | z_i] + 2E[(\hat{Y}_i^{EB} - \hat{Y}_i)(\hat{Y}_i - Y_i) | z_i]$$

$$= M_{1i} + M_{2i} + 2M_{3i},$$

where we show below that  $M_{3i} = 0$ .

To show that the cross product,  $M_{3i}$  goes to 0, recall the Bayes estimator is

$$E[\hat{Y}_i] = E[Y_i \mid z_i] \implies E[\hat{Y}_i - Y_i \mid z_i] = 0.$$

Consider

$$M_{3i} = E[(\hat{Y}_i^{EB} - \hat{Y}_i)(\hat{Y}_i - Y_i)|z_i]$$
  
=  $E\{(\hat{Y}_i^{EB} - \hat{Y}_i)E((\hat{Y}_i - Y_i) | z_i)|z_i\} = 0.$ 

## 3. Jackknife and Parametric Bootstrap Estimators of the MSPE

In this section, we propose two estimators for the MSPE of the EB predictor  $\hat{Y}_i^{\text{EB}}$ . First, we propose a jackknife estimator of the MSPE. Second, we propose a parametric bootstrap estimator of the MSPE. The expectation of the proposed measure of uncertainty based on both methods is correct up to the order  $O(m^{-1})$  for the EB predictor.

### 3.1. Jackknife Estimator of the MSPE

In this section, we propose a jackknife estimator of the MSPE of the EB predictor  $\hat{Y}_i^{\text{EB}}$ , denoted by  $\text{mspe}_J(\hat{Y}_i^{\text{EB}})$ . We prove the order of the bias of  $\text{mspe}_J(\hat{Y}_i^{\text{EB}})$  is correct up to the order  $O(m^{-1})$  under six regularity conditions. We propose the following jackknife estimator:

$$mspe_J(\hat{Y}_i^{EB}) = \hat{M}_{1i,J} + \hat{M}_{2i,J} \quad \text{where}$$
(6)

$$\hat{M}_{1i,J} = \hat{M}_{1i} - \frac{m-1}{m} \sum_{j=1}^{m} (\hat{M}_{1i} - \hat{M}_{1i(-j)}) \quad \text{and} \quad \hat{M}_{2i,J} = \frac{m-1}{m} \sum_{j=1}^{m} (\hat{Y}_{i}^{EB} - \hat{Y}_{i(-j)}^{EB})^{2},$$

where (-j) denotes all areas except the j-th area. Therefore, let

$$\hat{M}_{1i} := M_{1i}(\hat{\sigma}_{\nu}^{2}, \hat{\boldsymbol{\beta}}) 
= \exp\left\{\psi_{i}\hat{\gamma}_{i}\right\} \left[\exp\left\{\psi_{i}\hat{\gamma}_{i}\right\} - 1\right] \exp\left\{2\left[\hat{\gamma}_{i}z_{i} + (1 - \hat{\gamma}_{i})\boldsymbol{w}_{i}^{\mathsf{T}}\hat{\boldsymbol{\beta}}\right]\right\}, \tag{7}$$

$$\hat{M}_{1i(-j)} = \exp\left\{\psi_{i}\hat{\gamma}_{i(-j)}\right\} \left[\exp\left\{\psi_{i}\hat{\gamma}_{i(-j)}\right\} - 1\right] \exp\left\{2\left[\hat{\gamma}_{i(-j)}z_{i} + (1 - \hat{\gamma}_{i(-j)})\boldsymbol{w}_{i}^{\mathsf{T}}\hat{\boldsymbol{\beta}}_{(-j)}\right]\right\}, \text{ and }$$

$$\hat{Y}_{i(-j)}^{\mathrm{EB}} = \exp\left\{\hat{\gamma}_{i(-j)}z_{i} + (1 - \hat{\gamma}_{i(-j)})\boldsymbol{w}_{i}^{\mathsf{T}}\hat{\boldsymbol{\beta}}_{(-j)} + \frac{\psi_{i}\hat{\gamma}_{i(-j)}}{2}\right\}.$$

Note that for all  $[.]_{(-j)}$  cases, the  $\phi = (\beta, \sigma_{\nu}^2)^{\top}$  estimators should plug into the expressions where the data is based on all the areas other than j. We define some notation and then establish six regularity conditions used in Theorem 1. Let  $\ell(\cdot|z_i)$  denote the conditional likelihood function. We define the corresponding first, second, and third derivatives of the conditional likelihood function by  $\ell'_i(\phi|z_i)$ ,  $\ell''_i(\phi|z_i)$ , and  $\ell'''_i(\phi|z_i)$ , respectively. Now, assume the following six regularity conditions:

Condition 1. Define  $\phi^{\top} = (\beta, \sigma_{\nu}^2) \in \Theta$  where  $\Theta$  is a compact set such that  $\Theta \subseteq (\mathbb{R}^p, \mathbb{R}^+)$ .

Condition 2. Assume  $\hat{\phi}$  is a consistent estimator for  $\phi$ , i.e.  $\hat{\phi} \xrightarrow{p} \phi$ .

Condition 3. Assume  $\ell'_i(\phi|z_i)$  and  $\ell''_i(\phi|z_i)$  both exist for  $i=1,\ldots m$ , almost surely in probability.

Condition 4. Assume  $E\{\ell'_i(\phi|z_i)|\phi\}=0$  for  $i=1,\ldots,m$ .

Condition 5. Assume  $\ell_i''(\phi|z_i)$  is a continuous function of  $\phi$  for i=1,...,m, almost surely in probability, where  $E\{\ell_i''(\phi|z_i)\}$  is positive definite, uniformly bounded away from 0, and is a measurable function of  $z_i$ .

Condition 6. Assume  $E\{|\ell_i'(\phi|z_i)|^{4+\delta}\}$ ,  $E\{|\ell_i''(\phi|z_i)|^{4+\delta}\}$ , and  $E\{\sup_{c\in(-\epsilon,\epsilon)}|\ell_i'''(\phi+c|z_i)|^{4+\delta}\}$  are uniformly bounded for  $i=1,\ldots m$  under some  $\epsilon>0$  and  $\delta>0$ .

**Theorem 1.** Assume Conditions 1–6 hold. Then

$$E[\mathit{mspe}_{J}(\hat{Y}_{i}^{EB})] = \mathit{MSPE}(\hat{Y}_{i}^{EB}) + O(m^{-1}).$$

*Proof.* Define

$$\begin{split} E(\text{mspe}_{J}(\hat{Y}_{i}^{\text{EB}})) &\equiv E(\hat{M}_{1i,J} + \hat{M}_{2i,J}) \\ &= E\Big(\hat{M}_{1i} - \frac{m-1}{m} \sum_{j=1}^{m} [(\hat{M}_{1i} - \hat{M}_{1i(-j)})|z_{i}]\Big) \\ &+ \frac{m-1}{m} E\Big(\sum_{i=1}^{m} [(\hat{Y}_{i}^{\text{EB}} - \hat{Y}_{i(-j)}^{\text{EB}})^{2}|z_{i}]\Big). \end{split}$$

Also, define a remainder term  $r_i$  that is bounded in absolute value by  $R_i$  such that,

$$|r_i| \le \max\{1, |\ell'(\phi|z_i)|^3, |\ell''(\phi|z_i)|^3, |\ell'''(\phi|z_i)|^3\} \equiv R_i.$$

First, we prove  $\hat{M}_{1i,J}$  has a bias of order  $O(m^{-1})$ . Using a Taylor series expansion, we find that

$$\hat{M}_{1i} = M_{1i} + M_{1i}^{'\top}(\phi)(\hat{\phi} - \phi) + \frac{1}{2}M_{1i}^{"\top}(\phi)(\hat{\phi} - \phi)^2 + \frac{1}{6}M_{1i}^{"\top}(\phi^*)(\hat{\phi} - \phi)^3,$$

for  $\phi^*$  between  $\phi$  and  $\hat{\phi}$ . Also,  $M_{1i}^{'\top}(\phi)$ ,  $M_{1i}^{''\top}(\phi)$ , and  $M_{1i}^{'''\top}(\phi^*)$  stand for the first, second, and third derivatives of  $M_{1i}$  with respect to  $\phi$ . Let  $\hat{\phi}^{\top} = (\hat{\beta}, \hat{\sigma}_{\nu}^2)$ , and it follows that

$$\hat{M}_{1i} - \hat{M}_{1i(-j)} = \hat{M}_{1i}^{'\top}(\hat{\phi})(\hat{\phi} - \hat{\phi}_{(-j)}) + \frac{1}{2}\hat{M}_{1i}^{"\top}(\hat{\phi})(\hat{\phi} - \hat{\phi}_{(-j)})^2 + \frac{1}{6}\hat{M}_{1i}^{"\top}(\hat{\phi}_{(-j)}^*)(\hat{\phi} - \hat{\phi}_{(-j)})^3,$$

for some  $\hat{\phi}_{(-j)}^*$  between  $\hat{\phi}_{(-j)}$  and  $\hat{\phi}$ . In order to approximate the solution  $\hat{\phi}$  of the equation  $f(\tau) = \sum_{i=1}^m \ell'(\tau|z_i) = 0$  in iteration  $(\xi + 1)$ , we use Householder's method (Householder (1970), Theorem 4.4.1). See also Theorem 1 of Lohr and Rao (2009):

$$\tau_{\xi+1} = \tau_{\xi} - \frac{f(\tau_{\xi})}{f'(\tau_{\xi})} \left[ 1 + \frac{\tau_{\xi} f''(\tau_{\xi})}{2\{f'(\tau_{\xi})\}^2} \right].$$

By taking the initial value  $\tau_{\xi} = \phi$ , we have

$$\hat{\phi} - \phi = -\frac{\sum_{i=1}^{m} \ell_i'(\phi|z_i)}{\sum_{i=1}^{m} \ell_i''(\phi|z_i)} \left\{ 1 + \frac{\sum_{k=1}^{m} \ell_k'(\phi|z_k) \sum_{r=1}^{m} \ell_r'''(\phi|z_r)}{2(\sum_{k=1}^{m} \ell_k''(\phi|z_k))^2} \right\} + O_p(|\hat{\phi} - \phi|^3), \quad \text{and}$$

$$\hat{\phi} - \hat{\phi}_{(-j)} = \frac{\ell'_j(\hat{\phi}|z_j)}{\sum\limits_{k \neq j}^m \ell''_k(\hat{\phi}|z_k)} \left[ 1 - \frac{\ell'_j(\hat{\phi}|z_j) \sum\limits_{k \neq j}^m \ell'''_k(\hat{\phi}|z_k)}{2(\sum\limits_{k \neq j}^m \ell''_k(\hat{\phi}|z_k))^2} \right] + O_p(|\hat{\phi} - \hat{\phi}_{(-j)}|^3).$$

By taking conditional expectation and using Theorem 2.1 of Jiang et al. (2002), we find that

$$E(\hat{\phi} - \phi | z_i) = \frac{-\ell'_i(\phi | z_i) + \varphi}{\sum_{i=1}^m E\{\ell''_i(\phi | z_i)\}} + r_i o(m^{-1}),$$

where

$$\varphi = \frac{\sum_{j=1}^{m} E[\ell'_{j}(\phi|z_{j})\ell''_{j}(\phi|z_{j})]}{\sum_{j=1}^{m} E\{\ell''_{j}(\phi|z_{j})\}} - \frac{\sum_{j=1}^{m} \sum_{k=1}^{m} E[\ell'_{j}(\phi|z_{j})]^{2} E(\ell'''_{k}(\phi|z_{k}))}{2(\sum_{j=1}^{m} E\{\ell''_{j}(\phi|z_{j})\})^{2}},$$

$$\sum_{j\neq i}^{m} E(\hat{\phi} - \hat{\phi}_{(-j)}|z_i) = \frac{-\ell'_i(\phi|z_i) + \varphi}{\sum_{j=1}^{m} E\{\ell''_j(\phi|z_j)\}} + r_i o(m^{-1}), \text{ and}$$

$$E(\hat{\phi}_{(-i)} - \hat{\phi}|z_i) = \frac{\ell'_i(\phi|z_i)}{\sum_{j=1}^{m} E\{\ell''_j(\phi|z_j)\}} + r_i o(m^{-1}).$$

By combining the above results, we find that

$$E\{\hat{M}_{1i,J} - M_{1i}|z_i\} = -M'_{1i}(\phi|z_i)\ell'_i(\phi|z_i)/\varphi + r_i o(m^{-1}).$$
Hence,  $E(\hat{M}_{1i,J}) = M_{1i} + O(m^{-1}).$ 

Second, we prove  $\hat{M}_{2i}$  has a bias of order  $o(m^{-1})$ . Let

$$\hat{Y}_i^{\text{EB}} - \hat{Y}_{i(-j)}^{\text{EB}} := h(\hat{\phi}|z_i) - h(\hat{\phi}_{(-j)}|z_i),$$

and  $h(\phi|z_i) = E(Y_i^{\text{EB}}|z_i, \phi)$ . Using a Taylor series expansion, we find that

$$\hat{Y}_{i}^{\text{EB}} - \hat{Y}_{i(-j)}^{\text{EB}} = h'^{\top}(\hat{\phi}|z_{i})(\hat{\phi} - \hat{\phi}_{(-j)}) + \frac{1}{2}h''^{\top}(\hat{\phi}_{(-j)}^{*}|z_{i})(\hat{\phi} - \hat{\phi}_{(-j)})^{2},$$

where

$$h^{'\top}(\hat{\phi}|z_i) = \left(\frac{\partial h(\hat{\phi}|z_i)}{\partial \boldsymbol{\beta}}, \frac{\partial h(\hat{\phi}|z_i)}{\partial \sigma_{\nu}^2}\right), \quad h^{''\top}(\hat{\phi}|z_i) = \left(\frac{\partial (\partial h(\hat{\phi}|z_i))}{\partial^2 \boldsymbol{\beta}}, \frac{\partial (\partial h(\hat{\phi}|z_i))}{\partial^2 \sigma_{\nu}^2}\right),$$

and  $\hat{\phi}_{(-j)}^*$  is between  $\hat{\phi}_{(-j)}$  and  $\hat{\phi}$ . Using an additional Taylor series expansion, we find that

$$\sum_{j=1}^{m} E\{(\hat{Y}_{i(-j)}^{\text{EB}} - \hat{Y}_{i}^{\text{EB}})^{2} | z_{i}\} = \{h'^{\top}(\phi|z_{i})\}^{2} \times \frac{\sum_{j=1}^{m} E\{(\ell'_{j}(\phi|z_{j}))^{2}\}}{\varphi^{2}} + r_{i} o(m^{-1}). \text{ Similarly,}$$

$$E\{(\hat{Y}_{i}^{\text{EB}} - \hat{Y}_{i})^{2} | z_{i}\} = \{h'^{\top}(\phi|z_{i})\}^{2} \times \frac{\sum_{j=1}^{m} E\{(\ell'_{j}(\phi|z_{j}))^{2}\}}{\varphi^{2}} + r_{i} o(m^{-1}).$$

By combining the above results, we find that

$$E(\hat{M}_{2i,J}) = M_{2i} + o(m^{-1}).$$

Finally,

$$E(\text{mspe}_{J}(\hat{Y}_{i}^{\text{EB}})) = E(\hat{M}_{1i,J}) + E(\hat{M}_{2i,J})$$

$$= \{M_{1i} + O(m^{-1})\} + \{M_{2i} + o(m^{-1})\}$$

$$= M_{1i} + M_{2i} + O(m^{-1}).$$

Hence,  $E[\text{mspe}_J(\hat{Y}_i^{\text{EB}})] = \text{MSPE}(\hat{Y}_i^{\text{EB}}) + O(m^{-1}).$ 

## 3.2. Parametric Bootstrap Estimator of the MSPE

In this section, we propose a parametric bootstrap estimator of the MSPE of the EB predictor  $\hat{Y}_i^{\text{EB}}$ , which we denote it by  $\text{mspe}_B(\hat{Y}_i^{\text{EB}})$ . We prove that the order of the bias is correct up to order  $O(m^{-1})$ . Specifically, we extend Butar and Lahiri (2003) to find a parametric bootstrap of our proposed EB predictor. To introduce the parametric bootstrap method, consider the following bootstrap model:

$$z_{i}^{\star}|\boldsymbol{w}_{i}^{\star}, \nu_{i}^{\star} \stackrel{ind}{\sim} N(\boldsymbol{w}_{i}^{\star\top}\hat{\boldsymbol{\beta}} + \nu_{i}^{\star}, \psi_{i})$$
$$\boldsymbol{w}_{i}^{\star} \stackrel{ind}{\sim} N_{p}(\boldsymbol{W}_{i}, \Sigma_{i})$$
$$\nu_{i}^{\star} \stackrel{ind}{\sim} N(0, \hat{\sigma}_{\nu}^{2}). \tag{8}$$

Recall that from Definition 1, MSPE( $\hat{Y}_i^{\text{EB}}$ ) =  $M_{1i} + E[(\hat{Y}_i^{\text{EB}} - \hat{Y}_i)^2 | z_i]$  since  $M_{3i} = 0$ . We use the parametric bootstrap twice. First, we use it to estimate  $M_{1i}$  in order to correct the bias of  $\hat{M}_{1i} := M_{1i}(\hat{\sigma}_{\nu}^2, \hat{\boldsymbol{\beta}})$  (see Eq. (7)). Second, we use it to estimate  $E[(\hat{Y}_i^{\text{EB}} - \hat{Y}_i)^2 | z_i]$ . More specifically, we propose to estimate  $M_{1i}$  by  $2M_{1i}(\hat{\sigma}_{\nu}^2, \hat{\boldsymbol{\beta}}) - E_{\star}[M_{1i}(\hat{\sigma}_{\nu}^{\star 2}, \hat{\boldsymbol{\beta}}^{\star})|z_i^{\star}]$ , and  $E[(\hat{Y}_i^{\text{EB}} - \hat{Y}_i)^2 | z_i]$  by  $E_{\star}[(\hat{Y}_i^{\text{EB}\star} - \hat{Y}_i^{\text{EB}})^2 | z_i^{\star}]$ , where  $E_{\star}$  denotes that the expectation is computed with respect to model in Eq. (8) and  $\hat{Y}_i^{\text{EB}\star} = \exp\{\hat{\gamma}_i^{\star} z_i + (1 - \hat{\gamma}_i^{\star}) \boldsymbol{w}_i^{\top} \hat{\boldsymbol{\beta}}^{\star} + \psi_i \hat{\gamma}_i^{\star}/2\}$ . In addition,  $\hat{\gamma}_i^{\star} = (\hat{\sigma}_{\nu}^{\star 2} + \hat{\boldsymbol{\beta}}^{\star \top} \Sigma_i \hat{\boldsymbol{\beta}}^{\star})/(\hat{\sigma}_{\nu}^{\star 2} + \hat{\boldsymbol{\beta}}^{\star \top} \Sigma_i \hat{\boldsymbol{\beta}}^{\star} + \psi_i)$ , where  $\hat{\boldsymbol{\beta}}^{\star}$  and  $\hat{\sigma}_{\nu}^{\star 2}$  are estimators of  $\boldsymbol{\beta}$  and  $\sigma_{\nu}^2$  with respect to the parametric bootstrap model in Eq. (8).

Our proposed estimator of  $MSPE(\hat{Y}_i^{EB})$  is

$$mspe_{B}(\hat{Y}_{i}^{EB}) = 2M_{1i}(\hat{\sigma}_{\nu}^{2}, \hat{\boldsymbol{\beta}}) - E_{\star}[M_{1i}(\hat{\sigma}_{\nu}^{\star 2}, \hat{\boldsymbol{\beta}}^{\star})|z_{i}^{\star}] + E_{\star}[(\hat{Y}_{i}^{EB\star} - \hat{Y}_{i}^{EB})^{2}|z_{i}^{\star}], \tag{9}$$

which has bias of order  $O(m^{-1})$  as shown in the Theorem 2.

**Theorem 2.** Assume  $E_{\star}(\hat{\sigma}_{\nu}^{\star 2} - \hat{\sigma}_{\nu}^{2}) = O_{p}(m^{-1})$  and  $E_{\star}(\hat{\beta}^{\star} - \hat{\beta}) = O_{p}(m^{-1})$ . The bootstrap estimator of the MSPE has bias of order  $O(m^{-1})$ , i.e.

$$E[mspe_B(\hat{Y}_i^{EB})] = MSPE(\hat{Y}_i^{EB}) + O(m^{-1}).$$

*Proof.* Let

$$E_{\star}[M_{1i}(\hat{\sigma}_{\nu}^{\star 2}, \hat{\boldsymbol{\beta}}^{\star})|z_{i}^{\star}] = M_{1i}(\hat{\sigma}_{\nu}^{2}, \hat{\boldsymbol{\beta}}) + O_{p}(m^{-1}).$$

Assume that  $R_m^* = O_{p^*}(m^{-1})$  such that  $mR_m^*$  is bounded in probability under the parametric bootstrap model in Eq. (8). Consider the following Taylor series expansion:

$$\hat{Y}_i^{\text{EB}\star} - \hat{Y}_i^{\text{EB}} = (\hat{\phi}^{\star} - \hat{\phi})^{\top} h'(\hat{\phi}|z_i) + R_m^{\star}$$

such that  $\hat{\phi}^{\star \top} = (\hat{\beta}^{\star}, \hat{\sigma}_{\nu}^{\star 2}).$ 

Using an argument similar to the proof of Theorem 1,

$$E_{\star}[(\hat{Y}_{i}^{\text{EB}\star} - \hat{Y}_{i}^{\text{EB}})^{2}|z_{i}^{\star}] = \hat{M}_{2i} + o_{p}(m^{-1}) \quad \text{and} \quad E_{\star}[\hat{M}_{1i}^{\star}|z_{i}^{\star}] = \hat{M}_{1i} + O_{p}(m^{-1}).$$
 (10)

Substituting Eq. (10) into Eq. (9), we find that

$$mspe_B(\hat{Y}_i^{EB}) = 2\hat{M}_{1i} - [\hat{M}_{1i} + O_p(m^{-1})] + \hat{M}_{2i} + o_p(m^{-1})$$
$$= \hat{M}_{1i} + \hat{M}_{2i} + O_p(m^{-1}).$$

This suggests that

$$E[\text{mspe}_B(\hat{Y}_i^{\text{EB}})] = \text{MSPE}(\hat{Y}_i^{\text{EB}}) + O(m^{-1}).$$

4. Experiments

In this section, we investigate the performance of the EB predictors in comparison to the direct estimators through design-based and model-based simulation studies. In addition, we evaluate the MSPE estimators using both a jackknife and parametric bootstrap.

## 4.1. Design-Based Simulation Study

In this section, we consider a design-based simulation study using the CoG data set as described in Sec. 1.1.

## 4.1.1. Design-Based Simulation Setup

We describe the design-based simulation setup. The parameter of interest is average number of full-time employees per government at the state level from 2012 data set. The covariate is the average number of full-time employees per government at the state level from the 2007 data set. There are observed skewed patterns in the average number of full-time employees in both 2007 and 2012, which motivates our proposed framework.

For the response variable, we select a total sample of 7,000 governmental units proportionally allocated to the states and for the covariates, we select a total sample of 70,000 units and the survey-weighted averages were then calculated. The measurement error variance  $\Sigma_i$  was obtained from a Taylor series approximation, where  $\operatorname{Var}(x_i)$  was estimated from the formula of variance in simple random sampling without replacement at each state. The  $\psi_i$ 's were estimated by a Generalized Variance Function (GVF) method (see Fay and Herriot (1979)). We assume the sampling variances to be known throughout the estimation procedure.

For the design-based simulation, we draw 1,000 samples and estimate the parameters from each sample. We evaluate our proposed predictors by empirical MSE per each state i:

EMSE(
$$\hat{Y}_i$$
) =  $\frac{1}{R} \sum_{r=1}^{R} \left[ \hat{Y}_i^{(r)} - Y_i^{(r)} \right]^2$ ,

where R = 1,000 is the total number of replications, and  $\hat{Y}_i$  is the estimator of  $Y_i$ . In addition, when the parametric bootstrap it used, we take B = 1,000 bootstrap samples. We use the same number of replications and bootstrap samples in the design and model-based simulation studies.

#### 4.1.2. Design-Based Simulation Results

In this section, we provide the results of the design-based simulation study.

Investigating the performance of the proposed estimators Recall that the covariate of interest is the average number of full-time employees per government at the state level from 2007 data set, and we wish to predict the average number of full-time employees per government at the state level in 2012. To do so, we give the predictors for each state as well as their corresponding EMSE's in Tables 1 and 2. More specifically, we compare the following three estimators:

- 1)  $y_i$ : the direct estimator,
- 2)  $\tilde{Y}_i$ : the EB predictor, assuming the true covariate  $w_i$  and ignoring  $\Sigma_i$  in our model,
- 3)  $\hat{Y}_i^{\text{EB}}$ : the EB predictor, assuming the true covariate  $w_i$  has measurement error, where  $\Sigma_i$  is included in our model.

We observe that in most cases the  $\mathrm{EMSE}(\hat{Y}_i^{\mathrm{EB}})$  is smaller than the  $\mathrm{EMSE}(\tilde{Y}_i)$ . However, we observe that our proposed EB predictor does not always outperform the direct estimator, which we further explore in our model-based simulation studies in Sec. 4.2.

Jackknife versus parametric bootstrap estimators Next, we consider the performance of MSPE estimators, i.e., the jackknife and bootstrap, with respect to the true MSE, i.e.,  $\text{EMSE}(\hat{Y}_i^{\text{EB}})$  in Figure 1. The results are given on the logarithmic scale, and we observe that the distribution of jackknife is closer to the distribution of true MSE when compared to the bootstrap. Therefore, we recommend the jackknife given that it slightly overestimates the true MSE. As already mentioned, given that our proposed estimator does not uniformly beat the direct estimator in terms of the EMSE, we conduct a model-based simulation study in Sec. 4.2 to investigate this and provide further insight.

## 4.2. Model-Based Simulation Study

In this section, we describe our model-based simulation study to further investigate the performance of the proposed EB predictor  $\hat{Y}_i^{\text{EB}}$ . Second, we compare the proposed jackknife and parametric bootstrap estimators,  $\text{mspe}_J(\hat{Y}_i^{\text{EB}})$  and  $\text{mspe}_B(\hat{Y}_i^{\text{EB}})$ . Third, we investigate how often the variance estimates  $\hat{\sigma}_u^2$  are zero. Finally, we investigate how the regression parameter changes when  $\Sigma_i$  is misspecified. Our goal through this model-based simulation study is to understand how one could improve the EB predictor through future research, and to further understand its underlying behavior.

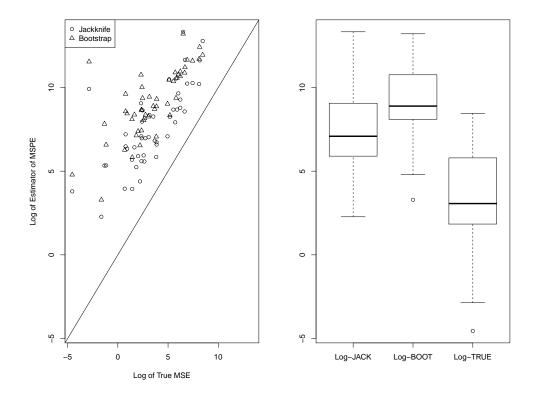


Figure 1: Left: The jackknife and the bootstrap estimators versus the true MSE (EMSE( $\hat{Y}_i^{\text{EB}}$ )), where the results are rescaled logarithmically. Right: Box plots of the jackknife and the bootstrap estimators and the true MSE (EMSE( $\hat{Y}_i^{\text{EB}}$ )), where the results are rescaled logarithmically. In general, the log of the jackknife is closer to the log of true MSE.

#### 4.2.1. Model-Based Simulation Setup

In this section, we provide the setup of our model-based simulation study in Table 3. This setup follows Eqs. (1) and (2). We are interested in comparing the following four estimators:

- 1)  $y_i$ : the direct estimator,
- 2)  $\hat{Y}_i$ : the EB predictor, assuming the true covariate  $W_i$
- 3)  $\tilde{Y}_i$ : the EB predictor, assuming the true covariate  $w_i$  and ignoring  $\Sigma_i$  in our model,
- 4)  $\hat{Y}_i^{\text{EB}}$ : the EB predictor, assuming the true covariate  $w_i$  has measurement error, where  $\Sigma_i$  is included in our model

We compare these four estimators (for each area i) using the empirical MSE:

EMSE(
$$\hat{Y}_i$$
) =  $\frac{1}{R} \sum_{r=1}^{R} \left[ \hat{Y}_i^{(r)} - Y_i^{(r)} \right]^2$ ,

where  $\hat{Y}_i$  is the estimator of  $Y_i$ .

In order to evaluate the jackknife and parametric bootstrap estimators of  $\hat{Y}_i^{\text{EB}}$ , we consider the relative bias, denoted by  $\text{RB}_J(\hat{Y}_i^{\text{EB}})$  and  $\text{RB}_B(\hat{Y}_i^{\text{EB}})$ , respectively. More specifically, the relative biases are defined as follows for each area i:

$$RB_{J}(\hat{Y}_{i}^{EB}) = \left\{ \frac{1}{R} \sum_{r=1}^{R} mspe_{J}^{(r)}(\hat{Y}_{i}^{EB(r)}) - EMSE(\hat{Y}_{i}^{EB}) \right\} / EMSE(\hat{Y}_{i}^{EB}),$$

$$RB_{B}(\hat{Y}_{i}^{EB}) = \left\{ \frac{1}{R} \sum_{r=1}^{R} mspe_{B}^{(r)}(\hat{Y}_{i}^{EB(r)}) - EMSE(\hat{Y}_{i}^{EB}) \right\} / EMSE(\hat{Y}_{i}^{EB}).$$

#### 4.2.2. Model-Based Simulation Results

In this section, we summarize our results of the model-based simulation study.

Investigating the performance of the proposed estimators. In this section, we investigate the performance of the proposed estimators. Table 4 provides the four estimators given in Sec. 4.2.1 with their empirical MSEs, where we average the results over all the small areas and re-scale them using the logarithmic scale. When k = 0, the MSE's for all EB predictors are the same since the term  $\Sigma_i$  vanishes and  $w_i$  is the same as  $W_i$ . Overall, as the value of k increases, the empirical MSE increases for almost all predictors. We observe there are cases in which the EB predictors cannot outperform the direct estimators based on the simulation results.

Table 4 illustrates that there are cases in which the EMSE( $\hat{Y}_i^{\text{EB}}$ ) is larger than the EMSE( $y_i$ ). In fact, the EB predictors cannot outperform the direct estimators due to propagated errors in the term  $\boldsymbol{\beta}^{\top}\Sigma_i\boldsymbol{\beta}$ , which is present in the term  $\gamma_i$  in the EB predictors through the simulations; see expression (3). Therefore, as the measurement error variance  $\Sigma_i$  increases, we have shown that the MSE of our proposed EB predictors can also increase. This is the main point that one should notice when using a log-model with measurement error. In order to prevent such behavior, a further adjustment should be made to the EB predictors, which we discuss in Sec. 5.

Jackknife versus parametric bootstrap estimators We compare the jackknife MSPE estimator of the EB predictor  $\hat{Y}_i^{\text{EB}}$  to that of the bootstrap using the relative bias (see Table 5). In addition, we consider box plots for the jackknife and bootstrap MSPE estimators of the EB predictor  $\hat{Y}_i^{\text{EB}}$ , where we compare these to box plots of the true values (see Figure 3). Both Table 5 and Figure 3 illustrate that the bootstrap receives a large number of negative values, which is due to the construction of  $\hat{M}_{1i}$ . Here, we find that the bootstrap grossly underestimates the true values, whereas the jackknife slightly overestimates the true values. This could be due to generating data from the normal distribution and the non-linear transformation in the model. Thus, we would recommend the jackknife in practice.

Amount of zeros for the estimates of  $\hat{\sigma}_u^2$  Here, we investigate the proportion of zero estimates for  $\sigma_\nu^2$  based on iteratively solving the Eqs. (4) and (5). Figure 2 illustrates that as the number of small areas increases, the magnitude of receiving zeros decreases. More specifically, we observe when m=20 and as k increases,  $\hat{Y}_i^{\text{EB}}$  and  $\hat{Y}_i$  tend to have a

proportion of zero estimates of  $\sigma_{\nu}^2$  between 0.3 and 0.5. When m=50 and as k increases,  $\hat{Y}_i^{\text{EB}}$  and  $\hat{Y}_i$  tend to have a proportion of zero estimates of  $\sigma_{\nu}^2$  between 0.15 and 0.4. When m=100 and as k increases,  $\hat{Y}_i^{\text{EB}}$  and  $\hat{Y}_i$  tend to have a proportion of zero estimates of  $\sigma_{\nu}^2$  between 0.05 and 0.3. When m=500 and as k increases,  $\hat{Y}_i^{\text{EB}}$  and  $\hat{Y}_i$  tend to have a proportion of zero estimates of  $\sigma_{\nu}^2$  between 0 and 0.05. One should be cautious of this in practical applications, and adjusting for this is of the interest of future work.

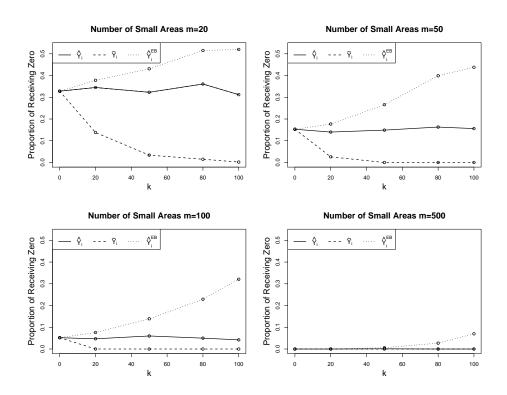


Figure 2: The proportion of zero estimates of  $\sigma_{\nu}^2$  from model-based simulation when we perform 1,000 replications of the simulation study for  $k=0,\ldots 100,\ m=20,50,100,500,$  and d=2.

The effect of misspecification of  $\Sigma_i$  on  $\beta$  We investigate the effect of mis-specifying the variance  $\Sigma_i$  on the estimation of the regression parameter  $\beta$ . To accomplish this, we conduct an empirical study based on the proposed model-based simulation study in Table 3 for the EB predictor  $\hat{Y}_i^{\text{EB}}$ . Assume  $\beta = 3$ , and we consider two sets of experiments for each value of k, which are summarized in Table 6. Recall that  $\Sigma_i \in \{0, d\}$ . Denote the first set of experiments by A1, B1, C1, D1 and E1, where we assume d = 2. Denote the misspecified value of d by  $d_{\text{mis}} = 4$ . Denote the second set of experiments by A2, B2, C2, D2 and E2, where we assume d = 4 and  $d_{\text{mis}} = 2$ . We conduct both sets of experiments for m = 20 and 500. For each experiment, we estimate the unknown parameter  $\beta$  under the followings: (1) the true value of d denoted by  $\hat{\beta}$  and (2) the misspecified value of  $d_{\text{mis}}$  denoted by  $\hat{\beta}_{\text{mis}}$ .

Then we compute the average absolute difference between the respective  $\beta$ 's by considering

the following:

$$100 \times \frac{1}{R} \sum_{r=1}^{R} \left| \hat{\beta}^{(r)} - \hat{\beta}_{\text{mis}}^{(r)} \right|.$$

In addition, we compute the magnitude of bias related to  $\hat{\beta}$  and  $\hat{\beta}_{mis}$  with respect to the true value of  $\beta = 3$  as follows

$$100 \times \frac{1}{R} \sum_{r=1}^{R} (\hat{\beta}^{(r)} - 3)$$
 and  $100 \times \frac{1}{R} \sum_{r=1}^{R} (\hat{\beta}^{(r)}_{\text{mis}} - 3)$ .

Table 6 illustrates that the overall misspecification of  $\Sigma_i$  leads to bias in  $\beta$ . When the magnitude of measurement error is zero (i.e. k=0), there is no difference between the estimated  $\beta$  using d or  $d_{\text{mis}}$ . On the other hand, when the magnitude of k increases and we have more uncertainty in the error variance  $\Sigma_i$ , values of  $\hat{\beta}$  and  $\hat{\beta}_{\text{mis}}$  diverge more from one another, and the magnitude of the bias increases. Also, we observe as the number of small areas increases, the value of bias decreases. One can resolve this bias issue by constructing an adaptive estimator for  $\hat{\beta}_{\text{mis}}$  in which its bias is corrected through some techniques such as bootstrap and develop a test of parameter specification.

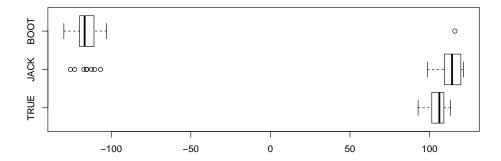


Figure 3: Comparing the distribution of the jackknife and bootstrap estimators with respect to the true MSE (EMSE( $\hat{Y}_i^{\text{EB}}$ )) from the model-based simulations. The results are logarithmically rescaled.

Table 1: Estimators and their empirical MSEs from CoG. Note that  $n_i$  is the sample size per state, and the MSEs are rescaled logarithmically.

				~	^ ED			^ ED.
i	State	$n_i$	$y_i$	$ ilde{Y}_i$	$\hat{Y}_i^{\mathrm{EB}}$	$\mathrm{EMSE}(y_i)$	$\mathrm{EMSE}(Y_i)$	$\mathrm{EMSE}(\hat{Y}_i^{\mathrm{EB}})$
1	RI	10	191.641	202.928	204.907	6.523	5.390	5.103
2	AK	14	132.301	120.112	123.684	4.501	6.153	5.793
3	NV	15	420.299	422.992	431.912	8.077	8.170	8.449
4	MD	19	824.939	784.779	794.784	8.050	5.524	6.503
5	DE	27	64.273	72.674	73.130	3.003	5.113	5.182
6	LA	40	363.838	342.056	343.344	6.017	0.845	-2.863
7	VA	40	560.881	526.612	530.160	6.669	8.265	8.148
8	NH	44	80.695	83.645	83.857	-3.994	2.254	2.387
9	UT	47	126.045	117.729	118.512	2.827	2.873	2.461
10	AZ	49	332.610	349.704	350.915	6.270	7.380	7.439
11	$\operatorname{CT}$	49	187.500	191.494	192.054	4.851	5.457	5.528
12	SC	53	231.790	221.881	222.574	5.158	6.279	6.218
13	WV	53	83.568	84.071	84.315	2.754	2.482	2.336
14	WY	55	43.084	39.629	39.795	2.493	3.873	3.824
15	VT	59	27.717	27.529	27.574	0.474	0.751	0.688
16	ME	65	38.892	40.713	40.789	2.966	1.899	1.840
17	NM	67	93.817	93.360	93.913	3.496	3.331	3.529
18	MA	70	237.066	231.213	231.999	4.218	1.741	2.310
19	TN	72	245.317	232.133	233.405	4.257	6.144	6.023
20	NC	76	372.264	357.791	359.375	5.846	6.997	6.700
21	MS	78	137.977	132.143	132.454	3.891	0.299	0.774
22	ID	92	46.450	45.642	45.784	2.451	1.910	2.016
23	AL	93	162.841	160.389	160.818	3.356	2.131	2.406
24	MT	99	23.296	22.457	22.508	0.461	1.481	1.433
25	KY	104	119.415	121.105	121.576	1.935	2.928	3.134
26	NJ	108	235.215	245.163	245.595	3.898	5.663	5.713
27	GA	109	255.141	243.862	244.539	5.686	6.696	6.648

Table 2: Estimators and their empirical MSEs from CoG. Note that  $n_i$  is the sample size per state, and the MSEs are rescaled logarithmically (continued).

i         State $n_i$ $y_i$ $\tilde{Y}_i$ $\hat{Y}_i^{EB}$ EMSE( $y_i$ )         EMSE( $\tilde{Y}_i$ )         EMSE( $\hat{Y}_i^{EB}$ )           28         AR         118         68.911         65.223         65.362         -3.160         2.719         2.646           29         OR         120         72.363         72.483         72.653         1.376         1.493         1.648           30         FL         124         377.822         365.035         366.658         7.658         8.148         8.092           31         OK         144         76.686         74.610         74.803         -2.447         1.155         0.926           32         WA         145         93.681         91.238         91.476         3.077         3.920         3.852           33         SD         153         14.757         14.078         14.142         -1.338         -3.583         -4.546           34         IA         155         56.219         55.686         55.790         -4.924         -0.962         -1.332           35         CO         184         74.373         71.579         71.908         0.789         2.907         2.746	<i>'</i>				O	·	,		
29         OR         120         72.363         72.483         72.653         1.376         1.493         1.648           30         FL         124         377.822         365.035         366.658         7.658         8.148         8.092           31         OK         144         76.686         74.610         74.803         -2.447         1.155         0.926           32         WA         145         93.681         91.238         91.476         3.077         3.920         3.852           33         SD         153         14.757         14.078         14.142         -1.338         -3.583         -4.546           34         IA         155         56.219         55.686         55.790         -4.924         -0.962         -1.332           35         CO         184         74.373         71.579         71.908         0.789         2.907         2.746           36         NE         197         33.713         31.017         31.215         1.326         -0.561         -1.167           37         IN         217         8.336         7.421         7.469         -1.704         -1.436         -1.644           49         MI	i	State	$n_i$	$y_i$	$ ilde{Y}_i$	$\hat{Y}_i^{ ext{EB}}$	$\mathrm{EMSE}(y_i)$	$\mathrm{EMSE}(\tilde{Y}_i)$	$\mathrm{EMSE}(\hat{Y}_i^{\mathrm{EB}})$
30         FL         124         377.822         365.035         366.658         7.658         8.148         8.092           31         OK         144         76.686         74.610         74.803         -2.447         1.155         0.926           32         WA         145         93.681         91.238         91.476         3.077         3.920         3.852           33         SD         153         14.757         14.078         14.142         -1.338         -3.583         -4.546           34         IA         155         56.219         55.686         55.790         -4.924         -0.962         -1.332           35         CO         184         74.373         71.579         71.908         0.789         2.907         2.746           36         NE         197         33.713         31.017         31.215         1.326         -0.561         -1.167           37         IN         217         75.994         78.712         78.830         0.619         0.609         0.775           38         ND         217         8.336         7.421         7.469         -1.704         -1.436         -1.644           49         WI	28	AR	118	68.911	65.223	65.362	-3.160	2.719	2.646
31         OK         144         76.686         74.610         74.803         -2.447         1.155         0.926           32         WA         145         93.681         91.238         91.476         3.077         3.920         3.852           33         SD         153         14.757         14.078         14.142         -1.338         -3.583         -4.546           34         IA         155         56.219         55.686         55.790         -4.924         -0.962         -1.332           35         CO         184         74.373         71.579         71.908         0.789         2.907         2.746           36         NE         197         33.713         31.017         31.215         1.326         -0.561         -1.167           37         IN         217         75.994         78.712         78.830         0.619         0.609         0.775           38         ND         217         8.336         7.421         7.469         -1.704         -1.436         -1.644           49         WI         246         61.304         61.320         61.451         2.486         2.495         2.569           41         NY <td>29</td> <td>OR</td> <td>120</td> <td>72.363</td> <td>72.483</td> <td>72.653</td> <td>1.376</td> <td>1.493</td> <td>1.648</td>	29	OR	120	72.363	72.483	72.653	1.376	1.493	1.648
32         WA         145         93.681         91.238         91.476         3.077         3.920         3.852           33         SD         153         14.757         14.078         14.142         -1.338         -3.583         -4.546           34         IA         155         56.219         55.686         55.790         -4.924         -0.962         -1.332           35         CO         184         74.373         71.579         71.908         0.789         2.907         2.746           36         NE         197         33.713         31.017         31.215         1.326         -0.561         -1.167           37         IN         217         75.994         78.712         78.830         0.619         0.609         0.775           38         ND         217         8.336         7.421         7.469         -1.704         -1.436         -1.644           39         MI         236         85.314         97.710         97.705         -1.220         4.945         4.944           40         WI         246         61.304         61.320         61.451         2.486         2.495         2.569           41         NY <td>30</td> <td><math>\operatorname{FL}</math></td> <td>124</td> <td>377.822</td> <td>365.035</td> <td>366.658</td> <td>7.658</td> <td>8.148</td> <td>8.092</td>	30	$\operatorname{FL}$	124	377.822	365.035	366.658	7.658	8.148	8.092
33         SD         153         14.757         14.078         14.142         -1.338         -3.583         -4.546           34         IA         155         56.219         55.686         55.790         -4.924         -0.962         -1.332           35         CO         184         74.373         71.579         71.908         0.789         2.907         2.746           36         NE         197         33.713         31.017         31.215         1.326         -0.561         -1.167           37         IN         217         75.994         78.712         78.830         0.619         0.609         0.775           38         ND         217         8.336         7.421         7.469         -1.704         -1.436         -1.644           39         MI         236         85.314         97.710         97.705         -1.220         4.945         4.944           40         WI         246         61.304         61.320         61.451         2.486         2.495         2.569           41         NY         270         280.982         278.784         283.973         6.457         6.275         6.681           42         MO	31	OK	144	76.686	74.610	74.803	-2.447	1.155	0.926
34         IA         155         56.219         55.686         55.790         -4.924         -0.962         -1.332           35         CO         184         74.373         71.579         71.908         0.789         2.907         2.746           36         NE         197         33.713         31.017         31.215         1.326         -0.561         -1.167           37         IN         217         75.994         78.712         78.830         0.619         0.609         0.775           38         ND         217         8.336         7.421         7.469         -1.704         -1.436         -1.644           39         MI         236         85.314         97.710         97.705         -1.220         4.945         4.944           40         WI         246         61.304         61.320         61.451         2.486         2.495         2.569           41         NY         270         280.982         278.784         283.973         6.457         6.275         6.681           42         MO         278         61.972         62.846         62.946         0.093         1.307         1.409           43         MN <td>32</td> <td>WA</td> <td>145</td> <td>93.681</td> <td>91.238</td> <td>91.476</td> <td>3.077</td> <td>3.920</td> <td>3.852</td>	32	WA	145	93.681	91.238	91.476	3.077	3.920	3.852
35         CO         184         74.373         71.579         71.908         0.789         2.907         2.746           36         NE         197         33.713         31.017         31.215         1.326         -0.561         -1.167           37         IN         217         75.994         78.712         78.830         0.619         0.609         0.775           38         ND         217         8.336         7.421         7.469         -1.704         -1.436         -1.644           39         MI         236         85.314         97.710         97.705         -1.220         4.945         4.944           40         WI         246         61.304         61.320         61.451         2.486         2.495         2.569           41         NY         270         280.982         278.784         283.973         6.457         6.275         6.681           42         MO         278         61.972         62.846         62.946         0.093         1.307         1.409           43         MN         289         42.025         45.747         45.727         -1.572         2.367         2.355           44         OH	33	SD	153	14.757	14.078	14.142	-1.338	-3.583	-4.546
36         NE         197         33.713         31.017         31.215         1.326         -0.561         -1.167           37         IN         217         75.994         78.712         78.830         0.619         0.609         0.775           38         ND         217         8.336         7.421         7.469         -1.704         -1.436         -1.644           39         MI         236         85.314         97.710         97.705         -1.220         4.945         4.944           40         WI         246         61.304         61.320         61.451         2.486         2.495         2.569           41         NY         270         280.982         278.784         283.973         6.457         6.275         6.681           42         MO         278         61.972         62.846         62.946         0.093         1.307         1.409           43         MN         289         42.025         45.747         45.727         -1.572         2.367         2.355           44         OH         302         108.261         111.756         111.905         2.364         3.820         3.864           45         KS </td <td>34</td> <td>IA</td> <td>155</td> <td>56.219</td> <td>55.686</td> <td>55.790</td> <td>-4.924</td> <td>-0.962</td> <td>-1.332</td>	34	IA	155	56.219	55.686	55.790	-4.924	-0.962	-1.332
37       IN       217       75.994       78.712       78.830       0.619       0.609       0.775         38       ND       217       8.336       7.421       7.469       -1.704       -1.436       -1.644         39       MI       236       85.314       97.710       97.705       -1.220       4.945       4.944         40       WI       246       61.304       61.320       61.451       2.486       2.495       2.569         41       NY       270       280.982       278.784       283.973       6.457       6.275       6.681         42       MO       278       61.972       62.846       62.946       0.093       1.307       1.409         43       MN       289       42.025       45.747       45.727       -1.572       2.367       2.355         44       OH       302       108.261       111.756       111.905       2.364       3.820       3.864         45       KS       309       30.804       30.253       30.321       1.859       2.253       2.208         46       CA       338       238.284       248.558       248.800       6.991       6.244       6.223	35	CO	184	74.373	71.579	71.908	0.789	2.907	2.746
38       ND       217       8.336       7.421       7.469       -1.704       -1.436       -1.644         39       MI       236       85.314       97.710       97.705       -1.220       4.945       4.944         40       WI       246       61.304       61.320       61.451       2.486       2.495       2.569         41       NY       270       280.982       278.784       283.973       6.457       6.275       6.681         42       MO       278       61.972       62.846       62.946       0.093       1.307       1.409         43       MN       289       42.025       45.747       45.727       -1.572       2.367       2.355         44       OH       302       108.261       111.756       111.905       2.364       3.820       3.864         45       KS       309       30.804       30.253       30.321       1.859       2.253       2.208         46       CA       338       238.284       248.558       248.800       6.991       6.244       6.223         47       TX       374       221.105       204.983       205.689       2.570       5.965       5.892	36	NE	197	33.713	31.017	31.215	1.326	-0.561	-1.167
39       MI       236       85.314       97.710       97.705       -1.220       4.945       4.944         40       WI       246       61.304       61.320       61.451       2.486       2.495       2.569         41       NY       270       280.982       278.784       283.973       6.457       6.275       6.681         42       MO       278       61.972       62.846       62.946       0.093       1.307       1.409         43       MN       289       42.025       45.747       45.727       -1.572       2.367       2.355         44       OH       302       108.261       111.756       111.905       2.364       3.820       3.864         45       KS       309       30.804       30.253       30.321       1.859       2.253       2.208         46       CA       338       238.284       248.558       248.800       6.991       6.244       6.223         47       TX       374       221.105       204.983       205.689       2.570       5.965       5.892         48       PA       386       81.653       83.551       83.668       2.888       3.628       3.666	37	IN	217	75.994	78.712	78.830	0.619	0.609	0.775
40       WI       246       61.304       61.320       61.451       2.486       2.495       2.569         41       NY       270       280.982       278.784       283.973       6.457       6.275       6.681         42       MO       278       61.972       62.846       62.946       0.093       1.307       1.409         43       MN       289       42.025       45.747       45.727       -1.572       2.367       2.355         44       OH       302       108.261       111.756       111.905       2.364       3.820       3.864         45       KS       309       30.804       30.253       30.321       1.859       2.253       2.208         46       CA       338       238.284       248.558       248.800       6.991       6.244       6.223         47       TX       374       221.105       204.983       205.689       2.570       5.965       5.892         48       PA       386       81.653       83.551       83.668       2.888       3.628       3.666	38	ND	217	8.336	7.421	7.469	-1.704	-1.436	-1.644
41       NY       270       280.982       278.784       283.973       6.457       6.275       6.681         42       MO       278       61.972       62.846       62.946       0.093       1.307       1.409         43       MN       289       42.025       45.747       45.727       -1.572       2.367       2.355         44       OH       302       108.261       111.756       111.905       2.364       3.820       3.864         45       KS       309       30.804       30.253       30.321       1.859       2.253       2.208         46       CA       338       238.284       248.558       248.800       6.991       6.244       6.223         47       TX       374       221.105       204.983       205.689       2.570       5.965       5.892         48       PA       386       81.653       83.551       83.668       2.888       3.628       3.666	39	MI	236	85.314	97.710	97.705	-1.220	4.945	4.944
42       MO       278       61.972       62.846       62.946       0.093       1.307       1.409         43       MN       289       42.025       45.747       45.727       -1.572       2.367       2.355         44       OH       302       108.261       111.756       111.905       2.364       3.820       3.864         45       KS       309       30.804       30.253       30.321       1.859       2.253       2.208         46       CA       338       238.284       248.558       248.800       6.991       6.244       6.223         47       TX       374       221.105       204.983       205.689       2.570       5.965       5.892         48       PA       386       81.653       83.551       83.668       2.888       3.628       3.666	40	WI	246	61.304	61.320	61.451	2.486	2.495	2.569
43       MN       289       42.025       45.747       45.727       -1.572       2.367       2.355         44       OH       302       108.261       111.756       111.905       2.364       3.820       3.864         45       KS       309       30.804       30.253       30.321       1.859       2.253       2.208         46       CA       338       238.284       248.558       248.800       6.991       6.244       6.223         47       TX       374       221.105       204.983       205.689       2.570       5.965       5.892         48       PA       386       81.653       83.551       83.668       2.888       3.628       3.666	41	NY	270	280.982	278.784	283.973	6.457	6.275	6.681
44       OH       302       108.261       111.756       111.905       2.364       3.820       3.864         45       KS       309       30.804       30.253       30.321       1.859       2.253       2.208         46       CA       338       238.284       248.558       248.800       6.991       6.244       6.223         47       TX       374       221.105       204.983       205.689       2.570       5.965       5.892         48       PA       386       81.653       83.551       83.668       2.888       3.628       3.666	42	MO	278	61.972	62.846	62.946	0.093	1.307	1.409
45       KS       309       30.804       30.253       30.321       1.859       2.253       2.208         46       CA       338       238.284       248.558       248.800       6.991       6.244       6.223         47       TX       374       221.105       204.983       205.689       2.570       5.965       5.892         48       PA       386       81.653       83.551       83.668       2.888       3.628       3.666	43	MN	289	42.025	45.747	45.727	-1.572	2.367	2.355
46       CA       338       238.284       248.558       248.800       6.991       6.244       6.223         47       TX       374       221.105       204.983       205.689       2.570       5.965       5.892         48       PA       386       81.653       83.551       83.668       2.888       3.628       3.666	44	ОН	302	108.261	111.756	111.905	2.364	3.820	3.864
47     TX     374     221.105     204.983     205.689     2.570     5.965     5.892       48     PA     386     81.653     83.551     83.668     2.888     3.628     3.666	45	KS	309	30.804	30.253	30.321	1.859	2.253	2.208
48 PA 386 81.653 83.551 83.668 2.888 3.628 3.666	46	CA	338	238.284	248.558	248.800	6.991	6.244	6.223
	47	TX	374	221.105	204.983	205.689	2.570	5.965	5.892
49 IL 567 64.650 67.278 67.172 1.490 3.110 3.064	48	PA	386	81.653	83.551	83.668	2.888	3.628	3.666
	49	IL	567	64.650	67.278	67.172	1.490	3.110	3.064

Table 3: Model-based simulation setup with definition of parameters and distributions

## Simulation Setup:

Generate  $W_i$  from a Normal(5,9) and  $\psi_i$  from a Gamma(4.5,2)

Take  $\theta_i = 3W_i + \nu_i$ ,  $z_i = \theta_i + e_i$ , and  $w_i = W_i + \eta_i$ 

 $\nu_i \sim \text{Normal}(0, \sigma_{\nu}^2), \ e_i \sim \text{Normal}(0, \psi_i), \ \text{and} \ \eta_i \sim \text{Normal}(0, \Sigma_i)$ 

Take  $y_i = \exp(z_i)$  and  $Y_i = \exp(\theta_i)$ 

### Parameter Definition:

Let m = 20, 50, 100, and 500 (number of small areas)

Let  $\sigma_{\nu}^2 = 2$  (for all cases)

Let  $k \in \{0, 20, 50, 80, \text{and } 100\}$ 

 $\Sigma_i \in \{0, d\}$ , where d = 2 or 4

Allow k% of the  $\Sigma_i$ 's randomly receive d and the rest 0.

Table 4: Estimators and their empirical MSEs from model-based simulations. The results are averaged over all the small areas and re-scaled logarithmically.

m	k	$y_i$	$\hat{Y}_i$	$ ilde{Y}_i$	$\hat{Y}_i^{\text{EB}}$	$\mathrm{EMSE}(y_i)$	$\mathrm{EMSE}(\hat{Y}_i)$	$\mathrm{EMSE}(\tilde{Y}_i)$	$\overline{\mathrm{EMSE}(\hat{Y}_i^{\mathrm{EB}})}$
20	0	46.766	50.626	50.626	50.626	102.088	111.09	111.09	111.09
	20	53.054	50.139	49.229	49.864	115.974	109.338	104.398	108.425
	50	42.232	41.174	42.464	43.110	93.496	91.163	92.948	94.223
	80	42.519	44.285	43.469	44.794	93.881	97.557	95.152	97.495
	100	44.682	41.81	45.073	46.331	99.13	92.161	99.938	102.48
50	0	49.624	47.732	47.732	47.732	110.061	106.167	106.167	106.167
	20	44.292	42.851	44.702	45.098	97.644	94.541	99.321	100.255
	50	45.512	44.677	46.071	47.59	99.615	98.56	101.351	105.547
	80	42.703	41.773	45.289	46.469	94.339	93.268	101.068	103.615
	100	43.83	43.201	44.779	45.68	97.144	95.961	98.347	100.319
100	0	42.635	42.241	42.241	42.241	94.625	92.802	92.802	92.802
	20	46.216	45.601	46.179	47.427	103.264	101.412	103.035	106.172
	50	50.93	45.982	49.347	48.519	113.343	103.08	109.718	108.214
	80	50.132	46.586	48.137	48.966	111.618	103.055	106.712	108.454
	100	44.925	44.711	48.009	49.031	100.996	100.764	107.472	109.515
500	0	47.338	45.253	45.253	45.253	107.275	103.465	103.465	103.465
	20	46.382	45.369	47.575	47.652	104.607	102.867	107.896	108.169
	50	53.045	46.854	50.662	49.126	119.208	106.396	114.378	110.006
	80	47.766	44.868	47.706	49.950	108.289	104.921	107.454	112.805
	100	48.586	45.313	49.372	50.449	109.795	103.378	111.069	113.218

Table 5: Comparison of the proposed jackknife and bootstrap estimators from model-based simulations. The results are averaged over all small ares. The results are rescaled by the logarithm of absolute value. Note, "\*" denote that the original value is negative.

m	k	$\mathrm{EMSE}(\hat{Y}_i^{\mathrm{EB}})$	$\mathrm{mspe}_J(\hat{Y}_i^{\mathrm{EB}})$	$\mathrm{mspe}_B(\hat{Y}_i^{\mathrm{EB}})$	$\mathrm{RB}_J(\hat{Y}_i^{\mathrm{EB}})$	$RB_B(\hat{Y}_i^{EB})$
20	0	111.09	120.731	118.478*	9.641	7.389*
	20	108.425	115.222	115.884*	6.796	7.459*
	50	94.223	107.245	111.255*	13.021	17.032*
	80	97.495	113.06	129.772*	15.565	32.277*
	100	102.48	115.536*	119.84*	13.056*	17.36*
50	0	106.167	112.318*	116.383*	6.154*	10.216*
	20	100.255	110.449	106.743*	10.194	6.489*
	50	105.547	115.365*	118.838*	9.818*	13.291*
	80	103.615	114.127	112.717*	10.512	9.102*
	100	100.319	110.454*	112.552*	10.134*	12.233*
100	0	92.802	98.67	102.932*	5.866	10.13*
	20	106.172	106.788*	108.623*	1.048*	2.534*
	50	108.214	119.666	120.032*	11.452	11.819*
	80	108.454	117.223*	119.418*	8.769*	10.964*
	100	109.515	119.028	117.071*	9.513	7.557*
500	0	103.465	108.38	110.455*	4.908	6.991*
	20	108.169	119.672	115.892	11.502	7.722
	50	110.006	121.191	125.754*	11.184	15.747*
	80	112.805	125.672*	129.761*	12.867*	16.955*
	100	113.217	123.037*	124.597*	9.819*	11.379*

Table 6: Percentage of bias related to the consequences of misspecifying the error variance  $\Sigma_i$  on  $\beta$  in the EB predictor  $\hat{Y}_i^{\rm EB}$  from model-based simulations. For all cases, we assume the true value for  $\beta$  is 3. Also,  $\sigma_{\nu}^2=2$  and m=20 (the smallest one) and 500 (the largest one).

m	k	Experiment	$\frac{1}{R} \sum_{r=1}^{R} \left  \hat{\beta}^{(r)} - \hat{\beta}_{\text{mis}}^{(r)} \right $	$\frac{1}{R} \sum_{r=1}^{R} \left( \hat{\beta}^{(r)} - 3 \right)$	$\frac{1}{R} \sum_{r=1}^{R} \left( \hat{\beta}_{\text{mis}}^{(r)} - 3 \right)$
20	0	$A1(d = 2, d_{\text{mis}} = 4)$ $A2(d = 4, d_{\text{mis}} = 2)$	0	-0.026 $-0.237$	-0.026 $-0.237$
	20	$B1(d = 2, d_{\text{mis}} = 2)$	6.034	0.314	-0.046
		$B2(d = 4, d_{\text{mis}} = 2)$	5.814	-0.634	-0.591
	50	$C1(d=2, d_{\text{mis}}=4)$	11.287	-0.184	-0.864
	0.0	$C2(d = 4, d_{\text{mis}} = 2)$	10.977	-0.157	-0.394
	80	$D1(d = 2, d_{\text{mis}} = 4)$ $D2(d = 4, d_{\text{mis}} = 2)$	19.641 $18.722$	0.842 $0.712$	2.177 $0.327$
	100	$E1(d=2, d_{\rm mis}=4)$	25.774	2.650	4.285
		$E2(d=4, d_{\rm mis}=2)$	25.967	4.020	2.937
500	0	$A1(d = 2, d_{\text{mis}} = 4)$ $A2(d = 4, d_{\text{mis}} = 2)$	0	0.185 $-0.082$	0.185 $-0.082$
	20	$A2(d = 4, d_{\text{mis}} = 2)$ $B1(d = 2, d_{\text{mis}} = 4)$	1.136	0.174	0.118
	20	$B1(d = 2, d_{\text{mis}} = 4)$ $B2(d = 4, d_{\text{mis}} = 2)$	1.107	0.010	0.009
	50	$C1(d=2, d_{\rm mis}=4)$	2.200	-0.138	0.026
		$C2(d=4, d_{\mathrm{mis}}=2)$	2.182	-0.044	0.006
	80	$D1(d = 2, d_{\text{mis}} = 4)$ $D2(d = 4, d_{\text{mis}} = 2)$	3.365 $3.386$	0.204 $-0.139$	-0.067 $-0.090$
	100	$E1(d = 2, d_{\text{mis}} = 2)$	5.086	0.174	-0.050 $0.152$
	_ 5 5	$E2(d = 4, d_{\text{mis}} = 2)$	4.941	-0.022	0.117

## 5. Discussion

In this paper, in order to stabilize the skewness and achieve normality in the response variable, we have proposed an area-level log-measurement error model on the response variable. In addition, we have proposed a measurement error model on the covariates. Second, under our proposed modeling framework, we derived the EB predictor of positive small area quantities subject to the covariates containing measurement error. Third, we proposed a corresponding estimate of MSPE using a jackknife and a bootstrap method, where we illustrated that the order of the bias is  $O(m^{-1})$ , where m is the number of small areas. Fourth, we have illustrated the performance of our methodology in both design-based simulation and model-based simulation studies, where the EMSE of the proposed EB predictor is not always uniformly better than that of the direct estimator. Our model-based simulation studies have provided further investigation and guidance on the behavior. For example, one fruitful area of future research would be providing a correction to the EB predictor to avoid for such behavior. One way to address this issue is by estimating  $\phi = (\beta, \sigma_{\nu}^2)^{\top}$  in such a way that its order of bias is smaller than  $O(m^{-1})$ . This could help to reduce the amount of propagated errors in the EB predictor  $\hat{Y}_i^{\text{EB}}$ . Another way, which is less theoretical burdensome is by estimating the covariate as we have assumed it follows a functional measurement error model rather than a structural one. We have studied the MSPE of the EB predictor using both the jackknife and bootstrap in simulation studies, where we have shown the jackknife estimator performs better than the bootstrap one under our log model.

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