Improved estimators in beta prime regression models

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Abstract

In this paper, we consider the beta prime regression model recently proposed by Bourguignon et al. (2018), which is tailored to situations where the response is continuous and restricted to the positive real line with skewed and long tails and the regression structure involves regressors and unknown parameters. We consider two different strategies of bias correction of the maximum-likelihood estimators for the parameters that index the model. In particular, we discuss bias-corrected estimators for the mean and the dispersion parameters of the model. Furthermore, as an alternative to the two analytically bias-corrected estimators discussed, we consider a bias correction mechanism based on the parametric bootstrap. The numerical results show that the bias correction scheme yields nearly unbiased estimates. An example with real data is presented and discussed.

Keywords: Beta prime distribution; Bias correction; Bootstrap; Dispersion covariates; Maximum-likelihood.

1 Introduction

The beta prime (BP) distribution (known as inverted beta distribution or beta distribution of the second kind as well) is a two-parameter distribution on the positive real line, which can be interpreted as the distribution of the odds ratio of a variable distributed according to the beta distribution, i.e., if X has a beta distribution with parameters α and β , then Y = X/(1-X) has a BP distribution with $\alpha > 0$ and $\beta > 0$ both shape parameters. We are adopting the parameterization for the BP distribution in terms of the mean and precision parameters which was proposed by Bourguignon et al. (2018). An advantage of using this parameterization is that we can introduce regression structures for each mean and precision parameters and the interpretation of the regression coefficients is straightforward in terms of them as in generalized linear models. Thus, the BP random variable Y (Bourguignon et al. (2018)) is defined as follows: Let Y be a random variable with probability density function (pdf) given by

$$f(y;\mu,\phi) = \frac{y^{\mu(\phi+1)-1}(1+y)^{-[\mu(\phi+1)+\phi+2]}}{B(\mu(1+\phi),\phi+2)}, \quad y > 0,$$
(1)

where $\mu>0$ and $\phi>0$ are mean and precision parameters, respectively, $B(\mu(1+\phi),\phi+2)=\Gamma(\mu(1+\phi))\Gamma(\phi+2)/\Gamma(\mu(1+\phi)+\phi+2)$ is the beta function and $\Gamma(\mu(1+\phi))=\int_0^\infty \omega^{\mu(1+\phi)-1}\mathrm{e}^{-\omega}\mathrm{d}\omega$

is the gamma function. From now on, we use the notation $Y \sim \mathrm{BP}(\mu, \phi)$ to indicate that Y is a random variable following a BP distribution. The mean and variance of Y are

$$\mathrm{E}[Y] = \mu \quad \mathrm{and} \quad \mathrm{Var}[Y] = \frac{\mu(1+\mu)}{\phi}.$$

Some features of the BP model are (Bourguignon et al., 2018): first, the variance function of the BP model assumes a quadratic form similar to the gamma distribution. However, the variance function of the proposed model is larger than the variance function of gamma distribution, which may be more appropriate in certain practical situations; second, the BP hazard rate function can have an upside-down bathtub or increasing depending on the parameter values. The most classical two-parameter distributions such as Weibull and gamma distributions have monotone hazard rate functions; third, the skewness and kurtosis of the BP distribution can be much larger than those of the gamma and inverse gaussian distributions; fourth, there are some stochastic representation of the BP random variable.

In the literature there are only a few works dealing with the BP distribution. McDonald (1987) discussed its properties and obtained the maximum likelihood (ML) estimates of the model parameters. Bias-corrected versions of the MLEs of the parameters that index the BP distribution were obtained by Stosić and Cordeiro (2009). It is worth mention that all the works related above have considered the usual parameterization of the BP distribution. Considering the parameterization we adopted, Bourguignon et al. (2018) used the ML method for estimating the parameters that index the BP regression model. However, as can be seen in Table 2 in Bourguignon et al. (2018), in small-sized samples, the ML estimators of these parameters (especially for precision structure) may be extremely biased. So, it is important consider alternative estimators with smaller biases when the number of observations are small.

Investigates how the maximum likelihood estimator behaves in small-sized sample, in particular bias analysis, is an important research area. In regular parametric statistical models the maximum likelihood estimator bias is generally of the order $\mathcal{O}(n^{-1})$ for large sample size n and are, in practice, usually ignored since that the asymptotic standard error is of order $O(n^{-1/2})$. When dealing with small-sized sample, however, bias can be a problematic issue, thus it can not be neglected. So, it is important to obtain bias correction in these cases. Bias reduction was studied by several authors. In uniparametric models, Bartlett (1953) obtained an expression for the $\mathcal{O}(n^{-1})$ bias from the maximum likelihood estimator. Assuming independent, but not necessarily identically distributed observations, Cox and Snell (1968) obtained a general expression for the $\mathcal{O}(n^{-1})$ bias of the maximum likelihood estimator in multiparametric models. This result has become widely used in the literature to obtain general expressions for the $\mathcal{O}(n^{-1})$ bias and to propose bias-corrected estimators in various parametric models. For instance, Lemonte et al. (2007), Cysneiros et al. (2010), Simas et al. (2011), Barreto-Souza and Vasconcellos (2011) and Melo et al. (2018).

Usually, the approach to obtain bias-corrected versions of the MLEs uses the second order bias. In this procedure the adjustment is made after the MLEs were computed. Additionally, an alternative approach was proposed by Firth (1993), who suggested that a bias reduction method by modifying the score function previous to obtain the parameter estimates. This method is called the preventive method and has been studied in parametric models where maximum likelihood estimates

can be unstable (infinite or belonging to the parametric space boundary) such as Bull et al. (2002), Sartori (2006), Kosmidis and Firth (2009), Kosmidis and Firth (2011) and Kosmidis (2014). Another possible way to perform bias correction is through bootstrap resampling, which requires no explicit derivation of the bias function. In this context, the main goal of this paper is to derive a closed-form expression for the second order biases of the ML estimators in the BP regression model which can be used to define bias corrected ML estimators to order $\mathcal{O}(n^{-1})$.

This paper is organized as follows: this introductory section. In Section 2, the BP regression model is introduced and some of its basic properties are outlined. In Section 3, we obtain the second order biases of the MLEs of the means of the responses and precision parameters of the model. Section 4 discusses the numerical results. In Section 5, we consider an empirical example. Finally, Section 6 concludes the paper.

2 Beta prime regression model

Consider n independent random variables Y_1, \ldots, Y_n where each Y_i , $i = 1, \ldots, n$ has BP distribution with pdf given by (1) with mean μ_i and precision parameter ϕ_i . Bourguignon et al. (2018) proposed the BP regression model which is defined by (1) and by two functional relations

$$g_1(\mu_i) = \eta_{1i} = \mathbf{x}_i^{\mathsf{T}} \boldsymbol{\beta} \quad \text{and} \quad g_2(\phi_i) = \eta_{2i} = \mathbf{z}_i^{\mathsf{T}} \boldsymbol{\nu},$$
 (2)

where $g_1: \mathbb{R} \to \mathbb{R}^+$ and $g_2: \mathbb{R} \to \mathbb{R}^+$ are strictly monotone, positive and at least twice differentiable link functions, η_{1i} and η_{2i} are the linear predictors, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)^\top$ ($\boldsymbol{\beta} \in \mathbb{R}^p$) and $\boldsymbol{\nu} = (\nu_1, \dots, \nu_q)^\top$ ($\boldsymbol{\nu} \in \mathbb{R}^q$, q < n - p) are unknown parameter vectors to be estimated, and $\mathbf{x}_i = (x_{i1}, \dots, x_{ip})^\top$ and $\mathbf{z}_i = (z_{i1}, \dots, z_{iq})^\top$ are observations on p and q known regressors, for $i = 1, \dots, n$. Additionally, we assume that the covariate matrices $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_n)^\top$ and $\mathbf{Z} = (\mathbf{z}_1, \dots, \mathbf{z}_n)^\top$ have rank p and q, respectively. Besides the interpretation of the regression coefficients being in terms of the mean and precision parameters, another advantage of the model proposed by (1) and (2) is that it is suitable for modeling asymmetric data, being an alternative to the generalized linear models when dealing with asymmetric dataset.

The log-likelihood function for (β, ν) given the observed values y_1, \ldots, y_n is

$$\ell(\boldsymbol{\beta}, \boldsymbol{\nu}) = \sum_{i=1}^{n} \ell(\mu_i, \phi_i), \tag{3}$$

being

$$\ell(\mu_i, \phi_i) = [\mu_i(1 + \phi_i) - 1] \log(y_i) - [\mu_i(1 + \phi_i) + \phi_i + 2] \log(1 + y_i) - \log[\Gamma(\mu_i(1 + \phi_i))] - \log[\Gamma(\phi_i + 2)] + \log[\Gamma(\mu_i(1 + \phi_i) + \phi_i + 2)];$$

 $\mu_i = g_1^{-1}(\eta_{1i})$ and $\phi_i = g_2^{-1}(\eta_{2i})$ are functions of β and ν , respectively, as defined in (2). A method for obtaining the parameters estimates of the BP model defined by (1) and (2) is described in details in Bourguignon et al. (2018). They consider the *gamlss* function for this purpose.

We assume that the log-likelihood function (3) satisfies the usual regularity conditions of large sample likelihood theory (see Cox and Hinkley, 1983). Thus, when n is large and under some regular conditions we have that

$$\left(egin{array}{c} \widehat{oldsymbol{eta}} \ \widehat{oldsymbol{
u}} \end{array}
ight) \overset{a}{\sim} \mathrm{N}_{p+q} \left(\left(egin{array}{c} oldsymbol{eta} \ oldsymbol{
u} \end{array}
ight), \mathrm{K}(\widehat{oldsymbol{eta}},\widehat{oldsymbol{
u}})^{-1}
ight),$$

where $\stackrel{a}{\sim}$ means "approximately distributed" and $\mathbf{K}(\widehat{\boldsymbol{\beta}},\widehat{\boldsymbol{\nu}})^{-1}$ is the inverse of Fisher's information matrix evaluated at $\widehat{\boldsymbol{\beta}}$ and $\widehat{\boldsymbol{\nu}}$, which can be approximated by $J(\widehat{\boldsymbol{\beta}},\widehat{\boldsymbol{\nu}})^{-1}$, where -J denotes the $(p+q)\times(p+q)$ Hessian matrix evaluated at $(\widehat{\boldsymbol{\beta}}^{\top},\widehat{\boldsymbol{\nu}}^{\top})^{\top}$. Fisher's information matrix $\mathbf{K}(\boldsymbol{\beta},\boldsymbol{\nu})$ is presented in Appendix A.

3 Bias correction of the MLEs

Let $\boldsymbol{\theta}=(\boldsymbol{\beta}^{\top},\boldsymbol{\nu}^{\top})^{\top}$ be the unknown parameter vector of the BP regression model. We now obtain an expression for the second order biases of the MLEs of the components of $\boldsymbol{\theta}$ using Cox and Snell's (Cox and Snell (1968)) general formula. In order to obtain this expression, we first introduce some notation. The lower subscripts r, s, t, u, \ldots and the upper subscripts R, S, T, U, \ldots denote, respectively, the components of $\boldsymbol{\beta}$ and $\boldsymbol{\nu}$ vectors. Therefore, the partional derivatives of the log-likelihood (3) with respect to the components of $\boldsymbol{\beta}$ and $\boldsymbol{\nu}$ are presented as $U_r = \partial \ell/\partial \beta_r$, $U_{rS} = \partial^2 \ell/\partial \beta_r \partial \nu_S$, $U_{rST} = \partial^3 \ell/\partial \beta_r \partial \nu_S \partial \nu_T$, etc. The moments of the log-likelihood derivatives are represented by $\kappa_{rs} = E(U_{rs})$, $\kappa_{r,s} = E(U_rU_s)$, $\kappa_{r,sT} = E(U_rU_{ST})$, etc, where all $\kappa's$ regard to a total covering the whole sample and are, in general, of order $\mathcal{O}(n^{-1})$. The moments derivatives are defined by $\kappa_{rs}^{(t)} = \partial \kappa_{rs}/\partial \beta_t$, $\kappa_{rs}^{(T)} = \partial \kappa_{rs}/\partial \nu_T$, etc. Finally, we denote the elements of the inverse of Fisher's information matrix $K(\boldsymbol{\beta},\boldsymbol{\nu})^{-1} = K(\boldsymbol{\theta})^{-1}$, which are $\mathcal{O}(n^{-1})$, as $\kappa^{r,s} = -\kappa^{rs}$, $\kappa^{r,s} = -\kappa^{Rs}$ and $\kappa^{R,S} = -\kappa^{RS}$.

From the general Cox and Snell's (1968) formula we can obtain the $\mathcal{O}(n^{-1})$ bias of the MLE for the ath component of the parameter vector $\widehat{\boldsymbol{\theta}} = (\widehat{\theta}_1, \dots, \widehat{\theta}_p, \widehat{\theta}_{p+1}, \dots, \widehat{\theta}_{p+q})^\top = (\widehat{\boldsymbol{\beta}}^\top, \widehat{\boldsymbol{\nu}}^\top)^\top$ as:

$$B_{\widehat{\theta}}(\theta_{a}) = \sum_{r,s,u} \kappa^{ar} \kappa^{su} \left\{ \kappa_{rs}^{(u)} - \frac{1}{2} \kappa_{rsu} \right\} + \sum_{R,s,u} \kappa^{aR} \kappa^{su} \left\{ \kappa_{Rs}^{(u)} - \frac{1}{2} \kappa_{Rsu} \right\}$$

$$+ \sum_{r,S,u} \kappa^{ar} \kappa^{Su} \left\{ \kappa_{rS}^{(u)} - \frac{1}{2} \kappa_{rSu} \right\} + \sum_{r,s,U} \kappa^{ar} \kappa^{sU} \left\{ \kappa_{rs}^{(U)} - \frac{1}{2} \kappa_{rsU} \right\}$$

$$+ \sum_{R,S,u} \kappa^{aR} \kappa^{Su} \left\{ \kappa_{RS}^{(u)} - \frac{1}{2} \kappa_{RSu} \right\} + \sum_{R,s,U} \kappa^{aR} \kappa^{sU} \left\{ \kappa_{Rs}^{(U)} - \frac{1}{2} \kappa_{RsU} \right\}$$

$$+ \sum_{r,S,U} \kappa^{ar} \kappa^{SU} \left\{ \kappa_{rS}^{(U)} - \frac{1}{2} \kappa_{rSU} \right\} + \sum_{R,S,U} \kappa^{aR} \kappa^{SU} \left\{ \kappa_{RS}^{(U)} - \frac{1}{2} \kappa_{RSU} \right\}.$$

$$(4)$$

From (5), we can observe that β and ν are not orthogonal, hence all terms in (4) must be considered. In order to save space, all cumulants needed to obtain (4) are given in Appendix B.

After a long algebra presented in details in Appendix B we achieve to the expressions for the second order biases of $\widehat{\beta}$ and $\widehat{\nu}$, given in matrix form respectively by

$$B_{\widehat{\beta}}(\beta) = K^{\beta\beta}X^{\top} [M_1P_{\beta\beta} + (M_2 + M_3)P_{\beta\nu} + M_5P_{\nu\nu}] + K^{\beta\nu}Z^{\top} [M_2P_{\beta\beta} + (M_4 + M_5)P_{\beta\nu} + M_6P_{\nu\nu}]$$

and

$$B_{\widehat{\nu}}(\nu) = K^{\beta\nu}X^{\top} [M_1 P_{\beta\beta} + (M_2 + M_3) P_{\beta\nu} + M_5 P_{\nu\nu}] + K^{\nu\nu}Z^{\top} [M_2 P_{\beta\beta} + (M_4 + M_5) P_{\beta\nu} + M_6 P_{\nu\nu}],$$

where $K^{\beta\beta}$, $K^{\beta\nu}$ and $K^{\nu\nu}$ represent matrices which components are respectively the (r,s)th, (r,S)th and (R,S)th elements of the inverse of Fisher's information matrix, M_1 to M_6 are presented in Appendix B, $P_{\beta\beta}$, $P_{\beta\nu}$ and $P_{\nu\nu}$ are vectors with the same $n\times 1$ dimension and which elements are the diagonal elements of $XK^{\beta\beta}X^{\top}$, $XK^{\beta\nu}Z^{\top}$ and $ZK^{\nu\nu}Z^{\top}$, respectively.

We now assume the $2n \times 1$ vector δ_1 defined as

$$\delta_1 = \begin{pmatrix} M_1 P_{\beta\beta} + (M_2 + M_3) P_{\beta\nu} + M_5 P_{\nu\nu} \\ M_2 P_{\beta\beta} + (M_4 + M_5) P_{\beta\nu} + M_6 P_{\nu\nu} \end{pmatrix},$$

and consider $K^{\beta*}=(K^{\beta\beta}\ K^{\beta\nu})$ and $K^{\nu*}=(K^{\nu\beta}\ K^{\nu\nu})$, the $p\times(p+q)$ upper and $q\times(p+q)$ lower blocks of the matrix $K(\beta,\nu)^{-1}$, respectively. Thus, we can express the second-order biases of $\widehat{\beta}$ and $\widehat{\nu}$ as

$$B_{\widehat{\boldsymbol{\beta}}}(\boldsymbol{\beta}) = K^{\boldsymbol{\beta}*}\mathbb{X}^{\top}\delta_{1} \quad \text{and} \quad B_{\widehat{\boldsymbol{\nu}}}(\boldsymbol{\nu}) = K^{\boldsymbol{\nu}*}\mathbb{X}^{\top}\delta_{1}.$$

From the expressions above, we can obtain in matrix form the second order bias of the MLE of the joint vector $\boldsymbol{\theta} = (\boldsymbol{\beta}^{\top}, \boldsymbol{\nu}^{\top})^{\top}$ expressed as

$$B_{\widehat{\boldsymbol{\theta}}}(\boldsymbol{\theta}) = (\mathbb{X}^{\top} \widetilde{\boldsymbol{K}} \mathbb{X})^{-1} \mathbb{X}^{\top} \delta_1.$$

Now, we define the bias-corrected estimator as

$$\widetilde{\boldsymbol{\theta}} = \widehat{\boldsymbol{\theta}} - B_{\widehat{\boldsymbol{\theta}}}(\widehat{\boldsymbol{\theta}}),$$

where $B_{\widehat{\boldsymbol{\theta}}}(\widehat{\boldsymbol{\theta}})$ is bias of the $\widehat{\boldsymbol{\theta}}$ with the unknown parameters replaced by their MLEs. Considering the assumptions assumed in Section 2, we have that the asymptotic distribution of $\boldsymbol{\theta}$ is $N_{p+q}(\boldsymbol{\theta}, \mathbf{J}(\boldsymbol{\theta})^{-1})$, where $\mathbf{J}(\boldsymbol{\theta}) = \mathbf{J}(\boldsymbol{\beta}, \boldsymbol{\nu})^{-1}$

A second approach to correct the second order bias of the MLE of $\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, \boldsymbol{\nu}^\top)^\top$ is considering the "preventive" method proposed by Firth (1993). This method basically consists of modify the original score function in order to remove the $\mathcal{O}(n^{-1})$ bias. The modified score function is given by

$$U^*(\boldsymbol{\theta}) = U(\boldsymbol{\theta}) - K(\boldsymbol{\theta}) B_{\widehat{\boldsymbol{\theta}}}(\boldsymbol{\theta}),$$

being $K(\theta)$ the information matrix and $B_{\widehat{\theta}}(\theta)$ the $\mathcal{O}(n^{-1})$ bias. Considering the BP regression model and replacing the expression obtained for $B_{\widehat{\theta}}(\theta)$, the modified score function has the following form:

$$U^*(\boldsymbol{\theta}) = U(\boldsymbol{\theta}) - \mathbb{X}^{\top} \delta_1.$$

The second order bias corrected MLEs $\check{\boldsymbol{\theta}}$ is the solution of $U^*(\boldsymbol{\theta}) = \mathbf{0}$. Also, $\check{\boldsymbol{\theta}}$ is asymptotically normal distributed as $\mathbf{N}_{p+q}(\boldsymbol{\theta}, \mathbf{J}(\boldsymbol{\theta})^{-1})$, with $\mathbf{J}(\boldsymbol{\theta})$ as given previously.

Another way to bias-correcting the MLEs of the regression parameters is by the bootstrap technique (see, for example, Efron and Tibshirani, 1993). In this paper, in order to reduce the computational burden, we shall adopt the warp-speed bootstrap method of (Giacomini et al., 2013) for evaluating the proposed resampling scheme. The warp-speed bootstrap method follows the steps described below. Instead of computing the MLEs for each Monte Carlo sample $r=1,2,\ldots,m$ (with m being the total number of Monte Carlo replications) on the basis of B bootstrap samples, just one resample (i.e. B=1) is generated from the assumed model with the parameters replaced by estimates of maximum likelihood computed using the original sample for each Monte Carlo sample and, hence, estimates of maximum likelihood, say $\hat{\theta}^*$, is computed for that sample. Therefore, the bootstrap bias estimates $\hat{\theta}$ is

$$B_{\widehat{\boldsymbol{\theta}}}(\widehat{\boldsymbol{\theta}}^*) = \widehat{\boldsymbol{\theta}}^* - \widehat{\boldsymbol{\theta}}.$$

By using the bootstrap bias estimate presented above, we arrive at the following bias-corrected, to order $\mathcal{O}(n^{-1})$, estimator:

$$\widetilde{\boldsymbol{\theta}}^b = 2\widehat{\boldsymbol{\theta}} - \widehat{\boldsymbol{\theta}}^*.$$

For a good discussion to the bootstrap method, see Efron and Tibshirani (1993, Chapter 16). Finally, it is worth mentioning that the idea behind the warp-speed bootstrap method is that taking just *one* bootstrap draw for each simulated sample is sufficient to provide a useful approximation to the bias of estimator. Applying this insight to Monte Carlo evaluation of bootstrap-based bias yields evaluation methods that work with M=1 (Giacomini et al., 2013). Due to the resulting dramatic computational savings, (Giacomini et al., 2013) called their method as "Warp-Speed" Monte Carlo method. Therefore, the bootstrap-based bias on the basis of warp-speed bootstrap method become a viable alternative to inferential improvements in small samples when there are impeditive or too costly analytical difficulties.

4 Numerical results

We now present a Monte Carlo simulation study to investigate and compare the performance of the MLEs along with their corrected versions proposed in this article in small and moderate-sized samples. We use a BP regression models with dispersion covariates and a log link. We consider the model

$$\log(\mu_i) = \beta_0 + \sum_{\ell=1}^p \beta_\ell x_{i\ell}$$
 and $\log(\phi_i) = \nu_0 + \sum_{\ell=1}^q \nu_\ell x_{i\ell}$, $i = 1, 2, \dots, n$,

where the true values of the parameters were taken as 1. The covariates values are taken as random draws from the $\mathcal{U}(0,1)$ distribution and their values were held constant throughout the simulations. We consider different values for the number of regression parameters (p and q) and the sample size (n=30,40 and 60). The number of Monte Carlo replicates was 10.000 and all the

simulations were performed using the R language (R Core Team, 2017). In each Monte Carlo replica, we computed the MLEs of the parameters, their corrected versions from the corrective method (Cox and Snell, 1968), preventive method (Firth, 1993), and the parametric version of the bootstrap method (Giacomini et al., 2013). In order to analyze the results, we computed, for each sample size and for each estimator, the mean of estimates, bias, variance and mean square error (MSE). The results are presented in Tables 1, 2 and 3 for p = q = 1, p = q = 2, and p = q = 3, respectively.

Tables 1-3 summarize the simulation results for the $\beta's$ and $\nu's$ varying the sample size n and the number of regression parameters (p and q). As can be seen in Tables 1-3, for most part of the parameters the estimated biases, in absolute value, of the original MLEs were larger than the others. In general, for the $\beta's$, the biases of preventive estimators were smaller than those of the corrective estimators and bootstrap estimators. For the ν 's, the biases of the bootstrap estimators were, in general, smaller than those of the corrective estimators and the preventive estimators. These performances are independent of the number of parameters to be estimated. For instance, in absolute value, when p=q=1 and n=30 the bias of the parameter β_0 were 0.0048 (MLE), 0.0009 (Cox-Snell), 0.0000 (Firth) and 0.0013 (p-boot) and the bias of the parameter ν_0 were 0.1044 (MLE), 0.0148 (Cox-Snell), 0.0083 (Firth) and 0.0064 (p-boot); see Table 1. However, for all parameters, in most cases the MSE of the corrective estimators were the smallest and the MSE of the bootstrap estimators were the largest, followed by the MSE of the preventive estimators. When we increase the sample size for the $\beta's$, the bootstrap estimators tends to shows the smallest bias, although they have the largest MSE. For instance, for p = q = 3 and n = 30 we have that the bias, in absolute value, for β_1 were 0.0026 (MLE), 0.0024 (Cox-Snell), 0.0016 (Firth) and 0.0031 (p-boot), while when n = 60 they were 0.0018 (MLE), 0.0016 (Cox-Snell), 0.0015 (Firth) and 0.0014 (p-boot); see Table 3. Comparing the results presented in each Table, we can observe that as the sample size increases, in general, the bias of the estimators reduces, as expected.

The previous findings are confirmed by the box plots shown in Fig. 1, which were obtained for sample size n=30. In summary, the bias of the MLEs, especially for the $\nu's$ parameters, are larger than the bias of the corrected estimators. Box plots for different values of n, p, and q (not shown) exhibited a similar pattern. Therefore, we recommend the use of method (Cox-Snell, Firth or parametric bootstrap) to reduce bias in small and moderate sample size.

5 An application

In order to illustrate the proposed methodology, in this section, we apply the estimation methods considered in the previous section to a real situation. We consider the real dataset used in Bonnail et al. (2016). The main purpose is to assess sediment quality using the freshwater clam Corbiculafluminea to determine its adequacy as a biomonitoring tool in relation to theoretical risk indexes and regulatory thresholds. The study contains 27 observations (small-sized sample), which measured, among other characteristics, the dry weight tissue of the clams (dry, in g), wet weight tissue (wet, in g), and the concentrations of caesium (cs) in the soft tissue. Such minerals were considered in 100 micrograms per liter $(100\mu g/L)$.

We adopted a BP model to fit the dry weight tissue of the clams; that is, we consider that

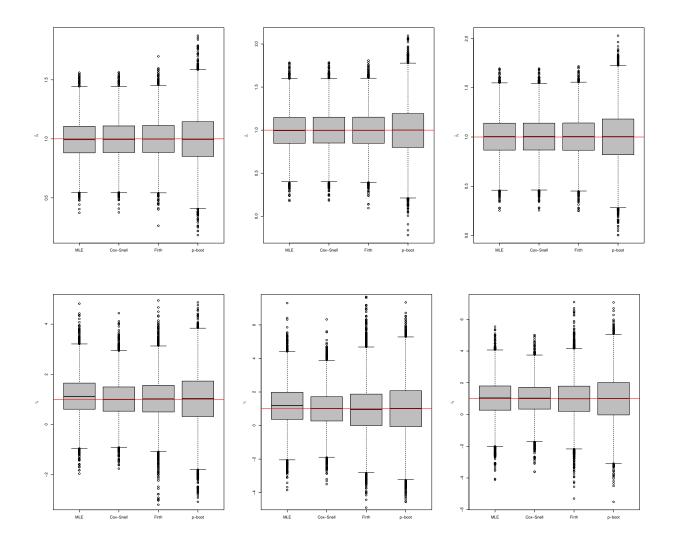


Figure 1: Box plots from 10.000 simulated estimates of β_0 , β_1 , β_2 , ν_0 , ν_1 and ν_2 for n=30.

Table 1: Simulation results for p = q = 1.

	n = 30				n = 40					n = 60			
Estimates	MLE	Cox-Snell	Firth	p-boot	MLE	Cox-Snell	Firth	p-boot	MLE	Cox-Snell	Firth	p-boot	
$\widehat{\beta}_0$	0.9952	0.9991	1.0000	0.9987	0.9952	0.9993	1.0000	0.9992	0.9965	0.9997	1.0001	0.9979	
Bias	-0.0048	-0.0009	-0.0000	-0.0013	-0.0048	-0.0007	0.0000	-0.0008	-0.0035	-0.0003	0.0001	-0.0021	
variance	0.0296	0.0297	0.0298	0.0540	0.0195	0.0196	0.0196	0.0363	0.0124	0.0124	0.0125	0.0237	
MSE	0.0296	0.0297	0.0298	0.0540	0.0195	0.0196	0.0196	0.0363	0.0124	0.0124	0.0125	0.0237	
\widehat{eta}_1	0.9955	1.0005	1.0019	1.0010	0.9988	1.0004	1.0007	1.0005	1.0004	1.0007	1.0007	1.0033	
Bias	-0.0045	0.0005	0.0019	0.0010	-0.0012	0.0004	0.0007	0.0005	0.0004	0.0007	0.0007	0.0033	
variance	0.0827	0.0824	0.0824	0.1509	0.0569	0.0568	0.0567	0.1070	0.0339	0.0339	0.0340	0.0660	
MSE	0.0827	0.0824	0.0825	0.1509	0.0569	0.0568	0.0567	0.1070	0.0339	0.0339	0.0340	0.0660	
$\widehat{ u_0}$	1.1044	1.0148	1.0083	1.0064	1.0872	1.0116	1.0094	1.0079	1.0624	1.0068	1.0045	1.0111	
Bias	0.1044	0.0148	0.0083	0.0064	0.0872	0.0116	0.0094	0.0079	0.0624	0.0068	0.0045	0.0111	
variance	0.6015	0.5083	0.6278	1.0849	0.3770	0.3412	0.3728	0.6987	0.2529	0.2372	0.2676	0.4702	
MSE	0.6124	0.5085	0.6278	1.0849	0.3847	0.3413	0.3729	0.6988	0.2568	0.2373	0.2676	0.4703	
$\widehat{ u_1}$	1.1215	1.0102	0.9939	1.0016	1.0699	1.0013	0.9913	0.9915	1.0320	0.9965	0.9941	0.9778	
Bias	0.1215	0.0102	-0.0061	0.0016	0.0699	0.0013	-0.0087	-0.0085	0.0320	-0.0035	-0.0059	-0.0222	
variance	1.7683	1.4470	1.8151	3.0861	0.9782	0.8591	0.9475	1.7650	0.6134	0.5640	0.6468	1.1392	
MSE	1.7831	1.4471	1.8152	3.0861	0.9831	0.8591	0.9475	1.7651	0.6144	0.5640	0.6468	1.1397	

Table 2: Simulation results for p = q = 2.

		n =	30			n =	40			n =	60	
Estimates	MLE	Cox-Snell	Firth	p-boot	MLE	Cox-Snell	Firth	p-boot	MLE	Cox-Snell	Firth	p-boot
$\widehat{\beta}_0$	0.9934	0.9966	0.9992	0.9981	0.9934	0.9976	0.9991	0.9999	0.9958	1.0003	1.0013	1.0006
Bias	-0.0066	-0.0034	-0.0008	-0.0019	-0.0066	-0.0024	-0.0009	-0.0001	-0.0042	0.0003	0.0013	0.0006
variance	0.0273	0.0274	0.0286	0.0480	0.0205	0.0205	0.0206	0.0378	0.0130	0.0130	0.0130	0.0245
MSE	0.0274	0.0274	0.0286	0.0480	0.0206	0.0205	0.0206	0.0378	0.0130	0.0130	0.0130	0.0245
\widehat{eta}_1	0.9970	0.9995	0.9994	0.9989	1.0018	1.0023	1.0025	0.9995	1.0019	1.0020	1.0018	0.9992
Bias	-0.0030	-0.0005	-0.0006	-0.0011	0.0018	0.0023	0.0025	-0.0005	0.0019	0.0020	0.0018	-0.0008
variance	0.0494	0.0493	0.0508	0.0872	0.0356	0.0355	0.0356	0.0660	0.0225	0.0225	0.0225	0.0435
MSE	0.0494	0.0493	0.0508	0.0872	0.0356	0.0355	0.0356	0.0660	0.0225	0.0225	0.0225	0.0435
\widehat{eta}_2	1.0040	1.0047	1.0044	1.0017	1.0027	1.0017	1.0013	1.0006	0.9999	0.9967	0.9963	0.9971
Bias	0.0040	0.0047	0.0044	0.0017	0.0027	0.0017	0.0013	0.0006	-0.0001	-0.0033	-0.0037	-0.0029
variance	0.0410	0.0409	0.0420	0.0721	0.0340	0.0339	0.0341	0.0618	0.0290	0.0290	0.0293	0.0549
MSE	0.0410	0.0409	0.0420	0.0721	0.0340	0.0339	0.0341	0.0618	0.0290	0.0290	0.0293	0.0549
$\widehat{ u_0}$	1.1367	1.0232	1.0213	1.0123	1.0753	1.0068	1.0112	0.9864	1.0602	1.0088	1.0100	0.9980
Bias	0.1367	0.0232	0.0213	0.0123	0.0753	0.0068	0.0112	-0.0136	0.0602	0.0088	0.0100	-0.0020
variance	0.6454	0.5492	0.7384	1.1125	0.6011	0.5218	0.7964	1.0588	0.3988	0.3587	0.4839	0.7352
MSE	0.6641	0.5498	0.7388	1.1127	0.6068	0.5219	0.7966	1.0590	0.4025	0.3588	0.4840	0.7352
$\widehat{ u_1}$	1.1884	1.0039	0.9675	1.0051	1.1069	1.0147	0.9922	1.0119	1.0253	0.9927	0.9881	1.0021
Bias	0.1884	0.0039	-0.0325	0.0051	0.1069	0.0147	-0.0078	0.0119	0.0253	-0.0073	-0.0119	0.0021
variance	1.5315	1.2260	2.1793	2.5333	0.8943	0.7649	0.9137	1.5950	0.5878	0.5286	0.6083	1.0811
MSE	1.5670	1.2260	2.1804	2.5333	0.9058	0.7652	0.9137	1.5952	0.5885	0.5286	0.6085	1.0811
$\widehat{ u_2}$	1.0311	1.0220	1.0017	0.9895	1.0974	1.0113	0.9926	1.0143	1.0901	1.0082	0.9948	1.0039
Bias	0.0311	0.0220	0.0017	-0.0105	0.0974	0.0113	-0.0074	0.0143	0.0901	0.0082	-0.0052	0.0039
variance	1.3491	1.0920	1.5705	2.2549	1.0802	0.9325	1.3096	1.8828	0.7401	0.6612	0.8356	1.3574
MSE	1.3500	1.0925	1.5705	2.2550	1.0897	0.9326	1.3097	1.8830	0.7482	0.6613	0.8356	1.3574

 $\mathtt{dry}_i \sim \mathtt{BP}(\mu_i,\phi_i)$ with systematic components given by

$$\begin{split} \log(\mu_i) &= \beta_0 + \beta_1 \mathtt{wet}_i + \beta_2 \mathtt{cs}_i \\ \log(\phi_i) &= \nu_0 + \nu_1 \mathtt{wet}_i, \quad i = 1, \dots, 27. \end{split}$$

An R implementation for obtaining MLEs along with their corrected versions proposed in this

Table 3: Simulation results for p = q = 3.

	n = 30				n = 40			n = 60				
Estimates	MLE	Cox-Snell	Firth	p-boot	MLE	Cox-Snell	Firth	p-boot	MLE	Cox-Snell	Firth	p-boot
$\widehat{\beta}_0$	0.9959	0.9989	1.0028	0.9973	0.9985	1.0002	1.0017	1.0011	0.9973	1.0011	1.0023	0.9994
Bias	-0.0041	-0.0011	0.0028	-0.0027	-0.0015	0.0002	0.0017	0.0011	-0.0027	0.0011	0.0023	-0.0006
variance	0.0249	0.0249	0.0267	0.0418	0.0301	0.0301	0.0324	0.0531	0.0111	0.0111	0.0112	0.0204
MSE	0.0249	0.0249	0.0267	0.0418	0.0301	0.0301	0.0324	0.0531	0.0111	0.0111	0.0112	0.0204
\widehat{eta}_1	1.0026	1.0024	1.0016	1.0031	0.9983	0.9996	1.0002	0.9975	1.0018	1.0016	1.0015	1.0014
Bias	0.0026	0.0024	0.0016	0.0031	-0.0017	-0.0004	0.0002	-0.0025	0.0018	0.0016	0.0015	0.0014
variance	0.0354	0.0354	0.0372	0.0598	0.0249	0.0249	0.0261	0.0443	0.0135	0.0135	0.0138	0.0249
MSE	0.0354	0.0354	0.0372	0.0598	0.0249	0.0249	0.0261	0.0443	0.0136	0.0135	0.0138	0.0249
\widehat{eta}_2	1.0019	1.0027	1.0018	1.0047	0.9995	0.9998	0.9995	0.9993	0.9993	0.9975	0.9972	0.9994
Bias	0.0019	0.0027	0.0018	0.0047	-0.0005	-0.0002	-0.0005	-0.0007	-0.0007	-0.0025	-0.0028	-0.0006
variance	0.0309	0.0309	0.0327	0.0528	0.0235	0.0234	0.0245	0.0413	0.0167	0.0167	0.0171	0.0307
MSE	0.0309	0.0309	0.0327	0.0528	0.0235	0.0234	0.0245	0.0413	0.0167	0.0167	0.0171	0.0307
\widehat{eta}_3	0.9937	0.9943	0.9942	0.9947	0.9978	0.9987	0.9983	0.9997	0.9997	0.9980	0.9974	0.9990
Bias	-0.0063	-0.0057	-0.0058	-0.0053	-0.0022	-0.0013	-0.0017	-0.0003	-0.0003	-0.0020	-0.0026	-0.0010
variance	0.0397	0.0397	0.0419	0.0659	0.0309	0.0309	0.0336	0.0556	0.0190	0.0190	0.0192	0.0347
MSE	0.0398	0.0397	0.0420	0.0659	0.0309	0.0309	0.0336	0.0556	0.0190	0.0190	0.0192	0.0347
$\widehat{\nu_0}$	1.1515	1.0091	1.0325	1.0023	1.0786	1.0051	1.0316	0.9976	1.0479	1.0049	1.0138	0.9899
Bias	0.1515	0.0091	0.0325	0.0023	0.0786	0.0051	0.0316	-0.0024	0.0479	0.0049	0.0138	-0.0101
variance	0.5818	0.4711	1.6123	0.9480	0.4827	0.4123	0.8403	0.8484	0.3339	0.3001	0.6093	0.5945
MSE	0.6048	0.4712	1.6134	0.9480	0.4889	0.4123	0.8413	0.8484	0.3362	0.3001	0.6095	0.5946
$\widehat{ u_1}$	1.1387	0.9982	0.9634	0.9994	1.1418	0.9729	0.9457	0.9954	1.0161	0.9977	0.9981	1.0051
Bias	0.1387	-0.0018	-0.0366	-0.0006	0.1418	-0.0271	-0.0543	-0.0046	0.0161	-0.0023	-0.0019	0.0051
variance	1.2110	0.9163	2.4586	1.8249	0.7643	0.6403	1.3250	1.3258	0.4878	0.4270	0.5802	0.8715
MSE	1.2303	0.9164	2.4599	1.8249	0.7844	0.6411	1.3280	1.3259	0.4881	0.4270	0.5802	0.8715
_												
$\widehat{\nu_2}$	1.0566	1.0589	1.0382	1.0115	1.0808	0.9918	0.9678	1.0014	1.0969	1.0066	0.9867	1.0172
Bias	0.0566	0.0589	0.0382	0.0115	0.0808	-0.0082	-0.0322	0.0014	0.0969	0.0066	-0.0133	0.0172
variance	1.1501	0.8767	1.5965	1.7398	0.8195	0.6709	1.3184	1.3820	0.5622	0.4854	0.8178	0.9851
MSE	1.1533	0.8802	1.5979	1.7400	0.8260	0.6710	1.3195	1.3820	0.5716	0.4855	0.8180	0.9854
^	4.466-	4.00:-	0.056:	4.000	4.005.0	1.00**	4 0 44 -	4.046-	4 00	4.04	0.004:	0.0000
$\widehat{\nu_3}$	1.1683	1.0215	0.9504	1.0233	1.0929	1.0866	1.0413	1.0185	1.0920	1.0114	0.9911	0.9990
Bias	0.1683	0.0215	-0.0496	0.0233	0.0929	0.0866	0.0413	0.0185	0.0920	0.0114	-0.0089	-0.0010
variance	1.0738	0.8799	2.3169	1.7278	0.9323	0.7420	2.0245	1.4912	0.4791	0.4213	0.5501	0.8672
MSE	1.1021	0.8803	2.3194	1.7283	0.9409	0.7495	2.0262	1.4916	0.4876	0.4214	0.5502	0.8672

article related to the data used is available at GitHub BiasBPR¹ repository.

Table 4 presents the maximum likelihood estimates along with their corrected versions and the corresponding estimates of asymptotic standard errors in parentheses. Note that for the parameters that model the precision, $\widehat{\nu}_0$ and $\widehat{\nu}_1$, the maximum likelihood estimates are smaller than the bias corrected ones, while for the parameters that model the mean, $\widehat{\beta}_0$, $\widehat{\beta}_1$ and $\widehat{\beta}_2$, the estimates are quite close.

In the Table 5, we present the relative changes (RCs). The RCs are calculated from $RC(\widehat{\theta}) = |(\widehat{\theta} - \widehat{\theta}_0)/\widehat{\theta}_0| \times 100\%$, where $\widehat{\theta}$ denotes the MLE of θ and $\widehat{\theta}_0$ denotes the bias-corrected MLE of θ . From Table 5, the bias-corrected MLEs for ν_0 and ν_1 present similar results. In contrast to the Cox-Snell and p-boot bias-corrected estimators, the preventive method (Firth) gives estimates that dramatically change for β_1 and β_2 . For example, the second-order bias is 36.585% of the total amount of the MLE of β_2 . Thus, this real example illustrates that bias corrections can have a great

https://github.com/sesiommedeiros/BiasBPR

Table 4: Estimated values of the parameters with estimated asymptotic standard errors in parenthesis.

Estimates	MLE	Cox-Snell	Firth	p-boot
$\widehat{\beta}_0$	-1.5550	-1.5550	-1.5596	-1.5526
	(0.0224)	(0.0252)	(0.0257)	(0.0254)
\widehat{eta}_1	-0.0221	-0.0221	-0.0193	-0.0231
	(0.0105)	(0.0118)	(0.0121)	(0.0120)
\widehat{eta}_2	-0.0182	-0.0183	-0.0287	-0.0203
	(0.1243)	(0.1393)	(0.1418)	(0.1406)
$\widehat{ u}_0$	1.4536	1.5362	1.5881	1.5950
	(0.9059)	(0.9060)	(0.9060)	(0.9060)
$\widehat{ u}_1$	5.1014	4.9186	4.8675	4.8728
	(0.5896)	(0.5897)	(0.5897)	(0.5897)

effect on the conclusions.

Table 5: Relative changes for each parameter.

Estimator	$RC(\widehat{\beta}_0)$	$RC(\widehat{\beta}_1)$	$RC(\widehat{\beta}_2)$	$RC(\widehat{\nu}_0)$	$RC(\widehat{\nu}_1)$
Cox-Snell	0.0000	0.0000	0.5464	5.3769	3.7165
Firth	0.2950	14.508	36.585	8.4692	4.8053
p-boot	0.1546	4.3290	10.345	8.8652	4.6914

6 Concluding remarks

In this paper, we have examined a wide range of estimators for the unknown parameter vector of the BP regression model. In particular, we have derived a closed-form expression, in matrix form, for the second order biases of the ML estimators of the parameters that index the BP regression model proposed by Bourguignon et al. (2018). For this, we use the expressions obtained through Cox and Snell's (Cox and Snell, 1968) formulae and Firth's (Firth, 1993) estimating equation. We also considered a bias correction based on parametric bootstrap. The numerical evidence here presented shows that our proposed estimators has good finite-sample behavior, even when the sample size is small. For the mean structure, we observe that the MLE presents a very small bias (even in small samples). In this case, it is not necessary to use the bias-corrected estimators. However, for the precision structure, we observe that the MLE can become considerably biased and, therefore, we strongly recommend its bias correction. This behavior was also observed in the application to the real data set presented, therefore, we strongly recommend that practitioners use these corrected estimators when modeling data using the BP regression model. Finally, we have applied our proposed estimators to a real data.

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Appendix A

In this Appendix, we presente the Fisher's information matrix for the BP regression model, which is expressed in matrix form as

$$\mathbf{K}(\boldsymbol{\beta}, \boldsymbol{\nu}) = \begin{pmatrix} \mathbf{X}^{\top} K_{\beta\beta} \mathbf{X} & \mathbf{X}^{\top} K_{\beta\nu} \mathbf{Z} \\ \mathbf{Z}^{\top} K_{\nu\beta} \mathbf{X} & \mathbf{Z}^{\top} K_{\nu\nu} \mathbf{Z} \end{pmatrix}, \tag{5}$$

where

$$K_{\beta\beta} = \operatorname{diag}\left\{ (1+\phi_i)^2 a_i \left(\frac{\partial \mu_i}{\partial \eta_{1i}}\right)^2 \right\}, K_{\nu\nu} = \operatorname{diag}\left\{ b_i \left(\frac{\partial \phi_i}{\partial \eta_{2i}}\right)^2 \right\},$$

$$K_{\beta\nu} = K_{\nu\beta}^{\mathsf{T}} = \operatorname{diag}\left\{ (1+\phi_i)[a_i\mu_i - \psi^{(1)}(\mu_i(1+\phi_i) + \phi_i + 2)] \frac{\partial \mu_i}{\partial \eta_{1i}} \frac{\partial \phi_i}{\partial \eta_{2i}} \right\},$$

with $a_i = \psi^{(1)}(\mu_i(1+\phi_i)) - \psi^{(1)}(\mu_i(1+\phi_i) + \phi_i + 2)$ and $b_i = \mu_i^2\psi^{(1)}(\mu_i(1+\phi_i)) - (1+\mu_i)^2\psi^{(1)}(\mu_i(1+\phi_i) + \phi_i + 2) + \psi^{(1)}(\phi_1 + 2)$.

We can rewrite the Fisher's information matrix given in (5). For this, let K be a $2n \times 2n$ matrix and X be a $2n \times (p+q)$ matrix defined, respectively, as

$$\widetilde{\boldsymbol{K}} = \begin{pmatrix} K_{\beta\beta} & K_{\beta\nu} \\ K_{\nu\beta} & K_{\nu\nu} \end{pmatrix}.$$

and

$$\mathbb{X} = \begin{pmatrix} X & 0 \\ 0 & Z \end{pmatrix}.$$

Thus, we have that

$$\mathbf{K}(\boldsymbol{\beta}, \boldsymbol{\nu}) = \mathbb{X}^{\top} \widetilde{K} \mathbb{X}.$$

Appendix B

In this Appendix, we present the cumulants and derivatives needed to obtain the second order bias of $\widehat{\beta}$ and $\widehat{\nu}$. In addiction, we describe in details how to obtain $B(\widehat{\beta})$ and $B(\widehat{\nu})$ from Eq. (4). In order to present the cumulants in a summarized form, consider the following quantities:

$$a_{i} = \psi^{(1)}(\mu_{i}(1+\phi_{i})) - \psi^{(1)}(\mu_{i}(1+\phi_{i}) + \phi_{i} + 2),$$

$$b_{i} = \mu_{i}^{2}\psi^{(1)}(\mu_{i}(1+\phi_{i})) - (1+\mu_{i})^{2}\psi^{(1)}(\mu_{i}(1+\phi_{i}) + \phi_{i} + 2) + \psi^{(1)}(\phi_{1} + 2),$$

$$c_{i} = \psi^{(2)}(\mu_{i}(1+\phi_{i})) - \psi^{(2)}(\mu_{i}(1+\phi_{i}) + \phi_{i} + 2),$$

$$d_{i} = (1+\mu_{i})^{2}\psi^{(2)}(\mu_{i}(1+\phi_{i}) + \phi_{i} + 2) - \mu_{i}^{2}\psi^{(2)}(\mu_{i}(1+\phi_{i})),$$

$$e_{i} = (1+\mu_{i})^{3}\psi^{(2)}(\mu_{i}(1+\phi_{i}) + \phi_{i} + 2) - \mu_{i}^{3}\psi^{(2)}(\mu_{i}(1+\phi_{i})) - \psi^{(2)}(\phi_{i} + 2).$$

So, the cumulants needed here are presented as follow:

$$\begin{split} \kappa_{rs} &= -\sum_{i=1}^{n} (1+\phi_{i})^{2} a_{i} \left(\frac{\partial \mu_{i}}{\partial \eta_{1i}}\right)^{2} x_{ir} x_{is}, \\ \kappa_{rS} &= -\sum_{i=1}^{n} (1+\phi_{i}) [a_{i}\mu_{i} - \psi^{(1)}(\mu_{i}(1+\phi_{i}) + \phi_{i} + 2)] \frac{\partial \mu_{i}}{\partial \eta_{1i}} \frac{\partial \phi_{i}}{\partial \eta_{2i}} x_{ir} z_{iS}, \\ \kappa_{RS} &= -\sum_{i=1}^{n} b_{i} \left(\frac{\partial \phi_{i}}{\partial \eta_{2i}}\right)^{2} z_{iR} z_{iS}, \\ \kappa_{rsu} &= -\sum_{i=1}^{n} (1+\phi_{i})^{2} \left\{ (1+\phi_{i})c_{i} \left(\frac{\partial \mu_{i}}{\partial \eta_{1i}}\right)^{3} + 3a_{i} \frac{\partial \mu_{i}}{\partial \eta_{1i}} \frac{\partial^{2} \mu_{i}}{\partial \eta_{1i}^{2}} \right\} x_{ir} x_{is} x_{iu}, \\ \kappa_{rsU} &= -\sum_{i=1}^{n} (1+\phi_{i}) \left\{ 2a_{i} + (1+\phi_{i})c_{i}\mu_{i} - (1+\phi_{i})\psi^{(2)}(\mu_{i}(1+\phi_{i}) + \phi_{i} + 2) \right\} \left(\frac{\partial \mu_{i}}{\partial \eta_{1i}}\right)^{2} \frac{\partial \phi_{i}}{\partial \eta_{2i}} x_{ir} x_{is} z_{iU} \\ &+ \sum_{i=1}^{n} (1+\phi_{i}) \left\{ \psi^{(1)}(\mu_{i}(1+\phi_{i}) + \phi_{i} + 2) - a_{i}\mu_{i} \right\} \frac{\partial^{2} \mu_{i}}{\partial \eta_{1i}^{2}} \frac{\partial \phi_{i}}{\partial \eta_{2i}} x_{ir} x_{is} z_{iU}, \\ \kappa_{rSU} &= \sum_{i=1}^{n} \left\{ \psi^{(1)}(\mu_{i}(1+\phi_{i}) + \phi_{i} + 2) - a_{i}\mu_{i} \right\} \left[(1+\phi_{i}) \frac{\partial \mu_{i}}{\partial \eta_{1i}} \frac{\partial^{2} \phi_{i}}{\partial \eta_{2i}^{2}} + 2 \left(\frac{\partial \phi_{i}}{\partial \eta_{2i}}\right)^{2} \frac{\partial \mu_{i}}{\partial \eta_{1i}} \right] x_{ir} z_{iS} z_{iU} \\ &+ \sum_{i=1}^{n} (1+\phi_{i}) d_{i} \frac{\partial \mu_{i}}{\partial \eta_{1i}} \left(\frac{\partial \phi_{i}}{\partial \eta_{2i}}\right)^{2} x_{ir} z_{iS} z_{iU}, \\ \kappa_{RSU} &= \sum_{i=1}^{n} \left\{ e_{i} \left(\frac{\partial \phi_{i}}{\partial \eta_{2i}}\right)^{3} - 3b_{i} \frac{\partial^{2} \phi_{i}}{\partial \eta_{2i}^{2}} \frac{\partial \phi_{i}}{\partial \eta_{2i}} \right\} z_{iR} z_{iS} z_{iU}. \end{split}$$

Taking the derivative of the cumulants with respect to the model parameters, we have

$$\begin{split} \kappa_{rs}^{(u)} &= -\sum_{i=1}^{n} (1+\phi_{i})^{2} \left\{ (1+\phi_{i})c_{i} \left(\frac{\partial \mu_{i}}{\partial \eta_{1i}} \right)^{3} + 2a_{i} \frac{\partial \mu_{i}}{\partial \eta_{1i}} \frac{\partial^{2} \mu_{i}}{\partial \eta_{2i}^{2}} \right\} x_{ir} x_{is} x_{iu}, \\ \kappa_{rs}^{(U)} &= -\sum_{i=1}^{n} \left\{ (1+\phi_{i})^{2} [c_{i}\mu_{i} - \psi^{(2)}(\mu_{i}(1+\phi_{i}) + \phi_{i} + 2)] + 2(1+\phi_{i})a_{i} \right\} \left(\frac{\partial \mu_{i}}{\partial \eta_{1i}} \right)^{2} \frac{\partial \phi_{i}}{\partial \eta_{2i}} x_{ir} x_{is} z_{iU}, \\ \kappa_{RS}^{(u)} &= \sum_{i=1}^{n} \left\{ d_{i}(1+\phi_{i}) + 2\psi^{(1)}(\mu_{i}(1+\phi_{i}) + \phi_{i} + 2) - 2a_{i}\mu_{i} \right\} \frac{\partial \mu_{i}}{\partial \eta_{1i}} \left(\frac{\partial \phi_{i}}{\partial \eta_{2i}} \right)^{2} z_{iR} z_{iS} x_{iu}, \\ \kappa_{RS}^{(U)} &= \sum_{i=1}^{n} \left\{ e_{i} \left(\frac{\partial \phi_{i}}{\partial \eta_{2i}} \right)^{3} - 2b_{i} \frac{\partial \phi_{i}}{\partial \eta_{2i}} \frac{\partial^{2} \phi_{i}}{\partial \eta_{2i}^{2}} \right\} z_{iR} z_{iS} z_{iU}, \\ \kappa_{rS}^{(u)} &= -\sum_{i=1}^{n} (1+\phi_{i}) \left\{ (1+\phi_{i})\mu_{i}c_{i} + a_{i} - (1+\phi_{i})\psi^{(2)}(\mu_{i}(1+\phi_{i}) + \phi_{i} + 2) \right\} \left(\frac{\partial \mu_{i}}{\partial \eta_{1i}} \right)^{2} \frac{\partial \phi_{i}}{\partial \eta_{2i}} x_{ir} x_{iu} z_{iS} \\ &- \sum_{i=1}^{n} (1+\phi_{i}) \left\{ a_{i}\mu_{i} - \psi^{(1)}(\mu_{i}(1+\phi_{i}) + \phi_{i} + 2) \right\} \frac{\partial^{2} \mu_{i}}{\partial \eta_{1i}^{2}} \frac{\partial \phi_{i}}{\partial \eta_{2i}} x_{ir} x_{iu} z_{iS}, \\ \kappa_{rS}^{(U)} &= \sum_{i=1}^{n} \left\{ (1+\phi_{i})d_{i} - a_{i}\mu_{i} + \psi^{(1)}(\mu_{i}(1+\phi_{i}) + \phi_{i} + 2) \right\} \frac{\partial \mu_{i}}{\partial \eta_{1i}} \left(\frac{\partial \phi_{i}}{\partial \eta_{2i}} \right)^{2} x_{ir} z_{iS} z_{iU} \\ &+ \sum_{i=1}^{n} (1+\phi_{i}) \left\{ \psi^{(1)}(\mu_{i}(1+\phi_{i}) + \phi_{i} + 2) - a_{i}\mu_{i} \right\} \frac{\partial \mu_{i}}{\partial \eta_{1i}} \frac{\partial^{2} \phi_{i}}{\partial \eta_{1i}} x_{ir} z_{iS} z_{iU}. \end{split}$$

Consider now the following diagonal matrices:

$$\begin{split} M_1 &= \operatorname{diag}\left(-\frac{(1+\phi_i)^2}{2}\left[(1+\phi_i)c_i\left(\frac{\partial\mu_i}{\partial\eta_{1i}}\right)^3 + a_i\frac{\partial\mu_i}{\partial\eta_{1i}}\frac{\partial^2\mu_i}{\partial\eta_{1i}^2}\right]\right),\\ M_2 &= \operatorname{diag}\left(\frac{(1+\phi_i)}{2}\left[-a_i\mu_i + \psi^{(1)}(\mu_i(1+\phi_i) + \phi_i + 2)\right]\frac{\partial^2\mu_i}{\partial\eta_{1i}^2}\frac{\partial\phi_i}{\partial\eta_{2i}}\right.\\ &\quad - \frac{(1+\phi_i)^2}{2}\left[c_i\mu_i - \psi^{(2)}(\mu_i(1+\phi_i) + \phi_i + 2)\right]\left(\frac{\partial\mu_i}{\partial\eta_{1i}}\right)^2\frac{\partial\phi_i}{\partial\eta_{2i}}\right),\\ M_3 &= \operatorname{diag}\left(-\frac{(1+\phi_i)}{2}\left\{\left[2a_i + (1+\phi_i)c_i\mu_i - (1+\phi_i)\psi^{(2)}(\mu_i(1+\phi_i) + \phi_i + 2)\right]\left(\frac{\partial\mu_i}{\partial\eta_{1i}}\right)^2\frac{\partial\phi_i}{\partial\eta_{2i}}\right.\\ &\quad + \left.\left[\psi^{(1)}(\mu_i(1+\phi_i) + \phi_i + 2) - a_i\mu_i\right]\frac{\partial^2\mu_i}{\partial\eta_{1i}^2}\frac{\partial\phi_i}{\partial\eta_{2i}}\right\}\right),\\ M_4 &= \operatorname{diag}\left(\frac{1}{2}\left\{\left[(1+\phi_i)d_i + 2\psi^{(1)}(\mu_i(1+\phi_i) + \phi_i + 2) - 2a_i\mu_i\right]\frac{\partial\mu_i}{\partial\eta_{1i}}\left(\frac{\partial\phi_i}{\partial\eta_{2i}}\right)^2\right.\\ &\quad - \left.(1+\phi_i)\left[\psi^{(1)}(\mu_i(1+\phi_i) + \phi_i + 2) - a_i\mu_i\right]\frac{\partial\mu_i}{\partial\eta_{1i}}\frac{\partial^2\phi_i}{\partial\eta_{2i}^2}\right\}\right),\\ M_5 &= \operatorname{diag}\left(\frac{(1+\phi_i)}{2}\left[d_i\left(\frac{\partial\phi_i}{\partial\eta_{2i}}\right)^2\frac{\partial\mu_i}{\partial\eta_{1i}} + \psi^{(1)}(\mu_i(1+\phi_i) + \phi_i + 2) - a_i\mu_i\right]\frac{\partial\mu_i}{\partial\eta_{1i}}\frac{\partial^2\phi_i}{\partial\eta_{2i}^2}\right),\\ M_6 &= \operatorname{diag}\left(\frac{1}{2}\left[e_i\left(\frac{\partial\phi_i}{\partial\eta_{2i}}\right)^3 - b_i\frac{\partial\phi_i}{\partial\eta_{2i}}\frac{\partial^2\phi_i}{\partial\eta_{2i}^2}\right]\right). \end{split}$$

Considering the matrices defined above, we have that m_{ij} represent the ith element of M_j . So, we

have

$$\kappa_{rs}^{(u)} - \frac{1}{2}\kappa_{rsu} = \sum_{i=1}^{n} m_{1i}x_{ir}x_{is}x_{iu},
\kappa_{Rs}^{(u)} - \frac{1}{2}\kappa_{Rsu} = \sum_{i=1}^{n} m_{2i}x_{is}x_{iu}z_{iR},
\kappa_{rs}^{(u)} - \frac{1}{2}\kappa_{rsu} = \sum_{i=1}^{n} m_{2i}x_{ir}x_{iu}z_{iS},
\kappa_{rs}^{(u)} - \frac{1}{2}\kappa_{rsu} = \sum_{i=1}^{n} m_{3i}x_{ir}x_{is}z_{iU},
\kappa_{Rs}^{(u)} - \frac{1}{2}\kappa_{Rsu} = \sum_{i=1}^{n} m_{4i}x_{iu}z_{iR}z_{iS},
\kappa_{Rs}^{(u)} - \frac{1}{2}\kappa_{Rsu} = \sum_{i=1}^{n} m_{5i}x_{is}z_{iU}z_{iR},
\kappa_{rs}^{(U)} - \frac{1}{2}\kappa_{rsu} = \sum_{i=1}^{n} m_{5i}x_{ir}z_{iS}z_{iU},
\kappa_{Rs}^{(U)} - \frac{1}{2}\kappa_{Rsu} = \sum_{i=1}^{n} m_{6i}z_{iR}z_{iS}z_{iU},
\kappa_{Rs}^{(U)} - \frac{1}{2}\kappa_{Rsu} = \sum_{i=1}^{n} m_{6i}z_{iR}z_{iS}z_{iU},$$

We now obtain the terms from (4), presenting in detail the algebra to obtain the first term of this expression. To calculate the other terms, we follow the same logic.

$$\sum_{r,s,u} \kappa^{ar} \kappa^{su} \left\{ \kappa_{rs}^{(u)} - \frac{1}{2} \kappa_{rsu} \right\} = \sum_{r,s,u} \left(\kappa^{ar} \kappa^{su} \sum_{i=1}^{n} m_{1i} x_{ir} x_{is} x_{iu} \right) \\
= \sum_{i=1}^{n} m_{1i} \left(\sum_{r} x_{ir} \kappa^{ar} \right) \left(\sum_{s,u} x_{ir} \kappa^{su} x_{iu} \right) \\
= \sum_{i=1}^{n} m_{1i} \left(\sum_{r} x_{ir} \kappa^{ar} \right) \boldsymbol{\delta}_{i}^{\top} (X \kappa^{\beta \beta} X^{\top}) \boldsymbol{\delta}_{i} \\
= \boldsymbol{\delta}_{a}^{\top} \sum_{i=1}^{n} \kappa^{a\beta} X^{\top} \boldsymbol{\delta}_{i} m_{1i} \boldsymbol{\delta}_{i}^{\top} (X \kappa^{\beta \beta} X^{\top}) \boldsymbol{\delta}_{i} \\
= \boldsymbol{\delta}_{a}^{\top} \kappa^{a\beta} X^{\top} M_{1} P_{\beta \beta},$$

being $\kappa^{a\beta}$ the matrix $\kappa^{\beta\beta}$ if $a=1,\ldots,p$ and $\kappa^{\nu\beta}$ if $a=p+1,\ldots,q$, $\boldsymbol{\delta}_a$ ($\boldsymbol{\delta}_i$) an $n\times 1$ vector with a one in the ath (ith) position. Also, the vector $P_{\beta\beta}$ is presented in Section 3. Likewise, we have the remaining quantities expressed by

$$\sum_{R,s,u} \kappa^{aR} \kappa^{su} \left\{ \kappa_{Rs}^{(u)} - \frac{1}{2} \kappa_{Rsu} \right\} = \boldsymbol{\delta}_{a}^{\mathsf{T}} \kappa^{a\nu} Z^{\mathsf{T}} M_{2} P_{\beta\beta},$$

$$\sum_{r,S,u} \kappa^{ar} \kappa^{Su} \left\{ \kappa_{rS}^{(u)} - \frac{1}{2} \kappa_{rSu} \right\} = \boldsymbol{\delta}_{a}^{\mathsf{T}} \kappa^{a\beta} X^{\mathsf{T}} M_{2} P_{\beta\nu},$$

$$\sum_{r,s,U} \kappa^{ar} \kappa^{sU} \left\{ \kappa_{rs}^{(U)} - \frac{1}{2} \kappa_{rsU} \right\} = \boldsymbol{\delta}_{a}^{\mathsf{T}} \kappa^{a\beta} X^{\mathsf{T}} M_{3} P_{\beta\nu},$$

$$\sum_{R,S,u} \kappa^{aR} \kappa^{Su} \left\{ \kappa_{RS}^{(u)} - \frac{1}{2} \kappa_{RSu} \right\} = \boldsymbol{\delta}_{a}^{\mathsf{T}} \kappa^{a\nu} Z^{\mathsf{T}} M_{4} P_{\beta\nu},$$

$$\sum_{R,s,U} \kappa^{aR} \kappa^{sU} \left\{ \kappa_{Rs}^{(U)} - \frac{1}{2} \kappa_{RsU} \right\} = \boldsymbol{\delta}_{a}^{\mathsf{T}} \kappa^{a\nu} Z^{\mathsf{T}} M_{5} P_{\beta\nu},$$

$$\sum_{r,S,U} \kappa^{ar} \kappa^{SU} \left\{ \kappa_{rS}^{(U)} - \frac{1}{2} \kappa_{rSU} \right\} = \boldsymbol{\delta}_{a}^{\mathsf{T}} \kappa^{a\beta} X^{\mathsf{T}} M_{5} P_{\nu\nu},$$

$$\sum_{R,S,U} \kappa^{aR} \kappa^{SU} \left\{ \kappa_{RS}^{(U)} - \frac{1}{2} \kappa_{RSU} \right\} = \boldsymbol{\delta}_{a}^{\mathsf{T}} \kappa^{a\beta} Z^{\mathsf{T}} M_{6} P_{\nu\nu},$$

where $\kappa^{a\nu}$ is the matrix $\kappa^{\beta\nu}$ if $a=1,\ldots,p$ and $\kappa^{\nu\nu}$ if $a=p+1,\ldots,q$ and the vectors $P_{\beta\nu}$ and $P_{\nu\nu}$ were presented in Section 3.