On Hilberg's Law and Its Links with Guiraud's Law

Łukasz Dębowski
Institute of Computer Science
Polish Academy of Sciences
ldebowsk@ipipan.waw.pl

Abstract

Hilberg (1990) supposed that finite-order excess entropy of a random human text is proportional to the square root of the text length. Assuming that Hilberg's hypothesis is true, we derive Guiraud's law, which states that the number of word types in a text is greater than proportional to the square root of the text length. Our derivation is based on some mathematical conjecture in coding theory and on several experiments suggesting that words can be defined approximately as the nonterminals of the shortest context-free grammar for the text. Such operational definition of words can be applied even to texts deprived of spaces, which do not allow for Mandelbrot's "intermittent silence" explanation of Zipf's and Guiraud's laws. In contrast to Mandelbrot's, our model assumes some probabilistic long-memory effects in human narration and might be capable of explaining Menzerath's law.

Keywords: excess entropy, grammar-based compression, Guiraud's law, Zipf's law

1 Introduction

Over a decade ago, Hilberg (1990) reinterpreted Shannon's (1950) well-known experimental data and formulated a novel hypothesis concerning the entropy of human language. The hypothesis states that block entropy H(n) of a text drawn from natural language production, except for disputable constant and linear terms, is proportional to the square root of the text length n measured in phonemes (or letters),

$$H(n) \approx h_0 + h_\mu n^\mu + h n,\tag{1}$$

where $\mu \approx 1/2$. For brevity, we call relation (1) Hilberg's law. Hilberg's publication appeared in a technical journal of telecommunications. It was popularized among natural scientists by Ebeling (Ebeling and Nicolis, 1991; Ebeling and Pöschel, 1994) and stimulated some discussions (Bialek et al., 2001; Crutchfield and Feldman, 2003; Shalizi, 2001; Dębowski, 2001, 2004a).

In this article, we shall discuss some interaction between Hilberg's law and the better known Guiraud's and Zipf's laws. Empirical Guiraud's law (Guiraud, 1954) states that the number of orthographic word types V in a text behaves like

$$V \propto N^{\rho}$$
, (2)

where ρ < 1 is constant and *N* is the length of the text measured in orthographic word tokens. On the other hand, Zipf's-Mandelbrot's law (Zipf, 1935, 1949; Mandelbrot, 1954) states that any text obeys relation

$$c(w) \propto \frac{1}{r(w)^B},$$
 (3)

where B > 1 is constant, frequency c(w) is the count of word w in the text, and rank r(w) is the position of word w in the list of words sorted in descending order by c(w).

We do not know to what extent Hilberg's law is valid. Formula (1) presupposes some stationary probabilistic model of the entire natural language production, which is a highly hypothetical entity itself. Nevertheless, we would like to argue that some form of Guiraud's law can be deduced from equation (1). Strictly speaking, assuming that Hilberg's law is true for all n, we shall only infer some lower bound for the growth of the vocabulary size. Despite that restriction, we think that our explanation of Guiraud's law can be more linguistically plausible than the famous joint derivation of Guiraud's and Zipf's laws provided by Mandelbrot (1953). The latter derivation is known also as "intermittent silence" explanation (Miller, 1957; Li, 1998).

Hilberg's law concerns the probabilistic distribution of arbitrary phoneme or letter strings, i.e. the law constrains the distribution of all human texts. On the other hand, both Guiraud's and Zipf's laws concern the distribution of individual words in texts. Saying that Guiraud's law can be deduced from Hilberg's law, we presuppose some procedure which transforms the distribution of phoneme strings (i.e. texts) into the corresponding distribution of words. In some naive approach, we could assume that the text is a string of phonemes or spaces and the words are the space-to-space strings of phonemes. In fact, "intermittent silence" explanation assumes that the text is a string of probabilistically independent random tokens taking the values of spaces and phonemes. Given this assumption and the space-to-space definition of word, Mandelbrot deduced Zipf's law, and hence Guiraud's law can be deduced as well (Kornai, 2002).

Unfortunately, "intermittent silence" explanation cannot be applied to natural language. We know that the occurrences of phonemes in the language production exhibit some strong probabilistic dependence and there are no definite spaces between the words in human speech (Jelinek, 1997). If we want to derive Zipf's law from the distribution of mere phoneme strings, we must use some definition of word tokens which could be applied to the text deprived of spaces and which would match empirically the definition of word tokens given by spelling conventions or by semantic considerations.

Some well-defined tokenization of the space-deprived text into word-like strings can be given by grammar-based text compression (Kieffer and Yang, 2000). In grammar-based compression, the text is represented as a special context-free grammar, called an admissible grammar. That class of context-free grammars should not be confused with phrase structure grammars: The nonterminals of admissible grammars correspond to fixed strings of phonemes rather than to part-of-speech classes. Each admissible grammar gives some tokenization of the text into hierarchically structured word-like strings being the nonterminal tokens. It was empirically confirmed that for the grammars which approximate the shortest admissible grammar for a human text, the nonterminals usually correspond to the orthographic words (de Marcken, 1996; Nevill-Manning, 1996).

We will show that the expected number of nonterminal types for the shortest admissible grammar cannot be less than proportional to so called finite-order excess entropy of the random text. It is some mathematical result based on a line of theorems and one unproved conjecture. On the other hand, if Hilberg's hypothesis is true then the finite-order excess entropy of the text

is roughly proportional to the square root of the text length. The close empirical correspondence between the nonterminals and the orthographic words allows us to claim that Hilberg's law implies some lower bound for the vocabulary growth, i.e. some form of Guiraud's law.

The rest of this article fills in the details of the deductions and empirical observations mentioned in the previous paragraphs:

- In section 2, we introduce the definitions of stationary distribution, block entropy, excess entropy, and infinitary distributions. We sketch the history of Hilberg's law and the general research of block entropy for natural language production.
- In section 3, we introduce the concepts of admissible and irreducible grammars. We also discuss some empirical evidence that the shortest admissible grammar matches largely the linguistic tokenization for the human text.
- In section 4, we relate block entropy to the expected length of irreducible grammar-based codes. Assuming some mathematical conjecture, we show that the expected total length of the non-initial productions of the shortest grammar cannot be less than finite-order excess entropy.
- In section 5, we discuss Guiraud's law in detail and we argue that Hilberg's law explains it better than the assumption of "intermittent silence". Some arguments for Hilberg's law explanation are: (i) non-randomness of texts, (ii) empirical detectability of word boundaries and internal structures, (iii) possibility of explaining Menzerath's law, and (iv) significant variation of word frequencies across different texts.

2 Excess entropy and Hilberg's law

Let us imagine some infinite sequence of characters, e.g.

where subsequence _a_rose_is_a_hose_is is repeated infinitely to fix our imagination. For such an (infinite) sequence we can compute the relative frequency of any (finite) string which appears in that sequence.

For example, let us define probability $\mathbf{P}(\mathtt{rose})$ as the relative frequency of string rose in the infinite sequence (4). We shall do it in two steps. Let a_i stand for the ith character of (4), i.e. $a_1 = \mathtt{t}, \ a_2 = \mathtt{h}, \ a_3 = \mathtt{e}, \ a_4 = _, \ a_5 = \mathtt{r}$ etc. We will write the finite substrings as $a_{m:n} := (a_m, a_{m+1}, ..., a_n)$. The relative frequency $\mathbf{P}(\mathtt{rose}; n)$ of string rose in string $a_{1:n}$ is the number of all positions $a_i, \ 1 \le i \le n$, where string rose starts divided by n. For any equality relation \mathfrak{p} let us define $[\![\mathfrak{p}]\!] = 1$ if \mathfrak{p} is true and $[\![\mathfrak{p}]\!] = 0$ if \mathfrak{p} is false. Thus, $\mathbf{P}(\mathtt{rose}; k)$ can be expressed as

$$\mathbf{P}(\text{rose};k) := \frac{1}{k} \sum_{i=1}^{k} [a_{i:i+3} = \text{rose}],$$
 (5)

where := means definition. We have $\mathbf{P}(\text{rose};1) = 0$, $\mathbf{P}(\text{rose};5) = 1/5$, $\mathbf{P}(\text{rose};10) = 1/10$, $\mathbf{P}(\text{rose};30) = 2/30$ and so on.

Let us define probability P(rose) as the limit of relative frequencies of string rose in the initial substrings of (4). So we will write

$$\mathbf{P}(\mathsf{rose}) := \lim_{n \to \infty} \mathbf{P}(\mathsf{rose}; n). \tag{6}$$

Every 20th character in sequence (4) is a position where string rose starts, so $\mathbf{P}(\text{rose}) = 1/20$. Analogically, we can define probability $\mathbf{P}(v)$ for any string v,

$$\mathbf{P}(v) := \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \left[a_{i:i+\text{len}\,v-1} = v \right], \tag{7}$$

where len v is the number of characters in v. Hence, for (4) we obtain not only $\mathbf{P}(\mathtt{t}) = 0$ (there are no t's), $\mathbf{P}(\mathtt{s}) = 1/5$ (two in ten characters are s), and $\mathbf{P}(\mathtt{e}) = 1/10$ but also $\mathbf{P}(\mathtt{e_is_a}) = 1/10$, $\mathbf{P}(\mathtt{a_rose_is_a_hose}) = 1/20$, and $\mathbf{P}(\mathtt{a_rose_is_a_rose}) = 0$.

Now let us take some general sequence $(a_1, a_2, a_3, ...)$. Let \mathbb{V} be the finite set of characters that appear in that sequence. Let \mathbb{V}^+ be the set of all finite strings formed by concatenating the characters in \mathbb{V} . For any sequence $(a_1, a_2, a_3, ...)$ such that limit (7) exists for each string $v \in \mathbb{V}^+$, probability function **P** satisfies relations

$$0 \le \mathbf{P}(v) \le 1, \quad \sum_{a \in \mathbb{V}} \mathbf{P}(a) = 1, \quad \sum_{a \in \mathbb{V}} \mathbf{P}(av) = \mathbf{P}(v) = \sum_{a \in \mathbb{V}} \mathbf{P}(va). \tag{8}$$

We will call any function **P** satisfying conditions (8) for all $v \in \mathbb{V}^+$ a stationary distribution.¹ It is an open question whether for any stationary distribution **P** exists such $(a_1, a_2, a_3, ...)$ that we have (7) for all $v \in \mathbb{V}^+$.

Let \mathbb{V}^n be the set of all *n*-character long strings. We define block entropy H(n) of any stationary distribution **P** as the entropy of strings of length n,

$$H(n) := -\sum_{v \in \mathbb{V}^n} \mathbf{P}(v) \log_2 \mathbf{P}(v). \tag{9}$$

We also put H(0) := 0 for algebraic convenience.

For any stationary distribution **P** block entropy H(n) is a nonnegative, growing, and concave function of n (Crutchfield and Feldman, 2003), i.e.,

$$H(n) \ge 0, \quad H'(n) \ge 0, \quad H''(n) \le 0,$$
 (10)

where

$$H'(n) := H(n) - H(n-1), \quad H''(n) := H(n) - 2H(n-1) + H(n-2).$$
 (11)

Because of inequalities (10), we can define entropy rate as

$$h := \lim_{n \to \infty} H(n)/n = \lim_{n \to \infty} H'(n) \ge 0.$$
(12)

If entropy rate satisfies h > 0 then H(n) grows almost linearly against the string length n for very long strings, $H(n) \approx hn$. We can ask how fast H(n) approaches hn. The departure of H(n) from the linear growth is known as excess entropy.

Finite-order excess entropies E(n) are some functions of H(n) and H(2n),

$$E(n) := 2H(n) - H(2n) = -\sum_{k=2}^{n} (k-1)H''(k) - \sum_{k=n+1}^{2n} (2n-k+1)H''(k).$$
 (13)

¹Stationary distributions are the distributions of stationary stochastic processes (Upper, 1997). For simplicity, we avoid the mathematical terms of stochastic processes, random variables and probabilistic spaces (Billingsley, 1979; Kallenberg, 1997). Since we do not need these notions to present the core reasonings, we ignore them to make the article as elementary as possible.

So defined functions are nonnegative and growing, i.e., $E(n) \ge E(n-1) \ge 0$. Crutchfield and Feldman (2003) proved that (total) excess entropy E can be defined equivalently as

$$E := \lim_{n \to \infty} E(n) = \lim_{n \to \infty} [H(n) - hn]. \tag{14}$$

We also have inequality

$$E = -\sum_{k=2}^{\infty} (k-1)H''(k) \ge -\sum_{k=2}^{\infty} H''(k) = H(1) - h.$$
 (15)

Let vu be the concatenation of strings v and u. We will say that stationary distribution \mathbf{P} is an IID distribution if

$$\mathbf{P}(vu) = \mathbf{P}(v)\mathbf{P}(u) \tag{16}$$

for all strings $v, u \in \mathbb{V}^+$. (IID stands for independent identically distributed random variables.) Distributions **P** can be IID even for some quite ordered underlying sequences $(a_1, a_2, a_3, ...)$. For instance, **P** given through (7) is IID for the sequence of digits of consecutive natural numbers $(a_1, a_2, a_3, ...) = (1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 0, 1, 1, ...)$, which is called Champernowne sequence (Li and Vitányi, 1993). Anyway, we do not expect that we could obtain IID distribution **P** if we substituted some collection of human texts for sequence $(a_1, a_2, a_3, ...)$.

For any IID distribution **P** we have H(n) = nH(1) so h = H(1) and E = 0. Conversely, if H(1) - h > 0 or E > 0, then distribution **P** cannot be IID. For the extreme departures from the IID case, we have h = 0 or $E = \infty$. Stationary distributions exhibiting h = 0 are called deterministic while the distributions obeying $E = \infty$ are called infinitary (Crutchfield and Feldman, 2003). In appendix B, we present some properties of infinitary distributions which could be important for their possible applications in quantitative and computational linguistics but which are not so relevant for the main reasoning of this article.

Let us assume that we could obtain some definite stationary distribution **P** through formula (7) if we substituted the infinite concatenation of some human texts for $(a_1, a_2, a_3, ...)$. We will call such an infinite sequence $(a_1, a_2, a_3, ...)$ natural language production. Research in the hypothetical stationary distribution of natural language production has attracted many scientists. The first one to work in this area was Shannon (1950). He tried to estimate block entropy using the guessing method and assuming some correspondence between particular instances of English texts and the hypothetical random English language production. Shannon published some estimates of H(n) for strings of n consecutive letters, where $n \le 100$.

Shannon was not convinced of any particular asymptotics of block entropy H(n) for the natural language production (Hilberg, 1990) but the later researchers in quantitative linguistics tried to model H(n) by some simple formulae. For example, Hoffmann and Piotrovskij (1979) proposed a model of exponential convergence,

$$H(n)/n = (h_0 - h) \exp[-n/n_0] + h.$$
 (17)

Petrova (1973) fitted model (17) to French language data and obtained $1/n_0 \in (0.24, 0.33)$.

On the other hand, Hilberg (1990) replotted the original plot of H(n) vs. n by Shannon (1950) into a log-log scale and observed that a simple square-root dependence fits all the data points,

$$H(n) \propto n^{\mu}, \quad \mu \approx 1/2, \quad n \le 100.$$
 (18)

For our convenience, we will call Hilberg's law an algebraic relation which is slightly more general than Hilberg's original hypothesis (18). We will say that Hilberg's law holds for any stationary distribution **P** if only relation (1) holds with $\mu \approx 1/2$ and $h_{\mu} > 0$ for any n. For such definition, Hilberg's law is independent of any hypothesis on the particular value of entropy rate h and the constant term h_0 .

While Shannon estimated block entropy using the guessing method, Ebeling and his collaborators tried to estimate the asymptotics of H(n) by counting n-tuples in the samples of various symbolic sequences. Using improved entropy estimators, the researchers fitted the general formula (1) with $\mu \approx 1/2$ for natural language texts and $\mu \approx 1/4$ for classical music transcripts. For English and German texts H(n) could be safely estimated for $n \le 30$ characters with $h_0 \approx 0$, $h_\mu \approx 3.1$ bits and $h \approx 0.4$ bits (Ebeling and Nicolis, 1992; Ebeling and Pöschel, 1994). In contrast, Shannon's guessing data, reinterpreted by Hilberg (1990), suggest that equation (1) can be extrapolated at least for $n \le 100$.

It is important to note that the estimation of block entropy H(n) based on the naive estimation of probabilities $\mathbf{P}(v)$ for all strings v of length n is expensive in the input data. In order to estimate the value of H(n), we need a sample of length about $2^{H(n)}$ (Herzel et al., 1994). If we try to make shortcuts, we assume some particular properties of the unknown stationary distribution \mathbf{P} . Even Shannon's (1950) guessing method need not give the reliable estimates of H(n) for the language production if the probabilistic language model internalized by the experimental subjects differs from the model estimated from the corpus (Bod et al., 2003; Hug, 1997).

Let us note that for the block entropy of formula (1), finite-order excess entropies are

$$E(n) \approx h_0 + (2 - 2^{\mu})h_{\mu}n^{\mu}. \tag{19}$$

If relations (19) hold with $0 < \mu < 1$ for any n then the total excess entropy is $E = \infty$. Hence, every stationary distribution exhibiting Hilberg's law is infinitary.

At the moment, we have no clear idea how one could verify if Hilberg's law holds for the hypothetical stationary distribution of the language production. Nevertheless, we can provide some mixed inductive and deductive arguments that Hilberg's law implies some phenomena that can be observed in human language.

3 Words and the shortest grammars

In the following sections, we shall argue that Hilberg's law can explain some quantitative laws concerning the distribution of word types in the language production. Nevertheless, before we can speak of any distribution of words in a finite string of phonemes or letters, we need to delimit the word tokens themselves. If the words are some objective entities of the language, there should be some method for identifying the boundaries between the words in a sufficiently long string of phoneme or letter tokens even if we delete the spaces between words and ignore the lexicon.

Let us take some text deprived of spaces, e.g.

$$v = \text{shouldawoodchuckchuckifawoodchuckcouldchuckwood.}$$
 (20)

We can express our knowledge of word tokens describing string v by means of a two-level context-free grammar

$$G = \left\{ \begin{array}{ll} b_0 \mapsto b_5 b_1 b_7 b_6 b_2 b_1 b_7 b_4 b_6 b_3, & b_1 \mapsto \mathbf{a}, \\ b_2 \mapsto \mathrm{if}, & b_3 \mapsto \mathrm{wood}, & b_4 \mapsto \mathrm{could}, \\ b_5 \mapsto \mathrm{should}, & b_6 \mapsto \mathrm{chuck}, & b_7 \mapsto \mathrm{woodchuck} \end{array} \right\}. \tag{21}$$

Symbols b_i are called nonterminals. For each b_i there is some production rule $(b_i \mapsto g_i) \in G$. On the other hand, the typewriter-typed symbols, which have no productions rules in the grammar, will be called terminals. Nonterminal b_0 is called the initial symbol of the grammar. If we recursively substitute productions g_i for all nonterminals b_i where $(b_i \mapsto g_i) \in G$, then b_0 expands into string v with the requested tokenization into the words. Namely,

$$v = \overline{\text{should } \overline{\text{a}} \text{ woodchuck } \overline{\text{chuck } \overline{\text{if } \overline{\text{a}} \text{ woodchuck } \overline{\text{could } \overline{\text{chuck wood}}}}}$$

where notation \overline{g} means that G contains rule $b_i \mapsto g$ for some $i \neq 0$ (de Marcken, 1996).

Of course, if we were not given any previous knowledge of English lexicon, we could propose other tokenizations for text (20). For instance,

$$G = \left\{ \begin{array}{ll} b_0 \mapsto \mathrm{sh} b_1 b_4 b_2 \mathrm{if} b_4 \mathrm{c} b_1 b_2 b_3, & b_1 \mapsto \mathrm{ould}, \\ b_2 \mapsto \mathrm{chuck}, & b_3 \mapsto \mathrm{wood}, & b_4 \mapsto \mathrm{a} b_3 b_2 \end{array} \right\}$$
(22)

yields

$$v = \operatorname{sh} \overline{\operatorname{ould}} \overline{\operatorname{awood} \overline{\operatorname{chuck}}} \overline{\operatorname{chuck}} \overline{\operatorname{if}} \overline{\operatorname{awood}} \overline{\operatorname{chuck}} \operatorname{could} \overline{\operatorname{chuck}} \overline{\operatorname{wood}}.$$

In the extreme, we could define b_0 as the entire string v or each b_i , $i \neq 0$, as a single letter. Since we ignore English lexicon, we need some purely formal criterion for deciding what grammars G are good for arbitrary strings v and what are not.

Let us state some formal definitions. Context-free grammar G will be called a grammar (more precisely, admissible grammar) for string v (cf. Kieffer and Yang, 2000) if:

- 1. For each nonterminal b_i there is exactly one production g_i such that $(b_i \mapsto g_i) \in G$.
- 2. Nonterminal b_0 expands into v if we recursively substitute productions g_i for all b_i .

The set of all admissible grammars for v will be denoted by F(v). Each grammar $G \in F(v)$ is allowed to produce only one derivation, which is the finite text v itself. In contrast, context-free grammars producing a single infinite derivation are known as L-systems.

Some a priori criterion for deciding which admissible grammars approximate the correct tokenizations of texts makes use of the principle of minimum description length (Rissanen, 1978; Lehman and Shelat, 2002). Define the length len g_i of production g_i as the total number of its terminal and nonterminal symbols, e.g. len $\operatorname{sh} b_1 b_4 b_2 \operatorname{if} b_4 \operatorname{c} b_1 b_2 b_3 = 12$ and len $\operatorname{ab}_3 b_2 = 3$. According to the principle of minimum description length, the best grammar for string v is grammar $G^{\text{MDL}}(v)$ having the minimal length,

$$G^{\text{MDL}}(v) := \underset{G \in F(v)}{\operatorname{arg\,min\,len}} G, \tag{23}$$

where the length of a grammar is the total length of all its productions,

$$\operatorname{len} G := \sum_{(b_i \mapsto g_i) \in G} \operatorname{len} g_i. \tag{24}$$

Strictly speaking, there can be more than one grammar having the minimal length, so object $G^{\mathrm{MDL}}(v)$ is slightly indeterminate.

Grammar $G^{\mathrm{MDL}}(v)$ usually cannot be computed in a reasonable amount of time but there is a multitude of heuristic algorithms which compute grammars whose lengths approximate len $G^{\mathrm{MDL}}(v)$ (Lehman, 2002; Lehman and Shelat, 2002). Various algorithms for computing the approximations of $G^{\mathrm{MDL}}(v)$ usually perform some kind of local search on set F(v) and output so called irreducible grammars. Grammar G is called irreducible (Kieffer and Yang, 2000, section 3.2) if:

- 1. Each nonterminal expands recursively into a different string of terminals.
- 2. Each nonterminal except for b_0 appears at least twice in productions g_i .
- 3. There is no string y of len $y \ge 2$ which appears more than once in productions g_i .

It can be shown that there is an irreducible grammar for ν whose length equals $\min_{G \in F(\nu)} \operatorname{len} G$. Hence, we can assume that $G^{\text{MDL}}(v)$ is irreducible.

Various algorithms for computing the irreducible approximations of $G^{\mathrm{MDL}}(v)$ have been tested empirically on natural language data. Wolff (1980), Nevill-Manning (1996), and de Marcken (1996) reported that those algorithms return quite sound representations of English texts. The nonterminals of some irreducible approximations of $G^{\mathrm{MDL}}(v)$ can be interpreted as syllables, morphemes, words, and fixed phrases. Some of the heuristic algorithms identify the correct boundaries of about 90% of orthographic words in the Brown corpus, in a text deprived of spaces, capitalization, and punctuation (de Marcken, 1996). Here is an example of the computed tokenization given by de Marcken:

The results of the automatic tokenization are especially impressive for strongly isolating languages, such as English and Chinese (de Marcken, 1996). The same algorithms need not be so effective for highly inflective languages, where numerous orthographic alternations occur within the morphological stems (e.g. for Polish). The pursuit for better tokenization algorithms cannot be separated from the quest for the data compression algorithms which identify the inflectional paradigms (Goldsmith, 2001) or the abstract phrase syntax structures (Nowak et al., 2000).

The shortest grammar and excess entropy

Let us denote the set of the non-initial rules of grammar G as $G_0 := G \setminus \{b_0 \mapsto g_0\}$, where $A \setminus B$ is the difference of sets A and B. We will call G_0 the vocabulary of G. The length of the vocabulary is defined as

$$len G_0 := \sum_{(b_i \mapsto g_i) \in G_0} len g_i = len G - len g_0.$$
(25)

We use notation $\operatorname{len} G_0^{\mathrm{MDL}}(v) := \operatorname{len} G^{\mathrm{MDL}}(v) - \operatorname{len} g_0^{\mathrm{MDL}}(v)$ respectively. If the average length of the word-like productions g_i , $i \neq 0$, does not depend significantly on the text then we may suppose that $G_0^{\text{MDL}}(v)$ is proportional to the number of word types in text v. In fact, we can observe an analog of Guiraud's law (2). If we look at the data published by Nevill-Manning (1996, figure 3.12 (b), p. 69), we can observe empirical proportionality

$$len G_0^{SEQUITUR}(\nu) \propto (len \nu)^{\alpha}, \tag{26}$$

where $1/2 < \alpha < 1$ and $G_0^{\rm SEQUITUR}(v)$ is some approximation of $G_0^{\rm MDL}(v)$ computed by the algorithm called SEQUITUR.

In this section, we would like to present some general theoretical result. We shall relate the length of $G_0^{\text{MDL}}(v)$ to the finite-order excess entropy. It is well known that there are intimate relations between block entropy and the expected lengths of some codes used in data compression. In particular, Kieffer and Yang (2000) discuss the concept of grammar-based codes, which represent strings $v \in \mathbb{V}^+$ as uniquely decodable binary strings $C(v) \in \{0,1\}^+$ by the mediation of the admissible grammars.

Let $F = \bigcup_{v \in \mathbb{V}^+} F(v)$ be the set of admissible grammars for all strings. Function $C : \mathbb{V}^+ \to \{0,1\}^+$ is called a grammar-based code if

$$C(v) = \mathbf{B}(G^{C}(v)), \tag{27}$$

where grammar transform G^C computes grammar $G^C(v) \in F(v)$ and grammar encoder **B** represents any grammar $G \in F$ as a unique binary string $\mathbf{B}(G) \in \{0,1\}^+$.

Let us introduce the expected length of code C for the strings of length n drawn from stationary distribution \mathbf{P} ,

$$H^{C}(n) := \sum_{v \in \mathbb{V}^{n}} \mathbf{P}(v) \cdot \text{len} C(v).$$
(28)

Code C is called universal (more precisely, weakly minimax universal) if

$$H^{C}(n) \ge H(n), \tag{29}$$

$$\lim_{n \to \infty} H^{C}(n)/n = \lim_{n \to \infty} H(n)/n \tag{30}$$

for any stationary distribution **P**. See Cover and Thomas (1991, sections 5.1–6 and 12.10) for a general background in information and coding theory.

Additionally, let us call C an irreducible code if for each input string $v \in \mathbb{V}^+$, grammar $G^C(v)$ is irreducible. Kieffer and Yang (2000, theorem 8) prove the following result:

Theorem 1 There exists such grammar encoder **B** that any irreducible code of form (27) is weakly minimax universal.

It is a very strong and profound theorem. In particular, code $MDL(v) := \mathbf{B}(G^{MDL}(v))$ is universal since the shortest grammar $G^{MDL}(v)$ is irreducible. Theorem 1 can be used to prove universality of the modified SEQUITUR code by Nevill-Manning (Kieffer and Yang, 2000, section 6.2). Universality of the famous Lempel-Ziv code, however, is proved differently since it is not an irreducible code and it uses a different grammar encoder (Cover and Thomas, 1991, section 12.10).

It has been checked empirically that codes whose grammars are shorter usually enjoy shorter lengths. For instance, Grassberger (2002) compressed 135 GB of English text and obtained compression rates (in bits per character) $\text{len} LZ(v)/\text{len} v \approx 2.6$ for Lempel-Ziv code LZ and $\text{len} NSRPS(v)/\text{len} v \approx 1.8$ for some heuristic irreducible code NSRPS. Other researchers reported comparable results (de Marcken, 1996).

By analogy to definition (13) of finite-order excess entropy E(n), let us introduce the expected excess code length

$$E^{C}(n) := 2H^{C}(n) - H^{C}(2n)$$

$$= \sum_{v,u \in \mathbb{V}^{n}} \mathbf{P}(vu) \left[\operatorname{len} C(v) + \operatorname{len} C(u) - \operatorname{len} C(vu) \right]. \tag{31}$$

Theorem 2 For any weakly minimax universal code C inequality

$$E^C(n) \ge E(n) \tag{32}$$

is true for infinitely many n. (See appendix A for the proof.)

Inequality (32) is valid in particular for C = MDL or for any irreducible code.

Now, we shall link the expected excess code length $E^{\mathrm{MDL}}(n)$ with the length of MDL vocabulary. Let $L^m(v) := \mathrm{len}\,G_0^{\mathrm{MDL}}(v)$ be the length of the shortest grammar and $L_0^m(v) := \mathrm{len}\,G_0^{\mathrm{MDL}}(v)$ be the length of its vocabulary. Define $L^{>1}(v)$ as the maximal length of a string which appears in string v at least twice.

Theorem 3 We have inequalities

$$L^m(v) \le \text{len } v, \tag{33}$$

$$L^{m}(v), L^{m}(u) \le L^{m}(vu) + L^{>1}(vu),$$
 (34)

$$0 \le L^{m}(v) + L^{m}(u) - L^{m}(vu) \le L_{0}^{m}(vu) + L^{>1}(vu). \tag{35}$$

(See appendix A for the proof.)

Inequality (35) states that the vocabulary length for the shortest grammar cannot be roughly less than the excess length of the shortest grammar. In a slightly heuristic reasoning, we shall argue that the excess length of the shortest grammar multiplied by a slowly growing function cannot be less than the excess length of code MDL. In order to do it we need some pretty strong symmetrical bound for the length of code MDL in terms of the length of the shortest grammar.

It is known that function **B** of Theorem 1 satisfies len $\mathbf{B}(G) \le \gamma(\text{len } G)$, where $\gamma(n) := n \cdot (c + \log n)$ for some constant c (Kieffer and Yang, 2000, section 4). The following symmetrical bound for code MDL seems probable:

Conjecture 4 There is inequality

$$|\operatorname{len} \mathrm{MDL}(v) - \gamma(L^m(v))| \le f_2(L^m(v)), \tag{36}$$

where $\gamma(n) := n \cdot f_1(n)$ and functions $f_i \ge 0$ satisfy $0 \le f_i(n+1) - f_i(n) \le c_i/n$ for some constants c_i .

Now we can give a bound for the excess length of code MDL in terms of the excess length of the shortest grammar.

Theorem 5 *If Conjecture 4 is true then*

len MDL(v) + len MDL(u) - len MDL(vu)

$$\leq \left[L^{m}(v) + L^{m}(u) - L^{m}(vu) + d_{1}\right] \left[d_{2} + c_{1}\log \ln vu + c_{1}\frac{L^{>1}(vu)}{L^{m}(vu)}\right] + c_{1}L^{>1}(vu), \quad (37)$$

where $d_1 = 3c_2/c_1$ and $d_2 = \max(f_1(1), f_2(1)c_1/c_2)$. (See appendix A for the proof.)

Recall that $H^{\mathrm{MDL}}(n)/n = \sum_{v \in \mathbb{V}^n} \mathbf{P}(v) \cdot L^m(v)/\operatorname{len} v$ approaches entropy rate h for $n \to \infty$ by Theorem 1. We may speculate that h > 0 for the language production. Let us assume a stronger statement, namely, that

$$len v \le d_3 L^m(v)$$
(38)

for some constant d_3 and (almost) every human text v. On the other hand, notice that $L^{>1}(v) \le \ln v$ follows by definition of $L^{>1}(v)$. By these two inequalities, we have $L^{>1}(vu)/L^m(vu) \le d_3$.

Combining the latter with (37) and (35) gives

$$\operatorname{lenMDL}(v) + \operatorname{lenMDL}(u) - \operatorname{lenMDL}(vu)$$

$$\leq \left[L_0^m(vu) + L^{>1}(vu) + d_1 \right] \left[d_4 + c_1 \log \operatorname{len} vu \right], \tag{39}$$

where $d_4 := d_2 + c_1(d_3 + 1)$. Averaging (39) with $\mathbf{P}(vu)$ for $v, u \in \mathbb{V}^n$, we obtain

$$\left[L_0^m[2n] + L^{>1}[2n] + d_1\right] \left[d_4 + c_1 \log(2n)\right] \ge E^{\text{MDL}}(n), \tag{40}$$

where

$$L_0^m[n] := \sum_{v \in \mathbb{V}^n} \mathbf{P}(v) \cdot \ln L_0^m(v), \quad L^{>1}[n] := \sum_{v \in \mathbb{V}^n} \mathbf{P}(v) \cdot \ln L^{>1}(v). \tag{41}$$

By inequality (40) and Theorem 2, we also have

$$\left[L_0^m[2n] + L^{>1}[2n] + d_1\right] \left[d_4 + c_1 \log(2n)\right] \ge E(n) \tag{42}$$

for infinitely many n. In particular, if stationary distribution \mathbf{P} obeys Hilberg's law (1) then inequality

$$L_0^m[n] + L^{>1}[n] \ge \operatorname{const} \cdot n^{\mu} / \log n \tag{43}$$

holds for infinitely many n by equation (19).

5 Hilberg's law and Guiraud's law

In this section, we would like to make the final step in deriving Guiraud's law from relation (43). First, let us have a closer look at Guiraud's and Zipf's laws. It is widely-known that if Zipf's law (3) holds with the same B for all N then Guiraud's law (2) is satisfied with $\rho = 1/B$ for large N, cf. Kornai (2002, section 3.2) or Ferrer i Cancho and Solé (2001).

In fact, the number of word types V and the number of word tokens N can be computed given the word frequencies,

$$V = \sum_{w:c(w)>0} 1, \quad N = \sum_{w:c(w)>0} c(w), \tag{44}$$

so any relation between V and N is a function of the exact distribution of frequencies c(w). The converse is not true. In general, frequency c(w) cannot be computed given only w, V, and N since different texts usually have different keywords. Still, we may seek for hypothetical derivations of formula (3) given formula (2) and some additional assumptions.

One could ask if Guiraud's law or Zipf's law do hold with the same ρ or B for texts of various size and origin. The answer is complex. For instance, Kornai (2002, section 2.5) discusses Guiraud's law extensively and according to the plot in his article value $\rho \approx 0.75$ holds perfectly for samples of sizes $N \in \left[1.4 \cdot 10^5, 1.8 \cdot 10^7\right]$ drawn from San Jose Mercury News corpus. Such value of ρ would correspond to $B \approx 1.33$ if formula (3) with constant B held for all word ranks. Nevertheless, if we investigate the rank-frequency plot for so large collections of texts, we encounter a different regularity.

Ferrer i Cancho and Solé (2001) discovered that parameter B in formula (3) depends on word rank r(w). For multi-author corpora there are two regimes where B is almost constant. Namely, we have

$$B = \begin{cases} B_1, & 0 \le r(w) \le R_1, \\ B_2, & R_1 \le r(w), \end{cases}$$
 (45)

where $B_2 < B_1 \approx 1$. Let us note that for sufficiently short text collections (those with $V < R_1$) only one of two regimes can be observed. For single-author corpora and $r(w) \ge R_1$, we have an exponential decay of c(w) rather than a power-law.

In another case of some multi-author collection of English texts counting $1.8 \cdot 10^8$ word tokens, Montemurro and Zanette (2002) reported $B_1 \approx 1$, $B_2 \approx 2.3$ and $R_1 \approx 6000$. The investigated collection is only 10 times larger than SJMN corpus surveyed by Kornai. If formula (3) with constant $B \approx 2.3$ held for all word ranks then we would have Guiraud's law (2) with $\rho \approx 0.43$. Anyway, if there are two regimes of B, like in (45), then we could obtain Guiraud's law (2) with $\rho \approx 0.75$ for all N if also parameter R_1 depends on the text length N. Until we have more experimental data on the dependence between N and R_1 , we can be only sure that there is inequality

$$V > \operatorname{const} \cdot N^{0.43}. \tag{46}$$

Let V(v) be the number of orthographic word types in text v and N(v)—the number of orthographic word tokens therein. If we assume that the mean length of the word tokens in text v does not change substantially with v then text length N(v) measured in orthographic words is proportional to text length len v measured in phonemes or letters,

$$N(v) \propto \text{len } v.$$
 (47)

In view of section 3, we may suppose that the number of orthographic word types V(v) is proportional to the number of the production rules in the shortest grammar $G^{\mathrm{MDL}}(v)$, cf. Nevill-Manning (1996, figure 3.12 (c) vs. (a), p. 69). If the mean length of the non-initial productions does not change substantially against v then the number of the rules is proportional to length $L_0^m(v)$ of the vocabulary of the shortest grammar $G^{\mathrm{MDL}}(v)$, cf. Nevill-Manning (1996, figure 3.12 (a) vs. (b), p. 69). Resuming, we would have proportionality

$$V(v) \propto L_0^m(v). \tag{48}$$

Assuming relations (47) and (48), we can restate Guiraud's law (46) as

$$L_0^m(v) \ge \operatorname{const} \cdot (\operatorname{len} v)^{0.43},\tag{49}$$

which resembles relation (26) reported by Nevill-Manning. Except for the effects of averaging and the negligible length $L^{>1}(\nu)$ of the longest substring appearing more than once, inequality (49) is implied by inequality (43) with the very rough estimate $\mu \approx 1/2$ done by Hilberg. We could say that Hilberg's law can be some explanation of Guiraud's law. Let us discuss the plausibility of such explanation.

Zipf's law is often understood as a specific algebraic relationship between the counts and ranks of various objects—not necessarily words. In such generalization, Zipf's law is observed also out of the linguistic domain, e.g. in income distribution (Pareto, 1897). We do not know if one can find a general explanation of Zipf's law both in linguistic and non-linguistic contexts. Explaining Zipf's law in the purely linguistic context seems somehow easier. One needs "only" to assign some reasonable relative frequency $\mathbf{P}(v)$ to every string v of phonemes and then to define how any finite string v should be cut into words. The existence or nonexistence of relation (3) should follow by pure mathematical deduction from these two assumptions.

That idea inspired Mandelbrot (1953) to formulate some classical explanation of Zipf's law. His assumptions are:

1. Stationary distribution **P** is an IID distribution, i.e. it satisfies (16).

2. Set V of atomic symbols is the set of phonemes and spaces. The word tokens in any text are defined as the space-to-space strings of phonemes.

Given these assumptions Mandelbrot derived Zipf's law for space-to-space words and hence Guiraud's law can be inferred as well. In fact, Mandelbrot did not discuss Guiraud's law but, as we have said, Zipf's law does imply Guiraud's law automatically. Mandelbrot's explanation assuming the existence of "intermittent silences" was quoted or rediscovered by many researchers, e.g. by Belevitch (1956), Miller (1957), Bell et al. (1990) and Li (1992). There is some historical summary of that literature done by Li (1998).

Although Mandelbrot's explanation of Zipf's and Guiraud's laws earned some popularity among natural scientists, we should stress that both of its assumptions are false with respect to the intended application to natural language. First, we would object to modeling human language production by an IID distribution. Second, Mandelbrot's definition of word is biased by the spelling conventions of the most popular alphabetic scripts which use blank spaces to separate words. No regular "intermittent silences" appear in the spoken versions of the corresponding ethnic languages (Jelinek, 1997). That phenomenon is a challenge for automatic speech recognition and it motivated some interest in the shortest admissible grammars as a means for restoring the boundaries between the words (de Marcken, 1996).

In this article, we present another explanation of Guiraud's law. Our assumptions are:

- 1. Stationary distribution \mathbf{P} exhibits Hilberg's law (1) for all n.
- 2. We may assume that V is a set of phonemes only. The word tokens in any text are defined as the nonterminal tokens of the shortest admissible grammar.

We think that the derivation of Guiraud's law based on Hilberg's law is better linguistically justified than the classical explanation by Mandelbrot. There are several reasons for that claim:

- 1. The new explanation assumes that human narration exhibits strong probabilistic dependence, it is not a IID distribution. In appendix B, we recall that no infinitary distribution **P** can be modeled by a stationary hidden Markov chain with a finite number of hidden states. This fact can have some important implications for computational linguistics (Jelinek, 1997).
- 2. The new explanation does not assume the pre-existence of spaces between the words in the natural language production. Children can learn the correct tokenization of speech into the words even if they do not know yet what the words are.
- 3. Space-to-space words for the IID distributions do not have any definite internal structure. It is no longer true for the new explanation. The nonterminals of the shortest grammar exhibit the internal structure of recursive rule productions. Such nonterminals have well-defined parts. Without any change of the model, we can speak not only of Guiraud's and Zipf's laws for the words but we can also discuss laws which relate words to their elements. Some example of the latter is Menzerath's law, which states that the longer the word is the shorter its constituents are (Menzerath, 1928; Altmann, 1980). By means of the grammar-based codes one can define the structure of word-like objects and investigate many quantitative linguistic laws not only for the language production but also for any other stationary distributions.
- 4. Stationary distribution is called ergodic (roughly) if the relative frequency of any fixed word does not vary significantly across different texts. By some theorem, every IID distribution is ergodic (Dębowski, 2005, chapter 4). Nevertheless, empirical studies do not

corroborate Mandelbrot's assumption that language production **P** is ergodic. The mere existence of concept "the keywords of the text" reflects the fact that different texts use different vocabularies systematically. Words, once they appear in some text, tend to reappear. Let us stress that some significant variation of the word frequencies *can* be modelled by non-ergodic stationary distributions. Many non-ergodic stationary distributions are infinitary (Dębowski, 2005, chapters 4 and 5), see also appendix B. It is an interesting question whether Hilberg's law (1) implies non-ergodicity of stationary distribution **P**. Some further discussion of Hilberg's law and non-ergodic distributions could give us insight where to seek general quantitative laws in the intertext variability of language. Any such laws would be of great importance to computational linguistics as well.

6 Conclusions

In this article, we have discussed some implications of Hilberg's (1990) hypothesis on the entropy of natural language production. That hypothesis states that finite-order excess entropy E(n) of the n-letter strings is proportional to the square root of n. So far, the proportionality has been roughly verified only for $n \le 50$. On the other hand, we have argued that Hilberg's hypothesis, when extrapolated to n of the text length magnitude, provides a better explanation of Guiraud's law than the classical explanation based on the existence of "intermittent silences" (Mandelbrot, 1953).

The new explanation is based on two points. First, we observe that the tokenization of a text into orthographic words and their morphemes matches largely the production rules of the shortest admissible grammar for the text. Second, we use some partially heuristic, but largely deductive, mathematical reasoning to argue that the length of the non-initial production rules of the shortest grammar cannot be less than finite-order excess entropy.

In the future research, the rough match of the linguistically-motivated tokenizations and the tokenizations given by the shortest grammars should be surveyed as one of the fundamental problems of quantitative linguistics. One should survey Zipf's, Guiraud's, and Menzerath's laws for the nonterminals of the admissible grammars and the orthographic words simultaneously across a large range of text sizes and languages. Proportionalities (47) and (48) should be verified as well.

It seems that the existence of a rich formal structure in the natural language production is reflected by its high total excess entropy E rather than by simply positive entropy gain H(1) - h. We think that the further discussion of Hilberg's hypothesis can improve the quality of statistical language models both in quantitative and computational linguistics, see appendix B and our doctoral dissertation (Dębowski, 2005).

Since the shortest admissible grammars reproduce also the internal structure of words, the behavior of excess entropy might be linked not only with Guiraud's and Zipf's laws but also with Menzerath's law. The shortest grammars can be used as the *definition* of words and their constituents in any symbolic string (Nevill-Manning, 1996). Adopting such a definition, empirical researchers can survey the form of Guiraud's, Zipf's, and Menzerath's laws also in the non-linguistic symbolic data (such as DNA). Last but not least, mathematicians can prove some rigorous theorems.

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A Proofs

Proof of Theorem 2: For any function f we have identity

$$\sum_{k=0}^{m-1} \left[2f(2^k n) - f(2^{k+1} n) \right] \cdot \frac{1}{2^{k+1}} = f(n) - \frac{f(2^m n)}{2^m n} \cdot n \tag{50}$$

for each finite m. Hence, if (30) is true then we obtain

$$H(n) - hn = \sum_{k=0}^{\infty} \left[2H(2^k n) - H(2^{k+1} n) \right] \cdot \frac{1}{2^{k+1}} = \sum_{k=0}^{\infty} \frac{E(2^k n)}{2^{k+1}}, \tag{51}$$

$$H^{C}(n) - hn = \sum_{k=0}^{\infty} \left[2H^{C}(2^{k}n) - H^{C}(2^{k+1}n) \right] \cdot \frac{1}{2^{k+1}} = \sum_{k=0}^{\infty} \frac{E^{C}(2^{k}n)}{2^{k+1}}.$$
 (52)

Because of inequality (29), we have $H(n) - hn \le H^{C}(n) - hn$ so

$$\sum_{k=0}^{\infty} \frac{E(2^k n)}{2^{k+1}} \le \sum_{k=0}^{\infty} \frac{E^C(2^k n)}{2^{k+1}}.$$
(53)

If we put $n = 2^p M$ with any p and some fixed M then (53) yields

$$\sum_{k=p}^{\infty} \frac{E^{C}(2^{k}M) - E(2^{k}M)}{2^{k+1}} \ge 0.$$
 (54)

Assume that $E^C(2^kM) - E(2^kM) \ge 0$ holds only for finitely many k. Then we would have $E^C(2^kM) - E(2^kM) < 0$ for all $k \ge p$ and some p. Hence, we would have

$$\sum_{k=n}^{\infty} \frac{E^C(2^k M) - E(2^k M)}{2^{k+1}} < 0. \tag{55}$$

Since (55) stays in contradiction with (54), our assumption that $E^C(2^kM) - E(2^kM) \ge 0$ only for finitely many k was false. We must have $E^C(2^kM) - E(2^kM) \ge 0$ for infinitely many k, and this is exactly inequality (32) which we were to prove.

Proof of Theorem 3: In order to prove (33), notice that $G = \{b_0 \mapsto v\}$ is a grammar for v. Its length satisfies $\text{len } v = \text{len } G \leq \text{len } G^{\text{MDL}}(v)$ by (24) and (23).

Now, let us prove (34) and (35). Since vocabulary $G_0^{\mathrm{MDL}}(vu)$ cannot beat vocabularies $G_0^{\mathrm{MDL}}(v)$ and $G_0^{\mathrm{MDL}}(u)$ in the efficient representation of any strings v and u respectively, we observe inequalities

$$len G^{\text{MDL}}(v) < len g_L + len G_0^{\text{MDL}}(vu),$$
(56)

$$\operatorname{len} G^{\mathrm{MDL}}(u) \le \operatorname{len} g_R + \operatorname{len} G_0^{\mathrm{MDL}}(vu), \tag{57}$$

where $G_0^{\text{MDL}}(vu) \cup \{b_0 \mapsto g_L\}$ and $G_0^{\text{MDL}}(vu) \cup \{b_0 \mapsto g_R\}$ are some grammars for v and u respectively. Analogically,

$$len G^{\text{MDL}}(vu) \le len G^{\text{MDL}}(v) + len G^{\text{MDL}}(u)$$
(58)

since $G_0^{\mathrm{MDL}}(v) \cup G_0^{\mathrm{MDL}}(u) \cup \left\{b_0 \mapsto g_0^{\mathrm{MDL}}(v)g_0^{\mathrm{MDL}}(u)\right\}$ is a grammar for vu.

Assume that g_L and g_R are obtained by splitting the initial production $g_0^{\text{MDL}}(vu)$ into two parts and recursively expanding the nonterminal at the border if necessary. That is, we have either $g_L g_R = g_0^{\text{MDL}}(vu)$ or $g_L = y_L x_L$, $g_R = x_R y_R$, and $g_0^{\text{MDL}}(vu) = y_L b_i y_R$, where nonterminal b_i expands recursively into string $x_L x_R \in \mathbb{V}^+$. Grammar $G^{\text{MDL}}(vu)$ is irreducible so we must have $\text{len } x_L x_R \leq L^{>1}(vu)$, where $L^{>1}(vu)$ is the maximal length of a string which appears in string vu at least twice. Thus,

$$\left| \operatorname{len} g_L + \operatorname{len} g_R - \operatorname{len} g_0^{\text{MDL}}(vu) \right| \le L^{>1}(vu). \tag{59}$$

By (59), adding (56) and (57) yields

$$len G^{\text{MDL}}(v) + len G^{\text{MDL}}(u) \le len g_0^{\text{MDL}}(vu) + 2 len G_0^{\text{MDL}}(vu) + L^{>1}(vu)
= len G^{\text{MDL}}(vu) + len G_0^{\text{MDL}}(vu) + L^{>1}(vu).$$
(60)

In fact, we can rewrite (60) and (58) as (35). By (59), we also have $\log_L, \log_R \leq \log_0^{\text{MDL}}(vu) + L^{>1}(vu)$. Inserting these two inequalities into (56) and (57) respectively yields (34).

Proof of Theorem 5: According to Conjecture 4, we have

$$len MDL(v) + len MDL(u) - len MDL(vu)
< \gamma(L^m(v)) + \gamma(L^m(u)) - \gamma(L^m(vu)) + f_2(L^m(v)) + f_2(L^m(u)) + f_2(L^m(vu))$$
(61)

By $0 \le f_i(n+1) - f_i(n) \le c_i/n$ and (34), there is

$$f_i(n) \le f_i(1) + \sum_{k=2}^n c_i/k < f_i(1) + c_i \log n, \tag{62}$$

$$f_i(L^m(v)) \le f_i(L^m(vu)) + c_i L^{>1}(vu)/L^m(vu).$$
 (63)

Hence by (33),

$$\gamma(L^{m}(v)) + \gamma(L^{m}(u)) - \gamma(L^{m}(vu))
\leq \left[L^{m}(v) + L^{m}(u) - L^{m}(vu)\right] f_{1}(L^{m}(vu)) + c_{1}\left[L^{m}(v) + L^{m}(u)\right] \frac{L^{>1}(vu)}{L^{m}(vu)}
= \left[L^{m}(v) + L^{m}(u) - L^{m}(vu)\right] \left[f_{1}(L^{m}(vu)) + c_{1}\frac{L^{>1}(vu)}{L^{m}(vu)}\right] + c_{1}L^{>1}(vu)
\leq \left[L^{m}(v) + L^{m}(u) - L^{m}(vu)\right] \left[f_{1}(\operatorname{len} vu) + c_{1}\frac{L^{>1}(vu)}{L^{m}(vu)}\right] + c_{1}L^{>1}(vu).$$
(64)

On the other hand,

$$f_{2}(L^{m}(v)) + f_{2}(L^{m}(u)) + f_{2}(L^{m}(vu)) \leq 3f_{2}(L^{m}(vu)) + 2c_{2}L^{>1}(vu)/L^{m}(vu)$$

$$\leq 3\left[f_{2}(\operatorname{len}vu) + c_{2}\frac{L^{>1}(vu)}{L^{m}(vu)}\right]. \tag{65}$$

Inserting (64), (65), and (62) into (61) we obtain (37).

B Some properties of infinitary distributions

Infinitary distributions seem to be a new interesting class of the stochastic models for human narration. The mathematics of excess entropy is just being developed, cf. Dębowski (2005) for an overview. Our program is to bring together some advanced results of mathematics (measure-theoretic probability theory, coding theory) and some quantitative linguistic intuitions. We can give a linguistic interpretation to some mathematical theorems and a formal language to express some vague hypotheses about the obscure nature of probabilistic language models.

We would like to mention four facts about infinitary distributions which can be important for quantitative and computational linguistics in the view of Hilberg's hypothesis. These are:

- 1. There are infinitary distributions which are not deterministic stationary distributions. That is, total excess entropy $E = \infty$ does not imply entropy rate h = 0.
- 2. All stationary distributions which consist in a random description of some infinite random object must be infinitary and nonergodic (Dębowski, 2005, chapter 5).
 - Hence, we may suppose that $E = \infty$ holds for the stationary distribution of the language production because almost every human text refers systematically to a different and potentially infinite fictitious world.
- 3. For some infinitary distributions, value $\mathbf{P}(v)$ can be computed for every string v by some finite procedure, cf. Berthé (1994) and Gramss (1994).
- 4. No infinitary distribution can be represented by a finite-state hidden Markov model (HMM), cf. Crutchfield and Feldman (2003), Upper (1997), Cover and Thomas (1991, section 2.8, data processing inequality).
 - In spite of their inadequacy as the models of infinitary distributions, finite-state HMMs are the standard heuristic models of natural language engineering. It happens so only for the necessity of the effective search for the most probable hidden states. Some well-known applications of HMMs are automatic speech recognizers (Jelinek, 1997) and trigram part-of-speech taggers (Manning and Schütze, 1999; Dębowski, 2004b). It was observed that the error rate of trigram taggers decreases as a negative power of the size of the training data. When we increase the training data size ten times, the error rate diminishes only by half (Megyesi, 2001). In fact, such power-law decay of the error rate can be also some consequence of Hilberg's law (Bialek et al., 2001).

The lack of space disallows us to exactly explain the terminology and the reasons for the mathematical facts mentioned above. We will try to popularize some ideas of our thesis among the linguistic audience in the next articles.