# Fitting Ranked English and Spanish Letter Frequency Distribution in U.S. and Mexican Presidential Speeches

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#### **ABSTRACT**:

The limited range in its abscissa of ranked letter frequency distributions causes multiple functions to fit the observed distribution reasonably well. In order to critically compare various functions, we apply the statistical model selections on ten functions, using the texts of U.S. and Mexican presidential speeches in the last 1-2 centuries. Dispite minor switching of ranking order of certain letters during the temporal evolution for both datasets, the letter usage is generally stable. The best fitting function, judged by either least-square-error or by AIC/BIC model selection, is the Cocho/Beta function. We also use a novel method to discover clusters of letters by their observed-over-expected frequency ratios.

### 1 Introduction

Although morphemes, not letters, are usually considered to be the smallest linguistic unit, studying statistics of letter usage has its own merit. For example, information on letter frequency is essential in cryptography for deciphering a substitution code (Friedman 1976), and "frequency analysis" was used in as early as the 9th century by the Arab scientist al-Kindi for the purpose of decryption (Mrayati et al., 2003).

An efficient design of a communication code also depends crucially on the letter frequency. The shortest Morse code is reserved to letters that are the most common: one dot for letter e and one dash for letter t, both letters being the most frequent in English. The same principle is also behind the design of minimum-redundancy code by Huffman (Huffman 1952).

The initial motivation for the "QWERTY" mechanical typewriter design is to keep the most common letters far away in the keyboard so that metal bars would not jam for a fast typist (David 1985). Even in modern times, the digraph (letter pairs) frequency is an important piece of information for keyboard design (Zhai et al. 1999).

In all these examples, a quantitative description of letter usage frequency is important. Unlike the ranked word frequency distribution, which is well characterized by a simple powerlaw function or Zipf's law (Zipf 1935), it is not clear whether a universal fitting function exists despite a claim of such a function (the logarithmic function) in (Kanter and Kessler 1995).

In this paper, we aim at critically examining various functional forms of fitting rankfrequency distribution of letters, ranging from simple to more complicated ones with two or three free parameters. The dataset used is the historical U.S. and Mexican presidential speeches. The presidential speeches are readily available (see another study where the Italian presidential "end of year" addresses are used (Tuzzy et al. 2009)), they also offer an opportunity for investigations of temporal patterns in letter usage.

The ranked word frequency distributions studied by George Zipf have extremely long tails, due to the presence of low-frequency words (such as hapax legomena). As a result, logarithmic transformation is usually applied to the x-axis (as well as the y-axis). The double logarithmic transformation is also justified by the expectation of a power-law function, as it will lead to a linear regression. This linear fitting in log-log scale may have its pitfall (Clauset et al. 2009), one being the uneven distribution of points along the log-transformed x-axis.

For ranked letter frequency data, the finite number of alphabets sets an upper bound for the rank, and there is no large number of rare events which is an important theoretical issue in modeling the word rank-frequency distribution (Baayen 2001). On the other hand, the limited range of abscissa may make it hard to distinguish different fitting functions. Since power-law function is not expected to be the best fitting function, double logarithmic transformation is not necessary, and we will fit the data in linear-linear scale. No longer linear fittings, the curve fitting is carried out by nonlinear least-square (Bates and Watts 1988).

Statistical models with a larger number of free parameters will guarantee to fit the data better than a model subset with fewer number of parameters. To compare the performance of models with different number of parameters, a penalty should be imposed on the extra number of parameters. Towards this end, we apply the standard model selection technique with Akaike Information criterion or AIC (Akaike 1974; Burnham and Anderson 2002) and Bayesian Information Criterion or BIC (Schwarz 1978) to compare various functions used to fit the ranked letter frequency data.

# 2 Data

US presidential inaugural speeches: In order to take into account of any possible letter usage trend in time, we use the US Presidential Inaugural Speech texts for the 44 presidents in the last 200 years. The data is downloaded from the *The American Presidency Project* from the University of California at Santa Barbara site (*http://www.presidency.ucsb.edu/*). Multiple inaugural speeches from the same person are combined into one, including the nonconsecutive presidency of Grover Cleveland. Five presidents did not give an inaugural speech (John Tyler, Millard Fillmore, Andrew Johnson, Chester Arthur, Gerald Ford). The final dataset consists of 38 text files.

Mexico presidential addresses to the congress: For Spanish texts, we selected the 19 Mexican presidents' report to congress (Informes Presidenciales) from 1914 to 2006. (http://www.diputados.gob.mx/cedia/sia/re\_info.htm ) Again, addresses by the same president are combined into one text file. Some presidential texts are much shorter than others due

to two possible reasons: either did the president only present one address (the typical number of addresses is 6), such as Adolfo de la Huerta (1920) and Emilio Portes Gil (1929), or the president gave shorter reports, such as Ernesto Zedillo Ponce de León (1995-2000) and Vicente Fox Quesada (2001-2006).

# 3 Letter frequencies and their temporal trends

Fig.1(A) shows the English letter frequency of the 38 US president's speeches, separated by the century. The letter e remains the most commonly used English letter with little change in its frequency. However there seems to be a trend of less usage of letter t, and more usage of letter w in the 20th century as compared to the 19th century.

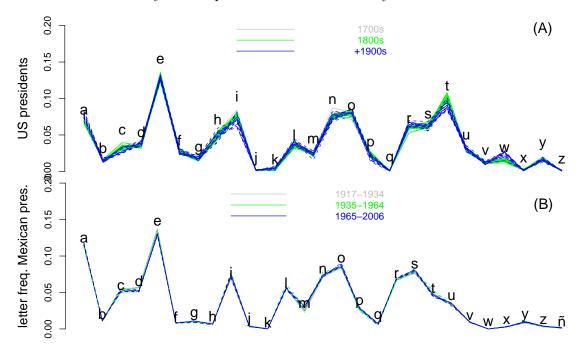


Figure 1: English (A) and Spanish (B) letter frequencies (unranked, in alphabetic order) for 38 U.S. presidential inaugural speeches and 19 Mexican presidents' report to congress. Letter frequencies of each president's speech are linked by a line, and different time periods are drawn separately (U.S. president speech: 1789-1800, shifting 1801-1901 by 0.02, shifting 1905-now by 0.04; Mexico president speech: 1919-1934, shifting 1935-1964 by 0.02, shifting 1965-2005 by 0.04). Due to a larger sample size, the fluctuation of frequency from president to president in Spanish texts is much smaller than that in English texts.

Similar frequencies of Spanish letters in the 19 Mexican presidents' addresses are shown

in Fig.1(B). Letters with accent (the acute accent for the vocals, the umlaut for the letter ü and the tilde for the ñ) are counted separatedly but later combined because they do not really represent different letters. The 19 files are arbitrarily split into three groups: the first 7 presidents (from 1917 to 1934), the next 5 presidents (from 1935 to 1964), and the last 7 presidents (from 1965 to 2006). These three groups are separatedly drawn in Fig.1(B). The narrowing of the variations of letter frequency in Fig.1(B) as compared to Fig.1(A) is due to the larger sample sizes in Spanish texts.

In Table 1 English letters are sorted by their frequency of usage, from common to rare, for the 38 US president speeches. Again, e and t are consistently ranked as number 1 and 2 (with the exception of Clinton's speech, where o is ranked second), but the ranking order of a and iseem to change with time: in older speeches (e.g. before year of 1890), i is ranked higher than a, after 10 more presidents where i and a were used about equally, then the order is reversed for newer speeches (e.g. after the year 1960).

Table 2 shows the corresponding sorting of Spanish letters in the 19 Mexico president addresses. The sequence *eaosinr* consistently appears at the head of the string. However, the order of d and l has been switched from dl in the first half of 20th century (until president Rodríuez whose term ended in 1934) to ld in the second half of the century (since president Alemán whose term started in 1946).

To confirm the observation from Fig.1 and Tables 1,2, in Fig.2 we directly plot the English letter frequencies of t, w, a, i and Spanish letter frequencies of d, l, m. Indeed, there is higher usage of w and lesser usage of t in recent US president speeches, and the relative order of a and t was switching from year 1889 to 1957. For Mexican president addresses, the letter l overcomes d in the last few decades. There is also an upward trend for the usage of Spanish letter m.

Despite these interesting trends of a few letters for the last two hundreds of years for English and one hundred of years in Spanish, the overall letter frequencies remain more or less stable. We combine all 38 English files into one (and 19 Spanish files into one) to examine the rank frequency distribution.

	name	year(s) of speech	sorting of alphabets	num
1	Washington	$1789,\!1793$	et in oa shr cd lum f py bwg v x j q k z	7710
2	Adams	1797	etnio a srhdlc fumpgy b v w x j k q z	11281
3	Jefferson	1801,1805	eto in a schldcufmpwgybvk x jzq	18701
4	Madison	1809,1813	eto in a schldcu fpmg w by v k x j z q	11572
5	Monroe	1817,1821	et in oar shdcluf pm wyg bv x k j z q	37522
6	Adams	1825	eto in a srhdlc fup mgy b v w x j q k z	14572
7	Jackson	1829,1833	eto in arshld cufmpy bgwvx kjqz	11372
8	Van Buren	1837	eto in a srhlduc fpmywgvbx kjqz	19215
9	Harrison	1841	eto in arshed lfum pyg bwvx kjz q	40526
10	Polk	1845	etonias rhdlcufmpygbwvxjqkz	23475
11	Taylor	1849	eto in a schlcdufm py bg wv x k j z q	5413
12	Pierce	1853	et in oar shlduc fmpygwbvk x jqz	16406
13	Buchanan	1857	etion a srhlcduf pmywgvbxqjkz	13696
14	Lincoln	1861,1865	eto in a srhldcu fpmywbgvk xjqz	19340
15	Grant	1869,1873	eto in arshld cufmpgywbvxkqjz	11476
16	Hayes	1877	eto in a schlcduf pmyg bw v j q x k z	12171
17	Garfield	1881	etonias rhlduc fmpgwy b v k x j q z	14477
18	Cleveland	1885	eto in a srhdlcufpmyg bwvxkz jq	18480
19	Harrison	1889	etonia srhldcufpmwygbvkxjqz	21394
20	Mckinley	1897,1901	etnoiar shlduc fpmyg bwvx kjz q	30179
21	T.Roosevelt	1905	eto a inrshlduf wcg bpm vy k x j z q	4480
22	Taft	1909	eto in a srhdcl fumpgywbvk x jqz	26272
23	Wilson	$1913,\!1917$	eto an is rhd luc fwpmgyvb kjqxz	14360
24	Harding	1921	etnio arsldh cufmwpgy bvk xzjq	16508
25	Coolidge	1925	etonairshldcufmpwybgvxkjqz	19482
26	Hoover	1929	eto in arshld cufmpgywbvz x j kq	19256
27	F.D.Roosevelt	1933, 1937, 1941, 1945	eto a inrshldc fumpwy gvb kj xzq	25696
28	Truman	1949	eto a inrshldc fumpwgyvbkjqxz	11070
29	Eisenhower	1953, 1957	eto a inrshld f cumpwygbv kq j x z	18313
30	Kennedy	1961	eto anr sihld fuw cmgypbv kj x zq	6003
31	Johnson	1965	et a no ir shdluw cfmgy bpv kj x z q	6468
32	Nixon	1969,1973	eto an ir shldcuwfmpg by v kjq xz	17142
33	Carter	1977	eta on ir shldum wcfpg by v kjq xz	5459
34	Reagan	1981, 1985	etonarishdlumwcfgpybvkjxzq	22494
35	G.H.W.Bush	1989	eta on rishdluw cmg fy bpv kz j x q	9781
36	Clinton	1993,1997	eot anrishld cumwfpgy bv kjz xq	16915
37	G.W.Bush	2001,2005	etonairsdhlcufmwygpbvkjzqx	16759
38	Obama	2009	${ m eto} anrsihdlucwfmgypbvkjqxz$	10632

Table 1: The names of the 38 U.S. presidents, the years of their inaugural speech, the order of letters ranked by their frequency in the corresponding president's speech, and the total counts of letters.

	name	years of speech	sorting of alphabets	num
1	Carranza	$1917,\!1918,\!1919$	$eaos nirdl ctup mbgyvfqhj xz \ {\tilde n} kw$	539107
2	De la Huerta	1920	$eaos inrdlctup mbg fvyhqjz x \tilde{n} k w$	113057
3	Obregón	$1921,\!1922,\!1923,\!1924$	$eaos inrdlctup mbgyfvhqjxz \ {\tilde n} kw$	675552
4	Elías	$1925,\!1926,\!1927,\!1928$	$eaos inrdlctup mbgy fvqhjz x \tilde{n} kw$	700715
5	Portes Gil	1929	$eaos inrdlctup mbgyvfqhjzx \ {\tilde n} kw$	231873
6	Ortiz	1930, 1931, 1932	$eao is nrdlctup mbgv fyqhjz x \tilde{n} k w$	664319
7	Rodríuez	$1933,\!1934$	$eao is nrdlctup mbgyvfqhjz x \tilde{n} kw$	301745
8	Cárdenas	1935, 1936, 1937, 1938, 1939, 1940	$eaos inrldctup mbgvyfqhjxz \ {\rm \tilde{n}}kw$	402748
9	Ávila	1941, 1942, 1943, 1944, 1945, 1946	$eaos inrlcdtump by gvfqhjzx \tilde{n}kw$	734540
10	Alemán	1947, 1948, 1949, 1950, 1951, 1952	$eao is nrcltdum pby vgfhqzjx \tilde{n} kw$	549980
11	Ruiz	1953, 1954, 1955, 1956, 1957, 1958	$eaos inrldctump by gvfqhjz x \tilde{n} kw$	592550
12	López	1959, 1960, 1961, 1962, 1963, 1964	$eaos inrldctup mbgvy fhqz jx \tilde{n} kw$	712056
13	Díaz	1965, 1966, 1967, 1968, 1969, 1970	$eaos inrldctup mbgvyfqhzjx \tilde{n}kw$	785528
14	Echeverría	1971, 1972, 1973, 1974, 1975, 1976	$eaos inrldctum pbvgfyqhjzx \tilde{n}kw$	792338
15	López Portillo	1977, 1978, 1979, 1980, 1981, 1982	$eaos inrlcdtump by gvfqhz jx \tilde{n} kw$	684658
16	De la Madrid	1983, 1984, 1985, 1986, 1987, 1988	$eao is nrlcdtum pby vgfhqzjx \tilde{n} kw$	761274
17	Salinas	1989, 1990, 1991, 1992, 1993, 1994	$eaos inrlcdtum pbvyg fhqz xj \tilde{n} kw$	624933
18	Zedillo	1995, 1996, 1997, 1998, 1999, 2000	$eaos inrlcdtump bgyvfqhz jx \tilde{n} kw$	282463
19	Fox	2001,2002,2003,2004,2005	$eaos inrldctump bgyvfqhz jx \tilde{n} kw$	311429

Table 2: The last names of the 19 Mexican presidents, the years when they addressed the congress, the order of letters ranked by their frequency in the corresponding president's address, and the total counts of letters.

# 4 Fitting ranked letter frequency distributions

We used ten different functions to fit the ranked letter frequency distribution in US presidential inaugural speeches that is averaged over all 38 presidents, and Mexican presidential addresses to the congress averaged over 19 presidents. Here is a list of these functions (fdenotes the normalized letter frequency, r denotes the rank: r = 1 for most frequent letter

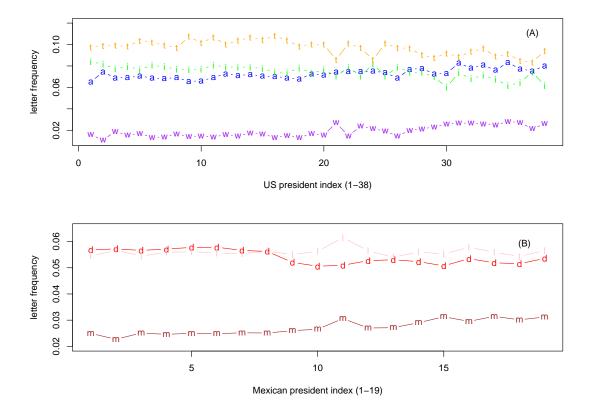


Figure 2: Temporal change of frequency in selected letters. (A) Letters t, i, a, w in 38 U.S. presidential speeches. (B) Letters d, l, m in 19 Mexican presidential speeches.

and r = 26 (or 27) for the rarest letter, and n = 26, 27 is the maximum rank value):

Gusein-Zade : 
$$f = C \log \frac{n+1}{r}$$
 (1)

power-law : 
$$f = \frac{C}{r^a}$$
 (2)

exponential : 
$$f = Ce^{-ar}$$
 (3)

logarithmic : 
$$f = C - a \log(r)$$
 (4)

Weibull : 
$$f = C \left( \log \frac{n+1}{r} \right)^a$$
 (5)

quadratic logarithmic : 
$$f = C - a \log(r) - b (\log(r))^2$$
 (6)

Yule : 
$$f = C \frac{b}{r^a}$$
 (7)

Menzerath-Altmann/Inverse-Gamma : 
$$f = C \frac{e^{-b/r}}{r^a}$$
 (8)

Cocho/Beta : 
$$f = C \frac{(n+1-r)^b}{r^a}$$
 (9)

1 /

Frappat : 
$$f = C + br + ce^{-ar}$$
 (10)

Since f is the normalized frequency,  $\sum_{i=1}^{n} f_i = 1$ , which adds a constraint on one parameter. The parameter under constraint is labeled as C whose value is generally of no interest to us. Besides C, the number of free (adjustable) parameters in these fitting functions ranges from 0 (Gusein-Zade) to 3 (Frappat). The power-law, exponential, logarithmic, and Weibull functions have 1 free parameter, quadratic-logarithmic, Yule, Menzerath-Altmann/Inverse-Gamma, and Cocho/Beta, functions have 2 free parameters, as discussed in (Li et al. 2010)

The power-law (Eq.(2)) and exponential function (Eq.(3)) are often the first group of function to be tested, due to their simplicity and widespread applicability. The zero-free-parameter function (Gusein-Zade) in Eq.(1) (Gusein-Zade 1987, 1988; Borodovsky and Gusein-Zade 1989) actually corresponds to the exponential cumulative distribution, and the Weibull function (Eq.(5)) (Nabeshima and Gunji 2004) corresponds to the stretched exponential cumulative distribution. The conversion from cumulative distribution to rank distribution of these two functions are discussed in details in (Li et al. 2010).

The logarithmic function (Eq.(4)) is an extension of the Gusein-Zade function  $C \log(n + 1) - C \log(r)$  by allowing the coefficient of  $\log(r)$  term to be independently fitted. Then the quadratic logarithmic function is an extension of the logarithmic function by adding one extra term. The logarithmic function is mentioned in (Kanter and Kessler 1995; Vlad et al. 2000), whereas quadratic logarithmic function has not been used to the best our knowledge.

The three two-parameter functions used are all attempts to modify the power-law function: Yule function (Yule 1925; Martindale et al. 1996) uses an exponential function  $(b^r)$ , Menzerath-Altmann or inverse-Gamma function (Altmann 1980) uses an exponential function of the inverse of rank  $(e^{-b/r})$ , and Cocho or Beta function (Mansilla et al. 2007; Naumis and Cocho 2008; Martínez-Mekler et al. 2009) uses a power-law function of the reverse rank  $((n+1-r)^b)$ . The 3-parameter function in Eq.(8) proposed in (Frappat et al. 2003; Frappat and Sciarrino 2006) is to add a linear trend over the exponential function.

All x and y relationship in Eqs.(1-10) are non-linear. It is possible to transform variables or introduce new variables to carry out the fitting by multiple linear regression. For example, after define  $y' = \log(f)$ ,  $x'_1 = \log(r)$ ,  $x'_2 = \log(n+1-r)$ , the Cocho/Beta function is equivalent to a multiple regression  $y' = c_0 + c_1 x_1 + c_2 x_2$ , where the regression coefficients can be converted back to the parameters used in Eq.(7):  $C = e^{c_1}$ ,  $a = -c_1$ ,  $b = c_2$ . The data-fitting result in the transformed variable, however, is generally not identical to the result in its original nonlinear form. Our method is to first use the multiple linear regression in the transformed version, if possible, in order to obtain a rough estimation of the parameter values. Then these values are used as the initial condition for nonlinear least-square iteration (using the *nls* function (Bates and Watts 1988) in R: *http://www.r-project.org/*).

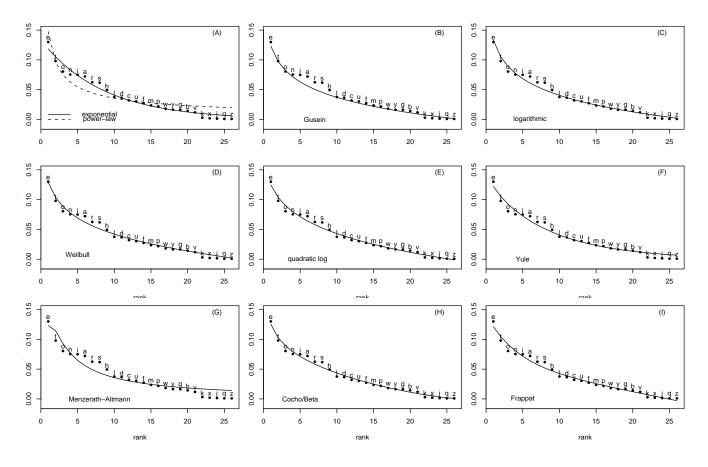


Figure 3: Fitting ranked English letter frequency of U.S. presidential speech by ten different functions: (A) power-law (a = 0.616) and exponential function (a = 0.118); (B) Gusein function (C = 0.0374); (C) logarithmic function (a = 0.0401); (D) Weibull function (a = 0.935); (E) quadratic logarithmic function (a = 0.0280, b = 0.00325); (F) Yule function (a = 0.0543, b = 0.897); (G) Menzerath-Altmann/Inverse-Gamma function (a = -1.05, b = -1.31); (H) Cocho/Beta function (a = 0.210, b = 1.35); (I) Frappat function (a = 0.245, b = -0.00242, c = 0.0813). The fitting performance measured by SSE and AIC/BIC is shown in Table 3.

Fig.3 shows the nonlinear least-square fitting of English letter ranked frequencies with all ten functions in Eqs.(1-10), and Fig.4 shows the result for Spanish ranked letter frequencies.

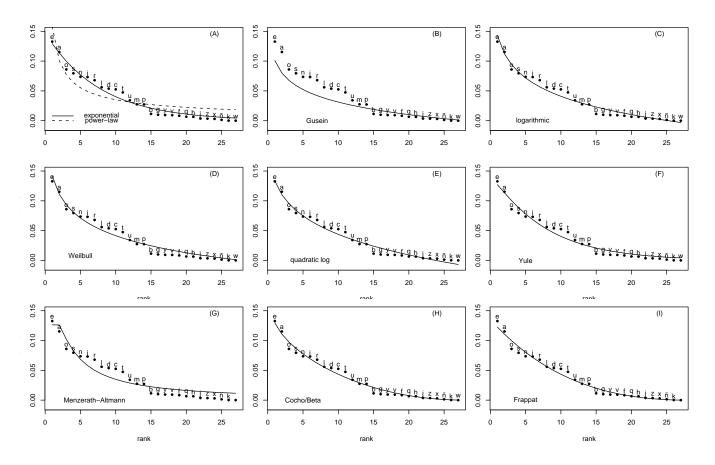


Figure 4: Fitting ranked Spanish letter frequency of Mexican presidents' speech to congress by ten different functions: (A) power-law (a = 0.653) and exponential function (a = 0.130); (B) Gusein function (C = 0.0303); (C) logarithmic function (a = 0.0443); (D) Weibull function (a = 1.05); (E) quadratic logarithmic function (a = 0.0306, b = 0.00362); (F) Yule function (a = -0.0333, b = 0.873); (G) Menzerath-Altmann/Inverse-Gamma function (a = -1.22, b = -1.69); (H) Cocho/Beta function (a = 0.115, b = 2.04); (I) Frappat function (a = 0.0592, b = 0.00315, c = 0.276). The fitting performance measured by SSE and AIC/BIC is shown in Table 3.

The first impression of Figs.3,4 is that all functions seem to fit the ranked letter frequency well, with the exception of power-law and Menzerath-Altmann functions. Is it possible to further distinguish those with even better fitting performance? That is the issue to be addressed in the next section.

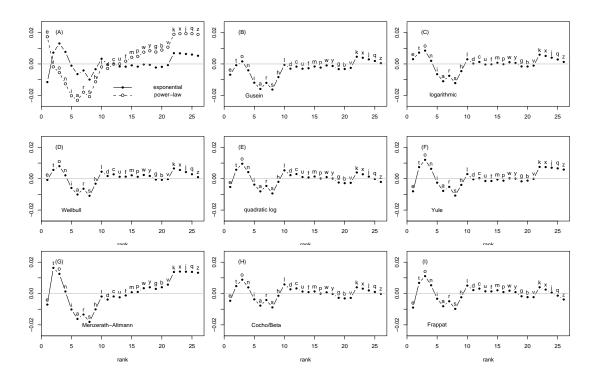


Figure 5: Fitting errors (residual, deviance),  $y_{(r)} - f(r)$ , of the ten functions used in Fig.3 for U.S. presidential speeches.

# 5 Comparison of the fitting performance

How well a function f fits the data can be measured by the sum of squared errors (residuals) SSE:

$$SSE = \sum_{i=1}^{n} (y_i - \hat{f}(x_i))^2$$
(11)

where the parameters of the function are estimated by least-square or maximum likelihood method. It is not correct to compare two functions with different number of parameters, as the function with more parameters has more freedom to adjusting in order to achieve a higher fitting performance. In the extreme example, a function with unlimited number of parameters can fit a finite dataset perfectly: this overfitting situation is called saturation.

To compare two functions with different number of parameters, the Akaike Information Criterion (AIC) (Akaike 1974) and Bayesian Information Criterion (BIC) (Schwarz 1978) can be used for model selection. Both criteria discount the (log) maximum likelihood of the fitting model by a term proportional to the number of parameters (p): AIC uses the term

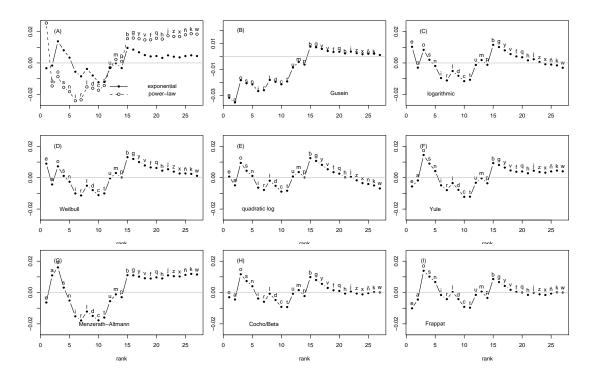


Figure 6: Fitting errors (residual, deviance),  $y_{(r)} - f(r)$ , of the ten functions used in Fig.4 for Mexican presidents' speech to congress.

2p, and BIC uses the term  $\log(n)p$  (where *n* is the sample size). Maximizing the discounted maximum likelihood is our criterion for the best model (equivalent to minimizing AIC or BIC) (Burnham and Anderson 2002).

In regression models (linear or nonlinear), there is a simple relationship between AIC/BIC and SSE if we assume the variance of errors is unknown (and has to be estimated from the data), and if we assume the variance of the error is the same for all data points (details are in Appendix).

Table 3 shows the AIC model selection result for the fitting in Fig.3 and Fig.4. The best function for both English and Spanish, selected by either AIC or BIC, is the Cocho/Beta function (Fig.3(H), Fig.3(H)). The second best function is the quadratic logarithmic function (Fig.3(E), Fig.4(E)). For English text, these functions are followed by Weibull, logarithmic, and Frappat functions. For Spanish texts, the two best functions are followed by Frappat, logarithmic, and exponential functions.

A single SSE value does not tell us whether there exist systematic deviations (e.g., larger

function	Eq.	p	English		Spanish			
			SSE	$\Delta$ AIC	$\Delta$ BIC	SSE	$\Delta$ AIC	$\Delta$ BIC
Gusein-Zade	1	0	0.00106	20.2	17.7	0.00670	57.3	54.8
power-law	2	1	0.00461	60.3	59.0	0.00721	61.3	60.0
exponential	3	1	.000814	15.2	14.0	0.00118	12.5	11.2
logarithmic	4	1	.000635	8.75	7.49	0.00115	11.7	10.4
Weibull	5	2	.000559	7.45	7.45	0.00136	18.2	18.2
quadratic log	6	2	.000460	2.40	2.40	.000915	7.59	7.59
Yule	7	2	.000788	16.4	16.4	0.00117	14.3	14.3
Menzerath-Altmann/Inverse-Gamma	8	2	0.00251	46.5	46.5	0.00340	43.0	43.0
Cocho/Beta	9	2	.000420	0	0	.000691	0	0
Frappat	10	3	.000587	10.7	12.0	.000838	7.20	8.49

 Table 3: Regression diagnosis and model selection of ten functions on English and Spanish letter rank-frequency plots.

deviations at high rank numbers). To address this question, Fig.5 and Fig.6 show the deviation at any rank number for all fitting functions, for English and Spanish respectively. It is interesting that functions with better fitting performance all have a similar pattern in rank-specific deviation.

#### 6 Piecewise functions

The zero-parameter Gusein-Zade function corresponds to a simple exponential cumulative distribution (CD) (for more discussions, see (Li et al. 2010)):

$$CD = 1 - \frac{r}{n+1} = 1 - e^{-f/C}.$$
(12)

In other words, the proportion of values that are larger than  $f_0$  is equal to  $e^{-f_0/C}$ . Since Gusein-Zade function (Eq.(1)) can also be written as  $C = f/\log[(n+1)/r]$ , if we plot  $f_i/\log((n+1)/r_i)$  against  $r_i$   $(i = 1, 2, \dots, n)$ , this function predicts a plateau.

Fig.7 shows  $f_i/\log[(n+1)/r_i]$  as a function of rank, for both English (black) and Spanish (red) letters. Surprisingly, instead of a plateau, we see step functions. For English letters, the top 21 letters (*etoniarshldcufmpwygbv*) form the first group, and the next 5 letters (*kxjqz*)

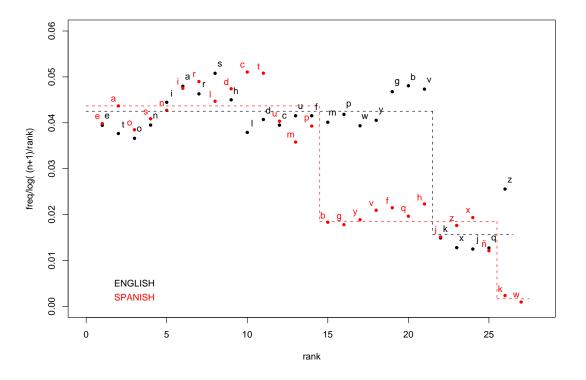


Figure 7: An alternative form of Gusein-Zade function is  $f/\log(\frac{n+1}{r}) = C$ , and its validity can be checked by plotting  $f/\log(\frac{n+1}{r})$  versus the rank r.

form the second one. The average plateau height of the first group in Fig.7 is 0.0425, that of the second group is 0.0157.

For Spanish letters in Fig.7, three groups appear in a step function. The two rarest letters (kw) are very different from others (average plateau height is 0.00165). This is a known fact as k and w are only used in foreign words. The top 14 letters (*eaosnirldctump*) are in one group (average height of 0.0437), and the next 11 letters (*bgyvfqhjzxñ*) form the second group (height is 0.0185). When the Spanish data is compared to the English data, it is interesting that the plateau height of the two groups are similar across the language, whereas the number of letters in the lower-plateau is much larger in Spanish than in English.

The result of Fig.7 indicates that we may construct a piecewise Gusein-Zade function to fit the ranked letter frequency distribution. It should be noted that the number of parameters in a piecewise Gusein-Zade function is no longer zero. For two-piece function, three parameters are estimated: plateau height of the first  $(C_1)$  and the second segment  $(C_2 \neq C_1)$ , and the partition position in x-axis  $(r_0)$ . This minus the normalization constraint leads to 2 free parameters. This 3-parameter (2 of them are free) piecewise function can be written as:

$$f = \begin{cases} C_1 log \frac{n+1}{r} & \text{if } r < r_0 \\ C_2 log \frac{n+1}{r} & \text{if } r \ge r_0 \end{cases}$$
(13)

For the English letter data in Fig.3,  $r_0$  is chosen at 22, least square regression leads to  $C_1 = 0.04065$  and  $C_2 = 0.01394$ , and SSE= 0.000578. For Spanish letters, with 2-segment function partition at  $r_0 = 15$ ,  $C_1 = 0.0424$  and  $C_2 = 0.01897$ , and SSE= 0.000539. Using 3-segment function, SSE is improved only slightly to 0.000537. These results are comparable to the best SSE results obtained by the Beta function (Table 3).

#### 7 Discussion

So far we have not considered space as a "letter". The number of space is simply equal to the number of words  $(N_{space} = N_{word})$ , and the space frequency is  $p_{space} = N_{space}/(N_{space} + N_{letter})$ . For the US presidential speeches, the averaged  $p_{space}$  is 0.174. For Mexican presidential speeches, the averaged  $p_{space}$  is 0.162. There is a mild upward trend for  $p_{space}$  in US presidential speeches, but such a temporal pattern is missing in Mexican texts.

When the "space" is considered as a symbol, its frequency is higher than any other single letters. The rank-frequency plot with space symbol can still be fit perfectly by the Cocho/Beta function (result not shown). The Cocho/Beta is still the best function than others. However, the fitted coefficient values can be quite different when space-symbol is included. For example, for English texts, a = 0.21 and b = 1.35 without the space, but a = 0.50 and b = 0.875 with the space symbol.

Due to the limited range of abscissa, many functions seem to fit the ranked letter frequency distribution very well, and any subtle change might disturb the relative performance among fitting functions. Take the  $-\log(r)$  type functions for example, we have considered three similar functions already, Eq.(1), Eq.(4), Eq.(5), and Eq.(6). The quadratic logarithmic function Eq.(6) clearly outperforms Eq.(1) and Eq.(4), and competes with Cocho/Beta function to become the best fitting function. We notice that in Gusein-Zade's original publication (Gusein-Zade 1988), he proposed a function of the form  $f = (1/r + 1/(r+1) \cdots 1/n)/n$ , which

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also appeared in (Gamow and Yčas 1955) after a random division of unit length problem by John von Neumann. That function can be approximated by  $a - \log(r)$  function.

The piecewise plateaus revealed by Fig.7 seem to partition alphabets into discrete groups. For English, rare letters k,x,j,q,z form their own group, with frequencies much lower than expected by the  $\log((n + 1)/r)$  function. For Spanish, besides the well know letter group of k,w, we found another group with letters  $b,g,y,v,f,q,h, j,z,x,\tilde{n}$ . The height of the first plateau is about twice that of the second plateau, for both English and Spanish. One hypothesis is that these lower-than-expected rare alphabets were originally paired as one letter, then each ancestral letter was split into two letters. Two such pairs can be imagined for English (discard z), and five pairs for Spanish (discard  $\tilde{n}$ ).

Of the ten functions used in this paper, some explicitly include the number of letters, n, as part of the modeling, whereas others do not. Those with n include Gusein-Zade, Weibull, and Cocho/Beta. For some linguistic data, the value of n is fundamentally undecided, for example, the number of words in a language. It is argued that word distribution should be better modeled by "large number of rare events" (LNRE) (Baayen 2001). One consequence of LNRE is that the number of words n increases with the text length (followed the Heaps' law (Heaps, 1978)), making the value of n uncertain. Fortunately, in letter frequencies, the value of n is independent of the text length.

There might be deeper reasons why Cocho/Beta outperforms nine other functions in fitting our data. It was suggected that when a new random variable is constructed by allowing both addition and subtraction of independent and identically distributed random variables, but within certain range, the new random variable follows the Cocho/Beta distribution (Beltrán del Río et al. 2010). Perhaps Cocho/beta function is a limiting functional form for ranked data under a very general condition.

In conclusion, we use ten functions to fit the English and Spanish ranked letter frequency distribution obtained from the US and Mexican presidential speeches. Cocho/Beta function is the best fitting function among the ten, judged by sum of errors (SSE) and Akaike information criterion (AIC). The quadratic logarithmic function is a close second best. We also discover a grouping of letters in both English and Spanish. The rarer-than-expected group in English consists of two pairs of letters whereas that in Spanish consists of five pairs. There is a third, even-rarer-than-expected letter group in Spanish with k, w, consistent with the fact that these are only used for foreign words. Besides the Cocho/Beta and quadratic logarithmic function, it is not conclusive whether other functions follow a universal relative fitting performance order. Needless to say, studying letter frequencies in other languages could potentially answer this question.

# Appendix: Relationship between AIC/BIC and SSE

Akaike information criterion is defined as:  $AIC = -2 \log \hat{L} + 2p$ , where  $\hat{L}$  is maximized likelihood, p is the number of parameter in the statistic model. When a dataset is fitted by a model, if the error is normally distributed, the likelihood of the model is (n is the number of samples,  $\sigma$  is the standard deviation of the normal distribution for the error,  $\{y_i\}$  are the data points, and  $\{\hat{y}_i\}$  are the fitted value):

$$L = \prod_{i=1}^{n} \frac{e^{-(y_i - \hat{y})^2 / 2\sigma^2}}{\sqrt{2\pi\sigma^2}} = \frac{e^{-\sum_{i=1}^{n} (y_i - \hat{y})^2 / 2\sigma^2}}{(2\pi\sigma^2)^{n/2}}$$
(14)

The  $\sum (y_i - \hat{y}_i)^2$  term can be called SSE (sum of squared errors).

If the error variance is unknown, it can be estimated from the data:

$$\hat{\sigma}^2 = \frac{SSE}{n} \tag{15}$$

Replacing  $\sigma$  by the estimated  $\hat{\sigma}$ , we obtained the maximized likelihood, which after log is (Venables and Ripley 2002):

$$\log(\hat{L}) = C - \frac{n}{2}\log(\hat{\sigma}^2) = C - \frac{n}{2}\log(SSE/n)$$
(16)

then,

$$AIC = n \log(SSE/n) + 2 \cdot p + const.$$
<sup>(17)</sup>

and

$$BIC = n \log(SSE/n) + \log(n) \cdot p + const.$$
<sup>(18)</sup>

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