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Using Indicators in Finite Termination Procedures

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Using Indicators in Finite Termination Procedures

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ABSTRACT

The presence of bounded variables complicates finite termination procedures in interior-point methods for linear programming. In our numerical experiments, we found that satisfying the upper bound constraints was the main obstacle to computing an exact solution of a linear program. To prevent the computed solution from violating the bound constraints, one approach

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incorporates nearest bound information into a projection model through an affine scaling transformation. This works well in practice but may introduce ill-conditioning due to the potential presence of infinitesimal weights, particularly for variables near a bound.

In this paper, we investigate the role of Tapia indicators in finite termination procedures. Using Tapia indicators, we identify variables in the active set, remove them from the subproblem, and solve a lower dimensional projection problem. Numerical evidence suggests that using Tapia indicators to identify variables in the active set in tandem with an affine scaling transformation results in the fewest iterations needed to compute an exact solution of a linear program.

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1 Introduction

We consider one class of finite termination procedures called optimal face identification methods. Optimal face identification methods identify the face upon which the objective function attains its optimal value. The optimal face is uniquely defined by the active set, the set of variables which are at a bound at the solution. Once the active set has been identified, the exact solution of a linear program can be obtained by computing an interior feasible point on the optimal face. For a survey of optimal face identification methods, see Williams, El-Bakry, and Tapia [32].

Adding optimal face identification methods to the interior-point framework can lead to computational savings and highly accurate solutions. Moreover, a point on the optimal face can be used to generate an optimal basic solution in strongly polynomial time, see for example Megiddo [18], Bixby and Saltzman [4], Andersen and Ye [2], and Andersen [3]. Knowledge of the optimal face in sensitivity analysis in the context of interior-point methods was assumed by Adler and Monteiro [1]; Monteiro and Mehrotra [22]; Jansen, Roos, and Terlaky [13, 14]; Jansen, Roos, Terlaky, and Vial [15]; and Greenberg [10, 11].

In this paper, we predict the active set of a linear program and remove the corresponding variables from the optimal face identification problem. We compare the efficiency of this approach with the incorporation of bound information through an affine scaling transformation.

1.1 The Linear Programming Problem

We consider linear programs of the form

$$\begin{aligned} & \text{minimize} && c^T x \\ & \text{subject to} && Ax = b, \\ & && l \leq x \leq u, \end{aligned} \tag{1}$$

where $c, x \in \mathbf{R}^n$, $b \in \mathbf{R}^m$, $A \in \mathbf{R}^{m \times n}$ ($m \leq n$) and A has full rank m . The vector $l \in \mathbf{R}^n$ represents the vector of lower bounds and $u \in \mathbf{R}^n$ represents the vector of upper bounds for the vector x . Without loss of generality, we assume all the variables have lower bounds of zero and finite upper bounds. Problem (1) written in standard form is

$$\begin{aligned} & \text{minimize} && c^T x \\ & \text{subject to} && Ax = b, \\ & && x + s = u, \\ & && x, s \geq 0, \end{aligned} \tag{2}$$

where $s \in \mathbf{R}^n$ is the primal slack vector. The inequality $x \geq 0$ denotes component-wise nonnegativity.

The Karush-Kuhn-Tucker(KKT) conditions for (2) are

$$F(x, y, z, s, w) = \begin{pmatrix} Ax - b \\ x + s - u \\ A^T y + z - w - c \\ XZe \\ SWe \end{pmatrix} = 0, \quad (x, z, s, w) \geq 0, \quad (3)$$

where $X = \text{diag}(x)$, $Z = \text{diag}(z)$, $S = \text{diag}(s)$, $W = \text{diag}(w)$, and e is the n -vector of all ones. The vector $y \in \mathbf{R}^m$ is the vector of Lagrange multipliers corresponding to the equality constraints, $z \in \mathbf{R}^n$ is the Lagrange multiplier vector corresponding to the lower bound constraints, and $w \in \mathbf{R}^n$ is the Lagrange multiplier vector corresponding to the upper bound constraints.

The Jacobian of (3) is

$$F'(x, y, z, s, w) = \begin{pmatrix} A & 0 & 0 & 0 & 0 \\ I & 0 & 0 & I & 0 \\ 0 & A^T & I & -I & 0 \\ Z & 0 & X & 0 & 0 \\ 0 & 0 & 0 & W & S \end{pmatrix}. \quad (4)$$

The vectors x and s are feasible for the primal problem if $Ax = b$, $x + s = u$, and $(x, s) \geq 0$. We say that z and w are feasible for the dual problem if there exists y such that (y, z, w) is feasible for the dual constraint, $A^T y + z - w = c$. A point (x, y, z, s, w) is said to be strictly feasible if it satisfies $Ax = b$, $x + s = u$, $A^T y + z - w = c$, and $(x, z, s, w) > 0$.

We denote the solution set of (3) as

$$S = \{(x, y, z, s, w) : F(x, y, z, s, w) = 0, \quad (x, z, s, w) \geq 0\}.$$

If a solution satisfies $x + z > 0$ and $s + w > 0$, in addition to $XZe = 0$ and $SWe = 0$, then this solution is said to satisfy the strict complementarity condition or strict complementarity. Given a feasible point (x, y, z, s, w) we see that $\|F(x, y, z, s, w)\|_1 = x^T z + s^T w$. It can be shown that the expression $x^T z + s^T w$ is equal to the duality gap, which vanishes at any solution.

1.2 Algorithmic Framework

It is well-known that the Kojima, Mizuno, and Yoshise [16] primal-dual interior-point method for linear programming can be viewed as perturbed and damped Newton's method on the first order optimality conditions. In this section, we describe an infeasible primal-dual Newton interior-point method since we implement our finite termination procedures in an infeasible interior-point method.

Algorithm 1 (Infeasible Primal-Dual Interior-Point Algorithm)

Given $v^0 = (x^0, y^0, z^0, s^0, w^0)$ with $(x^0, z^0, s^0, w^0) > 0$, for $k = 0, 1, \dots$, do

1. Choose $\sigma^k \in (0, 1)$ and set $\mu^k = ((x^k)^T z^k + (s^k)^T w^k)/2n$.
2. Solve for the step Δv^k

$$F'(v^k)\Delta v^k = -F(v^k) + \sigma^k \mu \bar{e} \text{ where } \mu \geq 0$$

3. Choose $\tau^k \in (0, 1)$ and set $\alpha^k = \min(1, \tau^k \hat{\alpha}^k)$, where

$$\hat{\alpha}^k = \frac{-1}{\min((X^k)^{-1}\Delta x^k, (Z^k)^{-1}\Delta z^k, (S^k)^{-1}\Delta s^k, (W^k)^{-1}\Delta w^k)}$$

4. Let $v^{k+1} := v^k + \alpha^k \Delta v^k$.
5. Test for convergence.

In Step 2, $\bar{e} = (0, \dots, 0, 1, \dots, 1)^T$ with $2n + m$ zero components. The optimality conditions (3) are perturbed so that the Newton direction obtained from the perturbed KKT conditions does not point towards the boundary. If $\sigma^k = 0$ (i.e., no perturbation), global convergence may be precluded. See Proposition 3.1 of Gonzalez-Lima [17] and Tapia [28] for a proof. In Step 4, the Newton steps are damped to maintain strict positivity of the iterates.

For notational convenience, we introduce

$$\tilde{x} = (x, s) \in \mathbf{R}^{2n} \quad \text{and} \quad \tilde{z} = (z, w) \in \mathbf{R}^{2n}.$$

If $\mathcal{S} \neq \emptyset$, then the relative interior of \mathcal{S} , $ri(\mathcal{S})$, is nonempty. In this case, the solution set \mathcal{S} has the following structure (see El-Bakry, Tapia, and Zhang [6] for a proof): (i) all points in the relative interior satisfy strict complementarity; (ii) the zero-nonzero pattern of points in the relative interior is invariant. For any $(\tilde{x}^*, y^*, \tilde{z}^*)$ in the relative interior of the solution set of (2), we define the index sets $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{N}}$ as

$$\tilde{\mathcal{B}} = \{j : \tilde{x}_j^* > 0, 1 \leq j \leq 2n\} \text{ and } \tilde{\mathcal{N}} = \{j : \tilde{x}_j^* = 0, 1 \leq j \leq 2n\}.$$

Moreover,

$$\tilde{\mathcal{B}} \cup \tilde{\mathcal{N}} = \{1, \dots, 2n\} \text{ and } \tilde{\mathcal{B}} \cap \tilde{\mathcal{N}} = \emptyset.$$

Thus the sets $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{N}}$ define the optimal partition of the set $\{1, 2, \dots, 2n\}$. The optimal partition uniquely defines the optimal primal and dual faces.

The optimal primal face of (2) is

$$\Theta_p = \{\tilde{x} : Ax = b, x + s = u, \tilde{x} \geq 0, \tilde{x}_j = 0 \text{ } j \in \tilde{\mathcal{N}}\}$$

and the optimal dual face is

$$\Theta_d = \{(y, \tilde{z}) : A^T y + z - w = c, \tilde{z} \geq 0, \tilde{z}_j = 0 \text{ } j \in \tilde{\mathcal{B}}\}.$$

In the following sections, for $u \in \mathbf{R}^n$, we use the notation

$$\min u = \min_{1 \leq i \leq n} u_i.$$

The cardinality of set $\tilde{\mathcal{B}}$ is denoted by $|\tilde{\mathcal{B}}|$. Unless otherwise specified, $\|\cdot\|$ is the Euclidean norm.

The paper is organized as follows. We provide an historical overview of the optimal face identification problem in Section 2. In Section 3 we describe mathematical models to solve the optimal face identification problem. Indicators and their role in finite termination procedures are discussed in Section 4. We present computational results in Section 5 and concluding remarks in Section 6.

2 Background

In 1989, stopping tests to compute optimal solutions in interior-point methods for linear programming were proposed by Gay [9]. While these tests did not constitute a finite termination procedure because the primal and dual optimality checks were iterative methods, they were clearly predecessors of current optimal face identification methods. In 1992 Ye [34] popularized the study of finite termination in interior-point methods for linear programming. He was motivated by the fact that the simplex method for linear programming has the finite termination property and also by research activity in efficient algorithmic termination techniques. Ye [34] established a theoretical base for Gay's tests when they are added to primal-dual interior-point algorithms which generate iteration sequences that converge to strict complementarity solutions. The author proposed an orthogonal projection model to identify the optimal primal and dual faces. Mehrotra and Ye [21] developed a solution technique based on Gaussian elimination to compute an interior feasible point on the optimal primal and dual faces. Previously Tardos [29] used Gaussian elimination to calculate a feasible point on the optimal face of an integer program. Recently, Ye [35] proposed a weighted projection model which incorporates bound information, for the standard linear program with no upper bound constraints. Ye [35] proved for k sufficiently large his method when included in a feasible primal-dual interior-point method computed an exact solution of a linear program in finite time. Williams, El-Bakry, and Tapia [33] extended Ye's weighted projection model and analysis to linear programs with bounded variables.

The following lemma provides a theoretical basis for most finite termination procedures for linear programs of the form

$$\begin{aligned} & \text{minimize} && c^T x \\ & \text{subject to} && Ax = b, \\ & && x \geq 0. \end{aligned} \tag{5}$$

Lemma 2.1 (Güler-Ye [12]) *Let $\{(x^k, y^k, z^k)\}$ be an iteration sequence generated by an interior-point algorithm. Furthermore, let x^k and z^k satisfy*

$$\frac{\min(X^k Z^k e)}{(x^k)^T z^k / n} \geq \gamma \quad (6)$$

where $\gamma > 0$ and is independent of k . Then every limit point of $\{(x^k, z^k)\}$ satisfies the strict complementarity condition.

Lemma 2.1 is sufficient to guarantee that all limit points of the iteration sequence are in the relative interior of the solution set. Since the nonzero-zero pattern of points is invariant in the relative interior, the optimal primal and dual faces of (5) are uniquely defined.

3 Mathematical Models

Before describing our proposed modification, we briefly discuss projection models for (5) and define some notation. The projection models for linear program (5) are important because traditionally researchers have not explicitly included upper bound constraints into the subproblem. Instead they treat the upper bounds as side constraints.

We define the index sets \mathcal{B} and \mathcal{N} as

$$\mathcal{B} = \{j : x_j^* > 0, 1 \leq j \leq n\} \text{ and } \mathcal{N} = \{j : x_j^* = 0, 1 \leq j \leq n\}.$$

The index sets \mathcal{B} and \mathcal{N} define the optimal partition of (5). The columns of A corresponding to the indices of \mathcal{B} comprise the matrix $A_{\mathcal{B}}$. The matrix $A_{\mathcal{N}}$ is formed in an similar manner. The vector $x_{\mathcal{B}}$ represents the components of the vector x whose indices are in \mathcal{B} .

Ye [35] was the first to incorporate bound information into the optimal face identification model via an affine scaling transformation of the subproblem. For the standard linear program (5), Ye posed two projection problems

$$\begin{aligned} \min \quad & \frac{1}{2} \|(X_{\mathcal{B}}^k)^{-1}(x_{\mathcal{B}} - x_{\mathcal{B}}^k)\|^2 \\ \text{s.t.} \quad & A_{\mathcal{B}} x_{\mathcal{B}} = b \end{aligned} \quad (7)$$

and

$$\begin{aligned} \min \quad & \frac{1}{2} \|(Z_{\mathcal{N}}^k)^{-1} A_{\mathcal{N}}^T (y - y^k)\|^2 \\ \text{s.t.} \quad & A_{\mathcal{B}}^T y = c_{\mathcal{B}}, \end{aligned} \quad (8)$$

to find an interior solution on the optimal primal and dual faces, respectively. The primal model restricts movement in components that can least afford to deviate from $x_{\mathcal{B}}^k$ by placing large weights on the smaller components. Similarly, the dual model attempts to restrict movement of $z_{\mathcal{N}}^k$ components.

Williams [31] and Williams, El-Bakry, Tapia [33] extended Ye's work to linear programs with bounded variables. The authors proposed the following modified weighted projection model

$$\begin{aligned} \min \quad & \frac{1}{2} \|(D^k)^{-1}(x_{\mathcal{B}} - x_{\mathcal{B}}^k)\|^2 \\ \text{s.t.} \quad & A_{\mathcal{B}}x_{\mathcal{B}} = b, \end{aligned} \tag{9}$$

where

$$d_{jj}^k = \min(x_j^k, u_j - x_j^k) \quad \text{for } j \in \mathcal{B}, \tag{10}$$

as the optimal primal face identification problem and

$$\min_y \quad \frac{1}{2} \|D^k(A_{\mathcal{B}}^T y - w_{\mathcal{B}} - c_{\mathcal{B}})\|^2 \tag{11}$$

as the optimal dual face identification problem. Given formulation (11), one matrix factorization is needed to solve both the optimal primal and dual face identification problems as opposed to the two matrix factorizations required by (7) and (8). If no upper bounds exist, problem (9) reduces to problem (7).

Weighting the objective function by D^k penalizes the movement of the variables in the direction of their nearest bound. Therefore, if x_j^k for $j \in \mathcal{B}$ is close to its upper bound, the weight in (10) prevents the j th component of the solution vector $x_{\mathcal{B}}$ from violating its upper bound as well as its lower bound, which is the desired result. The use of weights to prevent bound violations is not novel. Plantenga [23], for example, proposed scaling the quadratic subproblem of a sequential quadratic programming algorithm for bound and equality constrained optimization using trust regions by D^k to prevent bound violations.

The affine scaling transformation in (9) has the following drawback. If any variable is 'close' to its upper bound, the scale D^k has the potential of introducing ill-conditioning into the subproblem. We want to minimize this possibility by removing these problematic variables from the subproblem. The difficulty arises in defining 'closeness' of a variable to its upper bound. The next section addresses this issue.

4 Identification of the Active Set

The term indicator denotes a function that identifies active constraints at the solution of a constrained optimization problem, see Tapia [27] and El-Bakry [5]. Commonly used indicators include variables as indicators, the primal-dual indicator, and the Tapia indicator. Recently, Facchinei, Fischer, and Kanzow [8] proposed an indicator based on growth functions to identify the active set of column sufficient linear complementarity problems. A similar indicator for general nonlinear programs was introduced in Facchinei, Fischer, and Kanzow [7].

An extensive numerical comparison of indicators as stopping criterion can be found in de Vreede [30]. For a thorough study of indicator theory, see El-Bakry [5] and El-Bakry, Tapia, and Zhang [6].

Tapia [27] used the following indicators to determine the active set in nonlinear constrained optimization problems. The Tapia indicators are

$$T_p(x_j^k) = \frac{x_j^{k+1}}{x_j^k} \quad \text{and} \quad T_d(z_j^k) = \frac{z_j^{k+1}}{z_j^k},$$

where $x_j^{k+1} = x_j^k + \beta^k \Delta x_j^k$ and $z_j^{k+1} = z_j^k + \beta^k \Delta z_j^k$. In [6] El-Bakry, Tapia, and Zhang showed the Tapia indicators have a 0-1 separation property and converge R -superlinearly to their terminal values.

It is well-known that the Tapia indicators are an effective computational tool for identifying the active set of problem (5), which can lead to reduction of problem size and computational savings. Because of the efficacy of the Tapia indicator in identifying variables which are zero on the solution set, it is the natural choice to identify variables at their upper bounds. Unfortunately the indicator, $T_p(x_j^k)$, does not differentiate between variables that are at their upper bounds and those which are strictly between their lower and upper bounds. Other indicators such as variables as indicators and the logarithmic Tapia indicators predict variables at the upper bounds with varying success. However, this drawback is easily remedied.

Let's consider the upper bound constraint

$$x + s = u.$$

We see that as $x_j^k \rightarrow u_j$, then $s_j^k \rightarrow 0$. Therefore determining variables at their upper bounds is the same as identifying which primal slack variables are zero.

The Tapia indicators for the primal slack variable, s , and its corresponding Lagrange multiplier, w , are

$$T(s_j^k) = \frac{s_j^{k+1}}{s_j^k} \quad \text{and} \quad T(w_j^k) = \frac{w_j^{k+1}}{w_j^k},$$

where $s_j^{k+1} = s_j^k + \beta^k \Delta s_j^k$ and $w_j^{k+1} = w_j^k + \beta^k \Delta w_j^k$.

We use the notation

$$\mathcal{U} = \{i : s_i = 0, 1 \leq i \leq n\}$$

to denote the primal slack variables which are zero at the solution of (2) and correspondingly the x variables which equal their upper bounds.

The following proposition shows that the Tapia indicators for the slack variables and the corresponding Lagrange multipliers for the upper bound constraints have a 0-1 separation property. For numerical experimentation, the 0-1 convergence limits of the Tapia indicators provide us with a theoretical basis for an indicator threshold value to use in active set identification.

Proposition 4.1 Consider a sequence of iterates $\{(x^k, y^k, z^k, s^k, w^k)\}$ generated by Algorithm 1. Assume

1. $(x^k)^T z^k + (s^k)^T w^k \rightarrow 0$;
2. $\min(X^k Z^k e, S^k W^k e) \geq \gamma \mu^k$, for all k and for some $\gamma \in (0, 1)$;
3. The algorithmic parameters are chosen such that

$$\sigma^k \rightarrow 0 \text{ and } \tau^k \rightarrow 1.$$

Then for $j = 1, \dots, n$

$$\lim_{k \rightarrow \infty} \frac{s_j^{k+1}}{s_j^k} = \begin{cases} 0 & j \in \mathcal{U} \\ 1 & j \notin \mathcal{U} \end{cases}$$

$$\lim_{k \rightarrow \infty} \left(1 - \frac{w_j^{k+1}}{w_j^k}\right) = \begin{cases} 0 & j \in \mathcal{U} \\ 1 & j \notin \mathcal{U} \end{cases}$$

where $s^{k+1} = s^k + \beta^k \Delta s^k$ and $w^{k+1} = w^k + \beta^k \Delta w^k$ for any $\beta^k \in [\alpha^k, 1]$ with α^k given in step 3 of Algorithm 1.

An alternative is to apply the Tapia indicator directly to the vector \tilde{x} , which includes the primal slack vector. We can then extract bound information from the terminal value of $T_p(\tilde{x}_j)$, but for our purposes it is more convenient to consider the slack vector separately.

In the following section, we describe numerical experiments performed to test the effectiveness of the proposed optimal face identification models in computing an exact solution of a linear program. We measure effectiveness in terms of the number of projection attempts that are needed to compute an exact solution of a linear program.

5 Computational Results

In our numerical experiments, we used the **LIPSOL** - Linear programming Interior-Point SOLver- package developed under the **MATLAB**³ environment. The software package, written by Zhang [36], implements an infeasible primal-dual predictor-corrector interior-point method. We selected 35 problems with upper bounds from the *netlib* suite of linear programming problems as our test set.

The initial matrix is scaled in an attempt to achieve row/column equilibration. Preprocessing deletes fixed variables, deletes zero rows and columns from the matrix A , solves equations of one variable, and shifts nonzero lower bounds to zero.

³**MATLAB** is a registered trademark of The MathWorks, Inc.

5.1 Methodology

We first attempt to compute an exact solution of a linear program, when

$$\frac{|c^T x^k - (b^T y^k - u^T w^k)|}{1 + |b^T y^k - u^T w^k|} \leq 10^{-8}.$$

Another option is to project when maximum relative error,

$$\max \left(\frac{\|Ax - b\|}{1 + \|b\|}, \frac{\|A^T y + z - w - c\|}{1 + \|c\|}, \frac{|c^T x - (b^T y - u^T w)|}{1 + |b^T y - u^T w|} \right) \leq 10^{-8}.$$

Mehrotra and Ye in [21], Mehrotra in ([19, 20]), as well as Andersen and Ye [2] used the Tapia indicators to identify the optimal partition. Specifically, they defined

$$\mathcal{B}^k = \left\{ j : \frac{|\Delta x_j^k|}{x_j^k} \leq \frac{|\Delta z_j^k|}{z_j^k} \right\}. \quad (12)$$

Mehrotra and Ye [21] proved that when \mathcal{B}^k was defined as in (12) the optimal partition could be identified in finite time for algorithms that generate iteration sequences that satisfy centrality measure (6). For numerical experiments, the authors used the Tapia indicators and variables as indicators in tandem.

Following their philosophy, we let

$$\mathcal{B}^k = \{j : z_j^k \leq 10^{-14} \text{ or } |\Delta x_j^k|/x_j^k \leq |\Delta z_j^k|/z_j^k\}. \quad (13)$$

Similarly,

$$\mathcal{U}^k = \{j : s_j^k \leq 10^{-14} \text{ or } |\Delta w_j^k|/w_j^k \leq |\Delta s_j^k|/s_j^k\}.$$

We then redefine \mathcal{B}^k as

$$\mathcal{B}^k := \mathcal{B}^k \setminus \mathcal{U}^k.$$

This step is necessary since the set in (13) may contain indices of variables at their upper bounds.

After setting

$$x_{\mathcal{U}} = u_{\mathcal{U}}, \quad x_{\mathcal{N}} = 0,$$

we solve

$$\begin{aligned} \min \quad & \frac{1}{2} \|(D^k)^{-1}(x_{\mathcal{B}} - x_{\mathcal{B}}^k)\|^2 \\ \text{s.t.} \quad & A_{\mathcal{B}} x_{\mathcal{B}} = b - A_{\mathcal{U}} x_{\mathcal{U}} \end{aligned} \quad (14)$$

and

$$\min_y \quad \frac{1}{2} \|D^k(A_{\mathcal{B}}^T y - w_{\mathcal{B}} - c_{\mathcal{B}})\|^2$$

for x_B and y . We then update z, s, w as follows.

Set $s = u - x$ and $\delta = c - A^T y$, then

$$z_j = \begin{cases} 0 & \text{if } \delta_j < 0 \\ \delta_j & \text{else} \end{cases} \quad \text{and } w_j = \begin{cases} -\delta_j & \text{if } \delta_j < 0 \\ 0 & \text{else.} \end{cases}$$

The update formula for the dual variables was used by Resende and Veiga in [25], Resende, Tsuchiya, and Veiga [26] and Portugal, Resende, Veiga, and Judice [24] to generate feasible dual variables in network flow problems.

If the computed solution is complementary and

$$\max \left(\frac{\|Ax - b\|}{1 + \|b\|}, \frac{\|A^T y + z - w - c\|}{1 + \|c\|}, \frac{|c^T x - (b^T y - u^T w)|}{1 + |b^T y - u^T w|} \right) \leq 10^{-11}$$

the algorithm terminates with an exact solution of a linear program. If not, the procedure is repeated at the next interior-point iteration.

We allow a maximum of six projection attempts to compute an exact solution.

5.2 Numerics

Table 1 illustrates problem size reduction, one of the benefits of identifying variables which are at their upper bounds. For problem *fit1d*, removal of variables at their bounds transforms the linear system from an undetermined system to a square system.

Table 1: Problem Size Reduction

Problem	$ B^k $	$ U^k $	$ B^k \setminus U^k $
boeing2	123	6	117
finnis	435	17	418
fit1d	377	353	24
greenbeb	1456	194	1262
pilot4	611	247	364
pilotnov	2114	1	2113
recipe	147	8	139
seba	442	3	439
vtpbase	135	32	103

We compare three projection models to compute an exact solution of a linear program. The first one is the orthogonal projection model where $D^k = I$, the second is Ye's weighted projection method (7) and the third is the modified weighted projection. The first two models

were developed by Ye [34, 35]. Column 1 of Tables 2 and 3 gives the number of failed calls (misses) to the finite termination procedure before the optimal face identification problem was solved to the desired accuracy. We consider a call a failure if the procedure does not generate a positive solution that satisfies the given feasibility tolerances. The second column gives the number of problems solved by the orthogonal projection model for the given number of misses. The third column contains the computational results of Ye's weighted projection model and the fourth column the modified weighted projection.

Table 2: Variables at Upper Bounds Not Removed

# of misses	Models		
	Orthogonal	Weighted	Modified Weighted
0	13	15	19
1	13	12	11
2	2	2	4
3	2	3	1
4	2	0	0
5	1	1	0
more than 5	2	2	0

Table 3: Variables at Bounds Removed

# of misses	Models		
	Orthogonal	Weighted	Modified Weighted
0	21	21	22
1	9	8	8
2	3	4	3
3	2	2	2

Identifying the variables at their upper bounds plays a crucial role in finite termination procedures. The orthogonal projection model computes an exact solution of a linear program with 60% fewer misses on the first attempt; the weighted projection model has approximately 40% fewer misses. The reduction in projection attempts represents a substantial savings in computational expense.

To further demonstrate the importance of identifying and removing variables at their upper bounds, in Table 4 we consider three problems, *etamacro*, *greenbea*, *nesm*, for which either 6 projection attempts were needed or the optimal face identification procedure failed to compute

an exact solution. Removing the variables at the upper bounds produces a dramatic decrease in the projection attempts. In particular, the problem *greenbea* requires 5 fewer projections.

Table 4: Orthogonal Projection Model

problem	Misses	
	Variables at Upper Bounds not Identified	Identified
etamacro	5	3
greenbea	> 5	0
nesm	> 5	2

It is important to point out that for these three problems the number of misses with variables at their upper bounds identified is the same as the modified weighted projection model without variables at upper bounds identified. This suggests that the modified scale simulates removal of variables at their upper bounds from the problem.

6 Concluding Remarks

Using Tapia indicators to identify variables at their upper bounds increases the efficiency of optimal face identification procedures. For best results we recommend the implementation of a finite termination procedure which includes Tapia indicators as well as an affine scale transformation which incorporates bound constraints.

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