# On the convergence of the forward-backward splitting method with linesearches 

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#### Abstract

In this paper we focus on the convergence analysis of the forward-backward splitting method for solving nonsmooth optimization problems in Hilbert spaces when the objective function is the sum of two convex functions. Assuming that one of the functions is Fréchet differentiable and using two new linesearches, the weak convergence is established without any Lipschitz continuity assumption on the gradient. Furthermore, we obtain many complexity results of cost values at the iterates when the stepsizes are bounded below by a positive constant.


Keywords: Armijo-type linesearch; Iteration complexity; Nonsmooth and convex optimization problems; Proximal gradient splitting method.

Mathematical Subject Classification (2010): 65K05, 90C25, 90C30.

## 1 Introduction

We are interested in solving problems of the following form:

$$
\begin{equation*}
\min f(x)+g(x) \text { subject to } x \in \mathcal{H} \tag{1}
\end{equation*}
$$

where $\mathcal{H}$ is a real Hilbert space with the inner product $\langle\cdot, \cdot\rangle$, and $f, g: \mathcal{H} \rightarrow \overline{\mathbb{R}}:=\mathbb{R} \cup\{+\infty\}$ are two proper lower semicontinuous convex functions in which $f$ is Fréchet differentiable on an open set containing the domain of $g$. The optimal solution set of this problem will be denoted by $S_{*}$. Recently problem (11) together with many variants of it has received much attention from optimization community due to its broad applications to many disciplines such as optimal control, signal processing, system identification, machine learning, and image analysis; see, e.g., [15, 16, 27] and the references therein. Many effective methods have been proposed to solve problem (11). Most of them keep using the idea of splitting $f$ and $g$ separately and taking the advantage of some Lipschitz assumption on the derivative of $f$ at each iteration. Here we focus our attention on the so-called forward-backward splitting method, which contains a forward gradient step of $f$ (an explicit step) followed by a backward proximal step of $g$ (an implicit step) for problem (1); see, e.g., [27].

[^0]In this work linesearches are used to eliminate the undesired Lipschitz assumption on the gradient of $f$ mostly imposed in the literature.

To describe and motivate our methods, let us recall here the so-called proximal operator prox ${ }_{g}:=$ $(\partial g+\mathrm{Id})^{-1}$, where $\partial g$ is the classical convex subdifferential of $g$ and Id is the identity operator in $\mathcal{H}$. Among many important properties of proximal operators, it is well-known that prox ${ }_{g}$ is welldefined with full domain, single-valued, and even nonexpansive; see, e.g., [3, 15, 16]. Furthermore, for any $\alpha>0, x$ is an optimal solution to problem (1) if and only if $x=\operatorname{prox}_{\alpha g}(x-\alpha \nabla f(x))$. This indeed motivates the construction of the iterative sequence forming the forward-backward iteration as following:

$$
\begin{equation*}
x^{k+1}:=\operatorname{prox}_{\alpha_{k} g}\left(x^{k}-\alpha_{k} \nabla f\left(x^{k}\right)\right) \tag{2}
\end{equation*}
$$

with positive stepsize $\alpha_{k}$. The iteration presented in (2) has been attracted extensive interests due to its simplicity and several important advantages. It is well-known that this method uses little storage, readily exploits the separable structure of problem (11), and is easily implemented to practical applications; see [5, 27, 29]. Moreover, scheme (2) may reduce to many popular optimization methods as particular cases including the projected gradient method for smooth constrained minimization; the proximal point method; the CQ algorithm for the split feasibility problem; the projected Landweber algorithm for constrained least squares; the iterative soft thresholding algorithm for linear inverse problems; decomposition methods for solving variational inequalities; and the simultaneous orthogonal projection algorithm for the convex feasibility problem; see, e.g., [4, 13, 17, 18, 32, 35, 36 and the references therein.

The convergence of the iteration (22) to an optimal solution of (11) is usually established under the assumption that the gradient of $f$ is Lipschitz continuous and the stepsize $\alpha_{k}$ is taken bounded below and less than some constant related with the Lipschitz modulus; see, e.g., [16, Theorem 3.4(i)]. In this case, the main machinery to prove the convergence and its complexity is based on the renowned Baillon-Haddad Theorem [3, Corollary 18.16]. When $\nabla f$ is Lipschitz continuous but somehow the Lipschitz constant is not known, finding the stepsize $\alpha_{k}$ that guarantees the convergence of (2) would be a challenge. However, the following linesearch proposed in [5] overcome this inconvenience: choosing the stepsize $\alpha_{k}$ in (2) as the largest $\alpha \in\left\{\sigma, \sigma \theta, \sigma \theta^{2}, \ldots,\right\}$ with constants $\sigma>0$ and $\theta \in(0,1)$ such that:

$$
\begin{equation*}
f\left(J\left(x^{k}, \alpha\right)\right) \leq f\left(x^{k}\right)+\left\langle\nabla f\left(x^{k}\right), J\left(x^{k}, \alpha\right)-x^{k}\right\rangle+\frac{1}{2 \alpha}\left\|x^{k}-J\left(x^{k}, \alpha\right)\right\|^{2} \tag{3}
\end{equation*}
$$

where $J\left(x^{k}, \alpha\right):=\operatorname{prox}_{\alpha g}\left(x^{k}-\alpha \nabla f\left(x^{k}\right)\right)$ and $\|\cdot\|$ is the norm induced by the inner product in $\mathcal{H}$. This linesearch is well-defined by taking the advantage of the Lipschitz assumption for $\nabla f$ again via the so-called Descent Lemma 3, Theorem 18.15(iii)]. As far as we observe, the theory of convergence and complexity for the forward-backward is almost complete under such a Lipschitz assumption. However, the Lipschitz condition fails in many natural circumstances; see, e.g., [14]. It is quite interesting to question the convergence of the method and its complexity without the Lipschitz assumption aforementioned. In [34 Tseng provided an evidence of positive answer even for more general problems of finding a zero point of the sum of two maximal monotone operators. His crucial approach motivates us to construct Method 1 for problem (1) in our Section 4. But working on the functionals ( $f$ and $g$ ) rather than just the maximal operators ( $\nabla f$ and $\partial g$ ) actually gives us much more convenience. Indeed, we completely relax an (expensive) extra projection step from Tseng's scheme and omit several unnatural assumptions in the main theorem [34, Theorem 3.4]. Moreover, in the spirit of linesearch on functionals like (3) and following some ideas presented in [7, 33, 37], we also introduce a new linesearch mainly used in our Method 3 in Section 5. Both Method 1 and

Method 3 guarantee weak convergence of their generated sequences to optimal solutions without imposing the Lipschitz assumption on $\nabla f$.

Another achievement of our work is the study on complexity of cost values at generated sequences, which are proved to converge to the infimum value of problem (11) even in the case when the set of optimal solutions is empty. It is worth mentioning that in order to obtain the rate $\mathcal{O}\left(k^{-1}\right)$ of the functional value $(f+g)\left(x^{k}\right)$ to the optimal cost, the gradient $\nabla f$ is usually supposed to be globally Lipschitz continuous in the classical forward-backward iteration [4, 5, 16, 27, 29. Here, in finite dimensions, we derive the better rate $o\left(k^{-1}\right)$ even with strictly weaker assumptions, for instance, $\nabla f$ only needs to be locally Lipschitz continuous for our Method 1 and Method 3. This partially generalizes several results in [21-23], in which the authors also derive the complexity $o\left(k^{-1}\right)$ for proximal point method (when $f \equiv 0$ ). Moreover, we present an interesting example of problem (1) with non-Lipschitz gradient where the stepsizes generated by both linesearches converge to zero and the complexity $o\left(k^{-1}\right)$ of the cost values remains valid. Furthermore, the rate $\mathcal{O}\left(k^{-2}\right)$ is also obtained for our Method 2, an accelerating version of Method 1 motivated from [5]. Again, global Lipschitz continuity on $\nabla f$ is lessened.

The paper is organized as follows. The next section presents some preliminary results that will be used throughout the paper. We also discuss here our standing assumptions for the problem which is somewhat natural for the lack of Lipschitz assumption aforementioned. Section 3 devotes to the two different linesearches for the forward-backward methods used in Sections 4 and 5. Weak convergence and complexity of the forward-backward method with the first linesearch are analyzed in Section 4. We also consider its accelerated version here. Section 5 provides a similar study for a variant of the forward-backward splitting method with the second linesearch. We complete the paper with some conclusion for further study.

## 2 Preliminary results

In this section we present some definitions and results needed for our paper. Let $h: \mathcal{H} \rightarrow \overline{\mathbb{R}}$ be a proper, lower semicontinuous (l.s.c.), and convex function. We denote the domain of $h$ by dom $h:=\{x \in \mathcal{H} \mid h(x)<+\infty\}$. For any $x \in \operatorname{dom} h$, the directional derivative of $h$ at $x$ in the direction $d$ is

$$
h^{\prime}(x ; d):=\lim _{t \rightarrow 0^{+}} \frac{h(x+t d)-h(x)}{t}
$$

which always exists (although it may be infinite). The subdifferential of $h$ at $x$ is defined by

$$
\begin{equation*}
\partial h(x):=\{v \in \mathcal{H} \mid\langle v, y-x\rangle \leq h(y)-h(x), y \in \mathcal{H}\} \tag{4}
\end{equation*}
$$

Fact 2.1 ( [3, Proposition 17.2] ). Let $h: \mathcal{H} \rightarrow \overline{\mathbb{R}}$ be a proper, l.s.c., and convex function. Then, for $x \in \operatorname{dom} h$ and $y \in \mathcal{H}$, the following hold:
(i) $h^{\prime}(x ; y)$ exists and $h^{\prime}(x ; y)=\inf _{t \in \mathbb{R}_{++}} \frac{h(x+t y)-h(x)}{t}$.
(ii) $h^{\prime}(x ; y-x)+h(x) \leq h(y)$.

Fact 2.2 ( [10, Theorem 4.7.1 and Proposition 4.2.1(i)] ). The subdifferential operator $\partial h$ is maximal monotone, i.e., it has no proper monotone extension in the graph inclusion sense. Moreover, the graph of $\partial h, \operatorname{Gph}(\partial h):=\{(x, v) \in \mathcal{H} \times \mathcal{H} \mid v \in \partial h(x)\}$ is demiclosed, i.e., if the sequence $\left(x^{k}, v^{k}\right)_{k \in \mathbb{N}} \subset \operatorname{Gph}(\partial h)$ satisfies that $\left(x^{k}\right)_{k \in \mathbb{N}}$ converges weakly to $x$ and $\left(v^{k}\right)_{k \in \mathbb{N}}$ converges strongly to $v$, then $(x, v) \in \operatorname{Gph}(\partial h)$.

Next we set the standing assumptions on the data of problem (1) used throughout the paper as follows:

A1 $f, g: \mathcal{H} \rightarrow \overline{\mathbb{R}}$ are two proper l.s.c. convex functions with $\operatorname{dom} g \subseteq \operatorname{dom} f$.
A2 The function $f$ is Fréchet differentiable on an open set containing dom $g$. The gradient $\nabla f$ is uniformly continuous on any bounded subset of $\operatorname{dom} g$ and maps any bounded subset of dom $g$ to a bounded set in $\mathcal{H}$.

Assumption A1 and the first part of Assumption A2 are popular and crucial for the well-definedness of the forward-backward iteration (2). It is easy to check that the second part of A2 is automatic when $\nabla f$ is Lipschitz continuous on dom $g$. However, Assumption A2 is not enough to guarantee the Lipschitz continuity of $\nabla f$. Indeed, the convex functions $f(x) \equiv\|x\|^{p}(1<p<+\infty, p \neq 2)$ and $g(x) \equiv 0, x \in \mathcal{H}$ satisfy all the conditions in $\mathbf{A 2}$ but $\nabla f$ is not globally Lipschitz continuous. When $\mathcal{H}$ is a finite-dimensional space and the domain of $g$ is closed, Assumption A2 actually means that $f$ is Fréchet differentiable on an open set containing dom $g$ and that its gradient is continuous on dom $g$. It is worth noting further that the closedness of $\operatorname{dom} g$ is broadly assumed for problem (11) in the literature including the case of optimization problems with geometric constraints, which can be written as (11) when $g$ is an indicator function; see, e.g., [27.

Proposition 2.3. Let $\mathcal{H}$ be a finite-dimensional space and let $f, g: \mathcal{H} \rightarrow \overline{\mathbb{R}}$ be two functions satisfying A1. Suppose that the closure of $\operatorname{dom} g$, denoted by $\mathrm{cl}(\operatorname{dom} g)$ is a subset of $\operatorname{dom} f, f$ is Fréchet differentiable on an open set containing $\mathrm{cl}(\operatorname{dom} g)$, and that its gradient $\nabla f$ is continuous on $\mathrm{cl}(\operatorname{dom} g)$. Then Assumption A2 is satisfied.

Consequently, if dom $g$ is closed then the validity of Assumption A2 is equivalent to the statement that $f$ is Fréchet differentiable on an open set containing $\operatorname{dom} g$ and its gradient is continuous on dom $g$.

Proof. To justify, suppose that $\operatorname{dim} \mathcal{H}<+\infty, \operatorname{cl}(\operatorname{dom} g) \subseteq X \subseteq \operatorname{dom} f, f$ is Fréchet differentiable on an open set containing $\mathrm{cl}(\operatorname{dom} g)$, and that $\nabla f$ is continuous on $\mathrm{cl}(\operatorname{dom} g)$. Take any bounded set $A$ of $\operatorname{dom} g$. Note that $\nabla f$ is uniformly continuous on the compact set $\mathrm{cl} A \subseteq \mathrm{cl}(\operatorname{dom} g)$ and thus on $A$ due to the classical Heine-Cantor Theorem. Since $\nabla f$ is continuous on cl (domg), it maps the compact set $\mathrm{cl} A \subseteq \mathrm{cl}(\operatorname{dom} g)$ to a compact set in $\mathcal{H}$. This verifies that $\nabla f(A)$ is bounded and completes the first part of the proposition.

Now suppose that dom $g$ is closed. It is easy to see that the validity of Assumption A2 implies that $\nabla f$ is continuous on dom $g$. This together with the first part of this proposition justifies the second part. The proof is completed.

Let us recall the proximal operator $\operatorname{prox}_{g}: \mathcal{H} \rightarrow \operatorname{dom} g$ with $\operatorname{prox}_{g}(z)=(\operatorname{Id}+\partial g)^{-1}(z), z \in \mathcal{H}$. It is well-known that the proximal operator is single-valued with full domain. Furthermore, note that

$$
\begin{equation*}
\frac{z-\operatorname{prox}_{\alpha g}(z)}{\alpha} \in \partial g\left(\operatorname{prox}_{\alpha g}(z)\right) \quad \text { for all } \quad z \in \mathcal{H}, \alpha \in \mathbb{R}_{++}:=\{t \in \mathbb{R} \mid t>0\} \tag{5}
\end{equation*}
$$

We also denote the forward-backward operator $J: \operatorname{dom} g \times \mathbb{R}_{++} \rightarrow \operatorname{dom} g \subset \mathcal{H}$ by

$$
\begin{equation*}
J(x, \alpha):=\operatorname{prox}_{\alpha g}(x-\alpha \nabla f(x)) \quad \text { for all } \quad x \in \operatorname{dom} g \subseteq \operatorname{dom} f, \alpha>0 . \tag{6}
\end{equation*}
$$

The following lemma is very useful for our further study.

Lemma 2.4 ( [24, Lemma 1]). Let $f, g: \mathcal{H} \rightarrow \overline{\mathbb{R}}$ be two functions satisfying Assumption A1, Then for any $x \in \operatorname{dom} g$ and $\alpha_{2} \geq \alpha_{1}>0$, we have

$$
\begin{equation*}
\frac{\alpha_{2}}{\alpha_{1}}\left\|x-J\left(x, \alpha_{1}\right)\right\| \geq\left\|x-J\left(x, \alpha_{2}\right)\right\| \geq\left\|x-J\left(x, \alpha_{1}\right)\right\| . \tag{7}
\end{equation*}
$$

Let us end the section by recalling the well-known concepts of so-called quasi-Fejér and Fejér convergence. The definition originates in 19 and has been elaborated further in [12, 25].
Definition 2.1. Let $S$ be a nonempty subset of $\mathcal{H}$. A sequence $\left(x^{k}\right)_{k \in \mathbb{N}}$ in $\mathcal{H}$ is said to be quasiFejér convergent to $S$ if and only if for all $x \in S$ there exists a sequence $\left(\epsilon_{k}\right)_{k \in \mathbb{N}}$ in $\mathbb{R}_{+}$such that $\sum_{k=0}^{\infty} \epsilon_{k}<+\infty$ and $\left\|x^{k+1}-x\right\|^{2} \leq\left\|x^{k}-x\right\|^{2}+\epsilon_{k}$ for all $k \in \mathbb{N}$. When $\left(\epsilon_{k}\right)_{k \in \mathbb{N}}$ is a null sequence, we say that $\left(x^{k}\right)_{k \in \mathbb{N}}$ is Fejér convergent to $S$.
Fact 2.5 ( [25, Theorem 4.1] ). If $\left(x^{k}\right)_{k \in \mathbb{N}}$ is quasi-Fejér convergent to $S$, then one has:
(i) The sequence $\left(x^{k}\right)_{k \in \mathbb{N}}$ is bounded.
(ii) If all weak accumulation points of $\left(x^{k}\right)_{k \in \mathbb{N}}$ belong to $S$, then $\left(x^{k}\right)_{k \in \mathbb{N}}$ is weakly convergent to a point in $S$.

## 3 The linesearches

In this section we present two different linesearches mainly used in the forward-backward methods proposed in Sections 4 and 5. The first one contains a backtracking procedure which computes at least one backward step (implicit step) inside the updating inner loop for finding the steplength. This linesearch is a particular case of the one proposed in [34] for solving inclusion problems. It will be used in Method 1 and Method 2 of Section 4.

Linesearch 1. Given $x, \sigma>0, \theta \in(0,1)$ and $\delta \in(0,1 / 2)$.
Input. Set $\alpha=\sigma$ and $J(x, \alpha):=\operatorname{prox}_{\alpha g}(x-\alpha \nabla f(x))$ with $x \in \operatorname{dom} g$.
While $\alpha\|\nabla f(J(x, \alpha))-\nabla f(x)\|>\delta\|J(x, \alpha)-x\|$ do
$\alpha=\theta \alpha$.
End While
Output. $\alpha$.
The well-definedness of Linesearch 1 follows from [34, Theorem 3.4(a)]. For the reader's convenience, we provide a different proof revealing that the convexity of $f$ is not necessary.
Lemma 3.1. If $x \in \operatorname{dom} g$ then Linesearch 1 stops after finitely many steps.
Proof. If $x \in S_{*}$ then $x=J(x, \sigma)$. Thus the linesearch stops with zero step and gives us the output $\sigma$. If $x \notin S_{*}$, by contradiction suppose that for all $\alpha \in \mathcal{P}:=\left\{\sigma, \sigma \theta, \sigma \theta^{2}, \ldots\right\}$,

$$
\begin{equation*}
\alpha\|\nabla f(J(x, \alpha))-\nabla f(x)\|>\delta\|J(x, \alpha)-x\| . \tag{8}
\end{equation*}
$$

When $\alpha \in \mathcal{P}$ is sufficiently closed to 0 , it follows from Lemma 2.4]that $J(x, \alpha)$ is uniformly bounded. Thus we get from (8) that $\|x-J(x, \alpha)\| \rightarrow 0$ as $\alpha \downarrow 0$ thanks to Assumption A2. The latter implies $\|\nabla f(J(x, \alpha))-\nabla f(x)\| \rightarrow 0$ when $\alpha \downarrow 0$ by Assumption A2 again. Thus we get from (8) that

$$
\begin{equation*}
\lim _{\alpha \downarrow 0} \frac{\|x-J(x, \alpha)\|}{\alpha}=0 . \tag{9}
\end{equation*}
$$

Employing (5) with $z=x-\alpha \nabla f(x)$ gives us that

$$
\frac{x-J(x, \alpha)}{\alpha} \in \nabla f(x)+\partial g(J(x, \alpha)) .
$$

By letting $\alpha \downarrow 0$ in the above inclusion and using (9), we get from the demiclosedness of $\operatorname{Gph}(\partial g)$ from Fact 2.2 that $0 \in \nabla f(x)+\partial g(x) \subseteq \partial(f+g)(x)$. This contradicts the assumption that $x$ is not an optimal solution to problem (1) and completes the proof of the lemma.

Next we propose the second backtracking procedure. In contrast to Linesearch 1, this linesearch demands only one evaluation of the backward step and uses it in all possible iterations. This is somehow an advantage of this linesearch, since in many practical problems computing the proximal operator many times may be very expensive. The linesearch is indeed a generalization of the one studied in [7] for solving the nonlinear constrained optimization problem $\left(g=\delta_{C}\right)$. We will employ it in Method 3 in Section 5 .

Linesearch 2. Given $x$ and $\theta \in(0,1)$.
Input. Set $\beta=1, J_{x}:=J(x, 1)=\operatorname{prox}_{g}(x-\nabla f(x))$ with $x \in \operatorname{dom} g$.
While $(f+g)\left(x-\beta\left(x-J_{x}\right)\right)>(f+g)(x)-\beta\left[g(x)-g\left(J_{x}\right)\right]-\beta\left\langle\nabla f(x), x-J_{x}\right\rangle+\frac{\beta}{2}\left\|x-J_{x}\right\|^{2}$
do
$\beta=\theta \beta$.
End While
Output. $\beta$.
Similarly to Linesearch 1 we also have finite termination for Linesearch 2. It is important to note that the well-definedness analysis is done without assuming the second part of A2 (uniform continuity and boundedness).

Lemma 3.2. If $x \in \operatorname{dom} g$ then Linesearch 2 stops after finitely many steps.
Proof. If $x \in S_{*}$ we have $x=J_{x}$. Thus the linesearch immediately gives us the output 1 without proceeding any step. If $x \notin S_{*}$, by contradiction let us assume that Linesearch 2 does not stop after finitely many steps. Thus for all $\beta \in \mathcal{Q}:=\left\{1, \theta, \theta^{2}, \ldots\right\}$, we have

$$
(f+g)\left(x-\beta\left(x-J_{x}\right)\right)>(f+g)(x)-\beta\left[g(x)-g\left(J_{x}\right)\right]-\beta\left\langle\nabla f(x), x-J_{x}\right\rangle+\frac{\beta}{2}\left\|x-J_{x}\right\|^{2} .
$$

It follows that

$$
\frac{(f+g)\left(x-\beta\left(x-J_{x}\right)\right)-(f+g)(x)}{\beta}+g(x)-g\left(J_{x}\right)+\left\langle\nabla f(x), x-J_{x}\right\rangle>\frac{1}{2}\left\|x-J_{x}\right\|^{2} .
$$

Taking $\beta \downarrow 0$ and using the Fréchet differentiability of $f$ and the convexity of $g$ give us that

$$
\begin{aligned}
\frac{1}{2}\left\|x-J_{x}\right\|^{2} & \leq\left\langle\nabla f(x), J_{x}-x\right\rangle+g^{\prime}\left(x ; J_{x}-x\right)+g(x)-g\left(J_{x}\right)+\left\langle\nabla f(x), x-J_{x}\right\rangle \\
& =g^{\prime}\left(x ; J_{x}-x\right)+g(x)-g\left(J_{x}\right) \leq 0
\end{aligned}
$$

where the last inequality follows from Fact 2.1(ii). Hence we have $x=J_{x}$, which readily implies that $x-\nabla f(x) \in \partial g(x)+x$, i.e., $0 \in \nabla f(x)+\partial g(x) \subseteq \partial(f+g)(x)$. This contradicts the assumption $x \notin S_{*}$.

## 4 The forward-backward method with Linesearch 1

This section devotes to the study of the forward-backward splitting method with Linesearch 1. We mainly derive the weak convergence of the generated sequences from this method and also obtain the same complexity of [5] Theorem 1.1] for the cost value sequences generated from the forwardbackward iteration under a weaker assumption than the Lipschitz one on $\nabla f$ usually imposed in the literature.

The following method has some similarities to the one proposed in 34 for maximal monotone operators. However, it completely relaxes an extra expensive projection step [34, Equation (2.3)] and seems to be more natural in comparison with the classical forward-backward splitting method (2).

## Method 1.

Initialization Step. Take $x^{0} \in \operatorname{dom} g, \sigma>0, \theta \in(0,1)$ and $\delta \in(0,1 / 2)$.
Iterative Step. Given $x^{k}$ set

$$
\begin{equation*}
x^{k+1}=J\left(x^{k}, \alpha_{k}\right):=\operatorname{prox}_{\alpha_{k} g}\left(x^{k}-\alpha_{k} \nabla f\left(x^{k}\right)\right) \tag{10}
\end{equation*}
$$

with $\alpha_{k}:=$ Linesearch $1\left(x^{k}, \sigma, \theta, \delta\right)$.
Stop Criteria. If $x^{k+1}=x^{k}$, then stop.
First note that from Lemma 3.1 that Linesearch 1 for finding the stepsize $\alpha_{k}$ in the above scheme is finite. Hence the choice of sequence $\left(x^{k}\right)_{k \in \mathbb{N}}$ in Method 1 is well-defined. Another important feature from the definition of Linesearch 1 useful for our analysis is the following inequality

$$
\begin{equation*}
\alpha_{k}\left\|\nabla f\left(x^{k+1}\right)-\nabla f\left(x^{k}\right)\right\| \leq \delta\left\|x^{k+1}-x^{k}\right\| \tag{11}
\end{equation*}
$$

Note further that if Method 1 stops at iteration $k$ then we have $x^{k}=\operatorname{prox}_{\alpha_{k} g}\left(x^{k}-\alpha_{k} \nabla f\left(x^{k}\right)\right)$ and consequently $x^{k} \in S_{*}$. Otherwise, we will mainly show that the sequence $\left(x^{k}\right)_{k \in \mathbb{N}}$ generated by this method is converging weakly to some optimal solution. Verifying this claim needs some auxiliary results as follows.

Proposition 4.1. Let $\alpha_{k}=$ Linesearch $\mathbb{1}\left(x^{k}, \sigma, \theta, \delta\right)$. For all $k \in \mathbb{N}$ and $x \in \operatorname{dom} g$, we have
(i) $\left\|x^{k}-x\right\|^{2}-\left\|x^{k+1}-x\right\|^{2} \geq 2 \alpha_{k}\left[(f+g)\left(x^{k+1}\right)-(f+g)(x)\right]+(1-2 \delta)\left\|x^{k+1}-x^{k}\right\|^{2}$;
(ii) $(f+g)\left(x^{k+1}\right)-(f+g)\left(x^{k}\right) \leq-\frac{(1-\delta)}{\alpha_{k}}\left\|x^{k+1}-x^{k}\right\|^{2}$.

Proof. First let us justify (i) by noting from (5) and (10) that

$$
\frac{x^{k}-x^{k+1}}{\alpha_{k}}-\nabla f\left(x^{k}\right)=\frac{x^{k}-J\left(x^{k}, \alpha_{k}\right)}{\alpha_{k}}-\nabla f\left(x^{k}\right) \in \partial g\left(J\left(x^{k}, \alpha_{k}\right)\right)=\partial g\left(x^{k+1}\right) .
$$

It follows from the convexity of $g$ that

$$
\begin{equation*}
g(x)-g\left(x^{k+1}\right) \geq\left\langle\frac{x^{k}-x^{k+1}}{\alpha_{k}}-\nabla f\left(x^{k}\right), x-x^{k+1}\right\rangle \quad \text { for all } \quad x \in \operatorname{dom} g . \tag{12}
\end{equation*}
$$

Since $f$ is convex, we also have

$$
\begin{equation*}
f(x)-f(y) \geq\langle\nabla f(y), x-y\rangle \quad \text { for all } \quad x \in \operatorname{dom} f, y \in \operatorname{dom} g . \tag{13}
\end{equation*}
$$

Summing (12) and (13) with any $x \in \operatorname{dom} g \subseteq \operatorname{dom} f$ and $y=x^{k} \in \operatorname{dom} g$ gives us the following expressions

$$
\begin{aligned}
(f+g)(x) \geq & f\left(x^{k}\right)+g\left(x^{k+1}\right)+\left\langle\frac{x^{k}-x^{k+1}}{\alpha_{k}}-\nabla f\left(x^{k}\right), x-x^{k+1}\right\rangle+\left\langle\nabla f\left(x^{k}\right), x-x^{k}\right\rangle \\
= & f\left(x^{k}\right)+g\left(x^{k+1}\right)+\frac{1}{\alpha_{k}}\left\langle x^{k}-x^{k+1}, x-x^{k+1}\right\rangle+\left\langle\nabla f\left(x^{k}\right), x^{k+1}-x^{k}\right\rangle \\
\geq & f\left(x^{k}\right)+g\left(x^{k+1}\right)+\frac{1}{\alpha_{k}}\left\langle x^{k}-x^{k+1}, x-x^{k+1}\right\rangle+\left\langle\nabla f\left(x^{k+1}\right), x^{k+1}-x^{k}\right\rangle \\
& -\left\|\nabla f\left(x^{k}\right)-\nabla f\left(x^{k+1}\right)\right\| \cdot\left\|x^{k+1}-x^{k}\right\| \\
\geq & f\left(x^{k}\right)+g\left(x^{k+1}\right)+\frac{1}{\alpha_{k}}\left\langle x^{k}-x^{k+1}, x-x^{k+1}\right\rangle+\left\langle\nabla f\left(x^{k+1}\right), x^{k+1}-x^{k}\right\rangle \\
& -\frac{\delta}{\alpha_{k}}\left\|x^{k+1}-x^{k}\right\|^{2},
\end{aligned}
$$

where the last inequality follows from (11). After rearrangement we get

$$
\begin{equation*}
\left\langle x^{k}-x^{k+1}, x^{k+1}-x\right\rangle \geq \alpha_{k}\left[f\left(x^{k}\right)+g\left(x^{k+1}\right)-(f+g)(x)+\left\langle\nabla f\left(x^{k+1}\right), x^{k+1}-x^{k}\right\rangle\right]-\delta\left\|x^{k+1}-x^{k}\right\|^{2} . \tag{14}
\end{equation*}
$$

Since $2\left\langle x^{k}-x^{k+1}, x^{k+1}-x\right\rangle=\left\|x^{k}-x\right\|^{2}-\left\|x^{k+1}-x\right\|^{2}-\left\|x^{k}-x^{k+1}\right\|^{2}$, we get from (14) that

$$
\begin{align*}
\left\|x^{k}-x\right\|^{2}-\left\|x^{k+1}-x\right\|^{2} \geq & 2 \alpha_{k}\left[f\left(x^{k}\right)+g\left(x^{k+1}\right)-(f+g)(x)\right] \\
& +2 \alpha_{k}\left\langle\nabla f\left(x^{k+1}\right), x^{k+1}-x^{k}\right\rangle+(1-2 \delta)\left\|x^{k}-x^{k+1}\right\|^{2} . \tag{15}
\end{align*}
$$

By using (13) with $x=x^{k}$ and $y=x^{k+1}$, we have $f\left(x^{k}\right)-f\left(x^{k+1}\right) \geq\left\langle\nabla f\left(x^{k+1}\right), x^{k}-x^{k+1}\right\rangle$. This together with (15) gives us that

$$
\left\|x^{k}-x\right\|^{2}-\left\|x^{k+1}-x\right\|^{2} \geq 2 \alpha_{k}\left[(f+g)\left(x^{k+1}\right)-(f+g)(x)\right]+(1-2 \delta)\left\|x^{k}-x^{k+1}\right\|^{2}
$$

which verifies (i). Note further that (ii) is a consequence of (i) when $x=x^{k}$. The proof is complete.

Proposition 4.1 (ii) shows that Method 1 is a descent method in the sense that the value of the cost function $f+g$ at each iteration is decreasing. Furthermore, it is easy to check from Proposition 4.1(i) that the generated sequence of Method 1 is Fejér convergent to the optimal solution set $S_{*}$ whenever $S_{*} \neq \emptyset$. This observation is indeed the center of the following main result of this section, where we prove the weak convergence of sequence $\left(x^{k}\right)_{k \in \mathbb{N}}$ in Method $\mathbf{1}$ and also $\left((f+g)\left(x^{k}\right)\right)_{k \in \mathbb{N}}$ is a minimizing sequence of $f+g$ without the Lipschitz assumption on $\nabla f$. To the best of our knowledge, this result improves [5, Theorem 1.2] and even the classical results for gradient method with linesearch; see, for instance, [8, Proposition 1.3.3] and [1. Moreover, we show that the sequence $\left((f+g)\left(x^{k}\right)\right)_{k \in \mathbb{N}}$ converges to the infimum value when the solution set is empty.

Theorem 4.2. Let $\left(x^{k}\right)_{k \in \mathbb{N}}$ and $\left(\alpha_{k}\right)_{k \in \mathbb{N}}$ be the sequences generated by Method 1 . The following statements hold:
(i) If $S_{*} \neq \emptyset$ then $\left(x^{k}\right)_{k \in \mathbb{N}}$ is weakly convergent to a point in $S_{*}$. Moreover,

$$
\begin{equation*}
\lim _{k \rightarrow \infty}(f+g)\left(x^{k}\right)=\min _{x \in \mathcal{H}}(f+g)(x) . \tag{16}
\end{equation*}
$$

(ii) If $S_{*}=\emptyset$ then we have

$$
\lim _{k \rightarrow \infty}\left\|x^{k}\right\|=+\infty \quad \text { and } \quad \lim _{k \rightarrow \infty}(f+g)\left(x^{k}\right)=\inf _{x \in \mathcal{H}}(f+g)(x) .
$$

Proof. Let us justify (i) by supposing that $S_{*} \neq \emptyset$. By applying Proposition 4.1(i) at any $x_{*} \in S_{*}$, we have

$$
\begin{align*}
\left\|x^{k}-x_{*}\right\|^{2}-\left\|x^{k+1}-x_{*}\right\|^{2} & \geq 2 \alpha_{k}\left[(f+g)\left(x^{k+1}\right)-(f+g)\left(x_{*}\right)\right]+(1-2 \delta)\left\|x^{k}-x^{k+1}\right\|^{2}  \tag{17}\\
& \geq(1-2 \delta)\left\|x^{k}-x^{k+1}\right\|^{2} \geq 0
\end{align*}
$$

It follows that the sequence $\left(x^{k}\right)_{k \in \mathbb{N}}$ is Fejér convergent to $S_{*}$ and thus is bounded by Fact 2.5(i). By using (17), we get

$$
\begin{aligned}
0 & \leq 2 \alpha_{k}\left[(f+g)\left(x^{k+1}\right)-(f+g)\left(x_{*}\right)\right] \leq\left\|x^{k}-x_{*}\right\|^{2}-\left\|x^{k+1}-x_{*}\right\|^{2} \\
& =\left(\left\|x^{k}-x_{*}\right\|+\left\|x^{k+1}-x_{*}\right\|\right) \cdot\left(\left\|x^{k}-x_{*}\right\|-\left\|x^{k+1}-x_{*}\right\|\right) \\
& \leq 2 M\left(\left\|x^{k}-x_{*}\right\|-\left\|x^{k+1}-x_{*}\right\|\right) \\
& \leq 2 M\left\|x^{k}-x^{k+1}\right\|,
\end{aligned}
$$

where $M:=\sup \left\{\left\|x^{k}-x_{*}\right\| \mid k \in \mathbb{N}\right\}<+\infty$. Hence the above inequalities lead us to

$$
\begin{equation*}
(f+g)\left(x^{k+1}\right)-(f+g)\left(x_{*}\right) \leq M \frac{\left\|x^{k}-x^{k+1}\right\|}{\alpha_{k}} . \tag{18}
\end{equation*}
$$

Due to the Fejér property of $\left(x^{k}\right)_{k \in \mathbb{N}}$ to $S_{*}$, the sequence $\left(\left\|x^{k}-x^{*}\right\|\right)_{k \in \mathbb{N}}$ is convergent. This together with (17) tells us that $\left\|x^{k}-x^{k+1}\right\| \rightarrow 0$ as $k \rightarrow \infty$.

Since $\left(x^{k}\right)_{k \in \mathbb{N}}$ is bounded, the set of its weak accumulation points is nonempty. Take any weak accumulation point $\bar{x}$ of $\left(x^{k}\right)_{k \in \mathbb{N}}$, we find a subsequence $\left(x^{n_{k}}\right)_{k \in \mathbb{N}}$ weakly converging to $\bar{x}$. Now let us split our further analysis into two distinct cases.

Case 1. Suppose that the sequence $\left(\alpha_{n_{k}}\right)_{k \in \mathbb{N}}$ defined in Method 1 does not converge to 0 . Hence there exist a subsequence (without relabelling) of $\left(\alpha_{n_{k}}\right)_{k \in \mathbb{N}}$ and $\alpha>0$ such that

$$
\begin{equation*}
\alpha_{n_{k}} \geq \alpha \tag{19}
\end{equation*}
$$

Since $\left(x^{k}\right)_{k \in \mathbb{N}}$ is bounded and $\left\|x^{k}-x^{k+1}\right\| \rightarrow 0$ as claimed above, we get from Assumption A2 that

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left\|\nabla f\left(x^{n_{k}}\right)-\nabla f\left(x^{n_{k}+1}\right)\right\|=0 \tag{20}
\end{equation*}
$$

Since $x^{n_{k}+1}=J\left(x^{n_{k}}, \alpha_{n_{k}}\right)$, it follows from (5) and (10) that

$$
\frac{x^{n_{k}}-\alpha_{n_{k}} \nabla f\left(x^{n_{k}}\right)-x^{n_{k}+1}}{\alpha_{n_{k}}} \in \partial g\left(x^{n_{k}+1}\right),
$$

which implies in turn the expression

$$
\begin{equation*}
\frac{x^{n_{k}}-x^{n_{k}+1}}{\alpha_{n_{k}}}+\nabla f\left(x^{n_{k}+1}\right)-\nabla f\left(x^{n_{k}}\right) \in \nabla f\left(x^{n_{k}+1}\right)+\partial g\left(x^{n_{k}+1}\right) \subseteq \partial(f+g)\left(x^{n_{k}+1}\right) . \tag{21}
\end{equation*}
$$

Note also that the subsequence $\left(x^{n_{k}+1}\right)_{k \in \mathbb{N}}$ converges weakly to $\bar{x}$ due to the fact that $\| x^{n_{k}}-$ $x^{n_{k}+1} \| \rightarrow 0$ as $k \rightarrow \infty$. By passing $k \rightarrow \infty$ in (21), we get from (19), (20), and Fact 2.2 that
$0 \in \partial(f+g)(\bar{x})$, which means $\bar{x} \in S_{*}$. Furthermore, since the sequence $\left((f+g)\left(x^{k}\right)\right)_{k \in \mathbb{N}}$ is decreasing due to Proposition 4.1(ii), (16) is a consequence of (18) and (19).

Case 2. Suppose now $\lim _{k \rightarrow \infty} \alpha_{n_{k}}=0$. Define $\hat{\alpha}_{n_{k}}:=\frac{\alpha_{n_{k}}}{\theta}>\alpha_{n_{k}}>0$ and $\hat{x}^{n_{k}}:=J\left(x^{n_{k}}, \hat{\alpha}_{n_{k}}\right)$. Due to Lemma 2.4 we have

$$
\left\|x^{n_{k}}-\hat{x}^{n_{k}}\right\|=\left\|x^{n_{k}}-J\left(x^{n_{k}}, \hat{\alpha}_{n_{k}}\right)\right\| \leq \frac{\hat{\alpha}_{n_{k}}}{\alpha_{n_{k}}}\left\|x^{n_{k}}-J\left(x^{n_{k}}, \alpha_{n_{k}}\right)\right\|=\frac{1}{\theta}\left\|x^{n_{k}}-x^{n_{k}+1}\right\|,
$$

which combines with the boundedness of $\left(x^{n_{k}}\right)_{k \in \mathbb{N}}$ to show that the sequence $\left(\hat{x}^{n_{k}}\right)_{k \in \mathbb{N}}$ is also bounded. It follows from the definition of Linesearch 1 that

$$
\begin{equation*}
\hat{\alpha}_{n_{k}}\left\|\nabla f\left(\hat{x}^{n_{k}}\right)-\nabla f\left(x^{n_{k}}\right)\right\|>\delta\left\|\hat{x}^{n_{k}}-x^{n_{k}}\right\| . \tag{22}
\end{equation*}
$$

Since $\hat{\alpha}_{n_{k}} \downarrow 0$ and both $\left(x^{n_{k}}\right)_{k \in \mathbb{N}}$ and $\left(\hat{x}^{n_{k}}\right)_{k \in \mathbb{N}}$ are bounded, (22) together with Assumption A2 tells us that $\lim _{k \rightarrow \infty}\left\|\hat{x}^{n_{k}}-x^{n_{k}}\right\|=0$ and thus $\left(\hat{x}^{n_{k}}\right)_{k \in \mathbb{N}}$ also weakly converges to $\bar{x}$. Thanks to Assumption A2 again, we have

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left\|\nabla f\left(\hat{x}^{n_{k}}\right)-\nabla f\left(x^{n_{k}}\right)\right\|=0 \tag{23}
\end{equation*}
$$

This and (22) imply that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \frac{1}{\hat{\alpha}_{n_{k}}}\left\|\hat{x}^{n_{k}}-x^{n_{k}}\right\|=0 \tag{24}
\end{equation*}
$$

Using (5) with $z=x^{n_{k}}-\hat{\alpha}_{n_{k}} \nabla f\left(x^{n_{k}}\right)$ gives us that

$$
\frac{x^{n_{k}}-\hat{\alpha}_{n_{k}} \nabla f\left(x^{n_{k}}\right)-\hat{x}^{n_{k}}}{\hat{\alpha}_{n_{k}}}+\nabla f\left(\hat{x}^{n_{k}}\right) \in \partial g\left(\hat{x}^{n_{k}}\right)+\nabla f\left(\hat{x}^{n_{k}}\right) \subseteq \partial(f+g)\left(\hat{x}^{n_{k}}\right) .
$$

By letting $k \rightarrow \infty$, we get from the latter, (23), (24), and Fact 2.2 that $0 \in \partial(f+g)(\bar{x})$, which means $\bar{x} \in S_{*}$. It remains to verify (16) in this case. Indeed, we get from Lemma 2.4 that

$$
\left\|x^{n_{k}}-\hat{x}^{n_{k}}\right\|=\left\|x^{n_{k}}-J\left(x^{n_{k}}, \frac{\alpha_{n_{k}}}{\theta}\right)\right\| \geq\left\|x^{n_{k}}-J\left(x^{n_{k}}, \alpha_{n_{k}}\right)\right\|=\left\|x^{n_{k}}-x^{n_{k}+1}\right\| .
$$

This together with (24) yields $\frac{\| x^{n_{k}-x^{n_{k}+1} \|}}{\alpha_{n_{k}}} \rightarrow 0$ as $k \rightarrow \infty$. Since $(f+g)\left(x^{k}\right)$ is decreasing due to Proposition 4.1(ii), we derive from the latter and (18) that
$0=\lim _{k \rightarrow \infty} M \frac{\left\|x^{n_{k}}-x^{n_{k}+1}\right\|}{\alpha_{n_{k}}} \geq \lim _{k \rightarrow \infty}(f+g)\left(x^{n_{k}}\right)-(f+g)\left(x_{*}\right)=\lim _{k \rightarrow \infty}(f+g)\left(x^{k}\right)-(f+g)\left(x^{*}\right) \geq 0$, which clearly ensures (16).

From both cases above, we have (16) and the fact that any weak accumulation point of $\left(x^{k}\right)_{k \in \mathbb{N}}$ is an element of $S_{*}$. Thanks to Fact [2.5(ii), the sequence $\left(x^{k}\right)_{k \in \mathbb{N}}$ weakly converges to some point in $S_{*}$. This verifies (i) of the theorem.

To justify (ii), suppose that $S_{*}=\emptyset$. Observe from the proof of (i) (without regarding (16), (17), and (18)) that if $\left(x^{k}\right)_{k \in \mathbb{N}}$ has any weak accumulation point then this point is an optimal solution as illustrated in both cases there. Since $S_{*}=\emptyset$, any subsequence of $\left(x^{k}\right)_{k \in \mathbb{N}}$ is unbounded and thus $\left\|x^{k}\right\| \rightarrow+\infty$ as $k \rightarrow \infty$. Furthermore, note that $s:=\lim _{k \rightarrow \infty}(f+g)\left(x^{k}\right) \geq \inf _{x \in \mathcal{H}}(f+g)(x)$, where $s$ exists due to fact that $\left((f+g)\left(x^{k}\right)\right)_{k \in \mathbb{N}}$ is decreasing by Proposition 4.1(ii). If $s>\inf _{x \in \mathcal{H}}(f+g)(x)$ then the following auxiliary set

$$
S_{\mathrm{lev}}\left(x^{0}\right):=\left\{x \in \operatorname{dom} g: \quad(f+g)(x) \leq(f+g)\left(x^{k}\right), \forall k \in \mathbb{N}\right\}
$$

is nonempty. By applying Proposition 4.1(i) at any $x \in S_{\text {lev }}\left(x^{0}\right)$, similarly to (17) we also have $\left(x^{k}\right)_{k \in \mathbb{N}}$ is Féjer convergent to $S_{\mathrm{lev}}\left(x^{0}\right)$. It follows from Fact [2.5(i) that the sequence $\left(x^{k}\right)_{k \in \mathbb{N}}$ is bounded, which is a contradiction. Hence we have $s=\inf _{x \in \mathcal{H}}(f+g)(x)$ and complete the proof of the theorem.

As discussed before Method 1, our method improves the scheme in 34 for the particular case that the two maximal monotone operators considered there are $\nabla f$ and $\partial g$ by relaxing completely an additional step. Our Theorem 4.2 also loosens some unnatural assumptions imposed in 34, Theorem 3.4(b)]. Furthermore, we obtain new information on the convergence of the cost values at generated sequences in this result.

### 4.1 Complexity analysis of Method 1

In this subsection we present complexity analysis of the iterates in Method (1) When the stepsizes generated by Linesearch 1 are bounded below by a positive number, our analysis shows that the expected error from the cost value at the $k$-th iteration to the optimal value is $\mathcal{O}\left(k^{-1}\right)$ in Hilbert spaces and $o\left(k^{-1}\right)$ in finite dimensions, which improves the complexity of the first-order algorithm presented in [5, Theorem 1.1]. It is worth emphasizing that the global Lipschitz continuity assumption on the gradient $\nabla f$ used in [5, Theorem 1.1] is sufficient but not necessary for the boundedness from below of the stepsizes aforementioned; see our Proposition 4.4 below. Since $\alpha_{k}>0$ for any $k \in \mathbb{N}$, this boundedness assumption actually means that $\lim \inf _{k \rightarrow \infty} \alpha_{k}>0$, which was used before in [34 for different purposes.

Theorem 4.3. Let $\left(x^{k}\right)_{k \in \mathbb{N}}$ and $\left(\alpha_{k}\right)_{k \in \mathbb{N}}$ be the sequences generated in Method 1 Suppose that $S_{*} \neq \emptyset$ and there exists $\alpha>0$ such that $\alpha_{k} \geq \alpha>0$ for all $k \in \mathbb{N}$. Then we have

$$
\begin{equation*}
(f+g)\left(x^{k}\right)-\min _{x \in \mathcal{H}}(f+g)(x) \leq \frac{1}{2 \alpha} \frac{\left[\operatorname{dist}\left(x^{0}, S_{*}\right)\right]^{2}}{k} \quad \text { for all } \quad k \in \mathbb{N} \text {. } \tag{25}
\end{equation*}
$$

If in addition $\operatorname{dim} \mathcal{H}<+\infty$ then

$$
\begin{equation*}
\lim _{k \rightarrow \infty} k\left[(f+g)\left(x^{k}\right)-\min _{x \in \mathcal{H}}(f+g)(x)\right]=0 . \tag{26}
\end{equation*}
$$

Proof. Pick any $x_{*} \in S_{*}$, Proposition 4.1(i) tells us that

$$
\begin{align*}
0 \geq(f+g)\left(x_{*}\right)-(f+g)\left(x^{\ell+1}\right) & \geq \frac{1}{2 \alpha_{\ell}}\left(\left\|x^{\ell+1}-x_{*}\right\|^{2}-\left\|x^{\ell}-x_{*}\right\|^{2}+(1-2 \delta)\left\|x^{\ell}-x^{\ell+1}\right\|^{2}\right) \\
& \geq \frac{1}{2 \alpha_{\ell}}\left(\left\|x^{\ell+1}-x_{*}\right\|^{2}-\left\|x^{\ell}-x_{*}\right\|^{2}\right) \tag{27}
\end{align*}
$$

for any $\ell \in \mathbb{N}$. Since $\alpha_{\ell} \geq \alpha$, we get from (27) that

$$
\begin{equation*}
0 \geq(f+g)\left(x_{*}\right)-(f+g)\left(x^{\ell+1}\right) \geq \frac{1}{2 \alpha}\left(\left\|x^{\ell+1}-x_{*}\right\|^{2}-\left\|x^{\ell}-x_{*}\right\|^{2}\right) \tag{28}
\end{equation*}
$$

Summing the above inequality over $\ell=0,1, \ldots, k-1$ implies that

$$
k(f+g)\left(x_{*}\right)-\sum_{\ell=0}^{k-1}(f+g)\left(x^{\ell+1}\right) \geq \frac{1}{2 \alpha}\left(\left\|x^{k}-x_{*}\right\|^{2}-\left\|x^{0}-x_{*}\right\|^{2}\right) .
$$

Since $(f+g)\left(x^{\ell}\right)$ is decreasing by Proposition 4.1(ii), the latter yields

$$
\begin{equation*}
k\left[(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right)\right] \leq \frac{1}{2 \alpha}\left(\left\|x_{*}-x^{0}\right\|^{2}-\left\|x^{k}-x_{*}\right\|^{2}\right) \leq \frac{1}{2 \alpha}\left\|x_{*}-x^{0}\right\|^{2} . \tag{29}
\end{equation*}
$$

Note that no matter how we choose $x_{*} \in S$, the optimal value $(f+g)\left(x_{*}\right)=\min _{x \in \mathcal{H}}(f+g)(x)$ is fixed. Hence we get from (29) that

$$
(f+g)\left(x^{k}\right)-\min _{x \in \mathcal{H}}(f+g)(x) \leq \frac{1}{2 \alpha} \inf _{y \in S_{*}} \frac{\left\|y-x^{0}\right\|^{2}}{k}=\frac{1}{2 \alpha} \frac{\left[\operatorname{dist}\left(x^{0}, S_{*}\right)\right]^{2}}{k},
$$

which verifies (25) and completes the first part of the theorem.
Now suppose additionally that $\operatorname{dim} \mathcal{H}<+\infty$, it follows from Theorem 4.2 that the sequence $\left(x^{k}\right)_{k \in \mathbb{N}}$ converges strongly to some $x_{*} \in S_{*}$, i.e., $\left\|x^{k}-x_{*}\right\| \rightarrow 0$ as $k \rightarrow \infty$. Take any $\varepsilon>0$, we find $K \in \mathbb{N}$ such that $\left\|x^{k}-x_{*}\right\| \leq \varepsilon$ for $k \geq K$. For any $\ell \geq K$ we get from the fact $\left\|x^{\ell}-x_{*}\right\| \geq\left\|x^{\ell+1}-x_{*}\right\|$ and (27) that

$$
\begin{align*}
0 \geq(f+g)\left(x_{*}\right)-(f+g)\left(x^{\ell+1}\right) & \geq \frac{1}{2 \alpha_{\ell}}\left(\left\|x^{\ell+1}-x_{*}\right\|+\left\|x^{\ell}-x_{*}\right\|\right) \cdot\left(\left\|x^{\ell+1}-x_{*}\right\|-\left\|x^{\ell}-x_{*}\right\|\right) \\
& \geq \frac{1}{\alpha_{\ell}}\left\|x^{\ell}-x_{*}\right\|\left(\left\|x^{\ell+1}-x_{*}\right\|-\left\|x^{\ell}-x_{*}\right\|\right)  \tag{30}\\
& \geq \frac{\varepsilon}{\alpha}\left(\left\|x^{\ell+1}-x_{*}\right\|-\left\|x^{\ell}-x_{*}\right\|\right) .
\end{align*}
$$

Now adding the above inequality over $\ell=K, K+1, \ldots, K+k-1$ gives us that

$$
k(f+g)\left(x_{*}\right)-\sum_{\ell=K}^{K+k-1}(f+g)\left(x^{\ell+1}\right) \geq \frac{\varepsilon}{\alpha}\left(\left\|x^{K+k}-x_{*}\right\|-\left\|x^{K}-x_{*}\right\|\right) \geq-\frac{\varepsilon}{\alpha}\left\|x^{K}-x_{*}\right\| \geq-\frac{\varepsilon^{2}}{\alpha} .
$$

Due to the decreasing property of $(f+g)\left(x^{\ell}\right)$ in Proposition 4.1(ii), we get from the latter that

$$
k\left[(f+g)\left(x_{*}\right)-(f+g)\left(x^{K+k}\right)\right] \geq-\frac{\varepsilon^{2}}{\alpha} .
$$

It follows that

$$
\begin{aligned}
\limsup _{k \rightarrow \infty} k\left[(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right)\right] & =\limsup _{k \rightarrow \infty}(K+k)\left[(f+g)\left(x^{K+k}\right)-(f+g)\left(x_{*}\right)\right] \\
& \leq \limsup _{k \rightarrow \infty} \frac{K+k}{k} \cdot \frac{\varepsilon^{2}}{\alpha}=\frac{\varepsilon^{2}}{\alpha} .
\end{aligned}
$$

Since this inequality holds for any $\varepsilon>0$, we have

$$
0 \leq \liminf _{k \rightarrow \infty} k\left[(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right)\right] \leq \limsup _{k \rightarrow \infty} k\left[(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right)\right] \leq 0,
$$

thanks to the fact that $x_{*} \in S_{*}$. Hence we obtain $\lim _{k \rightarrow \infty} k\left[(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right)\right]=0$, which verifies (26) and completes the proof of theorem.

It is worth mentioning that the rate $o\left(k^{-1}\right)$ was obtained [21-23] earlier when using the proximal point method to solve problem (1) with $f \equiv 0$. 1 Our result above could be considered an extension of some results in these papers, in particular, [22, Corollary 3.1] to the more general framework of (11) with linesearch. When the the stepsizes are not bounded below by a positive constant, we discuss the possible validity of the same complexity as follows.

[^1]Remark 4.1. The main question arising from the above theorem is that: Can we have the complexity $o\left(k^{-1}\right)$ of the difference $(f+g)\left(x^{k}\right)-\min _{x \in \mathcal{H}}(f+g)(x)$ when $\lim \inf _{k \rightarrow \infty} \alpha_{k}=0$ ? Suppose that $\left(x^{k}\right)_{k \in \mathbb{N}}$ (strongly) converges to some $x^{*} \in S_{*}$ in finite dimension; see our Theorem 4.2. By analyzing carefully the proof of (25) in Theorem 4.3, we observe that complexity $o\left(k^{-1}\right)$ remains when the following condition holds: there exists $\lambda \in[-1,1)$ such that

$$
\begin{equation*}
\limsup _{k \rightarrow \infty} \frac{\left\|x^{k}-x_{*}\right\|^{1+\lambda}}{\alpha_{k}}<+\infty, \tag{31}
\end{equation*}
$$

which may allow $\alpha_{k}$ to approach 0 . Indeed, suppose that (31) is satisfied with some $\lambda \in[-1,1$ ), we find $C>0$ and $K \in \mathbb{N}$ such that $\left\|x^{k}-x_{*}\right\|^{1+\lambda} \leq C \alpha_{k}$ for all $k>K$. For any $\varepsilon \in(0,1)$, there exists $K_{1}>K$ such that $\left\|x^{k}-x_{*}\right\|<\varepsilon$ for all $k \geq K_{1}$. Moreover, it is easy to prove the existence of some constant $D \geq 1$ so that

$$
\begin{equation*}
\left(\rho^{2}-1\right) \leq 2 D \rho^{1+\lambda}\left(\rho^{1-\lambda}-1\right) \text { for all } \rho \geq 1 . \tag{32}
\end{equation*}
$$

Note again that $\frac{\left\|x^{k}-x_{*}\right\|}{\left\|x^{k+1}-x_{*}\right\|} \geq 1$ due to the Fejér property of $\left(x^{k}\right)_{k \in \mathbb{N}}$ in Theorem 4.2(i). This together with (32) tells us that

$$
\left\|x^{k}-x^{*}\right\|^{2}-\left\|x^{k+1}-x^{*}\right\|^{2} \leq 2 D\left\|x^{k}-x^{*}\right\|^{1+\lambda}\left(\left\|x^{k}-x^{*}\right\|^{1-\lambda}-\left\|x^{k+1}-x^{*}\right\|^{1-\lambda}\right)
$$

Hence for any $\ell>K_{1}$ we get from (27) that

$$
\begin{aligned}
0 \geq(f+g)\left(x_{*}\right)-(f+g)\left(x^{\ell+1}\right) & \geq D \frac{\left\|x^{\ell}-x_{*}\right\|^{1+\lambda}}{\alpha_{\ell}}\left(\left\|x^{\ell+1}-x_{*}\right\|^{1-\lambda}-\left\|x^{\ell}-x_{*}\right\|^{1-\lambda}\right) \\
& \geq C D\left(\left\|x^{+1}-x_{*}\right\|^{1-\lambda}-\left\|x^{\ell}-x_{*}\right\|^{1-\lambda}\right) .
\end{aligned}
$$

By adding the above inequality over $\ell=K_{1}, K_{1}+1, \ldots, K_{1}+k-1$, we have

$$
k(f+g)\left(x_{*}\right)-\sum_{\ell=K_{1}}^{K_{1}+k-1}(f+g)\left(x^{\ell+1}\right) \geq C D\left(\left\|x^{k+K_{1}}-x_{*}\right\|^{1-\lambda}-\left\|x^{K_{1}}-x_{*}\right\|^{1-\lambda}\right)
$$

Due to the decreasing property of $(f+g)\left(x^{\ell}\right)$ in Proposition 4.1(ii), the latter implies that

$$
k\left[(f+g)\left(x_{*}\right)-(f+g)\left(x^{K_{1}+k}\right)\right] \geq-C D\left\|x^{K_{1}}-x_{*}\right\|^{1-\lambda} \geq-C D \varepsilon^{1-\lambda} .
$$

Thus we derive the following expressions

$$
\begin{aligned}
\limsup _{k \rightarrow \infty} k\left[(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right)\right] & =\limsup _{k \rightarrow \infty}\left(K_{1}+k\right)\left[(f+g)\left(x^{K_{1}+k}\right)-(f+g)\left(x_{*}\right)\right] \\
& \leq \limsup _{k \rightarrow \infty} \frac{K_{1}+k}{k} C D \varepsilon^{1-\lambda}=C D \varepsilon^{1-\lambda} .
\end{aligned}
$$

Since this inequality holds for any $\varepsilon>0$, we have $\lim _{\sup }^{k \rightarrow \infty}$ $k\left[(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right)\right] \leq 0$, which also verifies (26) due to the fact that $x_{*} \in S_{*}$.

It is clear that (31) holds when $\alpha_{k}$ is bounded below by a positive number. The following simple example shows the possible validity of (31) even when $\alpha_{k} \rightarrow 0$ as $k \rightarrow \infty$. Thus the complexity $o\left(k^{-1}\right)$ of the function values remains true in the example below. However, in general, checking (31) may be not trivial, since $x_{*}$ is unknown.

Example 4.1. Let

$$
f(x):=\frac{1}{1+p}|x|^{1+p} \quad \text { with } 0<p<1 \quad \text { and } \quad g(x)=\delta_{[0, \infty)}(x) .
$$

Then a unique solution for problem (11) is $x_{*}=0$. Note further that for any $x>0$, we have

$$
\begin{equation*}
J(x, \alpha)=P_{[0,+\infty)}\left(x-\alpha x^{p}\right)=\max \left\{x-\alpha x^{p}, 0\right\} \tag{33}
\end{equation*}
$$

To distinguish the iteration from the exponent in this example, we write $\left(x_{k}\right)_{k \in \mathbb{N}}$ instead of $\left(x^{k}\right)_{k \in \mathbb{N}}$. To avoid the trivial case, suppose that $x_{k}>0$ for all $k \in \mathbb{N}$, then we have

$$
0<x_{k+1}=x_{k}-\alpha_{k}\left(x_{k}\right)^{p}<x_{k} .
$$

It follows from Linesearch 1 that

$$
\begin{equation*}
\alpha_{k}\left|x_{k+1}^{p}-x_{k}^{p}\right| \leq \delta\left|x_{k+1}-x_{k}\right| . \tag{34}
\end{equation*}
$$

By mean value theorem, there exists $\eta \in[0,1]$ such that

$$
\left|x_{k+1}^{p}-x_{k}^{p}\right|=\left|x_{k+1}-x_{k}\right| \cdot p\left|\eta x_{k+1}+(1-\eta) x_{k}\right|^{p-1} \geq\left|x_{k+1}-x_{k}\right| \cdot p\left|x_{k}\right|^{p-1}
$$

This together with (34) gives us $\alpha_{k} \leq \delta p^{-1}\left|x_{k}\right|^{1-p} \rightarrow 0$ as $k \rightarrow \infty$, since $x_{k} \rightarrow 0$. Therefore, we may suppose without loss of generality that $\alpha_{k}<\sigma$ for all $k$. Define $\hat{\alpha}_{k}:=\frac{\alpha_{k}}{\theta}$ and $\hat{x}_{k+1}=J\left(x_{k}, \hat{\alpha}_{k}\right)$, it follows from the Linesearch 1 that

$$
\begin{equation*}
\hat{\alpha}_{k}\left|\hat{x}_{k+1}^{p}-x_{k}^{p}\right|>\delta\left|\hat{x}_{k+1}-x_{k}\right| . \tag{35}
\end{equation*}
$$

Note that $0 \leq \hat{x}_{k+1}<x_{k}$ by (33) and that

$$
0 \leq x_{k}^{p}-\hat{x}_{k+1}^{p} \leq x_{k}^{p}-x_{k}^{p-1} \hat{x}_{k+1}=x_{k}^{p-1}\left(x_{k}-\hat{x}_{k+1}\right)
$$

Combining this with (35) gives us that $\frac{\alpha_{k}}{\theta}\left|x_{k}\right|^{(p-1)}\left|x_{k}-\hat{x}_{k+1}\right| \geq \delta\left|\hat{x}_{k+1}-x_{k}\right|>0$, which implies that $\frac{\left|x_{k}-x_{*}\right|^{1-p}}{\alpha_{k}} \leq \frac{1}{\theta}$. This is exactly (31) with $\lambda=-p \in[-1,1)$.

Another natural question from Theorem 4.3 is that in which class of functions the stepsizes $\alpha_{k}$ are bounded below by a positive number. Next we show that this condition is satisfied under some mild Lipschitz continuity assumption of $\nabla f$. The first part of this result is not much surprising due to the similar achievement in [34, Theorem 3.4(a)]. However, the second part is a significant improvement when we replace the global Lipschitz continuity by the local one in finite dimensions.

Proposition 4.4. Let $\left(\alpha_{k}\right)_{k \in \mathbb{N}}$ be the sequence generated by Linesearch 1 on Method 1. The following statements hold:
(i) If the gradient of $f$ is globally Lipschitz continuous on $\operatorname{dom} g$ with constant $L>0$, then $\alpha_{k} \geq$ $\min \left\{\sigma, \frac{\delta \theta}{L}\right\}$ for all $k \in \mathbb{N}$.
(ii) Suppose that $\operatorname{dim} \mathcal{H}<+\infty$ and $S_{*} \neq \emptyset$. If $\nabla f$ is locally Lipschitz continuous at any $x \in S_{*}$ then there exists $x_{*} \in S_{*}$ such that

$$
\liminf _{k \rightarrow \infty} \alpha_{k} \geq \min \left\{\sigma, \frac{\delta \theta}{\mathcal{L}}\right\}
$$

where $\mathcal{L}>0$ is a Lipschitz constant of $\nabla f$ around $x_{*}$. Consequently, there exists $\alpha>0$ such that $\alpha_{k} \geq \alpha$ for all $k \in \mathbb{N}$.

Proof. To justify (i), suppose that $\nabla f$ is globally Lipschitz continuous with constant $L>0$. If $\alpha_{k}<\sigma$, define $\hat{\alpha}_{k}:=\frac{\alpha_{k}}{\theta}>0$ and $\hat{x}^{k}:=J\left(x^{k}, \hat{\alpha}_{k}\right)$. It follows from the definition of Linesearch 1 that

$$
\begin{equation*}
\hat{\alpha}_{k}\left\|\nabla f\left(\hat{x}^{k}\right)-\nabla f\left(x^{k}\right)\right\|>\delta\left\|\hat{x}^{k}-x^{k}\right\|, \tag{36}
\end{equation*}
$$

which yields $\left\|\hat{x}^{k}-x^{k}\right\| \neq 0$ for all $k \in \mathbb{N}$. Moreover, due to Lipschitz assumption on $\nabla f$, we get $\left\|\nabla f\left(x^{k}\right)-\nabla f\left(\hat{x}^{k}\right)\right\| \leq L\left\|x^{k}-\hat{x}^{k}\right\|$ for all $k \in \mathbb{N}$. Combining the latter inequality with (36) gives us that $\hat{\alpha}_{k} L>\delta$, i.e., $\alpha_{k} \geq \frac{\delta \theta}{L}$ when $\alpha_{k}<\sigma$. This clearly verifies (i).

To justify the second part, we suppose that $\operatorname{dim} \mathcal{H}<+\infty$, that $S_{*} \neq \emptyset$, and that $f$ is locally Lipschitz continuous at any point in $S_{*}$. By Theorem 4.2, $\left(x^{k}\right)_{k \in \mathbb{N}}$ converges (strongly) to some $x_{*} \in S_{*}$. Due to the local Lipschitz continuity of $\nabla f$ at $x_{*}$, there exist $\varepsilon, \mathcal{L}>0$ such that

$$
\begin{equation*}
\|\nabla f(x)-\nabla f(y)\| \leq \mathcal{L}\|x-y\| \quad \text { for all } \quad x, y \in \mathbb{B}_{\varepsilon}\left(x_{*}\right) \tag{37}
\end{equation*}
$$

where $\mathbb{B}_{\varepsilon}\left(x_{*}\right)$ is the closed ball in $\mathcal{H}$ with center $x_{*}$ and radius $\varepsilon$. Since $\left(x^{k}\right)_{k \in \mathbb{N}}$ is converging (strongly) to $x_{*}$, we find some $K \in \mathbb{N}$ satisfying that

$$
\begin{equation*}
\left\|x^{k}-x_{*}\right\| \leq \frac{\theta \varepsilon}{2+\theta}<\varepsilon \quad \text { for all } \quad k>K \tag{38}
\end{equation*}
$$

with $\theta \in(0,1)$ defined in Linesearch 1. Take any $k>K$, if $\alpha_{k}<\sigma$, similarly to the first part we define $\hat{\alpha}_{k}:=\frac{\alpha_{k}}{\theta}>0$ and $\hat{x}^{k}:=J\left(x^{k}, \hat{\alpha}_{k}\right)$. Thus we also have (36). It follows from Lemma 2.4 that

$$
\left\|x^{k}-\hat{x}^{k}\right\|=\left\|x^{k}-J\left(x^{k}, \hat{\alpha}_{k}\right)\right\| \leq \frac{\hat{\alpha}_{k}}{\alpha_{k}}\left\|x^{k}-J\left(x^{k}, \alpha_{k}\right)\right\|=\frac{1}{\theta}\left\|x^{k}-x^{k+1}\right\|,
$$

which together with (38) implies the following expression

$$
\left\|\hat{x}^{k}-x_{*}\right\| \leq\left\|\hat{x}^{k}-x^{k}\right\|+\left\|x^{k}-x_{*}\right\| \leq \frac{1}{\theta}\left\|x^{k}-x^{k+1}\right\|+\left\|x^{k}-x_{*}\right\| \leq \frac{1}{\theta} \cdot \frac{2 \theta \varepsilon}{2+\theta}+\frac{\theta \varepsilon}{2+\theta}=\varepsilon
$$

Hence we have $\hat{x}^{k} \in \mathbb{B}_{\varepsilon}\left(x_{*}\right)$ and derive from (37) and (38) that $\left\|\nabla f\left(x^{k}\right)-\nabla f\left(\hat{x}^{k}\right)\right\| \leq \mathcal{L}\left\|x^{k}-\hat{x}^{k}\right\|$. Combining this with (36) gives us that $\mathcal{L} \hat{\alpha}_{k} \geq \delta$, i.e., $\alpha_{k} \geq \frac{\delta \theta}{\mathcal{L}}$. It follows that $\alpha_{k} \geq \min \left\{\sigma, \frac{\delta \theta}{\mathcal{L}}\right\}$ for all $k>K$.
Finally, since $\alpha_{k}>0$ for $k \in \mathbb{N}$, we obtain that $\alpha_{k} \geq \alpha:=\min \left\{\alpha_{1}, \ldots, \alpha_{K}, \frac{\delta \theta}{\mathcal{L}}, \sigma\right\}>0$ and ensure the last part of the proposition. The proof is complete.

It is worth recalling that the assumption of Proposition $4.4(\mathrm{i})$ that $\nabla f$ is globally Lipschitz continuous on dom $g$ is also sufficient for Assumption A2. Assumptions of Proposition 4.4(ii) are certainly not enough to guarantee Assumption A2. However, there are many broad classes of functions satisfying all of them. For instance, when $\operatorname{dim} \mathcal{H}<+\infty$ and $\operatorname{dom} g$ is closed, a function $f$, which is differentiable with locally Lipschitz continuous gradient on dom $g$ satisfies all the requirements; see also Proposition 2.3.

Theorem 4.3 together with Proposition 4.4 and Theorems 4.6 leads us to the following result. Unlike [5. Theorem 1.1], we obtain better complexity $o\left(k^{-1}\right)$ with linesearches in finite dimensions for a broader class of functions.
Corollary 4.5. Let $\left(x^{k}\right)_{k \in \mathbb{N}}$ be the sequence generated by Method 1. Suppose that $S_{*} \neq \emptyset$.
(i) If the gradient of $f$ is globally Lipschitz continuous on $\operatorname{dom} g$, then we have

$$
(f+g)\left(x^{k}\right)-\min _{x \in \mathcal{H}}(f+g)(x)=\mathcal{O}\left(k^{-1}\right) .
$$

(ii) If $\operatorname{dim} \mathcal{H}<+\infty$ and the gradient of $f$ is locally Lipschitz continuous on $S_{*}$, then we have

$$
(f+g)\left(x^{k}\right)-\min _{x \in \mathcal{H}}(f+g)(x)=o\left(k^{-1}\right) .
$$

We obtain linear convergence when the stepsizes are bounded below by a positive number and either $f$ or $g$ is strongly convex. Recall that $h: \mathcal{H} \rightarrow \overline{\mathbb{R}}$ is strongly convex with constant $\mu>0$ if,

$$
h(x) \geq h(y)+\langle v, x-y\rangle+\frac{\mu}{2}\|x-y\|^{2} \quad \text { for all } \quad x \in \mathcal{H},(y, v) \in \operatorname{Gph} \partial h .
$$

Theorem 4.6. Let $\left(x^{k}\right)_{k \in \mathbb{N}}$ and $\left(\alpha_{k}\right)_{k \in \mathbb{N}}$ be the sequences generated in Method 1. Suppose that $S_{*} \neq \emptyset$, that there exists $\alpha>0$ satisfying $\alpha_{k} \geq \alpha>0$ for all $k \in \mathbb{N}$, and that either $f$ or $g$ is strongly convex with constant $\mu>0$. Then $S_{*}=\left\{x_{*}\right\}$ is singleton and

$$
\begin{equation*}
\left\|x^{k+1}-x_{*}\right\| \leq \frac{1}{\sqrt{1+\alpha \mu}} \cdot\left\|x^{k}-x_{*}\right\| \leq\left(\frac{1}{\sqrt{1+\alpha \mu}}\right)^{k+1}\left\|x^{0}-x_{*}\right\| \quad \forall k \in \mathbb{N}, \tag{39}
\end{equation*}
$$

i.e., the sequence $\left(x^{k}\right)_{k \in \mathbb{N}}$ converges (strongly) to $x_{*}$ with the linear rate $\frac{1}{\sqrt{1+\alpha \mu}}<1$.

Consequently, if either $f$ or $g$ is strongly convex, $\nabla f$ is locally Lipschitz continuous on $S_{*}$, and $\operatorname{dim} \mathcal{H}<+\infty$, then $\left(x^{k}\right)_{k \in \mathbb{N}}$ converges linearly to the unique optimal solution.

Proof. Since either $f$ or $g$ is strongly convex with constant $\mu>0, f+g$ is also strongly convex with constant $\mu>0$. It follows that $S_{*}$ is singleton (i.e., $S_{*}=\left\{x_{*}\right\}$ ). Moreover, using Proposition 4.1(i) with $x=x_{*}$ and the strong convexity of $f+g$ gives us that

$$
\begin{aligned}
\left\|x^{k}-x_{*}\right\|^{2} & \geq\left\|x^{k+1}-x_{*}\right\|^{2}+2 \alpha_{k}\left[(f+g)\left(x^{k+1}\right)-(f+g)\left(x_{*}\right)\right] \\
& \geq\left\|x^{k+1}-x_{*}\right\|^{2}+\alpha_{k} \mu\left\|x^{k+1}-x_{*}\right\|^{2} \geq(1+\alpha \mu)\left\|x^{k+1}-x_{*}\right\|^{2} .
\end{aligned}
$$

It follows that

$$
\left\|x^{k+1}-x_{*}\right\| \leq \frac{1}{\sqrt{1+\alpha \mu}} \cdot\left\|x^{k}-x_{*}\right\| \leq\left(\frac{1}{\sqrt{1+\alpha \mu}}\right)^{k+1}\left\|x^{0}-x_{*}\right\|,
$$

which verifies (39) and thus completes the proof of the theorem.
Since the condition $x=J(x, \alpha)$ for $\alpha>0$ is necessary and sufficient for $x$ to be an optimal solution to problem (1), it is interesting to study the complexity of $\left\|x^{k}-J\left(x^{k}, \alpha_{k}\right)\right\|$ in our Method (1) The velocity of the convergence obtained below is not affected by the behavior of the stepsizes $\alpha_{k}$.

Theorem 4.7. Let $\left(x^{k}\right)_{k \in \mathbb{N}}$ and $\left(\alpha_{k}\right)_{k \in \mathbb{N}}$ be the sequences generated from Method 1. Then we have

$$
\begin{equation*}
\liminf _{k \rightarrow \infty} \sqrt{k} \cdot\left\|x^{k}-J\left(x^{k}, \alpha_{k}\right)\right\|=0 \tag{40}
\end{equation*}
$$

Proof. If (40) does not hold, then we may find a number $\varepsilon>0$ such that for some fixed $K \in \mathbb{N}$ large enough, we have $\left\|x^{k}-J\left(x^{k}, \alpha_{k}\right)\right\| \geq \frac{\varepsilon}{\sqrt{k}}$ for all $k \geq K$. Thus,

$$
\begin{equation*}
\sum_{k=K}^{\infty}\left\|x^{k}-J\left(x^{k}, \alpha_{k}\right)\right\|^{2} \geq \varepsilon^{2} \sum_{k=K}^{\infty} \frac{1}{k}=+\infty . \tag{41}
\end{equation*}
$$

On the other hand, using (10) and Proposition 4.1(ii), we get, for all $k \geq K$,

$$
\begin{aligned}
\left\|x^{k}-J\left(x^{k}, \alpha_{k}\right)\right\|^{2} & =\left\|x^{k}-x^{k+1}\right\|^{2} \leq \frac{\alpha_{k}}{1-\delta}\left[(f+g)\left(x^{k}\right)-(f+g)\left(x^{k+1}\right)\right] \\
& \leq \frac{\sigma}{1-\delta}\left[(f+g)\left(x^{k}\right)-(f+g)\left(x^{k+1}\right)\right]
\end{aligned}
$$

where we have used in the last inequality that $\alpha_{k} \leq \sigma$ for all $k \in \mathbb{N}$, which follows from Linesearch 1. Hence, we have

$$
\sum_{k=K}^{\infty}\left\|x^{k}-J\left(x^{k}, \alpha_{k}\right)\right\|^{2} \leq \frac{\sigma}{1-\delta}\left[(f+g)\left(x^{K}\right)-(f+g)\left(x_{*}\right)\right]<+\infty
$$

which contradicts (41). The proof is complete.

### 4.2 A fast multistep forward-backward method with Linesearch 1

In the spirit of the classical work of Nesterov [31] many accelerated multistep versions have been proposed in the literature for the forward-backward iteration, but to the best of our knowledge all of them have to employ the global Lipschitz continuity assumption on $\nabla f$; see, e.g., [4,5,29]. In this subsection, by following these ideas and assuming no Lipschitz continuity on $\nabla f$, we present a fast version of the proximal forward-backward method with Linesearch 1, improving the convergence result of Theorem [.3] for Method 1]. In [4,5],29] this kind of fast versions usually demands Lipschitz assumption over $\nabla f$ to establish convergence of this method. Here we modify the method by adding a linesearch and an extra projection step in (43) below to avoid the requirements aforementioned. For simplicity, we suppose $\Omega:=\operatorname{dom} g$ is closed in this section.

## Method 2.

Initialization Step. Take $x^{-1}=x^{0} \in \operatorname{dom} g, t_{0}=1, \theta \in(0,1), \alpha_{-1}=\sigma$ and $\delta \in(0,1 / 2)$.
Iterative Step. Given $t_{k}$ and $x^{k}$, set

$$
\begin{align*}
t_{k+1} & =\frac{1+\sqrt{1+4 t_{k}^{2}}}{2}  \tag{42}\\
y^{k} & =x^{k}+\left(\frac{t_{k}-1}{t_{k+1}}\right)\left(x^{k}-x^{k-1}\right), \quad \tilde{y}^{k}=P_{\Omega}\left(y^{k}\right)  \tag{43}\\
x^{k+1} & =J\left(\tilde{y}^{k}, \alpha_{k}\right):=\operatorname{prox}_{\alpha_{k} g}\left(\tilde{y}^{k}-\alpha_{k} \nabla f\left(\tilde{y}^{k}\right)\right) \tag{44}
\end{align*}
$$

with $\alpha_{k}:=$ Linesearch $\mathbb{1}\left(\tilde{y}^{k}, \alpha_{k-1}, \theta, \delta\right)$.
Stop Criteria. If $x^{k+1}=\tilde{y}^{k}$, then stop.
Note that from (43) and (44), $\tilde{y}^{k}$ and $x^{k}$ belong to $\operatorname{dom} g$ for all $k \in \mathbb{N}$ and as a direct consequence of Lemma 3.1, $\alpha_{k}$ satisfying (45) is always positive and nonincreasing. Moreover, it is similar to Method 1 that if $x^{k+1}=\tilde{y}^{k}$ then $x^{k+1}$ is an optimal solution. An important inequality for our further study from Linesearch 1 is

$$
\begin{equation*}
\alpha_{k}\left\|\nabla f\left(x^{k+1}\right)-\nabla f\left(\tilde{y}^{k}\right)\right\| \leq \delta\left\|x^{k+1}-\tilde{y}^{k}\right\| \tag{45}
\end{equation*}
$$

with $\delta \in(0,1 / 2)$. We also need some auxiliary results before establishing the convergence results.

Lemma 4.8. The positive sequence $\left(t_{k}\right)_{k \in \mathbb{N}}$ generated by Method 2 via (42) satisfies, for all $k \in \mathbb{N}$,
(i) $\frac{1}{t_{k}} \leq \frac{2}{k+1}$;
(ii) $t_{k+1}^{2}-t_{k+1}=t_{k}^{2}$.

Proof. The proof easily follows by induction argument.
Proposition 4.9. Let $\alpha_{k}$ be defined in Method 2 and $x \in \operatorname{dom} g$. Then we have

$$
\begin{equation*}
(f+g)(x)-(f+g)\left(x^{k+1}\right) \geq \frac{1}{2 \alpha_{k}}\left(\left\|x^{k+1}-x\right\|^{2}-\left\|y^{k}-x\right\|^{2}\right) \quad \text { for all } \quad k \in \mathbb{N} \text {. } \tag{46}
\end{equation*}
$$

Proof. First note from (5) with $z=\tilde{y}^{k}-\alpha_{k} \nabla f\left(\tilde{y}^{k}\right)$ that $\frac{\tilde{y}^{k}-x^{k+1}}{\alpha_{k}}-\nabla f\left(\tilde{y}^{k}\right) \in \partial g\left(x^{k+1}\right)$. Then,

$$
\begin{equation*}
g(x)-g\left(x^{k+1}\right) \geq\left\langle\frac{\tilde{y}^{k}-x^{k+1}}{\alpha_{k}}-\nabla f\left(\tilde{y}^{k}\right), x-x^{k+1}\right\rangle \tag{47}
\end{equation*}
$$

for all $x \in \operatorname{dom} g$. The convexity of $f$ implies that

$$
\begin{equation*}
f(x)-f(y) \geq\langle\nabla f(y), x-y\rangle \quad \text { for all } \quad x \in \operatorname{dom} f \text { and } y \in \operatorname{dom} g . \tag{48}
\end{equation*}
$$

By summing (47) and (48) with $y=\tilde{y}^{k} \in \Omega=\operatorname{dom} g$, we obtain that

$$
\begin{aligned}
(f+g)(x) \geq & f\left(\tilde{y}^{k}\right)+g\left(x^{k+1}\right)+\left\langle\frac{\tilde{y}^{k}-x^{k+1}}{\alpha_{k}}-\nabla f\left(\tilde{y}^{k}\right), x-x^{k+1}\right\rangle+\left\langle\nabla f\left(\tilde{y}^{k}\right), x-\tilde{y}^{k}\right\rangle \\
= & f\left(\tilde{y}^{k}\right)+g\left(x^{k+1}\right)+\frac{1}{\alpha_{k}}\left\langle\tilde{y}^{k}-x^{k+1}, x-x^{k+1}\right\rangle+\left\langle\nabla f\left(\tilde{y}^{k}\right), x^{k+1}-\tilde{y}^{k}\right\rangle \\
= & f\left(\tilde{y}^{k}\right)+g\left(x^{k+1}\right)+\frac{1}{\alpha_{k}}\left\langle\tilde{y}^{k}-x^{k+1}, x-x^{k+1}\right\rangle+\left\langle\nabla f\left(\tilde{y}^{k}\right)-\nabla f\left(x^{k+1}\right), x^{k+1}-\tilde{y}^{k}\right\rangle \\
& +\left\langle\nabla f\left(x^{k+1}\right), x^{k+1}-\tilde{y}^{k}\right\rangle \\
\geq & f\left(\tilde{y}^{k}\right)+g\left(x^{k+1}\right)+\frac{1}{\alpha_{k}}\left\langle\tilde{y}^{k}-x^{k+1}, x-x^{k+1}\right\rangle-\frac{\delta}{\alpha_{k}}\left\|x^{k+1}-\tilde{y}^{k}\right\|^{2} \\
& +\left\langle\nabla f\left(x^{k+1}\right), x^{k+1}-\tilde{y}^{k}\right\rangle,
\end{aligned}
$$

where the last inequality follows from (45). Rearranging the inequality gives us that

$$
\begin{align*}
\left\langle\tilde{y}^{k}-x^{k+1}, x^{k+1}-x\right\rangle \geq & \alpha_{k}\left[f\left(\tilde{y}^{k}\right)+g\left(x^{k+1}\right)-(f+g)(x)\right]-\delta\left\|x^{k+1}-\tilde{y}^{k}\right\|^{2} \\
& +\alpha_{k}\left\langle\nabla f\left(x^{k+1}\right), x^{k+1}-\tilde{y}^{k}\right\rangle . \tag{49}
\end{align*}
$$

Observe that $2\left\langle\tilde{y}^{k}-x^{k+1}, x^{k+1}-x\right\rangle=\left\|\tilde{y}^{k}-x\right\|^{2}-\left\|x^{k+1}-x\right\|^{2}-\left\|\tilde{y}^{k}-x^{k+1}\right\|^{2}$. By combining the above equality with (49), we have

$$
\begin{align*}
\left\|\tilde{y}^{k}-x\right\|^{2}-\left\|x^{k+1}-x\right\|^{2} \geq & 2 \alpha_{k}\left[f\left(\tilde{y}^{k}\right)+g\left(x^{k+1}\right)-(f+g)(x)+\left\langle\nabla f\left(x^{k+1}\right), x^{k+1}-\tilde{y}^{k}\right\rangle\right] \\
& +(1-2 \delta)\left\|\tilde{y}^{k}-x^{k+1}\right\|^{2} \\
\geq & 2 \alpha_{k}\left[f\left(\tilde{y}^{k}\right)+g\left(x^{k+1}\right)-(f+g)(x)+\left\langle\nabla f\left(x^{k+1}\right), x^{k+1}-\tilde{y}^{k}\right\rangle\right] . \tag{50}
\end{align*}
$$

It follows from (48) with $x=\tilde{y}^{k}$ and $y=x^{k+1}$ that $f\left(\tilde{y}^{k}\right)-f\left(x^{k+1}\right) \geq\left\langle\nabla f\left(x^{k+1}\right), \tilde{y}^{k}-x^{k+1}\right\rangle$, which together with (50) implies

$$
\begin{aligned}
\left\|\tilde{y}^{k}-x\right\|^{2}-\left\|x^{k+1}-x\right\|^{2} & \geq 2 \alpha_{k}\left[f\left(\tilde{y}^{k}\right)+g\left(x^{k+1}\right)-(f+g)(x)+f\left(x^{k+1}\right)-f\left(\tilde{y}^{k}\right)\right] \\
& =2 \alpha_{k}\left[(f+g)\left(x^{k+1}\right)-(f+g)(x)\right] .
\end{aligned}
$$

Since $\left\|\tilde{y}^{k}-x\right\| \leq\left\|y^{k}-x\right\|$ for all $x \in \operatorname{dom} g$ due to (43), we get from the latter (46) and complete the proof of the proposition.

In the next result we establish a better complexity for Method 2 than Method 1 in Theorem 4.3 under a similar assumption.

Theorem 4.10. Let $\left(x^{k}\right)_{k \in \mathbb{N}}$ and $\left(\alpha_{k}\right)_{k \in \mathbb{N}}$ be the sequences generated in Method 2. Suppose that $S_{*} \neq \emptyset$ and there is $\alpha>0$ such that $\alpha_{k} \geq \alpha>0$ for all $k \in \mathbb{N}$. Then we have
$(f+g)\left(x^{k}\right)-\min _{x \in \mathcal{H}}(f+g)(x) \leq \frac{\frac{2}{\alpha} \cdot\left(\left\|x^{0}-x_{*}\right\|^{2}+2 \sigma\left[(f+g)\left(x^{0}\right)-\min _{x \in \mathcal{H}}(f+g)(x)\right]\right)}{(k+1)^{2}}$ for all $k \in \mathbb{N}$.
Proof. To justify, pick any $x_{*} \in S_{*}$. By Lemma 4.8(i) and the convexity of $g$, we have $t_{k+1} \geq 1$ and thus $x:=t_{k+1}^{-1} x_{*}+\left(1-t_{k+1}^{-1}\right) x^{k} \in \operatorname{dom} g$. Applying Proposition 4.9 for this $x$ gives us that

$$
\begin{aligned}
& \frac{1}{2 \alpha_{k}}\left(\left\|x^{k+1}-\left(t_{k+1}^{-1} x_{*}+\left(1-t_{k+1}^{-1}\right) x^{k}\right)\right\|^{2}-\left\|y^{k}-\left(t_{k+1}^{-1} x_{*}+\left(1-t_{k+1}^{-1}\right) x^{k}\right)\right\|^{2}\right) \\
& \leq(f+g)\left(t_{k+1}^{-1} x_{*}+\left(1-t_{k+1}^{-1}\right) x^{k}\right)-(f+g)\left(x^{k+1}\right) \\
& \leq t_{k+1}^{-1}(f+g)\left(x_{*}\right)+\left(1-t_{k+1}^{-1}\right)(f+g)\left(x^{k}\right)-(f+g)\left(x^{k+1}\right)
\end{aligned}
$$

After rearrangement, we obtain

$$
\begin{aligned}
& \left(1-t_{k+1}^{-1}\right)\left[(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right)\right]-\left[(f+g)\left(x^{k+1}\right)-(f+g)\left(x_{*}\right)\right] \\
& \geq \frac{1}{2 \alpha_{k} t_{k+1}^{2}}\left(\left\|t_{k+1} x^{k+1}-\left(x_{*}+\left(t_{k+1}-1\right) x^{k}\right)\right\|^{2}-\left\|t_{k+1} y^{k}-\left(x_{*}+\left(t_{k+1}-1\right) x^{k}\right)\right\|^{2}\right) .
\end{aligned}
$$

By multiplying by $t_{k+1}^{2}$ to the above inequality and using (43) and Lemma 4.8(ii), we have

$$
\begin{aligned}
& \frac{1}{2 \alpha_{k}}\left(\left\|t_{k+1} x^{k+1}-\left(x_{*}+\left(t_{k+1}-1\right) x^{k}\right)\right\|^{2}-\left\|t_{k+1} y^{k}-\left(x_{*}+\left(t_{k+1}-1\right) x^{k}\right)\right\|^{2}\right) \\
& =\frac{1}{2 \alpha_{k}}\left(\left\|t_{k+1} x^{k+1}-\left(t_{k+1}-1\right) x^{k}-x_{*}\right\|^{2}-\left\|t_{k} x^{k}-\left(t_{k}-1\right) x^{k-1}-x_{*}\right\|^{2}\right) \\
& \leq\left(t_{k+1}^{2}-t_{k+1}\right)\left[(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right)\right]-t_{k+1}^{2}\left[(f+g)\left(x^{k+1}\right)-(f+g)\left(x_{*}\right)\right] \\
& =t_{k}^{2}\left[(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right)\right]-t_{k+1}^{2}\left[(f+g)\left(x^{k+1}\right)-(f+g)\left(x_{*}\right)\right] .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \left\|t_{k} x^{k}-\left(t_{k}-1\right) x^{k-1}-x_{*}\right\|^{2}-\left\|t_{k+1} x^{k+1}-\left(t_{k+1}-1\right) x^{k}-x_{*}\right\|^{2} \\
& \geq 2 \alpha_{k}\left(t_{k+1}^{2}\left[(f+g)\left(x^{k+1}\right)-(f+g)\left(x_{*}\right)\right]-t_{k}^{2}\left[(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right)\right]\right) \\
& \geq 2 \alpha_{k+1} t_{k+1}^{2}\left[(f+g)\left(x^{k+1}\right)-(f+g)\left(x_{*}\right)\right]-2 \alpha_{k} t_{k}^{2}\left[(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right)\right],
\end{aligned}
$$

where the last inequality follows from the facts that $\alpha_{k} \geq \alpha_{k+1}=\boldsymbol{L i n e s e a r c h} \mathbf{1}\left(\tilde{y}^{k}, \alpha_{k}, \theta, \delta\right)$ and $(f+g)\left(x^{k+1}\right)-(f+g)\left(x_{*}\right) \geq 0$. Reordering the above inequality and applying it inductively yield

$$
\begin{aligned}
& 2 \alpha_{k+1} t_{k+1}^{2}\left[(f+g)\left(x^{k+1}\right)-(f+g)\left(x_{*}\right)\right] \\
& \leq\left\|t_{k+1} x^{k+1}-\left(t_{k+1}-1\right) x^{k}-x_{*}\right\|^{2}+2 \alpha_{k+1} t_{k+1}^{2}\left[(f+g)\left(x^{k+1}\right)-(f+g)\left(x_{*}\right)\right] \\
& \leq\left\|t_{k} x^{k}-\left(t_{k}-1\right) x^{k-1}-x_{*}\right\|^{2}+2 \alpha_{k} t_{k}^{2}\left[(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right)\right] \\
& \leq \ldots \leq\left\|t_{0} x^{0}-\left(t_{0}-1\right) x^{-1}-x_{*}\right\|^{2}+2 \alpha_{0} t_{0}^{2}\left[(f+g)\left(x^{0}\right)-(f+g)\left(x_{*}\right)\right] \\
& =\left\|x^{0}-x_{*}\right\|^{2}+2 \alpha_{0}\left[(f+g)\left(x^{0}\right)-(f+g)\left(x_{*}\right)\right],
\end{aligned}
$$

which readily imply $2 \alpha_{k} t_{k}^{2}\left[(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right)\right] \leq\left\|x^{0}-x_{*}\right\|^{2}+2 \sigma\left[(f+g)\left(x^{0}\right)-(f+g)\left(x_{*}\right)\right]$. Using this inequality together with Lemma 4.8(i) gives us that

$$
\begin{aligned}
(f+g)\left(x^{k}\right)-\min _{x \in \mathcal{H}}(f+g)(x) & \leq \frac{1}{2 \alpha_{k} t_{k}^{2}}\left(\left\|x^{0}-x_{*}\right\|^{2}+2 \sigma\left[(f+g)\left(x^{0}\right)-\min _{x \in \mathcal{H}}(f+g)(x)\right]\right) \\
& \leq \frac{\frac{2}{\alpha} \cdot\left(\left\|x^{0}-x_{*}\right\|^{2}+2 \sigma\left[(f+g)\left(x^{0}\right)-\min _{x \in \mathcal{H}}(f+g)(x)\right]\right)}{(k+1)^{2}}
\end{aligned}
$$

for all $x_{*} \in S_{*}$ and thus verifies (46). The proof of the theorem is complete.
This theorem shows that the expected error of the iterates generated by Method 2 after $k$ iterations is $\mathcal{O}\left(k^{-2}\right)$ when the stepsizes are bounded below by a positive constant. Similarly to Proposition 4.4, we prove in the next result that such a requirement is satisfied under global Lipschitz assumption on the gradient of $f$. The complexity $o\left(k^{-2}\right)$ for the accelerated scheme similarly to (42)-(45) has been obtained recently in [2, 11] under the global Lipschitz assumption. It would be interesting to combine their techniques with ours to derive similar complexity under the weaker assumption of local Lipschitz continuity as in Proposition 4.4 (ii).

Proposition 4.11. Let $\left(\alpha_{k}\right)_{k \in \mathbb{N}}$ be the sequence generated by Linesearch 1 on Method 2. If the gradient of $f$ is globally Lipschitz continuous on $\operatorname{dom} g$ then there exists some $\alpha>0$ such that $\alpha_{k} \geq \alpha$ for all $k \in \mathbb{N}$.

Proof. Suppose that $\nabla f$ is globally Lipschitz continuous on $\operatorname{dom} g$ with constant $L>0$. Since $\alpha_{k}$ is nonnegative and decreasing, $\lim _{k \rightarrow \infty} \alpha_{k}=\alpha$ exists. If $\alpha<\frac{\delta \theta}{L}$, we may find $K \in \mathbb{N}$ such that $\alpha_{k}<\frac{\delta \theta}{L}$ for all $k>K$. Define further $\hat{\alpha}_{k}:=\frac{\alpha_{k}}{\theta}>0$, and $\hat{y}^{k}:=J\left(\tilde{y}^{k}, \hat{\alpha}_{k}\right)=\operatorname{prox}_{\hat{\alpha}_{k} g}\left(\tilde{y}^{k}-\hat{\alpha}_{k} \nabla f\left(\tilde{y}^{k}\right)\right) \in \operatorname{dom} g$. If $\alpha_{k}<\alpha_{k-1}$ for $k>K$, it follows from the definition of Linesearch 1 that

$$
\begin{equation*}
\hat{\alpha}_{k}\left\|\nabla f\left(\hat{y}^{k}\right)-\nabla f\left(\tilde{y}^{k}\right)\right\|>\delta\left\|\hat{y}^{k}-\tilde{y}^{k}\right\| . \tag{51}
\end{equation*}
$$

Due to the fact $\nabla f$ is Lipschitz continuous on $\operatorname{dom} g$ with constant $L$, we get from (51) that $\hat{\alpha}_{k} L\left\|\tilde{y}^{k}-\hat{y}^{k}\right\|>\delta\left\|\tilde{y}^{k}-\hat{y}^{k}\right\|$. Thus $\alpha_{k} \geq \frac{\delta \theta}{L}$, which is a contradiction. Hence $\alpha_{k} \geq \alpha_{k-1}$, i.e., $\alpha_{k}=\alpha_{k-1}$ for all $k>K$. This tells us that $\alpha_{K}=\alpha>0$ whenever $\alpha<\frac{\delta \theta}{L}$. Thus we always have $\alpha>0$ and complete the proof.

Let us complete the section with a direct consequence of the above proposition and Theorem 4.10.

Corollary 4.12. Let $\left(x^{k}\right)_{k \in \mathbb{N}}$ be the sequence generated by Method 2. Suppose that $S_{*} \neq \emptyset$ and the gradient of $f$ is Lipschitz continuous on dom $g$. Then we have

$$
(f+g)\left(x^{k}\right)-\min _{x \in \mathcal{H}}(f+g)(x)=\mathcal{O}\left((k+1)^{-2}\right) .
$$

## 5 The forward-backward method with Linesearch 2

Method 1 requires to evaluate the resolvent of $\partial g$ inside Linesearch 1 at each step of the iteration. When the proximal step is not easy to compute, Method 1 may be inefficient. To overcome this drawback, we propose here a modification of the forward-backward method by using Linesearch 2 , which involves only one computation of the resolvent of $\partial g$ for all steps of this linesearch. We also prove that the sequence generated by this method is weakly convergent to a solution of problem (11).

## Method 3.

Initialization Step. Take $x^{0} \in \operatorname{dom} g$ and $\theta \in(0,1)$.
Iterative Step. Set

$$
\begin{align*}
J_{k} & =\operatorname{prox}_{g}\left(x^{k}-\nabla f\left(x^{k}\right)\right)  \tag{52}\\
x^{k+1} & =x^{k}-\beta_{k}\left(x^{k}-J_{k}\right) \tag{53}
\end{align*}
$$

with $\beta_{k}:=$ Linesearch $\mathbf{2}\left(x^{k}, \theta\right)$.
Stop Criteria. If $x^{k+1}=x^{k}$, then stop.
Thanks to Lemma 3.2 and the convexity of $g$, we note that $x^{k} \in \operatorname{dom} g$ inductively. Moreover, it follows from Linesearch 2 that

$$
\begin{equation*}
(f+g)\left(x^{k+1}\right) \leq(f+g)\left(x^{k}\right)-\beta_{k}\left[g\left(x^{k}\right)-g\left(J_{k}\right)\right]-\beta_{k}\left\langle\nabla f\left(x^{k}\right), x^{k}-J_{k}\right\rangle+\frac{\beta_{k}}{2}\left\|x^{k}-J_{k}\right\|^{2} . \tag{54}
\end{equation*}
$$

Next we obtain some similar results for Method 3 to the ones in Section 3 for Method 1. The following proposition is corresponding to Proposition 4.1.

Proposition 5.1. Let $x \in \operatorname{dom} g$. Then we have

$$
\left\|x^{k+1}-x\right\|^{2} \leq\left\|x^{k}-x\right\|^{2}+2\left[(f+g)\left(x^{k}\right)-(f+g)\left(x^{k+1}\right)\right]+2 \beta_{k}\left[(f+g)(x)-(f+g)\left(x^{k}\right)\right], \quad \forall k \in \mathbb{N} .
$$

Proof. Fix any $x \in \operatorname{dom} g$ and set $A_{k}:=\left\|x^{k+1}-x^{k}\right\|^{2}+\left\|x^{k}-x\right\|^{2}-\left\|x^{k+1}-x\right\|^{2}=2\left\langle x^{k}-x^{k+1}, x^{k}-x\right\rangle$. Moreover, we get from (53) that

$$
\begin{aligned}
\frac{A_{k}}{2 \beta_{k}} & =\left\langle x^{k}-J_{k}, x^{k}-x\right\rangle=\left\langle\nabla f\left(x^{k}\right), x^{k}-x\right\rangle+\left\langle x^{k}-J_{k}-\nabla f\left(x^{k}\right), x^{k}-x\right\rangle \\
& =\left\langle\nabla f\left(x^{k}\right), x^{k}-x\right\rangle+\left\langle x^{k}-J_{k}-\nabla f\left(x^{k}\right), J_{k}-x\right\rangle+\left\langle x^{k}-J_{k}-\nabla f\left(x^{k}\right), x^{k}-J_{k}\right\rangle \\
& =\left\langle\nabla f\left(x^{k}\right), x^{k}-x\right\rangle+\left\langle x^{k}-J_{k}-\nabla f\left(x^{k}\right), J_{k}-x\right\rangle-\left\langle\nabla f\left(x^{k}\right), x^{k}-J_{k}\right\rangle+\left\|x^{k}-J_{k}\right\|^{2} .
\end{aligned}
$$

Observe from (52) that $x^{k}-\nabla f\left(x^{k}\right)-J_{k} \in \partial g\left(J_{k}\right)$. By applying (4) and (54) to the above expression, we have

$$
\begin{aligned}
\frac{A_{k}}{2 \beta_{k}} & \geq f\left(x^{k}\right)-f(x)+g\left(J_{k}\right)-g(x)-\left\langle\nabla f\left(x^{k}\right), x^{k}-J_{k}\right\rangle+\left\|x^{k}-J_{k}\right\|^{2} \\
& \geq f\left(x^{k}\right)+g\left(J_{k}\right)-(f+g)(x)+\frac{1}{\beta_{k}}\left[(f+g)\left(x^{k+1}\right)-(f+g)\left(x^{k}\right)\right]+g\left(x^{k}\right)-g\left(J_{k}\right)+\frac{1}{2}\left\|x^{k}-J_{k}\right\|^{2} \\
& =\left[(f+g)\left(x^{k}\right)-(f+g)(x)\right]+\frac{1}{\beta_{k}}\left[(f+g)\left(x^{k+1}\right)-(f+g)\left(x^{k}\right)\right]+\frac{1}{2}\left\|x^{k}-J_{k}\right\|^{2} .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left\|x^{k+1}-x\right\|^{2} \leq & \left\|x^{k}-x\right\|^{2}+\left\|x^{k+1}-x^{k}\right\|^{2}-\beta_{k}\left\|x^{k}-J_{k}\right\|^{2}+2\left[(f+g)\left(x^{k}\right)-(f+g)\left(x^{k+1}\right)\right] \\
& +2 \beta_{k}\left[(f+g)(x)-(f+g)\left(x^{k}\right)\right]
\end{aligned}
$$

Since $x^{k+1}-x^{k}=\beta_{k}\left(J_{k}-x^{k}\right)$ by (53) and $\beta_{k}^{2} \leq \beta_{k}$, we conclude that

$$
\begin{aligned}
\left\|x^{k+1}-x\right\|^{2} \leq & \left\|x^{k}-x\right\|^{2}+\left(\beta_{k}^{2}-\beta_{k}\right)\left\|x^{k}-J_{k}\right\|^{2}+2\left[(f+g)\left(x^{k}\right)-(f+g)\left(x^{k+1}\right)\right] \\
& +2 \beta_{k}\left[(f+g)(x)-(f+g)\left(x^{k}\right)\right] \\
\leq & \left\|x^{k}-x\right\|^{2}+2\left[(f+g)\left(x^{k}\right)-(f+g)\left(x^{k+1}\right)\right]+2 \beta_{k}\left[(f+g)(x)-(f+g)\left(x^{k}\right)\right]
\end{aligned}
$$

as desired. The proof is complete.
It is worth noting that using Proposition 5.1 with $x=x^{k} \in \operatorname{dom} g$ gives us that

$$
\begin{equation*}
(f+g)\left(x^{k}\right)-(f+g)\left(x^{k+1}\right) \geq \frac{1}{2}\left\|x^{k+1}-x^{k}\right\|^{2} \geq 0 \tag{55}
\end{equation*}
$$

which shows that Method 3 is also a descent method.
Next we establish the main result of this section whose statement is similar to Theorem 4.2,
Theorem 5.2. Let $\left(x^{k}\right)_{k \in \mathbb{N}}$ be the sequence generated by Method 3. The following statements hold:
(i) If $S_{*} \neq \emptyset$ then $\left(x^{k}\right)_{k \in \mathbb{N}}$ is quasi-Fejér convergent to $S_{*}$ and weakly converges to a point in $S_{*}$.
(ii) If $S_{*}=\emptyset$ then we have

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left\|x^{k}\right\|=+\infty \quad \text { and } \quad \lim _{k \rightarrow \infty}(f+g)\left(x^{k}\right)=\inf _{x \in \mathcal{H}}(f+g)(x) . \tag{56}
\end{equation*}
$$

Proof. To justify (i), suppose that $S_{*} \neq \emptyset$. By employing Proposition 5.1 at $x=x_{*} \in S_{*} \subseteq \operatorname{dom} g$, we have

$$
\begin{equation*}
\left\|x^{k+1}-x_{*}\right\|^{2} \leq\left\|x^{k}-x_{*}\right\|^{2}+2\left[(f+g)\left(x^{k}\right)-(f+g)\left(x^{k+1}\right)\right] \quad \text { for all } \quad k \in \mathbb{N} . \tag{57}
\end{equation*}
$$

It follows from (55) that $\epsilon_{k}:=2\left[(f+g)\left(x^{k}\right)-(f+g)\left(x^{k+1}\right)\right] \geq 0$. Moreover, observe that

$$
\begin{aligned}
\sum_{k=0}^{\infty} \epsilon_{k} & =2 \sum_{k=0}^{\infty}\left[(f+g)\left(x^{k}\right)-(f+g)\left(x^{k+1}\right)\right] \leq 2\left[(f+g)\left(x^{0}\right)-\lim _{k \rightarrow \infty}(f+g)\left(x^{k+1}\right)\right] \\
& \leq 2\left[(f+g)\left(x^{0}\right)-(f+g)\left(x_{*}\right)\right]<+\infty
\end{aligned}
$$

This together with (57) tells us that the sequence $\left(x^{k}\right)_{k \in \mathbb{N}}$ is quasi-Fejér convergent to $S_{*}$ via Definition [2.1] By Fact 2.5(i), this sequence is bounded and hence it has weak accumulation points. Let $\bar{x}$ be a weak accumulation point of $\left(x^{k}\right)_{k \in \mathbb{N}}$. Hence there exists a subsequence $\left(x^{n_{k}}\right)_{k \in \mathbb{N}}$ of $\left(x^{k}\right)_{k \in \mathbb{N}}$ converging weakly to $\bar{x}$. Now we distinguish our analysis into two cases.

Case 1. The sequence $\left(\beta_{n_{k}}\right)_{k \in \mathbb{N}}$ does not converge to 0 , i.e., there exist some $\beta>0$ and a subsequence of $\left(\beta_{n_{k}}\right)_{k \in \mathbb{N}}$ (without relabelling) such that

$$
\begin{equation*}
\beta_{n_{k}} \geq \beta, \quad \forall k \in \mathbb{N} . \tag{58}
\end{equation*}
$$

By using Proposition 5.1 with $x=x_{*} \in S_{*}$, we get

$$
\beta_{k}\left[(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right)\right] \leq \frac{1}{2}\left(\left\|x^{k}-x_{*}\right\|^{2}-\left\|x^{k+1}-x_{*}\right\|^{2}\right)+(f+g)\left(x^{k}\right)-(f+g)\left(x^{k+1}\right) .
$$

Summing from $k=0$ to $m$ in the above inequality implies

$$
\begin{aligned}
\sum_{k=0}^{m} \beta_{k}\left[(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right)\right] & \leq \frac{1}{2}\left(\left\|x^{0}-x_{*}\right\|^{2}-\left\|x^{m+1}-x_{*}\right\|^{2}\right)+(f+g)\left(x^{0}\right)-(f+g)\left(x^{m+1}\right) \\
& \leq \frac{1}{2}\left\|x^{0}-x_{*}\right\|^{2}+(f+g)\left(x^{0}\right)-(f+g)\left(x_{*}\right)
\end{aligned}
$$

By taking $m \rightarrow \infty$ and using the fact that $(f+g)\left(x^{k}\right) \geq(f+g)\left(x_{*}\right)$, we obtain that

$$
\sum_{k=0}^{\infty} \beta_{n_{k}}\left[(f+g)\left(x^{n_{k}}\right)-(f+g)\left(x_{*}\right)\right] \leq \sum_{k=0}^{\infty} \beta_{k}\left[(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right)\right]<+\infty
$$

which together with (58) establishes that $\lim _{k \rightarrow \infty}(f+g)\left(x^{n_{k}}\right)=(f+g)\left(x_{*}\right)$. Since $f+g$ is lower semicontinuous on $\operatorname{dom} g$, it is also weakly l.s.c. due to the convexity of $f+g$. It follows from the last equality that

$$
(f+g)\left(x_{*}\right) \leq(f+g)(\bar{x}) \leq \liminf _{k \rightarrow \infty}(f+g)\left(x^{n_{k}}\right)=\lim _{k \rightarrow \infty}(f+g)\left(x^{n_{k}}\right)=(f+g)\left(x_{*}\right),
$$

which yields $(f+g)(\bar{x})=(f+g)\left(x_{*}\right)$ and thus $\bar{x} \in S_{*}$.
Case 2. $\lim _{k \rightarrow \infty} \beta_{k}=0$. Define $\hat{\beta}_{k}:=\frac{\beta_{k}}{\theta}>0$ and

$$
\begin{equation*}
\hat{y}^{k}:=x^{k}-\hat{\beta}_{k}\left(x^{k}-J_{k}\right)=\left(1-\hat{\beta}_{k}\right) x^{k}+\hat{\beta}_{k} J_{k} . \tag{59}
\end{equation*}
$$

It follows from the definition of Linesearch 2 that

$$
\begin{equation*}
(f+g)\left(\hat{y}^{k}\right)>(f+g)\left(x^{k}\right)-\hat{\beta}_{k}\left[g\left(x^{k}\right)-g\left(J_{k}\right)\right]-\hat{\beta}_{k}\left\langle\nabla f\left(x^{k}\right), x^{k}-J_{k}\right\rangle+\frac{\hat{\beta}_{k}}{2}\left\|x^{k}-J_{k}\right\|^{2} \tag{60}
\end{equation*}
$$

This together with (4) and (59) gives us that

$$
\begin{aligned}
0> & -\hat{\beta}_{k}\left\langle\nabla f\left(x^{k}\right), x^{k}-J_{k}\right\rangle+(f+g)\left(x^{k}\right)-(f+g)\left(\hat{y}^{k}\right)-\hat{\beta}_{k}\left[g\left(x^{k}\right)-g\left(J_{k}\right)\right]+\frac{\hat{\beta}_{k}}{2}\left\|x^{k}-J_{k}\right\|^{2} \\
= & -\hat{\beta}_{k}\left\langle\nabla f\left(x^{k}\right), x^{k}-J_{k}\right\rangle+f\left(x^{k}\right)-f\left(\hat{y}^{k}\right)+g\left(x^{k}\right)-g\left(\hat{y}^{k}\right)-\hat{\beta}_{k}\left[g\left(x^{k}\right)-g\left(J_{k}\right)\right]+\frac{\hat{\beta}_{k}}{2}\left\|x^{k}-J_{k}\right\|^{2} \\
\geq & -\hat{\beta}_{k}\left\langle\nabla f\left(x^{k}\right), x^{k}-J_{k}\right\rangle+\left\langle\nabla f\left(\hat{y}^{k}\right), x^{k}-\hat{y}^{k}\right\rangle+\frac{\hat{\beta}_{k}}{2}\left\|x^{k}-J_{k}\right\|^{2} \\
& +g\left(x^{k}\right)-\left(1-\hat{\beta}_{k}\right) g\left(x^{k}\right)-\hat{\beta}_{k} g\left(J_{k}\right)-\hat{\beta}_{k}\left[g\left(x^{k}\right)-g\left(J_{k}\right)\right] \\
= & \hat{\beta}_{k}\left\langle\nabla f\left(\hat{y}^{k}\right)-\nabla f\left(x^{k}\right), x^{k}-J_{k}\right\rangle+\frac{\hat{\beta}_{k}}{2}\left\|x^{k}-J_{k}\right\|^{2} .
\end{aligned}
$$

We obtain that

$$
\frac{\hat{\beta}_{k}}{2}\left\|x^{k}-J_{k}\right\|^{2}<\hat{\beta}_{k}\left\|\nabla f\left(\hat{y}^{k}\right)-\nabla f\left(x^{k}\right)\right\| \cdot\left\|x^{k}-J_{k}\right\|,
$$

which yields

$$
\begin{equation*}
\frac{1}{2}\left\|x^{k}-J_{k}\right\| \leq\left\|\nabla f\left(\hat{y}^{k}\right)-\nabla f\left(x^{k}\right)\right\| . \tag{61}
\end{equation*}
$$

Since $\operatorname{prox}_{g}(\cdot)$ is nonexpansive, we get from (52) that $\left\|J_{k}-J_{0}\right\| \leq\left\|x^{k}-x^{0}\right\|+\left\|\nabla f\left(x^{k}\right)-\nabla f\left(x^{0}\right)\right\|$. Due to Assumption A2 and the boundedness of $\left(x^{k}\right)_{k \in \mathbb{N}}$, the latter tells us that $\left(J_{k}\right)_{k \in \mathbb{N}}$ is also bounded. This together with (59) and the fact $\beta_{k} \rightarrow 0$ implies that $\left\|\hat{y}^{k}-x^{k}\right\| \rightarrow 0$ as $k \rightarrow \infty$. Since $\nabla f$ is uniformly continuous on bounded sets, we get $\left\|\nabla f\left(\hat{y}^{k}\right)-\nabla f\left(x^{k}\right)\right\| \rightarrow 0$ as $k \rightarrow \infty$ and derive from (61) that

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left\|x^{k}-J_{k}\right\|=0 \tag{62}
\end{equation*}
$$

Since $\nabla f$ is uniformly continuous on bounded sets, (62) implies

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left\|\nabla f\left(x^{k}\right)-\nabla f\left(J_{k}\right)\right\|=0 \tag{63}
\end{equation*}
$$

Using (5) with $z=x^{k}-\nabla f\left(x^{k}\right)$ gives us that

$$
x^{k}-J_{k}+\nabla f\left(J_{k}\right)-\nabla f\left(x^{k}\right) \in \nabla f\left(J_{k}\right)+\partial g\left(J_{k}\right) \subseteq \partial(f+g)\left(J_{k}\right) .
$$

By passing to the limit over the subsequence $\left(n_{k}\right)_{k \in \mathbb{N}}$ in the above inclusion, we get from Fact 2.2, (62), and (63) that $0 \in \partial(f+g)(\bar{x})$, which implies $\bar{x} \in S_{*}$.

In all possible cases above, any weak accumulation point of $\left(x^{k}\right)_{k \in \mathbb{N}}$ belongs to $S_{*}$. Fact 2.5(ii) tells us that $\left(x^{k}\right)_{k \in \mathbb{N}}$ converges weakly to an optimal solution in $S_{*}$. Thus this completes the proof of (i). Moreover, the proof of part (ii) is quite similar to the arguments used to prove Theorem4.2(ii). We omit the detail and complete the proof.

From the view of (56) and also our Theorem 4.2, it is natural to question that whether

$$
\begin{equation*}
\lim _{k \rightarrow \infty}(f+g)\left(x^{k}\right)=\min _{x \in \mathcal{H}}(f+g)(x) \tag{64}
\end{equation*}
$$

in the case $S_{*} \neq \emptyset$. We do not know the answer in general, but when either $f+g$ is continuous on the $\operatorname{dom} g$ in finite dimensions or the sequence $\left(\beta_{k}\right)_{k \in \mathbb{N}}$ is bounded below by a positive constant, the equality (64) is true with some further complexity discussed in the next subsection.

### 5.1 Complexity analysis of Method 3

In this subsection we establish the complexity of Method 3 with a similar rate to Theorem 4.3 as follows.

Theorem 5.3. Let $\left(x^{k}\right)_{k \in \mathbb{N}}$ and $\left(\beta_{k}\right)_{k \in \mathbb{N}}$ be the sequences generated in Method 3. Suppose that $S_{*} \neq \emptyset$ and there is some $\beta>0$ satisfying $\beta_{k} \geq \beta>0$ for all $k \in \mathbb{N}$. Then for all $k \in \mathbb{N}$

$$
\begin{equation*}
(f+g)\left(x^{k}\right)-\min _{x \in \mathcal{H}}(f+g)(x) \leq \frac{1}{2 \beta} \frac{\left[\operatorname{dist}\left(x^{0}, S_{*}\right)\right]^{2}+2\left[(f+g)\left(x^{0}\right)-\min _{x \in \mathcal{H}}(f+g)(x)\right]}{k} \tag{65}
\end{equation*}
$$

If in addition $\operatorname{dim} \mathcal{H}<+\infty$ then we have

$$
\begin{equation*}
\lim _{k \rightarrow \infty} k\left[(f+g)\left(x^{k}\right)-\min _{x \in \mathcal{H}}(f+g)(x)\right]=0 . \tag{66}
\end{equation*}
$$

Proof. By using Proposition [5.1, at $\ell \in \mathbb{N}$ and $x_{*} \in S_{*}$, we get

$$
\begin{align*}
0 & \geq(f+g)\left(x_{*}\right)-(f+g)\left(x^{\ell+1}\right) \\
& \geq \frac{1}{2 \beta_{\ell}}\left(\left\|x^{\ell+1}-x_{*}\right\|^{2}-\left\|x^{\ell}-x_{*}\right\|^{2}+2\left[(f+g)\left(x^{\ell+1}\right)-(f+g)\left(x^{\ell}\right)\right]\right) \\
& \geq \frac{1}{2 \beta}\left(\left\|x^{\ell+1}-x_{*}\right\|^{2}-\left\|x^{\ell}-x_{*}\right\|^{2}+2\left[(f+g)\left(x^{\ell+1}\right)-(f+g)\left(x^{\ell}\right)\right]\right) \tag{67}
\end{align*}
$$

for all $\ell \in \mathbb{N}$. Summing the above inequality (67), over $\ell=0,1, \ldots, k-1$, we have

$$
\begin{align*}
& \sum_{\ell=0}^{k-1}\left[(f+g)\left(x_{*}\right)-(f+g)\left(x^{\ell+1}\right)\right] \geq \frac{1}{2 \beta}\left(\left\|x^{k}-x_{*}\right\|^{2}-\left\|x^{0}-x_{*}\right\|^{2}+2\left[(f+g)\left(x^{k}\right)-(f+g)\left(x^{0}\right)\right]\right) \\
& \geq \frac{1}{2 \beta}\left(\left\|x^{k}-x_{*}\right\|^{2}-\left\|x^{0}-x_{*}\right\|^{2}+2\left[(f+g)\left(x_{*}\right)-(f+g)\left(x^{0}\right)\right]\right) . \tag{68}
\end{align*}
$$

Noting that $(f+g)\left(x^{\ell+1}\right) \geq(f+g)\left(x^{\ell}\right)$ for all $\ell=0, \ldots, k-1$ by (55), we obtain from (68) that

$$
k\left[(f+g)\left(x_{*}\right)-(f+g)\left(x^{k}\right)\right] \geq \frac{1}{2 \beta}\left(\left\|x^{k}-x_{*}\right\|^{2}-\left\|x^{0}-x_{*}\right\|^{2}+2\left[(f+g)\left(x_{*}\right)-(f+g)\left(x^{0}\right)\right]\right),
$$

which clearly implies the following expression

$$
\begin{equation*}
(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right) \leq \frac{1}{2 \beta} \frac{\left\|x^{0}-x_{*}\right\|^{2}+2\left[(f+g)\left(x^{0}\right)-(f+g)\left(x_{*}\right)\right]}{k} \tag{69}
\end{equation*}
$$

for all $x^{*} \in S_{*}$. (65) is obtained.
To justify (66) when $\operatorname{dim} \mathcal{H}<+\infty$, suppose that $\left(x^{k}\right)_{k \in \mathbb{N}}$ converges (strongly) to some $x_{*} \in S_{*}$ by Theorem 5.2. Hence for any $\varepsilon>0$ there exists some $K>0$ such that

$$
\begin{equation*}
\left\|x^{k}-x_{*}\right\| \leq \varepsilon \quad \text { and } \quad(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right) \leq \varepsilon \quad \text { for all } \quad k \geq K \tag{70}
\end{equation*}
$$

where the second inequality follows from the recent estimate (69). Adding (67) for $\ell=K, K+$ $1, \ldots, K+k-1$ and noting that

$$
\sum_{\ell=K}^{K+k-1}\left[(f+g)\left(x_{*}\right)-(f+g)\left(x^{\ell+1}\right)\right] \geq \frac{1}{2 \beta}\left(\left\|x^{K+k}-x_{*}\right\|^{2}-\left\|x^{K}-x^{*}\right\|^{2}+2\left[(f+g)\left(x^{K+k}\right)-(f+g)\left(x^{K}\right)\right]\right) .
$$

Since $(f+g)\left(x^{\ell+1}\right) \geq(f+g)\left(x^{K+k}\right)$ for all $\ell=K, K+1, \ldots, K+k-1$ by (55), we get from the latter and (70) that

$$
k\left[(f+g)\left(x_{*}\right)-(f+g)\left(x^{K+k}\right)\right] \geq \frac{1}{2 \beta}\left(-\left\|x^{K}-x^{*}\right\|^{2}+2\left[(f+g)\left(x_{*}\right)-(f+g)\left(x^{K}\right)\right]\right) \geq \frac{1}{2 \beta}\left(-\varepsilon^{2}-2 \varepsilon\right) .
$$

It follows that

$$
\begin{aligned}
\limsup _{k \rightarrow \infty} k\left[(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right)\right] & =\limsup _{k \rightarrow \infty}(K+k)\left[(f+g)\left(x^{K+k}\right)-(f+g)\left(x_{*}\right)\right] \\
& \leq \limsup _{k \rightarrow \infty} \frac{K+k}{k} \cdot \frac{\varepsilon^{2}+2 \varepsilon}{2 \beta}=\frac{\varepsilon^{2}+2 \varepsilon}{2 \beta} .
\end{aligned}
$$

Since this inequality holds for any $\varepsilon>0$, we have $\limsup _{k \rightarrow \infty} k\left[(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right)\right] \leq 0$. Note that $(f+g)\left(x^{k}\right)-(f+g)\left(x_{*}\right) \geq 0$ for all $k \in \mathbb{N}$, we get (66) and thus complete the proof of theorem.

Similarly to Lemma 4.4, we present some sufficient conditions for the below boundedness by a positive constant of the stepsize generated by Linesearch [2,

Proposition 5.4. Let $\left(\beta_{k}\right)_{k \in \mathbb{N}}$ be the sequence generated by Linesearch 2 on Method 3. The following statements hold:
(i) If the gradient of $f$ is globally Lipschitz continuous on $\operatorname{dom} g$ with constant $L>0$, then $\beta_{k} \geq$ $\min \left\{1, \frac{\theta}{2 L}\right\}$ for all $k \in \mathbb{N}$.
(ii) Suppose that $\operatorname{dim} \mathcal{H}<+\infty$ and $S_{*} \neq \emptyset$. If $\nabla f$ is locally Lipschitz continuous at any $x \in S_{*}$ then there exists $x_{*} \in S_{*}$ such that

$$
\begin{equation*}
\liminf _{k \rightarrow \infty} \beta_{k} \geq \min \left\{1, \frac{\theta}{2 \mathcal{L}}\right\} \tag{71}
\end{equation*}
$$

where $\mathcal{L}>0$ is a Lipschitz constant of $\nabla f$ around $x_{*}$. Consequently, there exists $\beta>0$ such that $\beta_{k} \geq \beta$ for all $k \in \mathbb{N}$.

Proof. First let us verify (i) by supposing that the gradient of $f$ is globally Lipschitz continuous on $\operatorname{dom} g$ with constant $L>0$. Define $\hat{\beta}_{k}:=\frac{\beta_{k}}{\theta}>0$ and

$$
\begin{equation*}
\hat{y}^{k}:=\hat{\beta}_{k} J_{k}+\left(1-\hat{\beta}_{k}\right) x^{k}=x^{k}-\hat{\beta}_{k}\left(x^{k}-J_{k}\right) . \tag{72}
\end{equation*}
$$

If $\beta_{k}<1$, we get from Linesearch 2 that

$$
(f+g)\left(\hat{y}^{k}\right)>(f+g)\left(x^{k}\right)-\hat{\beta}_{k}\left[g\left(x^{k}\right)-g\left(J_{k}\right)\right]-\hat{\beta}_{k}\left\langle\nabla f\left(x^{k}\right), x^{k}-J_{k}\right\rangle+\frac{\hat{\beta}_{k}}{2}\left\|x^{k}-J_{k}\right\|^{2},
$$

which together with (721) and that $x^{k}-J_{k} \neq 0$ implies that $\hat{y}^{k} \neq x^{k}$. Furthermore, it is similar to (61) in the proof of Theorem 5.2 that $\frac{1}{2}\left\|x^{k}-J_{k}\right\| \leq\left\|\nabla f\left(\hat{y}^{k}\right)-\nabla f\left(x^{k}\right)\right\|$. Due to the Lipschitz continuity with constant $L$ of $\nabla f$, we get from the latter and (72) that

$$
\frac{1}{2}\left\|x^{k}-J_{k}\right\| \leq L\left\|x^{k}-\hat{y}^{k}\right\|=L \hat{\beta}_{k}\left\|x^{k}-J_{k}\right\| .
$$

Since $x^{k}-J_{k} \neq 0$, the inequality above yields $\hat{\beta}_{k} \geq \frac{1}{2 L}$ and thus $\beta_{k} \geq \frac{\theta}{2 L}$ when $\beta_{k}<1$. It follows that $\beta_{k} \geq \min \left\{1, \frac{\theta}{2 L}\right\}$ as desired.

To verify the second part, suppose that $\operatorname{dim} \mathcal{H}<+\infty, S_{*} \neq \emptyset$, and that $\nabla f$ is locally Lipschitz continuous at any $x \in S_{*}$. By Theorem [5.2, suppose that $\left(x^{k}\right)_{k \in \mathbb{N}}$ (strongly) converges to $x_{*} \in S_{*}$. Hence there exist $\varepsilon, \mathcal{L}>0$ such that

$$
\|\nabla f(x)-\nabla f(y)\| \leq \mathcal{L}\|x-y\| \quad \text { for all } \quad x, y \in \mathbb{B}_{\varepsilon}\left(x_{*}\right) .
$$

Since $x^{k} \rightarrow x_{*}$ as $k \rightarrow \infty$, we find $K>0$ such that $\left\|x^{k}-x_{*}\right\| \leq \varepsilon$ for all $k>K$. Pick any $k>K$, if $\beta_{k}<1$, define $\hat{\beta}_{k}=\frac{\beta_{k}}{\theta}>0$ and $\hat{y}^{k}=\hat{\beta}_{k} J_{k}+\left(1-\hat{\beta}_{k}\right) x^{k}$. Similarly to the above argument of the first part, we have $x^{k}-J_{k} \neq 0$ and

$$
\begin{equation*}
\frac{1}{2}\left\|x^{k}-J_{k}\right\| \leq\left\|\nabla f\left(\hat{y}^{k}\right)-\nabla f\left(x^{k}\right)\right\| . \tag{73}
\end{equation*}
$$

We consider two cases as in Theorem 5.2 as below:

Case 1. The sequence $\left(\beta_{k}\right)_{k \in \mathbb{N}}$ is bounded below by a positive number $\beta>0$. Thanks to (53) we have

$$
\left\|x^{k}-J_{k}\right\|=\frac{\left\|x^{k}-x^{k+1}\right\|}{\beta_{k}} \leq \frac{\left\|x^{k}-x^{k+1}\right\|}{\beta} \rightarrow 0
$$

as $k \rightarrow \infty$. It follows that $\left\|x^{k}-\hat{y}^{k}\right\|=\frac{\beta_{k}}{\theta}\left\|x^{k}-J_{k}\right\| \rightarrow 0$, which tells us that $\left(\hat{y}^{k}\right)_{k \in \mathbb{N}}$ is converging to $x_{*}$. Hence there exists $K_{1}>K$ such that $\hat{y}^{k} \in \mathbb{B}_{\varepsilon}\left(x_{*}\right)$ for all $k>K_{1}$. By combining this with (73), we derive

$$
\frac{1}{2}\left\|x^{k}-J_{k}\right\| \leq \mathcal{L}\left\|\hat{y}^{k}-x^{k}\right\|=\mathcal{L} \hat{\beta}_{k}\left\|x^{k}-J_{k}\right\| \quad \text { for all } \quad k>K_{1} .
$$

Since $\left\|x^{k}-J_{k}\right\| \neq 0$, the latter gives us that $\frac{1}{2} \leq \mathcal{L} \hat{\beta}_{k}$, i.e., $\beta_{k} \geq \frac{\theta}{2 \mathcal{L}}$ for all $k>K_{1}$.
Case 2. The sequence $\left(\beta_{k}\right)_{k \in \mathbb{N}}$ is not bounded below by a positive number $\beta$. Hence we may find a subsequence (no labeling) $\left(\beta_{k}\right)_{k \in \mathbb{N}}$ converging to 0 . It is similar to the proof of Case 2 in Theorem 5.2 that $\left(x^{k}\right)_{k \in \mathbb{N}}$ and $\left(J_{k}\right)_{k \in \mathbb{N}}$ are bounded. It follows that

$$
\lim _{k \rightarrow \infty}\left\|x^{k}-\hat{y}^{k}\right\|=\lim _{k \rightarrow \infty} \frac{\beta_{k}}{\theta}\left\|x^{k}-J_{k}\right\|=0
$$

Thus the sequence $\left(\hat{y}^{k}\right)_{k \in \mathbb{N}}$ is converging to $x_{*}$. Repeating the corresponding part in the proof of Case 1 above, we also have $\beta_{k} \geq \frac{\theta}{2 \mathcal{L}}$ for any large $k$, which is the contradiction.

From the analysis of both cases above, we find $K_{1}>0$ such that $\beta_{k} \geq \frac{\theta}{2 \mathcal{L}}$ if $\beta_{k}<1$ for any $k>K_{1}$. This means $\beta_{k} \geq \min \left\{1, \frac{\theta}{2 \mathcal{L}}\right\}$ for $k \geq K_{1}$. The proof is complete.

Let us complete the section by presenting a corresponding corollary to Corollary 4.5, which is easily derived from Theorem 5.3 and Proposition 5.4.

Corollary 5.5. Let $\left(x^{k}\right)_{k \in \mathbb{N}}$ be the sequence generated by Method 3. Suppose that $S_{*} \neq \emptyset$.
(i) If the gradient of $f$ is globally Lipschitz continuous on $\operatorname{dom} g$, then

$$
(f+g)\left(x^{k}\right)-\min _{x \in \mathcal{H}}(f+g)(x)=\mathcal{O}\left(k^{-1}\right) .
$$

(ii) If $\operatorname{dim} \mathcal{H}<+\infty$ and the gradient of $f$ is locally Lipschitz continuous on $S_{*}$, then we have

$$
(f+g)\left(x^{k}\right)-\min _{x \in \mathcal{H}}(f+g)(x)=o\left(k^{-1}\right) .
$$

## 6 Conclusions

In Hilbert spaces, it is well-known that convexity on both functions and global Lipschitz continuity on the gradient of $f$ are sufficient for providing convergence of the sequence generated by the forward-backward splitting methods for solving problem (1). However, the Lipschitz assumption is usually a restriction in many particular circumstances. In this work we dealt with weak convergence of the forward-backward splitting method for convex optimization problems by taking the advantage of the linesearches. This not only eliminates the serious drawback of estimating the Lipschitz constant to choose the stepsize in (2) but also establishes many complexity results without imposing the Lipschitz assumption. Our schemes through the linesearches provide rigorous and implementable ways of updating the iterates, which can be easily adapted for applications.

We hope that this study will serve as a basis for future research on other efficient variants of the forward-backward splitting iteration. In particular we find possibility to develop our methods to the descent coordinate gradient method 30 for solving structured convex optimization problems. Moreover, we discuss in separate papers the cases when $f$ or $g$ are nonconvex following the ideas exposed in [9] and even removing the differentiability of $f$ and adding dynamic choices of the stepsizes with conditional and deflected techniques combining the ideas in [6, 20, 26]. We are also looking to the incremental (sub)gradient method like [28] for problem (11), when $f$ is the sum of a large number of functions. An interesting project, suggested by a referee, that we are pursuing is to study possible complexity $o\left(k^{-2}\right)$ and the weak convergence of Method2 without assuming the global Lipschitz continuity on the gradient of the smooth function as in [2, 11].

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[^1]:    ${ }^{1}$ This important observation is pointed out from by one of the referees

