# The method of codifferential descent for convex and global piecewise affine optimization 

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#### Abstract

The class of nonsmooth codifferentiable functions was introduced by professor V.F. Demyanov in the late 1980s. He also proposed a method for minimizing these functions called the method of codifferential descent (MCD). However, until now almost no theoretical results on the performance of this method on particular classes of nonsmooth optimization problems were known. In the first part of the paper, we study the performance of the method of codifferential descent on a class of nonsmooth convex functions satisfying some regularity assumptions, which in the smooth case are reduced to the Lipschitz continuity of the gradient. We prove that in this case the MCD has the iteration complexity bound $\mathcal{O}(1 / \varepsilon)$. In the second part of the paper we obtain new global optimality conditions for piecewise affine functions in terms of codifferentials. With the use of these conditions we propose a modification of the MCD for minimizing piecewise affine functions (called the method of global codifferential descent) that does not use line search, and discards those "pieces" of the objective functions that are no longer useful for the optimization process. Then we prove that the MCD as well as its modification proposed in the article find a point of global minimum of a nonconvex piecewise affine function in a finite number of steps.


## 1 Introduction

An interesting approach to the analysis of nonsmooth functions based on the use of continuous approximations called codifferentials was proposed by Demyanov in [5, 6, 7]. He developed the codifferential calculus [9] (see [37, 12, 13] for its extensions and generalizations), and proposed a method for minimizing codifferentiable functions called the method of codifferential descent (MCD). This method was applied to some problems of cluster analysis [8, computational geometry 34, 33, calculus of variations 10, 11 and optimal control problems [17, 16. Hybrid methods for solving convex and DC (difference-of-convex) optimization problems combining the ideas of bundle methods and the MCD were proposed in [2, 3, 35]. A comprehensive convergence analysis of the MCD and some of its modifications was presented in the recent paper [14]. However, almost

[^0]nothing is known about the global performance of the MCD on particular classes of nonsmooth optimization problems apart from some results of numerical experiments.

The main goal of this article is to analyse the overall performance of the method of codifferential descent in two tractable cases. Namely, the first part of the paper is devoted to the analysis of this method in the convex case. Some of the most popular black-box methods of convex optimization are subgradient methods [27, 30, 23, 31, 36] and bundle methods [21, 26, 24, 22, 4]. However, these methods are relatively slow in the general case, since they require $\mathcal{O}\left(1 / \varepsilon^{2}\right)$ iterations to find an $\varepsilon$-optimal solution [27]. In the case when a certain information about the structure of the optimization problem under consideration is known, one can devise significantly faster methods (see, e.g., [28, 29]). In this article, we demonstrate that under some natural regularity assumptions the method of codifferential descent finds an $\varepsilon$-optimal solution in at most $\mathcal{O}(1 / \varepsilon)$ iterations, which is better than the iteration complexity bound for subgradient methods, despite the fact that the MCD is also a black-box method. On the other hand, it should be noted that the MCD utilises an oracle that provides significantly more information about the objective function than the one used by subgradient and bundle methods. Thus, in a sense, the MCD trades off the complexity of each call of the oracle for the better rate of convergence in comparison with subgradient methods.

The second part of the paper is devoted to the analysis of the method of codifferential descent in the piecewise affine case. As it was demonstrated via numerical simulation in [8], the MCD "jumps over" some points of local minimum of nonsmooth functions, and in some applications it is capable of finding a global minimizer of the objective function in spite of the fact that the MCD is a black-box local search method. To understand a reason behind this phenomenon we derive new global optimality conditions for piecewise affine functions in terms of codifferentials, which are significantly different from the ones obtained by Polyakova [32] or from the standard global optimality conditions for DC (difference-of-convex) optimization problems [19, 20]. It turns out that new conditions for global optimality are implicitly incorporated into the MCD. With the use of these conditions we propose a modification of the MCD for minimizing piecewise affine functions that, unlike the original method, does not use line search, and allows one to avoid unnecessary computations by discarding those "pieces" of the objective function that no longer provide useful information about the global behaviour of this function. Then we prove that the modified MCD as well as the MCD itself find a point of global minimum of a piecewise affine function in a finite number of steps, thus giving a first theoretical explanation for the ability of the MCD to find a globally optimal solution in some applications.

The paper is organized as follows. In Section 3 some new natural regularity assumptions on nonsmooth convex functions are introduced, and the performance of the MCD on the class of nonsmooth convex functions satisfying these assumptions is analysed. New necessary and sufficient global optimality conditions for piecewise affine functions in terms of codifferentials are obtained in Section 4. We utilise these conditions in order to propose a modification of the MCD , and to prove that this modification as well as the original method find a point of global minimum of a piecewise affine function in a finite number of steps. Finally, for reader's convenience, some basic definitions and results from
the codifferential calculus are given in Section 2.

## 2 Preliminaries

Let $\mathcal{H}$ be a real Hilbert space, and $U$ be a neighbourhood of a point $x \in \mathcal{H}$. Recall that a function $f: U \rightarrow \mathbb{R}$ is called codifferentiable at $x$, if there exist weakly compact convex sets $\underline{d} f(x), \bar{d} f(x) \subset \mathbb{R} \times \mathcal{H}$ such that for any $\Delta x \in \mathcal{H}$ one has

$$
\begin{aligned}
\left.\lim _{\alpha \rightarrow+0} \frac{1}{\alpha} \right\rvert\, f(x+\alpha \Delta x)-f(x) & -\max _{(a, v) \in \underline{d} f(x)}(a+\langle v, \Delta x\rangle) \\
& -\min _{(b, w) \in \bar{d} f(x)}(b+\langle w, \Delta x\rangle) \mid=0,
\end{aligned}
$$

and

$$
\begin{equation*}
\max _{(a, v) \in \underline{d} f(x)} a+\min _{(b, w) \in \bar{d} f(x)} b=0 . \tag{1}
\end{equation*}
$$

Here $\langle\cdot, \cdot\rangle$ is the inner product in $\mathcal{H}$, and we suppose that the space $\mathbb{R} \times \mathcal{H}$ is endowed with the norm $\|(a, v)\|^{2}=a^{2}+\|v\|_{\mathcal{H}}^{2}$ for any $(a, v) \in \mathbb{R} \times \mathcal{H}$. The pair $D f(x)=[\underline{d} f(x), \bar{d} f(x)]$ is called a codifferential of $f$ at $x$, the set $\underline{d} f(x)$ is called a hypodifferential of $f$ at $x$, while the set $\bar{d} f(x)$ is referred to as a hyperdifferential of $f$ at $x$. Let us note that the function $f$ is codifferentiable at $x$ if and only if its increment $f(x+\Delta x)-f(x)$ can be locally approximated by the difference of two convex functions, i.e. by a DC function (see [13, Example 3.10] for more details). Hence, in particular, any function that can be represented as the difference of convex functions is codifferentiable.

It is easy to see that a codifferential of $f$ at $x$ is not unique. Therefore, it seems natural to single out a codifferential of $f$ at $x$ that has some useful additional properties. At first, let us note that without loss of generality [9, 14] one can suppose that

$$
\begin{equation*}
\max _{(a, v) \in \underline{d} f(x)} a=\min _{(b, w) \in \bar{d} f(x)} b=0 \tag{2}
\end{equation*}
$$

(cf. (11)). At second, recall that $f$ is said to be continuously codifferentiable at $x$, if $f$ is codifferentiable at every point in a neighbourhood of $x$, and there exists a codifferential mapping $D f(\cdot)=[\underline{d} f(\cdot), \bar{d} f(\cdot)]$ defined in a neighbourhood of $x$, and such that the multifunctions $\underline{d} f(\cdot)$ and $\bar{d} f(\cdot)$ are Hausdorff continuous at $x$. This codifferential mapping $D \overline{f(\cdot)}$ is called continuous at $x$. Similarly, a function $f: \mathcal{H} \rightarrow \mathbb{R}$ is called continuously codifferentiable on a set $A \subset \mathcal{H}$, if $f$ is codifferentiable at every point $x \in A$, and the exists a continuous codifferential mapping $D f(\cdot)$ defined on $A$, i.e. a codifferential mapping $D f(\cdot)$ such that the corresponding multifunctions $\underline{d} f: A \rightrightarrows \mathbb{R} \times \mathcal{H}$ and $\bar{d} f: A \rightrightarrows \mathbb{R} \times \mathcal{H}$ are Hausdorff continuous on $A$. Let us note that the set of all those nonsmooth functions that are continuously codifferentiable on a given convex set $A$ is closed under all standard algebraic operations, the pointwise maximum and minimum of finite families of functions, as well as the composition with smooth functions. Furthermore, there exists simple and well-developed codifferential calculus [9, 13, 14].

One can check that if a function $f: U \rightarrow \mathbb{R}$ is codifferentiable at $x$, then $f$ is directionally differentiable at $x$, and the standard necessary condition for a

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Algorithm 1: The method of codifferential descent (MCD).
    Step 1. Choose \(\mu \geq 0\), a starting point \(x_{0} \in \mathcal{H}\), and set \(n:=0\).
    Step 2. Compute \(\underline{d} f\left(x_{n}\right)\) and \(\bar{d}_{\mu} f\left(x_{n}\right)\).
    Step 3. For any \(z=(b, w) \in \bar{d}_{\mu} f\left(x_{n}\right)\) compute \(\left(a_{n}(z), v_{n}(z)\right) \in \mathbb{R} \times \mathcal{H}\)
    by solving
\[
\min \|(a, v)\|^{2} \quad \text { s.t. }(a, v) \in \underline{d} f\left(x_{n}\right)+z
\]
```

Step 4. For any $z \in \bar{d}_{\mu} f\left(x_{n}\right)$ compute $\alpha_{n}(z) \geq 0$ by solving

$$
\min _{\alpha} f\left(x_{n}-\alpha v_{n}(z)\right) \quad \text { s.t. } \alpha \geq 0
$$

Step 5. Compute $z_{n} \in \bar{d}_{\mu} f\left(x_{n}\right)$ by solving

$$
\min _{z} f\left(x_{n}-\alpha_{n}(z) v_{n}(z)\right) \quad \text { s.t. } z \in \bar{d}_{\mu} f\left(x_{n}\right) .
$$

Set $x_{n+1}=x_{n}-\alpha_{n}\left(z_{n}\right) v_{n}\left(z_{n}\right), n:=n+1$, and go to Step 2 .
minimum $f^{\prime}(x, \cdot) \geq 0$ is satisfied if and only if

$$
\begin{equation*}
0 \in \underline{d} f(x)+\{(0, w)\} \quad \forall(0, w) \in \bar{d} f(x) \tag{3}
\end{equation*}
$$

(see [9, 14). Here $f^{\prime}(x, h)$ is the directional derivative of $f$ at $x$ in the direction $h$. A point $x$ satisfying optimality condition (3) is called an inf-stationary point of the function $f$. Note that the definition of inf-stationary point is independent of the choice of a codifferential, since the optimality condition $f^{\prime}(x, \cdot) \geq 0$ is invariant with respect to the choice of a codifferential.

One can utilise optimality condition (3) to design a numerical method for minimizing codifferentiable functions called the method of codifferential descent 9, 14]. Let a function $f: \mathcal{H} \rightarrow \mathbb{R}$ be codifferentiable (i.e. codifferentiable on $\mathcal{H}$ ), and $D f(\cdot)$ be its given codifferential mapping. For any $\mu \geq 0$ denote

$$
\bar{d}_{\mu} f(x)=\{(b, w) \in \bar{d} f(x) \mid b \leq \mu\}
$$

(cf. (22)). Let us note that in the definition of $\bar{d}_{\mu} f(x)$ it is sufficient to consider only extreme points $(b, w)$ of the hyperdifferential $\bar{d} f(x)$ (see [14]). A description of the original version of the method of codifferential descent (MCD) [9] is given in Algorithm 1

Note that in each iteration of the MCD one must perform line search in several directions (unless $\bar{d} f(\cdot) \equiv\{0\}$; see Step 4). One can verify that at least one of these directions is a descent direction of the function $f$, and $f\left(x_{n+1}\right)<$ $f\left(x_{n}\right)$ for all $n \in \mathbb{N} \cup\{0\}$. On the other hand, some of these directions might not be descent directions, i.e. the function $f$ may first increase and then decrease in these directions. This interesting feature of the MCD allows it to "jump over" some points of local minimum of the function $f$, provided the parameter $\mu>0$ is sufficiently large (see [8] for a particular example). However, no results on the convergence of the MCD to a global minimizer of the function $f$ are known. The main goal of this article is to shed some light on this problem. To this end, below we study the performance of the MCD in the case when $f$ is either convex

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Algorithm 2: The method of hypodifferential descent (MHD).
    Step 1 . Choose a starting point \(x_{0} \in \mathcal{H}, \sigma \in(0,1)\) and \(\gamma \in(0,1)\), and
    set \(n:=0\).
    Step 2. Compute \(\underline{d} f\left(x_{n}\right)\).
    Step 3 . Compute \(\left(a_{n}, v_{n}\right) \in \mathbb{R} \times \mathcal{H}\) by solving
\[
\min \|(a, v)\|^{2} \quad \text { s.t. }(a, v) \in \underline{d} f\left(x_{n}\right) .
\]
```

Step 4. Compute $k \in \mathbb{N} \cup\{0\}$ by solving

$$
\max _{k \in \mathbb{N} \cup\{0\}} \gamma^{k} \text { s.t. } f\left(x_{n}-\gamma^{k} v_{n}\right)-f\left(x_{n}\right) \leq-\gamma^{k} \sigma\left\|\left(a_{n}, v_{n}\right)\right\|^{2},
$$

and set $\alpha_{n}=\gamma^{k}$.
Step 5. Set $x_{n+1}=x_{n}-\alpha_{n} v_{n}, n:=n+1$, and go to Step 2.
or piecewise affine. For a comprehensive convergence analysis of the MCD and its modifications in the general case see [14].

## 3 The method of hypodifferential descent for convex optimization

In this section, we study the performance of the method of codifferential descent in the convex case. Let $f: \mathcal{H} \rightarrow \mathbb{R}$ be a convex function. As it was noted above, a function is codifferentiable if and only if its increment can be locally approximated by the difference of convex function (i.e. a DC function). If a codifferentiable function under consideration is convex, then it is natural to assume that its increment can be approximated by a convex function. In other words, it is natural to suppose that $f$ is hypodifferentiable, i.e. that there exists a codifferential mapping $D f(\cdot)$ such that $\bar{d} f(\cdot) \equiv\{0\}$. Furthermore, in this section we suppose that the function $f$ is continuously hypodifferentiable on $\mathcal{H}$, and consider only its continuous hypodifferential mapping $\underline{d} f(\cdot)$. Note that by (3) a point $x^{*}$ is a global minimizer of $f$ if and only if $0 \in \underline{d} f\left(x^{*}\right)$, since in the convex case $f^{\prime}\left(x^{*}, \cdot\right) \geq 0$ if and only if $x^{*}$ is a global minimizer of $f$.

When the MCD is applied to a hypodifferentiable convex function, one calls it the method of hypodifferential descent (MHD). Moreover, in the convex case one can utilise Armijo's step-size rule (cf. [14]). The scheme of the MHD for minimizing the function $f$ is given in Algorithm 2 ,

Let us note that by [14, Lemma 1] one has $f^{\prime}\left(x_{n},-v_{n}\right) \leq-\left\|\left(a_{n}, v_{n}\right)\right\|^{2}$. Hence by the definition of directional derivative for any sufficiently small $\alpha>0$ one has

$$
\begin{equation*}
f\left(x_{n}-\alpha v_{n}\right)-f\left(x_{n}\right) \leq \alpha \sigma f^{\prime}\left(x_{n},-v_{n}\right) \leq-\alpha \sigma\left\|\left(a_{n}, v_{n}\right)\right\|^{2} \tag{4}
\end{equation*}
$$

if $\left\|\left(a_{n}, v_{n}\right)\right\|>0$, i.e. $0 \notin \underline{d} f\left(x_{n}\right)$. Therefore, the step sizes $\alpha_{n}$ (see Step 4 of the MHD) are correctly defined, and $f\left(x_{n+1}\right)<f\left(x_{n}\right)$ for all $n \in \mathbb{N} \cup\{0\}$, provided $x_{n}$ is not a point of global minimum of the function $f$.

Our aim is to estimate a rate of convergence of the MHD for the function $f$. This problem is very complicated in the general case due to the nonuniqueness of hypodifferential mapping. A poor choice of a hypodifferential mapping might significantly slow down the convergence of the method. To overcome this difficulty we must assume that the chosen hypodifferential mapping somehow agrees with the convexity of the function $f$. The following definition provides a precise and natural formulation of this assumption.

Definition 3.1. Let $C \subseteq \mathcal{H}$ be a nonempty convex set. A hypodifferential mapping $\underline{d} f(\cdot)$ of the function $f$ is called amenable on $C$, if for any $x \in C$ and $(a, v) \in \underline{d} f(x)$ one has

$$
f(y)-f(x) \geq a+\langle v, y-x\rangle \quad \forall y \in C
$$

Clearly, if $f$ is continuously differentiable, then $\underline{d} f(\cdot)=\{(0, \nabla f(\cdot))\}$ is an amenable continuous hypodifferential mapping of the function $f$ on any convex set $C$, since

$$
f(y)-f(x) \geq\langle\nabla f(x), y-x\rangle \quad \forall x, y \in \mathcal{H}
$$

due to the convexity of the function $f$. Moreover, the amenability of hypodifferential mapping is preserved under addition and pointwise maximum.

Proposition 3.2. Let convex functions $f_{i}: \mathcal{H} \rightarrow \mathbb{R}$ be hypodifferentiable, and $\underline{d} f_{i}(\cdot)$ be their hypodifferential mappings that are amenable on a convex set $C \subseteq$ $\mathcal{H}, i \in I=\{1, \ldots, k\}$. Then

$$
\begin{equation*}
\underline{d} g(\cdot)=\sum_{i=1}^{k} \lambda_{i} \underline{d} f_{i}(\cdot) \tag{5}
\end{equation*}
$$

is a hypodifferential mapping of the function $g=\sum_{i=1}^{k} \lambda_{i} f_{i}$ that is amenable on $C$ (here $\lambda_{i} \geq 0$ ), and

$$
\begin{equation*}
\underline{d} u(\cdot)=\operatorname{co}\left\{\left(f_{i}(\cdot)-u(\cdot), 0\right)+\underline{d} f_{i}(\cdot) \mid 1 \leq i \leq k\right\} \tag{6}
\end{equation*}
$$

is a hypodifferential mapping of the function $u=\max _{i \in I} f_{i}$ that is amenable on $C$ as well.

Proof. Fix arbitrary $x, \Delta x \in \mathcal{H}$. By the definition of hypodifferentiable function for any $i \in I$ one has

$$
f_{i}(x+\alpha \Delta x)-f_{i}(x)=\max _{(a, v) \in \underline{d} f_{i}(x)}(a+\alpha\langle v, \Delta x\rangle)+o_{i}(\alpha),
$$

where $o_{i}(\alpha) / \alpha \rightarrow 0$ as $\alpha \rightarrow+0$. Hence

$$
\begin{align*}
g(x+\alpha \Delta x)-g(x) & =\sum_{i=1}^{k} \lambda_{i}\left(f_{i}(x+\alpha \Delta x)-f_{i}(x)\right) \\
& =\sum_{i=1}^{k} \lambda_{i} \max _{(a, v) \in \underline{d} f_{i}(x)}(a+\alpha\langle v, \Delta x\rangle)+\sum_{i=1}^{k} \lambda_{i} o_{i}(\alpha) . \tag{7}
\end{align*}
$$

Observe that

$$
\begin{equation*}
\sum_{i=1}^{k} \lambda_{i} \max _{(a, v) \in \underline{d} f_{i}(x)}(a+\alpha\langle v, \Delta x\rangle)=\max _{(a, v) \in \underline{d} g(x)}(a+\alpha\langle v, \Delta x\rangle), \tag{8}
\end{equation*}
$$

where $\underline{d} g(x)$ is defined as in (5). Consequently, (7) implies that

$$
\left|g(x+\alpha \Delta x)-g(x)-\max _{(a, v) \in \underline{d} g(x)}(a+\alpha\langle v, \Delta x\rangle)\right| \leq \sum_{i=1}^{k} \lambda_{i}\left|o_{i}(\alpha)\right| .
$$

Therefore $g$ is hypodifferentiable, and (5) is its hypodifferential mapping. Let us check that it is amenable on $C$. Indeed, fix $x, y \in C$ and $(a, v) \in \underline{d} g(x)$. By (5) there exists $\left(a_{i}, v_{i}\right) \in \underline{d} f_{i}(x)$ such that

$$
\begin{equation*}
(a, v)=\sum_{i=1}^{k} \lambda_{i}\left(a_{i}, v_{i}\right) . \tag{9}
\end{equation*}
$$

From the fact that the hypodifferentials $\underline{d} f_{i}(x)$ are amenable on $C$ it follows that

$$
f_{i}(y)-f_{i}(x) \geq a_{i}+\left\langle v_{i}, y-x\right\rangle \quad \forall i \in I
$$

Multiplying these inequalities by $\lambda_{i}$ and summing them up one obtains that
$g(y)-g(x)=\sum_{i=1}^{k}\left(\lambda_{i} f_{i}(y)-\lambda_{i} f_{i}(x)\right) \geq \sum_{i=1}^{k} \lambda_{i}\left(a_{i}+\left\langle v_{i}, y-x\right\rangle\right)=a+\langle v, y-x\rangle$,
where the last equality follows from (9). Thus, hypodifferential mapping (5) of the function $g$ is amenable.

Let us now turn to the function $u$. By the definition of hypodifferentiable function one has

$$
\begin{aligned}
u(x+\alpha \Delta x)-u(x) & =\max _{i \in I}\left(f_{i}(x+\alpha \Delta x)-u(x)\right) \\
& =\max _{i \in I}\left(f_{i}(x)-u(x)+\max _{(a, v) \in \underline{d} f_{i}(x)}(a+\alpha\langle v, \Delta x\rangle)+o_{i}(\alpha)\right)
\end{aligned}
$$

Consequently, taking into account the fact that

$$
\begin{equation*}
\max _{(a, v) \in \underline{d} u(x)}(a+\alpha\langle v, \Delta x\rangle)=\max _{i \in I}\left(f_{i}(x)-u(x)+\max _{(a, v) \in \underline{d} f_{i}(x)}(a+\alpha\langle v, \Delta x\rangle)\right) \tag{10}
\end{equation*}
$$

(here $\underline{d} u(x)$ is defined as in (6) ), and applying the inequality

$$
\begin{equation*}
\min _{i \in I} d_{i} \leq \max _{i \in I}\left(c_{i}+d_{i}\right)-\max _{i \in I} c_{i} \leq \max _{i \in I} d_{i} \tag{11}
\end{equation*}
$$

which is valid for any $c_{i}, d_{i} \in \mathbb{R}$, with $c_{i}=f_{i}(x)-u(x)+\max _{(a, v) \in \underline{d} f_{i}(x)}(a+$ $\alpha\langle v, \Delta x\rangle)$ and $d_{i}=o_{i}(\alpha)$ one obtains that

$$
\min _{i \in I} o_{i}(\alpha) \leq u(x+\alpha \Delta x)-u(x)-\max _{(a, v) \in \underline{d} u(x)}(a+\alpha\langle v, \Delta x\rangle) \leq \max _{i \in I} o_{i}(\alpha) .
$$

Hence with the use of the inequality $\min _{i \in I} d_{i} \geq-\max _{i \in I}\left|d_{i}\right|$ one gets

$$
\begin{equation*}
\left|u(x+\alpha \Delta x)-u(x)-\max _{(a, v) \in \underline{d} u(x)}(a+\alpha\langle v, \Delta x\rangle)\right| \leq \max _{i \in I}\left|o_{i}(\alpha)\right| \tag{12}
\end{equation*}
$$

which implies that the function $u$ is hypodifferentiable, and (6) is its hypodifferential mapping. Let us show that this mapping is amenable on $C$. Indeed, fix any $x, y \in C$ and $(a, v) \in \underline{d} u(x)$. By (6) there exist $\alpha_{i} \geq 0$ and $\left(a_{i}, v_{i}\right) \in \underline{d} f_{i}(x)$, $i \in I$ such that

$$
(a, v)=\sum_{i=1}^{k} \alpha_{i}\left(f_{i}(x)-u(x), 0\right)+\sum_{i=1}^{k} \alpha_{i}\left(a_{i}, v_{i}\right), \quad \sum_{i=1}^{k} \alpha_{i}=1
$$

With the use of the amenability of $\underline{d} f_{i}(x)$ on $C$ one gets that

$$
\begin{array}{r}
u(y)-u(x)=\max _{i \in I}\left(f_{i}(y)-u(x)\right) \geq \max _{i \in I}\left(f_{i}(x)-u(x)+a_{i}+\left\langle v_{i}, y-x\right\rangle\right) \\
\geq \sum_{i=1}^{k} \alpha_{i}\left(f_{i}(x)-u(x)+a_{i}+\left\langle v_{i}, y-x\right\rangle\right)=a+\langle v, y-x\rangle
\end{array}
$$

for any $y \in \mathcal{H}$, which implies the required result.
In the smooth case the rate of convergence of gradient methods for convex minimization is typically estimated under the assumption that the gradient of the objective function is globally Lipschitz continuous (cf. [27]). Therefore, it is natural to expect that in order to estimate the rate of convergence of the MHD in the nonsmooth case we have to utilise a generalization of this assumption.

Recall that if a function $f$ is differentiable, and its gradient is globally Lipschitz continuous with Lipschitz constant $L$, then

$$
|f(y)-f(x)-\langle\nabla f(x), y-x\rangle| \leq \frac{L}{2}\|y-x\|^{2} \quad \forall x, y \in \mathcal{H}
$$

(see, e.g., [27, Lemma 1.2.3]). We use this inequality as a basis for the generalization of the Lipschitz continuity assumption to the nonsmooth case.

Definition 3.3. Let $C \subseteq \mathcal{H}$ be a nonempty set. One says that a hypodifferential mapping $\underline{d} f(\cdot)$ is a Lipschitzian approximation of the function $f$ on the set $C$ with Lipschitz constant $L>0$, if

$$
\left|f(y)-f(x)-\max _{(a, v) \in \underline{d} f(x)}(a+\langle v, y-x\rangle)\right| \leq \frac{L}{2}\|y-x\|^{2} .
$$

for all $x, y \in C$.
From the proof of Proposition 3.2, it follows that the property of being a Lipschitzian approximation is preserved under addition and pointwise maximum (see (7) and (121). Namely, the following result holds true.

Proposition 3.4. Let convex functions $f_{i}: \mathcal{H} \rightarrow \mathbb{R}$ be hypodifferentiable, and $\underline{d} f_{i}(\cdot)$ be their hypodifferential mappings, $i \in I=\{1, \ldots, k\}$. Suppose that for any $i \in I$ the mapping $\underline{d} f_{i}(\cdot)$ is a Lipschitzian approximation of the function $f_{i}$ on a set $C \subseteq \mathcal{H}$ with Lipschitz constant $L_{i}>0$. Then the hypodifferential mapping (5) is a Lipschitzian approximation of the function $g=\sum_{i=1}^{k} \lambda_{i} f_{i}$ on the set $C$ with Lipschitz constant $L \leq \sum_{i=1}^{k}\left|\lambda_{i}\right| L_{i}$ (here $\lambda_{i} \in \mathbb{R}$ ), and (6) is a Lipschitzian approximation of the function $u=\max _{1 \leq i \leq k} f_{i}$ on the set $C$ with Lipschitz constant $L \leq \max _{1 \leq i \leq k} L_{i}$.

Proof. Fix any $x, y \in C$. With the use of (8) and Def. 3.3 one obtains that

$$
\begin{aligned}
\mid g(y)-g(x) & -\max _{(a, v) \in \underline{d} g(x)}(a+\langle v, y-x\rangle) \mid \\
& \leq \sum_{i=1}^{k}\left|\lambda_{i}\right| \mid f_{i}(y)-f_{i}(x)-\max _{(a, v) \in \underline{d} f_{i}(x)}(a+\langle v, y-x\rangle \mid \\
& \leq \sum_{i=1}^{k}\left|\lambda_{i}\right| \frac{L_{i}}{2}\|y-x\|^{2} .
\end{aligned}
$$

Therefore the hypodifferential mapping (5) is a Lipschitzian approximation of the function $g=\sum_{i=1}^{k} \lambda_{i} f_{i}$ on the set $C$ with Lipschitz constant $L \leq$ $\sum_{i=1}^{k}\left|\lambda_{i}\right| L_{i}$.

To prove the assertion for the function $u$, denote

$$
\begin{equation*}
\omega_{i}(y, x)=f_{i}(y)-f_{i}(x)-\max _{(a, v) \in \underline{d} f_{i}(x)}(a+\langle v, y-x\rangle) . \tag{13}
\end{equation*}
$$

By definition one has

$$
\begin{aligned}
u(y)-u(x) & =\max _{i \in I}\left(f_{i}(y)-u(x)\right) \\
& =\max _{i \in I}\left(f_{i}(x)-u(x)+\max _{(a, v) \in \underline{d} f_{i}(x)}(a+\langle v, y-x\rangle)+\omega_{i}(y, x)\right) .
\end{aligned}
$$

Subtracting $\max _{(a, v) \in \underline{d} u(x)}(a+\langle v, y-x\rangle)$ (see (10)), and applying inequality (11) with $c_{i}=f_{i}(x)-u(x)+\max _{(a, v) \in \underline{d} f_{i}(x)}(a+\langle v, y-x\rangle)$, and $d_{i}=\omega_{i}(y, x)$ one obtains that

$$
\min _{i \in I} \omega_{i}(y, x) \leq u(y)-u(x)-\max _{(a, v) \in \underline{d} u(x)}(a+\langle v, y-x\rangle) \leq \max _{i \in I} \omega_{i}(y, x)
$$

Therefore

$$
\left|u(y)-u(x)-\max _{(a, v) \in \underline{d} u(x)}(a+\langle v, y-x\rangle)\right| \leq \max _{i \in I}\left|\omega_{i}(y, x)\right| .
$$

From (13) and the fact that $\underline{d} f_{i}(\cdot)$ is a Lipschitzian approximation of the function $f_{i}$ on $C$ with Lipschitz constant $L_{i}$ it follows that

$$
\left|\omega_{i}(y, x)\right| \leq \frac{L_{i}}{2}\|y-x\|^{2}, \quad \max _{i \in I}\left|\omega_{i}(y, x)\right| \leq \frac{\max _{i \in I} L_{i}}{2}\|y-x\|^{2}
$$

which implies the required result.
Now, we can obtain an upper estimate of the rate of convergence of the MHD that coincides with the upper estimate of the rate of convergence of the standard gradient method in the convex case (see, e.g., [27, Theorem 2.1.14]). This result is not surprising since in the smooth case the MHD is reduced to the gradient method with Armijo's step-size rule. Let us note that the proof of the following theorem is a straightforward modification of the proof of the corresponding result for gradient methods to the nonsmooth case.

Theorem 3.5. Let $f$ be a closed convex function, the set

$$
S_{0}=\left\{x \in \mathcal{H} \mid f(x) \leq f\left(x_{0}\right)\right\}
$$

be bounded, and let the continuous hypodifferential mapping $\underline{d} f(\cdot)$ be amenable and bounded on the set $S_{0}$. Suppose also that $\underline{d} f(\cdot)$ is a Lipschitzian approximation of $f$ on the set $S_{\varepsilon}=\left\{x \in \mathcal{H} \mid \operatorname{dist}\left(x, S_{0}\right) \leq \varepsilon\right\}$ for some $\varepsilon>0$, and a sequence $\left\{x_{n}\right\}$ is generated by the MHD. Then there exists $\widehat{\alpha}>0$ such that $\alpha_{n} \geq \widehat{\alpha}$ for all $n \in \mathbb{N}$, and the following inequality holds true:

$$
\begin{equation*}
f\left(x_{n}\right)-f\left(x^{*}\right) \leq \frac{\left(f\left(x_{0}\right)-f\left(x^{*}\right)\right) R^{2}}{R^{2}+\left(f\left(x_{0}\right)-f\left(x^{*}\right)\right) \widehat{\alpha} \sigma n}=\mathcal{O}\left(\frac{1}{n}\right) \quad \forall n \in \mathbb{N} \tag{14}
\end{equation*}
$$

Here $x^{*}$ is a point of global minimum of $f$, and $R=1+\sup _{n \geq 0}\left\|x_{n}-x^{*}\right\|<+\infty$.
Proof. At first, let us note that $f$ attains a global minimum by [15, Prop. II.1.2], since $f$ is closed, the set $S_{0}$ is bounded, and $\mathcal{H}$ is a Hilbert space. Note also that $R=1+\sup _{n \geq 0}\left\|x_{n}-x^{*}\right\|$ is finite due to the facts that $\left\{x_{n}\right\} \subset S_{0}$, and $S_{0}$ is bounded (the validity of the inclusion follows from the inequality $f\left(x_{n+1}\right)<f\left(x_{n}\right)$; see (4) ).

Denote $\Phi_{n}(y)=\max _{(a, v) \in \underline{d} f\left(x_{n}\right)}(a+\langle v, y\rangle)$. Applying the necessary and sufficient condition for a minimum of a convex function on a convex set [15, Proposition II.2.1] one obtains that

$$
\begin{equation*}
a_{n} a+\left\langle v_{n}, v\right\rangle \geq\left\|\left(a_{n}, v_{n}\right)\right\|^{2} \quad \forall(a, v) \in \underline{d} f\left(x_{n}\right), \tag{15}
\end{equation*}
$$

where the pair $\left(a_{n}, v_{n}\right)$ is computed on Step 3 of the MHD. If $a_{n}=0$, then taking into account the fact that $a \leq 0$ for all $(a, v) \in \underline{d} f\left(x_{n}\right)$ (see (2)) one gets that

$$
\Phi_{n}\left(-v_{n}\right) \leq \max _{(a, v) \in \underline{d} f\left(x_{n}\right)}\left\langle v,-v_{n}\right\rangle \leq-\left\|\left(a_{n}, v_{n}\right)\right\|^{2}
$$

which with the use of the convexity of $\Phi_{n}$ and the equality $\Phi_{n}(0)=0$ (see (2)) implies that

$$
\begin{equation*}
\Phi_{n}\left(-\alpha v_{n}\right) \leq \alpha \Phi_{n}\left(-v_{n}\right)+(1-\alpha) \Phi_{n}(0) \leq-\alpha\left\|\left(a_{n}, v_{n}\right)\right\|^{2} \quad \forall \alpha \in[0,1] \tag{16}
\end{equation*}
$$

On the other hand, if $a_{n}<0$, then dividing (15) by $a_{n}$, and taking the maximum over all $(a, v) \in \underline{d} f\left(x_{n}\right)$ one obtains

$$
\Phi_{n}\left(\frac{1}{a_{n}} v_{n}\right) \leq-\frac{1}{\left|a_{n}\right|}\left\|\left(a_{n}, v_{n}\right)\right\|^{2}
$$

Applying the convexity of $\Phi_{n}$ and the equality $\Phi_{n}(0)=0$ again one obtains that

$$
\Phi_{n}\left(\frac{\alpha}{a_{n}} v_{n}\right) \leq \alpha \Phi_{n}\left(\frac{1}{a_{n}} v_{n}\right) \leq-\frac{\alpha}{\left|a_{n}\right|}\left\|\left(a_{n}, v_{n}\right)\right\|^{2} \quad \forall \alpha \in[0,1]
$$

Combining this inequality with (16) one gets that in either case

$$
\begin{equation*}
\Phi_{n}\left(-\alpha v_{n}\right) \leq-\alpha\left\|\left(a_{n}, v_{n}\right)\right\|^{2} \quad \forall \alpha \in\left[0, \frac{1}{\left|a_{n}\right|}\right] \tag{17}
\end{equation*}
$$

where $1 / 0=1$ by definition. Observe that the sequence $\left\{a_{n}\right\}$ is bounded by virtue of the facts that $\left\{x_{n}\right\} \subset S_{0}$, and the hypodifferential mapping $\underline{d} f(\cdot)$ is
bounded on $S_{0}$. Therefore, there exists $\varkappa \in(0,1]$ such that $\left|a_{n}\right|^{-1}>\varkappa>0$ for all $n \in \mathbb{N}$. Furthermore, from the boundedness of $\underline{d} f(\cdot)$ on $S_{0}$ it follows that there exists $K>0$ such that $\left\|v_{n}\right\| \leq K$ for all $n \in \mathbb{N}$. Hence, in particular, $x_{n}-\alpha v_{n} \in S_{\varepsilon}=\left\{x \in \mathcal{H} \mid \operatorname{dist}\left(x, S_{0}\right) \leq \varepsilon\right\}$ for any $\alpha \in[0, \varepsilon / K]$ and $n \in \mathbb{N}$.

Recall that $\underline{d} f(\cdot)$ is a Lipschitzian approximation of $f$ on $S_{\varepsilon}$. Therefore there exists $L>0$ such that

$$
f\left(x_{n}-\alpha v_{n}\right)-f\left(x_{n}\right)-\Phi_{n}\left(-\alpha v_{n}\right) \leq \frac{L \alpha^{2}}{2}\left\|v_{n}\right\|^{2} \quad \forall \alpha \in\left[0, \frac{\varepsilon}{K}\right]
$$

Hence and from (17) it follows that

$$
f\left(x_{n}-\alpha v_{n}\right)-f\left(x_{n}\right) \leq\left(-\alpha+\frac{L \alpha^{2}}{2}\right)\left\|\left(a_{n}, v_{n}\right)\right\|^{2} \quad \forall \alpha \in\left[0, \min \left\{\varkappa, \frac{\varepsilon}{K}\right\}\right] .
$$

Consequently, as it is easy to see, there exists $\widehat{\alpha}>0$ such that

$$
f\left(x_{n}-\widehat{\alpha} v_{n}\right)-f\left(x_{n}\right) \leq-\widehat{\alpha} \sigma\left\|\left(a_{n}, v_{n}\right)\right\|^{2} \quad \forall n \in \mathbb{N}
$$

(one can choose any $\widehat{\alpha} \leq \min \{2(1-\sigma) / L, \varkappa, \varepsilon / K\}$ ), which implies that

$$
\begin{equation*}
f\left(x_{n+1}\right)-f\left(x_{n}\right) \leq-\widehat{\alpha} \sigma\left\|\left(a_{n}, v_{n}\right)\right\|^{2}, \quad \alpha_{n} \geq \widehat{\alpha} \quad \forall n \in \mathbb{N} \tag{18}
\end{equation*}
$$

where $x_{n+1}=x_{n}-\alpha_{n} v_{n}$, and $\alpha_{n}$ is computed on Step 4 of the MHD. Note that one can set $\widehat{\alpha}=\gamma^{k}$ for a sufficiently large $k \in \mathbb{N}$. Then $\alpha_{n}=\gamma^{k_{n}}$ with $k_{n} \leq k$.

Denote $\Delta_{n}=f\left(x_{n}\right)-f\left(x^{*}\right)$, where $x^{*}$ is a point of global minimum of the function $f$. From the facts that the hypodifferential mapping $\underline{d} f(\cdot)$ is amenable, and $\left(a_{n}, v_{n}\right) \in \underline{d} f\left(x_{n}\right)$ (see Step 3 of the MHD) it follows that

$$
\Delta_{n} \leq-a_{n}+\left\langle v_{n}, x_{n}-x^{*}\right\rangle \leq\left\|\left(a_{n}, v_{n}\right)\right\|\left(1+\left\|x_{n}-x^{*}\right\|\right) \leq R\left\|\left(a_{n}, v_{n}\right)\right\|
$$

(recall that $R=1+\sup _{n>0}\left\|x_{n}-x^{*}\right\|$ ). Adding and subtracting $f\left(x^{*}\right)$ in (18), and estimating $\left\|\left(a_{n}, v_{n}\right)\right\|^{2}$ with the use of the inequality above one gets that

$$
\Delta_{n+1} \leq \Delta_{n}-\frac{\widehat{\alpha} \sigma}{R^{2}} \Delta_{n}^{2}
$$

Dividing this inequality by $\Delta_{n} \cdot \Delta_{n+1}$ one obtains

$$
\frac{1}{\Delta_{n+1}} \geq \frac{1}{\Delta_{n}}+\frac{\widehat{\alpha} \sigma}{R^{2}} \frac{\Delta_{n}}{\Delta_{n+1}} \geq \frac{1}{\Delta_{n}}+\frac{\widehat{\alpha} \sigma}{R^{2}}
$$

(note that $\Delta_{n+1} \leq \Delta_{n}$ due to the fact that $f\left(x_{n+1}\right) \leq f\left(x_{n}\right)$ ). Summing up these inequalities one gets

$$
\frac{1}{\Delta_{n+1}} \geq \frac{1}{\Delta_{0}}+\frac{\widehat{\alpha} \sigma}{R^{2}}(n+1) \quad \forall n \in \mathbb{N}
$$

which implies that (14) is valid.
Remark 3.1. Let us point out how $\widehat{\alpha}$ from the theorem above depends on the problem data. Let $K>0$ be such that $|a| \leq K$ and $\|v\| \leq K$ for all $(a, v) \in \underline{d} f(x)$ and $x \in S_{0}$. Then, as it was pointed out in the proof, one can set

$$
\widehat{\alpha}=\min \left\{\frac{\min \{1, \varepsilon\}}{K}, \frac{2(1-\sigma)}{L}\right\} .
$$

Furthermore, if $\varepsilon=+\infty$, then it is sufficient to suppose that $K>0$ is such that $|a| \leq K$ for any $(a, v) \in \underline{d} f(x)$ and $x \in S_{0}$. Note that in the smooth case one can define $\underline{d} f(\cdot)=\{(0, \nabla f(\cdot))\}$, which implies that $\widehat{\alpha}=2(1-\sigma) / L$, provided the gradient $\nabla f(\cdot)$ is globally Lipschitz continuous. Observe also that the theorem above remains valid in the case when instead of Armijo's step-size rule one finds $\alpha_{n}$ via the minimization of the function $\alpha \mapsto f\left(x_{n}-\alpha v_{n}\right)$.
Remark 3.2. Note that the rate of convergence of the MHD is better than the optimal rate of convergence of subgradient methods $\mathcal{O}(1 / \sqrt{n})$ [27, Sect. 3.2]. This is obviously due to the fact the oracle utilised by the MHD provides much more information about the objective function than just a single subgradient. On the other hand, each call of this oracle is significantly more expensive than the call of the oracle used in subgradient methods. Let us also note that one can utilise Nesterov's acceleration technique [27, Sect. 2.2] to design a faster method for minimizing hypodifferentiable convex functions than the MHD. However, this method must accumulate the Minkowski sum of the form $a_{1} \underline{d} f\left(y_{1}\right)+a_{2} \underline{d} f\left(y_{2}\right)+\ldots$ with some $a_{i} \in \mathbb{R}$ (cf. the optimal gradient method in [27]), which is unreasonable both in terms of memory consumption and computational effort. That is why we do not present an accelerated version of the MHD here.
Remark 3.3. Let $U \subset \mathcal{H}$ be a bounded open set such that $f$ is Lipschitz continuous on $U$. By [14, Example 4] for any $x \in U$ one has $f(x)=\max _{(a, v) \in C}(a+$ $\langle v, x\rangle$ ), where

$$
\begin{equation*}
C=\{(f(z)-\langle v, z\rangle, v) \in \mathbb{R} \times \mathcal{H} \mid v \in \partial f(z), z \in U\} \tag{19}
\end{equation*}
$$

and $\partial f(z)$ is the subdifferential of $f$ at $z$ in the sense of convex analysis. Therefore, for any $x, y \in U$ one has
$f(y)-f(x)=\max _{(a, v) \in C}(a+\langle v, y\rangle)-f(x)=\max _{(a, v) \in C}(a-f(x)+\langle v, x\rangle+\langle v, y-x\rangle)$
or, equivalently, $f(y)-f(x)=\max _{(a, v) \in \underline{d} f(x)}(a+\langle v, y-x\rangle)$, where

$$
\begin{equation*}
\underline{d} f(x)=\operatorname{cl} \operatorname{co}\{(f(z)-f(x)-\langle v, z-x\rangle, v) \in \mathbb{R} \times \mathcal{H} \mid v \in \partial f(z), z \in U\} \tag{20}
\end{equation*}
$$

(see (19)). Applying the fact that $f$ is Lipschitz continuous on $U$ one can verify that the multifunction $\underline{d} f(\cdot)$ is Hausdorff continuous and bounded on $U$. Note that this hypodifferential mapping is obviously a Lipschitzian approximation of $f$ on $U$. Furthermore, observe that from the inequality $f(y)-f(z) \geq\langle v, y-z\rangle$, where $y, z \in U$ and $v \in \partial f(z)$, it follows that

$$
f(y)-f(x) \geq f(z)-f(x)-\langle v, z-x\rangle+\langle v, y-x\rangle \quad \forall x, y, z \in U \forall v \in \partial f(z)
$$

With the use of this inequality and (20) one can check that the hypodifferential mapping (20) is amenable on $U$. Thus, if the sublevel set $S_{0}=\{x \in \mathcal{H} \mid f(x) \leq$ $\left.f\left(x_{0}\right)\right\}$ is bounded, and the exists $\varepsilon>0$ such that $f$ is Lipschitz continuous on $S_{\varepsilon}$, then there exists a hypodifferential mapping of $f$ (of the form (20)) satisfying the assumptions of Theorem 3.5. In particular, if $\mathcal{H}$ is finite dimensional, then the boundedness of the sublevel set $S_{0}$ guarantees that there exists a hypodifferential mapping of the function $f$ satisfying the assumptions of Theorem 3.5. Thus, at least from the theoretical point of view the assumptions of this theorem are not very restrictive.

## 4 Codifferential calculus and global piecewise affine optimization

The main goal of this section is to demonstrate that the method of codifferential descent finds a point of global minimum of a nonconvex piecewise affine function in a finite number of steps. To this end, we derive new necessary and sufficient conditions for a global minimum of a piecewise affine functions in terms of its codifferential, which significantly differ from the ones obtained in 32, and develop a modification of the MCD call the method of global codifferential descent.

### 4.1 Global codifferential and optimality conditions

From this point onwards we suppose that $\mathcal{H}=\mathbb{R}^{d}$, and write $\mathbb{R}^{d+1}$ instead of $\mathbb{R} \times \mathbb{R}^{d}$. We start with an auxiliary result for polyhedral convex functions.

Lemma 4.1. Let a function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ have the form $f(x)=\max _{i \in I}\left(a_{i}+\right.$ $\left\langle v_{i}, x\right\rangle$ ) for some $\left(a_{i}, v_{i}\right) \in \mathbb{R}^{d+1}$, where $I=\{1, \ldots, k\}$. Then $f(x) \geq 0$ for all $x \in \mathbb{R}^{d}$ if and only if either $0 \in C=\operatorname{co}\left\{\left(a_{i}, v_{i}\right) \mid i \in I\right\}$ or $f$ is bounded below and $a_{0}>0$, where

$$
\begin{equation*}
\left\{\left(a_{0}, v_{0}\right)\right\}=\arg \min \left\{\|(a, v)\|^{2} \mid(a, v) \in C\right\} . \tag{21}
\end{equation*}
$$

Proof. Let $f(x) \geq 0$ for all $x \in \mathbb{R}^{d}$. Arguing by reductio ad absurdum, suppose that $0 \notin C$, but $a_{0} \leq 0$. Applying the necessary and sufficient condition for a minimum of a convex function on a convex set [15, Proposition II.2.1] one obtains that

$$
\begin{equation*}
a_{0}\left(a-a_{0}\right)+\left\langle v_{0}, v-v_{0}\right\rangle \geq 0 \quad \forall(a, v) \in C . \tag{22}
\end{equation*}
$$

If $a_{0}=0$, then $v_{0} \neq 0$ (otherwise $0 \in C$ ), and $\left\langle v,-v_{0}\right\rangle \leq-\left\|v_{0}\right\|^{2}$ for any $(a, v) \in C$. Therefore, for all $\alpha \geq 0$ one has

$$
\begin{equation*}
f\left(-\alpha v_{0}\right)=\max _{i \in I}\left(a_{i}+\left\langle v_{i},-\alpha v_{0}\right\rangle\right) \leq \max _{i \in I}\left(a_{i}-\alpha\left\|v_{0}\right\|^{2}\right)=f(0)-\alpha\left\|v_{0}\right\|^{2} \tag{23}
\end{equation*}
$$

which contradicts the assumption that $f$ is nonnegative.
If $a_{0}<0$, then dividing (22) by $a_{0}$ one obtains that

$$
a+\left\langle v, \frac{1}{a_{0}} v_{0}\right\rangle \leq-\frac{1}{\left|a_{0}\right|}\left\|\left(a_{0}, v_{0}\right)\right\|^{2}<0 \quad \forall(a, v) \in C
$$

Taking the maximum over all $(a, v) \in C$ one gets that $f\left(a_{0}^{-1} v_{0}\right)<0$, which is impossible. Thus, $a_{0}>0$.

Let us prove the converse statement. If $0 \in C$, then for any $x \in \mathbb{R}^{d}$ one has $f(x)=\max _{(a, v) \in C}(a+\langle v, x\rangle) \geq 0+\langle 0, x\rangle=0$, i.e. the function $f$ is nonnegative. Arguing by reductio ad absurdum suppose now that $0 \notin C, f$ is bounded below, and $a_{0}>0$, but there exists $x \in \mathbb{R}^{d}$ such that $f(x)<0$.

Define $f^{*}=\inf _{x \in \mathbb{R}^{d}} f(x)$. By our assumptions $-\infty<f^{*}<0$. Our aim is to show that $\left(f^{*}, 0\right) \in C$. Then for any $\alpha \in[0,1]$ one has $(1-\alpha)\left(f^{*}, 0\right)+$ $\alpha\left(a_{0}, v_{0}\right) \in C$. Setting $\alpha=\left|f^{*}\right| /\left(\left|f^{*}\right|+a_{0}\right) \in(0,1)$ one gets that $\left(0, \alpha v_{0}\right) \in C$, which is impossible due to the definition of $\left(a_{0}, v_{0}\right)$ (see (21)), and the fact that $\left\|\left(0, \alpha v_{0}\right)\right\|^{2}<\left\|\left(a_{0}, v_{0}\right)\right\|^{2}$.

For any $\varepsilon>0$ there exists $x_{\varepsilon}$ such that $f\left(x_{\varepsilon}\right)<f^{*}+\varepsilon$. Hence by definition $0 \in \partial_{\varepsilon} f\left(x_{\varepsilon}\right)$, where $\partial_{\varepsilon} f\left(x_{\varepsilon}\right)$ is the $\varepsilon$-subdifferential of $f$ at $x_{\varepsilon}$. By [21, Example XI.3.5.3] one has

$$
\partial_{\varepsilon} f(x)=\left\{v \in \mathbb{R}^{d} \mid \exists(a, v) \in C: a+\langle v, x\rangle \geq f(x)-\varepsilon\right\} .
$$

Consequently, for any $\varepsilon>0$ there exists $a_{\varepsilon} \geq f\left(x_{\varepsilon}\right)-\varepsilon \geq f^{*}-\varepsilon$ such that $\left(a_{\varepsilon}, 0\right) \in C$. Observe that for any $(a, 0) \in C$ one has $f(x) \geq a$ for all $x \in \mathbb{R}^{d}$, which implies that $f^{*} \geq a$. Thus, $f^{*} \geq a_{\varepsilon} \geq f^{*}-\varepsilon$. Hence passing to the limit as $\varepsilon \rightarrow 0$, and taking into account the fact that the set $C$ is closed one obtains that $\left(f^{*}, 0\right) \in C$.

Corollary 4.2. Let all assumptions of Lemma 4.1 be valid, and suppose that $f$ is bounded below. Then $f(x) \geq 0$ for all $x \in \mathbb{R}^{d}$ if and only if $a_{0} \geq 0$.

Proof. If $f$ is nonnegative, then by Lemma 4.1 either $a_{0}>0$ or $0 \in C$. In the latter case, by definition one has $\left(a_{0}, v_{0}\right)=(0,0)$, i.e. $a_{0}=0$.

Suppose now that $a_{0} \geq 0$. If $a_{0}>0$, then $f$ is nonnegative by Lemma 4.1. Therefore, suppose that $a_{0}=0$. If $v_{0}=0$, then $0 \in C$ and, once again, $f$ is nonnegative by Lemma 4.1. On the other hand, if $v_{0} \neq 0$, then, as it was shown in the proof of the lemma (see (23)), $f$ is unbounded below, which contradicts our assumptions.

Remark 4.1. Let us note that the assumption on the boundedness below of the function $f$ cannot be discarded from Lemma 4.1 A simple counterexample is the function $f(x)=a+\langle v, x\rangle$ with $a>0$ and $v \neq 0$.

Now we turn to the study of piecewise affine functions. At first, let us recall the definition of piecewise affine function [25, 18]. A convex set $Q \subset \mathbb{R}^{d}$ is referred to as polyhedral, if it can be represented as the intersection of a finite family of closed halfspaces. A finite family $\sigma=\left\{Q_{1}, \ldots, Q_{k}\right\}, k \in \mathbb{N}$, of polyhedral sets is said to be a polyhedral partition of $\mathbb{R}^{d}$, if $\mathbb{R}^{d}=\cup_{i=1}^{k} Q_{i}$, int $Q_{i} \neq \emptyset$ for $1 \leq i \leq k$, and the interiors of the sets $Q_{i}$ are mutually disjoint. Finally, a function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is called piecewise affine, if there exists a polyhedral partition $\sigma=\left\{Q_{1}, \ldots, Q_{k}\right\}$ of $\mathbb{R}^{d}$ such that the restriction of $f$ to each $Q_{i}$ is an affine function.

Let $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ be a piecewise affine function. Then by [18, Theorem 3.1], there exist $\left(a_{i}, v_{i}\right) \in \mathbb{R}^{d+1}, i \in I=\{1, \ldots, l\}$, and $\left(b_{j}, w_{j}\right) \in \mathbb{R}^{d+1}, j \in J=$ $\{1, \ldots, s\}$, such that

$$
\begin{equation*}
f(x)=\max _{i \in I}\left(a_{i}+\left\langle v_{i}, x\right\rangle\right)+\min _{j \in J}\left(b_{j}+\left\langle w_{j}, x\right\rangle\right) \quad \forall x \in \mathbb{R}^{d} \tag{24}
\end{equation*}
$$

Define

$$
\begin{equation*}
\underline{f}(x)=\max _{i \in I}\left(a_{i}+\left\langle v_{i}, x\right\rangle\right), \quad \bar{f}(x)=\min _{j \in J}\left(b_{j}+\left\langle w_{j}, x\right\rangle\right) \tag{25}
\end{equation*}
$$

Then $f=\underline{f}-(-\bar{f})$ is a DC decomposition of the function $f$ (i.e. $f=\underline{f}-(-\bar{f})$ is a representation of the function $f$ as the difference of convex functions). Introduce the set-valued mappings

$$
\begin{align*}
& \underline{d} f(x)=\operatorname{co}\left\{\left(a_{i}-\underline{f}(x)+\left\langle v_{i}, x\right\rangle, v_{i}\right) \in \mathbb{R}^{d+1} \mid i \in I\right\}  \tag{26}\\
& \bar{d} f(x)=\operatorname{co}\left\{\left(b_{j}-\bar{f}(x)+\left\langle w_{j}, x\right\rangle, w_{j}\right) \in \mathbb{R}^{d+1} \mid j \in J\right\}
\end{align*}
$$

Then, as it is easy to see, for any $x, \Delta x \in \mathbb{R}^{d+1}$ one has

$$
\begin{align*}
f(x & +\Delta x)-f(x)=\underline{f}(x+\Delta x)+\bar{f}(x+\Delta x)-(\underline{f}(x)+\bar{f}(x)) \\
& =\max _{i \in I}\left(a_{i}+\left\langle v_{i}, x+\Delta x\right\rangle\right)+\min _{j \in J}\left(b_{j}+\left\langle w_{j}, x+\Delta x\right\rangle\right)-(\underline{f}(x)+\bar{f}(x)) \\
& =\max _{i \in I}\left(a_{i}-\underline{f}(x)+\left\langle v_{i}, x\right\rangle+\left\langle v_{i}, \Delta x\right\rangle\right) \\
& +\min _{j \in J}\left(b_{j}-\bar{f}(x)+\left\langle w_{j}, x\right\rangle+\left\langle w_{j}, \Delta x\right\rangle\right) \\
& =\max _{(a, v) \in \underline{d f}(x)}(a+\langle v, \Delta x\rangle)+\min _{(b, w) \in \bar{d} f(x)}(b+\langle w, \Delta x\rangle) . \tag{27}
\end{align*}
$$

Furthermore, for any $x \in \mathbb{R}^{d}$ one has

$$
\begin{equation*}
\max _{(a, v) \in \underline{d} f(x)} a=\max _{i \in I}\left(a_{i}+\left\langle v_{i}, x\right\rangle\right)-\underline{f}(x)=\underline{f}(x)-\underline{f}(x)=0, \tag{28}
\end{equation*}
$$

and, similarly, $\min _{(b, w) \in \bar{d} f(x)} b=0$. Thus, the pair $D f(x)=[\underline{d} f(x), \bar{d} f(x)]$ is a codifferential of $f$ at $x$. On the other hand, codifferential is defined as a local approximation of a nonsmooth function, while equality (27) holds true for all $x, \Delta x \in \mathbb{R}^{d+1}$, i.e. globally.

Definition 4.3. The pair $D f=[\underline{d} f, \bar{d} f]$ defined by (26) is called a global codifferential mapping (or simply global codifferential) of the function $f$ (associated with the DC decomposition $f=\underline{f}-(-\bar{f}))$. The multifunction $\underline{d} f$ is called $a$ global hypodifferential of $f$, while the multifunction $\bar{d} f$ is called a global hyperdifferential of $f$.

Note that a global codifferential mapping of a piecewise affine function is not unique, since there exists infinitely many DC decompositions of a piecewise affine function of the form (24). Let us also point out that a global codifferential mapping of a piecewise affine function was first implicitly utilised by Polyakova in 32.

With the use of the codifferential calculus 9, 13, 14, one can obtain some simple calculus rules for global codifferentials of piecewise affine functions.

Proposition 4.4. Let $f_{m}: \mathbb{R}^{d} \rightarrow \mathbb{R}, m \in M=\{1, \ldots, p\}$, be piecewise affine functions of the form $f_{m}=\underline{f}_{m}+\bar{f}_{m}$, where

$$
\underline{f}_{m}(x)=\max _{i \in I_{m}}\left(a_{m i}+\left\langle v_{m i}, x\right\rangle\right), \quad \bar{f}_{m}(x)=\min _{j \in J_{m}}\left(b_{m j}+\left\langle w_{m j}, x\right\rangle\right),
$$

and $I_{m}=\left\{1, \ldots, l_{m}\right\}, J_{m}=\left\{1, \ldots, s_{m}\right\}$. Let also $D f_{m}$ be the global codifferential mapping of the function $f_{m}$ associated with the DC decomposition $f_{m}=\underline{f}_{m}-\left(\bar{f}_{m}\right), m \in M$, and let $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ be a given function. Then the following statements hold true:

1. if $f(x)=a+\langle v, x\rangle$, then both $D f(\cdot) \equiv[\{(0, v)\},\{(0,0)\}]$ and $D f(\cdot) \equiv$ $[\{(0,0)\},\{(0, v)\}]$ are global codifferential mappings of the function $f$;
2. if $f=f_{1}+c$ for some $c \in \mathbb{R}$, then $D f=D f_{1}$;
3. if $f=\lambda f_{1}$, then $D f=\left[\lambda \underline{d} f_{1}, \lambda \bar{d} f_{1}\right]$ in the case $\lambda \geq 0$, and $D f=$ $\left[\lambda \bar{d} f_{1}, \lambda \underline{d} f_{1}\right]$ in the case $\lambda<0$;
4. if $f=\sum_{m=1}^{p} f_{m}$, then $D f=\left[\sum_{m=1}^{p} \underline{d} f_{m}, \sum_{m=1}^{p} \bar{d} f_{m}\right]$;
5. if $f=\max _{m \in M} f_{m}$, then

$$
D f(\cdot)=\left[\operatorname{co}\left\{\left(f_{m}(\cdot)-f(\cdot), 0\right)+\underline{d} f_{m}(\cdot)-\sum_{k \neq m} \bar{d} f_{k}(\cdot) \mid m \in M\right\}, \sum_{m=1}^{p} \bar{d} f_{m}(\cdot)\right]
$$

is a global codifferential mapping of $f$;
6. if $f=\min _{m \in M} f_{m}$, then

$$
D f(\cdot)=\left[\sum_{m=1}^{p} \underline{d} f_{m}(\cdot), \text { co }\left\{\left(f_{m}(\cdot)-f(\cdot), 0\right)+\bar{d} f_{m}(\cdot)-\sum_{k \neq m} \underline{d} f_{k}(\cdot) \mid m \in M\right\}\right]
$$

is a global codifferential mapping of $f$.
Proof. 1. Define

$$
\underline{f}^{\prime}(x)=a+\langle v, x\rangle, \bar{f}^{\prime}(x)=0, \quad \underline{f}^{\prime \prime}(x)=0, \bar{f}^{\prime \prime}(x)=a+\langle v, x\rangle .
$$

Then $f=\underline{f}^{\prime}-\left(-\bar{f}^{\prime}\right)$ and $f=\underline{f}^{\prime \prime}-\left(-\bar{f}^{\prime \prime}\right)$ are two DC decompositions of the function $f$. Applying the definition of global codifferential (26) one gets that $D f(\cdot)=[\{(0, v)\},\{(0,0)\}]$ is a global codifferential of $f$ associated with the first DC decomposition, while $D f(\cdot)=[\{(0,0)\},\{(0, v)\}]$ is a global codifferential of $f$ associated with the second DC decomposition.
(2. Define

$$
\underline{f}(x)=\underline{f}_{1}(x)+c=\max _{i \in I_{1}}\left(a_{1 i}+c+\left\langle v_{1 i}, x\right\rangle\right), \quad \bar{f}(x)=\bar{f}_{1}(x) .
$$

Then $f=\underline{f}-(-\bar{f}(x))$ is a DC decomposition of the function $f$. Applying the definition of global codifferential (26), and the fact that

$$
\left(a_{1 i}+c-\underline{f}(x)+\left\langle v_{1 i}, x\right\rangle, v_{1 i}\right)=\left(a_{1 i}-\underline{f}_{1}(x)+\left\langle v_{1 i}, x\right\rangle, v_{1 i}\right),
$$

one gets that $\underline{d} f(x)=\underline{d} f_{1}(x)$ and $\bar{d} f(x)=\bar{d} f_{1}(x)$, i.e. $D f=D f_{1}$ is a global codifferential of $f$ associated with the DC decomposition $f=\underline{f}-(-\bar{f})$ defined above.
(3) Let $\lambda \geq 0$. Define $\underline{f}(x)=\lambda \underline{f}(x)$ and $\bar{f}(x)=\lambda \bar{f}_{1}(x)$. Then $f=\lambda f_{1}=\underline{f}-(-\bar{f})$ is a DC decomposition of the function $f$. By definition

$$
\underline{f}(x)=\max _{i \in I_{1}}\left(\lambda a_{1 i}+\left\langle\lambda v_{1 i}, x\right\rangle\right), \quad \bar{f}(x)=\min _{j \in J_{1}}\left(\lambda b_{1 i}+\left\langle\lambda w_{1 i}, x\right\rangle\right) .
$$

Hence with the use of (26) and the fact that

$$
\begin{equation*}
\left(\lambda a_{1 i}-\underline{f}(x)+\left\langle\lambda v_{1 i}, x\right\rangle, \lambda v_{1 i}\right)=\lambda\left(a_{1 i}-\underline{f}_{1}(x)+\left\langle v_{1 i}, x\right\rangle, v_{1 i}\right) \tag{29}
\end{equation*}
$$

one gets that $\underline{d} f=\lambda \underline{d} f_{1}$ and $\bar{d} f=\lambda \bar{d} f_{1}$, i.e. $D f=\left[\lambda \underline{d} f_{1}, \lambda \bar{d} f_{1}\right]$ is a global codifferential of $f$ associated with the DC decomposition $f=\underline{f}-(-\bar{f})$ defined above.

Let now $\lambda<0$. Define $\underline{f}(x)=\lambda \bar{f}_{1}(x)$ and $\bar{f}(x)=\lambda \underline{f}_{1}(x)$. Then taking into account the fact the negative of a convex function is a concave function and
vice versa one obtains that $f=\lambda f_{1}=\underline{f}-(-\bar{f})$ is a DC decomposition of the function $f$. By definition one has

$$
\underline{f}(x)=\max _{j \in J_{1}}\left(\lambda b_{1 j}+\left\langle\lambda w_{1 j}, x\right\rangle\right), \quad \bar{f}(x)=\min _{i \in I_{1}}\left(\lambda a_{1 i}+\left\langle\lambda v_{1 i}, x\right\rangle\right)
$$

(recall that $\lambda<0$ ). Hence applying (26) and the fact that

$$
\left(\lambda b_{1 j}-\underline{f}(x)+\left\langle\lambda w_{1 j}, x\right\rangle, \lambda w_{1 j}\right)=\lambda\left(b_{1 j}-\bar{f}_{1}(x)+\left\langle w_{1 j}, x\right\rangle, w_{1 j}\right)
$$

one obtains that $\underline{d} f=\lambda \bar{d} f_{1}$ and $\bar{d} f=\lambda \underline{d} f_{1}$, i.e. $D f=\left[\lambda \bar{d} f_{1}, \lambda \underline{d} f_{1}\right]$ is a global codifferential of $f$ associated with the DC decomposition $f=\underline{f}-(-\bar{f})$ defined above.
4. Define

$$
\begin{equation*}
\underline{f}(x)=\sum_{m=1}^{p} \underline{f}_{m}(x), \quad \bar{f}(x)=\sum_{m=1}^{p} \bar{f}_{m}(x) \tag{30}
\end{equation*}
$$

Then $f=\underline{f}-(-\bar{f})$ is a DC decomposition of the function $f$ due to the fact that the sum of convex/concave functions is a convex/concave function. Note that
$\underline{f}(x)=\sum_{m=1}^{p} \max _{i \in I_{m}}\left(a_{m i}+\left\langle v_{m i}, x\right\rangle\right)=\max _{\left(i_{1}, \ldots, i_{p}\right) \in I_{1} \times \ldots \times I_{p}}\left(\sum_{m=1}^{p} a_{m i_{m}}+\left\langle\sum_{m=1}^{p} v_{m i_{m}}, x\right\rangle\right)$,
and a similar equality holds true for $\bar{f}(x)$. Hence with the use of (26), and the fact that

$$
\begin{aligned}
\left(\sum_{m=1}^{p} a_{m i_{m}}-\underline{f}(x)+\left\langle\sum_{m=1}^{p} v_{m i_{m}},\right.\right. & \left.x\rangle, \sum_{m=1}^{p} v_{m i_{m}}\right) \\
& =\sum_{m=1}^{p}\left(a_{m i_{m}}-\underline{f}_{m}(x)+\left\langle v_{m i_{m}}, x\right\rangle, v_{m i_{m}}\right)
\end{aligned}
$$

one obtains that $\underline{d} f(x)=\sum_{m=1}^{p} \underline{d} f_{m}(x)$ and $\bar{d} f(x)=\sum_{m=1}^{p} \bar{d} f_{m}(x)$, i.e. $D f=$ [ $\sum_{m=1}^{p} \underline{d} f_{m}, \sum_{m=1}^{p} \bar{d} f_{m}$ ] is a global codifferential of $f$ associated with the DC decomposition $f=\underline{f}-(-\bar{f})$ defined by (30).
55 Define

$$
\begin{equation*}
\underline{f}(x)=\max _{m \in M}\left(\underline{f}_{m}(x)-\sum_{k \neq m} \bar{f}_{m}(x)\right), \quad \bar{f}(x)=\sum_{m=1}^{p} \bar{f}_{m}(x) . \tag{31}
\end{equation*}
$$

Note that the function $\underline{f}$ is convex, since the maximum and the sum of convex functions is convex, while the function $\bar{f}$ is concave as the sum of concave functions. By definition

$$
f(x)=\max _{m \in M} f_{m}(x)=\max _{m \in M}\left(\underline{f}_{m}(x)+\bar{f}_{m}(x)\right)
$$

Adding and subtracting $\sum_{m=1}^{p} \bar{f}_{m}$ one obtains that

$$
f(x)=\max _{m \in M}\left(\underline{f}_{m}(x)-\sum_{k \neq m} \bar{f}_{m}(x)\right)+\sum_{m=1}^{p} \bar{f}_{m}(x)=\underline{f}(x)+\bar{f}(x),
$$

i.e. $f=\underline{f}-(-\bar{f})$ is a DC decomposition of the function $f$. Let us compute the global codifferential of $f$ associated with this DC decomposition.

From the proof of part (4) it follows that $\bar{d} f(x)=\sum_{m=1}^{p} \bar{d} f_{m}(x)$ (cf. (30) and (31)). Let us compute the global hypodifferential. By definition one has

$$
\begin{aligned}
& \underline{f}(x)=\max _{m \in M}(\underline{f} \\
&\left.=\max _{m \in M}(x)-\sum_{k \neq m} \bar{f}_{m}(x)\right) \\
& \max _{i_{m} \in I_{m}}\left(a_{m i_{m}}+\left\langle v_{m i_{m}}, x\right\rangle\right)-\sum_{k \neq m} \min _{j_{k}^{m} \in J_{k}}\left(b_{k j_{k}^{m}}+\left\langle w_{k j_{k}^{m}}, x\right\rangle\right) \\
&=\max \left(a_{m i_{m}}-\sum_{k \neq m} b_{k j_{k}^{m}}+\left\langle v_{m i_{m}}-\sum_{k \neq m} w_{k j_{k}^{m}}, x\right\rangle\right)
\end{aligned}
$$

where the last maximum is taken over all $i_{m} \in I_{m}, j_{k}^{m} \in J_{k}$, and $k, m \in M$. Hence and from (26) it follows that the first coordinate of a vector from $\underline{d} f(x)$ has the form

$$
a_{m i_{m}}-\sum_{k \neq m} b_{k j_{k}^{m}}-\underline{f}(x)+\left\langle v_{m i_{m}}-\sum_{k \neq m} w_{k j_{k}^{m}}, x\right\rangle .
$$

Adding and subtracting $\underline{f}_{m}(x)-\sum_{k \neq m} \bar{f}_{k}(x)$, and taking into account the fact that

$$
\begin{aligned}
\underline{f}_{m}(x)-\sum_{k \neq m} \bar{f}_{k}(x)-\underline{f}(x) & =\underline{f}_{m}(x)+\bar{f}_{m}(x)-\sum_{k=1}^{p} \bar{f}_{k}(x)-\underline{f}(x) \\
& =f_{m}(x)-\bar{f}(x)-\underline{f}(x)=f_{m}(x)-f(x)
\end{aligned}
$$

one obtains that

$$
\begin{aligned}
& a_{m i_{m}}-\sum_{k \neq m} b_{k j_{k}^{m}}-\underline{f}(x)+\left\langle v_{m i_{m}}-\sum_{k \neq m} w_{k j_{k}^{m}}, x\right\rangle \\
= & \left(f_{m}(x)-f(x)\right)+\left(a_{m i_{m}}-\underline{f}_{m}(x)+\left\langle v_{m i_{m}}, x\right\rangle\right)-\sum_{k \neq m}\left(b_{k j_{k}^{m}}-\bar{f}_{k}(x)+\left\langle w_{k j_{k}^{m}}, x\right\rangle\right) .
\end{aligned}
$$

Hence with the use of (26) one gets that

$$
\begin{gather*}
\underline{d} f(x)=\operatorname{co}\left\{\left(f_{m}(x)-f(x), 0\right)+\left(a_{m i_{m}}-\underline{f}_{m}(x)+\left\langle v_{m i_{m}}, x\right\rangle, v_{m i_{m}}\right)\right. \\
\left.-\sum_{k \neq m}\left(b_{k j_{k}^{m}}-\bar{f}_{k}(x)+\left\langle w_{k j_{k}^{m}}, x\right\rangle, w_{k j_{k}^{m}}\right) \mid i_{m} \in I_{m}, j_{k}^{m} \in J_{k}, k \in M, m \in M\right\} \tag{32}
\end{gather*}
$$

From the fact that by definition

$$
\begin{aligned}
\left(a_{m i_{m}}-\underline{f}_{m}(x)+\left\langle v_{m i_{m}}, x\right\rangle, v_{m i_{m}}\right) & \in \underline{d} f_{m}(x) \\
\left(b_{k j_{k}^{m}}-\bar{f}_{k}(x)+\left\langle w_{k j_{k}^{m}}, x\right\rangle, w_{k j_{k}^{m}}\right) & \in \bar{d} f_{k}(x)
\end{aligned}
$$

it follows that

$$
\begin{equation*}
\underline{d} f(x) \subseteq \operatorname{co}\left\{\left(f_{m}(x)-f(x), 0\right)+\underline{d} f_{m}(x)-\sum_{k \neq m} \bar{d} f_{k}(x) \mid m \in M\right\} \tag{33}
\end{equation*}
$$

To prove the converse inclusion fix $m \in M$, and note that taking the convex hull in (32) only over $i_{m} \in I_{m}$ one obtains that

$$
\left(f_{m}(x)-f(x), 0\right)+\underline{d} f_{m}(x)-\sum_{k \neq m}\left(b_{k j_{k}^{m}}-\bar{f}_{k}(x)+\left\langle w_{k j_{k}^{m}}, x\right\rangle, w_{k j_{k}^{m}}\right) \subseteq \underline{d} f(x) .
$$

Now, taking consecutively the convex hull over all $j_{k}^{m} \in J_{k}$ for each $k \neq m$ one gets that

$$
\left(f_{m}(x)-f(x), 0\right)+\underline{d} f_{m}(x)-\sum_{k \neq m} \bar{d} f_{k}(x) \subseteq \underline{d} f(x)
$$

Finally, taking the convex hull over all $m \in M$ one obtains that the inclusion opposite to (33) is valid, which implies the desired result.
6. Define

$$
\underline{f}(x)=\sum_{m=1}^{p} \underline{f}_{m}(x), \quad \bar{f}(x)=\min _{m \in M}\left(\bar{f}_{m}(x)-\sum_{k \neq m} \underline{f}_{m}(x)\right) .
$$

Clearly, the function $\underline{f}$ is convex, while the function $\bar{f}$ is concave. By definition

$$
f(x)=\min _{m \in M} f_{m}(x)=\min _{m \in M}\left(\underline{f}_{m}(x)+\bar{f}_{m}(x)\right) .
$$

Adding and subtracting $\sum_{m=1}^{p} \underline{f}_{m}$ one obtains that

$$
f(x)=\sum_{m=1}^{p} \bar{f}_{m}+\min _{m \in M}\left(\bar{f}_{m}(x)-\sum_{k \neq m} \underline{f}_{m}(x)\right)=\underline{f}(x)+\bar{f}(x),
$$

i.e. $f=\underline{f}-(-\bar{f})$ is a DC decomposition of the function $f$. Computing the global codifferential of the function $f$ associated with this DC decomposition in the same way as in part (5) one obtains the required result (alternatively, one can rewrite $f=-\max _{m \in M}\left(-f_{m}\right)$, and consecutively apply part (31) with $\lambda=-1$, part (5), and part (3) with $\lambda=-1$ again to obtain exactly the same result).

Remark 4.2. (i) Note that with the use of the proposition above one can compute DC decomposition (24) of a piecewise affine function (see [1] for more details). Namely, suppose that a global codifferential $D f(0)$ of $f$ at zero is known, $\underline{d} f(0)=$ $\operatorname{co}\left\{\left(a_{i}, v_{i}\right) \mid 1 \leq i \leq l\right\}$, and $\bar{d} f(0)=\operatorname{co}\left\{\left(b_{j}, w_{j}\right) \mid 1 \leq j \leq s\right\}$. Applying (27) with $x=0$ one obtains that

$$
\begin{aligned}
f(\Delta x)-f(0) & =\max _{(a, v) \in d f(0)}(a+\langle v, \Delta x\rangle)+\min _{(b, w) \in \bar{d} f(0)}(b+\langle w, \Delta x\rangle) \\
& =\max _{1 \leq i \leq l}\left(a_{i}+\left\langle v_{i}, \Delta x\right\rangle\right)+\min _{1 \leq j \leq s}\left(b_{j}+\left\langle w_{j}, \Delta x\right\rangle\right)
\end{aligned}
$$

Define

$$
\begin{equation*}
\underline{f}(x)=\max _{1 \leq i \leq l}\left(a_{i}+f(0)+\left\langle v_{i}, x\right\rangle\right), \quad \bar{f}(x)=\min _{1 \leq j \leq s}\left(b_{j}+\left\langle w_{j}, x\right\rangle\right) \tag{34}
\end{equation*}
$$

Then $f=\underline{f}-(-\bar{f})$ is a DC decomposition of the function $f$, i.e. there is a one-to-one correspondence between DC decompositions and global codifferentials of
piecewise affine functions. Let us also note that from the definition of global codifferential (26), and the equality

$$
\left(a_{i}-\underline{f}(y)+\left\langle v_{i}, y\right\rangle, v_{i}\right)=\left(a_{i}-\underline{f}(x)+\left\langle v_{i}, x\right\rangle, v_{i}\right)+\left(\underline{f}(x)-\underline{f}(y)+\left\langle v_{i}, y-x\right\rangle, 0\right)
$$

it follows that

$$
\begin{equation*}
\underline{d} f(y)=\left\{(a+\underline{f}(x)-\underline{f}(y)+\langle v, y-x\rangle, v) \in \mathbb{R}^{d+1} \mid(a, v) \in \underline{d} f(x)\right\} \tag{35}
\end{equation*}
$$

for all $x, y \in \mathbb{R}^{d}$, and a similar equality holds true for $\bar{d} f(\cdot)$. Thus, one can easily compute $D f(y)$ for any $y$, if $D f(x)$ for some $x$ is known.
(ii) It should be mentioned that the proposition above allows one to compute a global codifferential of a piecewise affine function $f$ without computing its DC decomposition. Nevertheless, in order to avoid rather lengthy computations at every point $x$ it seems most reasonable to compute $D f(0)$ first, then compute a DC decomposition of $f$ with the use of (34), and, finally, utilise (35) to compute $D f(x)$ at any point $x \in \mathbb{R}^{d}$.
(iii) Let us note that the proper choice of a global codifferential of the affine function $a+\langle v, x\rangle$ allows one to simplify the computation of a global codifferential of a piecewise affine function. The first global codifferential from part (1) of Prop. 4.4 is more suitable for the computation of a global codifferential of the maximum of affine functions, while the second one is more suitable in the case of the minimum. Indeed, if $f=\max _{1 \leq i \leq l} f_{i}$, where $f_{i}(x)=a_{i}+\left\langle v_{i}, x\right\rangle$, then applying part (5) of Prop. 4.4 one obtains

$$
\underline{d} f(x)=\operatorname{co}\left\{\left(a_{i}+\left\langle v_{i}, x\right\rangle-f(x), v_{i}\right) \mid 1 \leq i \leq l\right\}, \quad \bar{d} f(x)=\{(0,0)\}
$$

for $D f_{i}(\cdot)=\left[\left\{\left(0, v_{i}\right)\right\},\{(0,0)\}\right]$, while
$\underline{d} f(x)=\mathrm{co}\left\{\left(a_{i}+\left\langle v_{i}, x\right\rangle-f(x),-\sum_{k \neq i} v_{k}\right) \mid 1 \leq i \leq l\right\}, \quad \bar{d} f(x)=\left\{\left(0, \sum_{i=1}^{l} v_{i}\right)\right\}$ for $D f_{i}(\cdot)=\left[\{(0,0)\},\left\{\left(0, v_{i}\right)\right\}\right]$.

Let us derive new global optimality conditions for a piecewise affine function in terms of its global codifferential.

Theorem 4.5. Let $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ be a piecewise affine function of the form (24), $D f$ be its global codifferential mapping, and $x^{*} \in \mathbb{R}^{d}$ be a given point. Suppose also that $f$ is bounded below, and for any $j \in J$ define $z_{j}=\left(b_{j}-\bar{f}\left(x^{*}\right)+\right.$ $\left.\left\langle w_{j}, x^{*}\right\rangle, w_{j}\right) \in \bar{d} f\left(x^{*}\right)$, and

$$
\begin{equation*}
\left\{\left(a_{j}, v_{j}\right)\right\}=\arg \min \left\{\|(a, v)\|^{2} \mid(a, v) \in \underline{d} f\left(x^{*}\right)+z_{j}\right\} . \tag{36}
\end{equation*}
$$

Then $x^{*}$ is a point of global minimum of the function $f$ if and only if for any $j \in J$, one has $a_{j} \geq 0$ or, equivalently, for any $j \in J$ either $0 \in \underline{d} f\left(x^{*}\right)+z_{j}$ or $a_{j}>0$.
Proof. Applying equality (27) with $x=x^{*}$ and $\Delta x=x-x^{*}$, and the definition of global codifferential (26) one obtains that

$$
\begin{aligned}
& f(x)-f\left(x^{*}\right)= \max _{(a, v) \in \underline{d} f\left(x^{*}\right)}\left(a+\left\langle v, x-x^{*}\right\rangle\right) \\
&+\min _{j \in J}\left(b_{j}-\bar{f}\left(x^{*}\right)+\left\langle w_{j}, x^{*}\right\rangle+\left\langle w_{j}, x-x^{*}\right\rangle\right) \\
&=\min _{j \in J} \max _{(a, v) \in \underline{d} f\left(x^{*}\right)}\left(a+b_{j}-\bar{f}\left(x^{*}\right)+\left\langle w_{j}, x^{*}\right\rangle+\left\langle v+w_{j}, x-x^{*}\right\rangle\right)
\end{aligned}
$$

for all $x \in \mathbb{R}^{d}$. Hence for any $x \in \mathbb{R}^{d}$ one has

$$
\begin{equation*}
f(x)-f\left(x^{*}\right)=\min _{j \in J} \max _{(a, v) \in \underline{d} f\left(x^{*}\right)+z_{j}}\left(a+\left\langle v, x-x^{*}\right\rangle\right)=\min _{j \in J} g_{j}(x), \tag{37}
\end{equation*}
$$

where $g_{j}(x)=\max _{(a, v) \in \underline{d} f\left(x^{*}\right)+z_{j}}(a+\langle v, x\rangle)$. Therefore, $x^{*}$ is point of global minimum of $f$ if and only if for any $j \in J$ the function $g_{j}$ is nonnegative. Note that each function $g_{j}$ is bounded below due to the facts that $g_{j}(x) \geq$ $f(x)-f\left(x^{*}\right)$ for all $x \in \mathbb{R}^{d}$ (see (37)), and $f$ is bounded below. Consequently, applying Lemma 4.1 and Corollary 4.2 to the functions $g_{j}$ one obtains the desired result.

The necessary and sufficient conditions for global optimality in terms of global codifferential from Theorem4.5 along with the proof of Lemma 4.1 allow one to get a new perspective on the method of codifferential descent. As it was noted above, a function is codifferentiable if and only if its increment can be locally approximated by a DC function. In most applications a codifferential of a nonsmooth function is a pair of convex polytopes, i.e. the increment of this function can be locally approximated by a piecewise affine function. In a sense, in each iteration of the method of codifferential descent one verifies whether the global optimality conditions from Theorem4.5 are satisfied for a local piecewise affine approximation of the objective function, and then utilises the "global descent" directions $-v_{j}$ of the approximation (see (36) and the proof of the first part of Lemma 4.1) as search directions for the objective function. In the case when the objective function itself is piecewise affine, and its global codifferential mapping is known, one can propose a natural modification of the MCD in which instead of performing the line search one utilises the first component of the vector $\left(a_{j}, v_{j}\right)$ in order to define the step size.

### 4.2 The method of global codifferential descent

Let $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ be a piecewise affine function of the form (24), and $D f$ be its global codifferential mapping (see (26)). For any $x \in \mathbb{R}^{d}$ and $j \in J$ denote

$$
\begin{align*}
z_{j}(x) & =\left(b_{j}-\bar{f}(x)+\left\langle w_{j}, x\right\rangle, w_{j}\right) \in \bar{d} f(x),  \tag{38}\\
\left\{\left(a_{j}(x), v_{j}(x)\right)\right\} & =\arg \min \left\{\|(a, v)\|^{2} \mid(a, v) \in \underline{d} f(x)+z_{j}(x)\right\} . \tag{39}
\end{align*}
$$

Suppose that $x$ is not a point of global minimum of the function $f$, and choose an arbitrary $j \in J$. Applying the necessary and sufficient condition for a minimum of a convex function on a convex set [15, Proposition II.2.1] one obtains that

$$
a_{j}(x)\left(a-a_{j}(x)\right)+\left\langle v_{j}(x), v-v_{j}(x)\right\rangle \geq 0 \quad \forall(a, v) \in \underline{d} f(x)+z_{j}(x)
$$

If $a_{j}(x)<0$, then dividing this inequality by $a_{j}(x)$, taking the maximum over all $(a, v) \in \underline{d} f(x)+z_{j}(x)$, and applying (37) one obtains

$$
\begin{equation*}
f\left(x+\frac{1}{a_{j}(x)} v_{j}(x)\right)-f(x) \leq-\frac{1}{\left|a_{j}(x)\right|}\left\|\left(a_{j}(x), v_{j}(x)\right)\right\|^{2}<0 \tag{40}
\end{equation*}
$$

If $a_{j}(x)=0$, but $v_{j}(x) \neq 0$, then $\left\langle v,-v_{j}(x)\right\rangle \leq-\left\|v_{j}(x)\right\|^{2}$ for any $(a, v) \in$ $\underline{d} f(x)+z_{j}(x)$, which with the use of (37) implies that

$$
\begin{equation*}
f\left(x-\alpha v_{j}(x)\right)-f(x) \leq \max _{(a, v) \in \underline{d} f(x)+z_{j}(x)} a-\alpha\left\|v_{j}(x)\right\|^{2}, \tag{41}
\end{equation*}
$$

and the function $f$ is unbounded below. Thus, if $a_{j}(x)=0$, and $f$ is bounded below, then $v_{j}(x)=0$.

Finally, if $a_{j}(x)>0$, then the set $\underline{d} f(x)+z_{j}(x)$ is of no use to the optimization process. Indeed, by Lemma 4.1 one has

$$
\begin{equation*}
\max _{(a, v) \in \underline{d} f(x)+z_{j}(x)}(a+\langle v, y\rangle) \geq 0 \quad \forall y \in \mathbb{R}^{d} . \tag{42}
\end{equation*}
$$

Applying (37) one gets that

$$
f(y)-f(x)=\min _{k \in J} \max _{(a, v) \in \underline{d} f(x)+z_{k}(x)}(a+\langle v, y-x\rangle) \quad \forall y \in \mathbb{R}^{d} .
$$

From (42) it follows that for any $y$ such that $f(y)<f(x)$ the minimum in this equality cannot be achieved for $k=j$. Therefore

$$
f(y)-f(x)=\min _{k \in J \backslash\{j\}} \max _{(a, v) \in \underline{d} f(x)+z_{k}(x)}(a+\langle v, y-x\rangle)
$$

for any $y \in \mathbb{R}$ such that $f(y)<f(x)$. In other words, the index $j$ and the corresponding vector $\left(b_{j}, w_{j}\right)$ are not needed to compute $f(y)$ for any $y \in \mathbb{R}^{d}$ satisfying the inequality $f(y)<f(x)$.

Let us prove an even stronger statement. Namely, let us show that if $a_{j}(x) \geq$ 0 for some $x \in \mathbb{R}^{d}$, then the index $j$ can be discarded from consideration.

Lemma 4.6. Let $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ be a piecewise affine function of the form (24), and $D f$ be its global codifferential mapping. Suppose that $f$ is bounded below, and for some $j \in J$ and $x \in \mathbb{R}^{d}$ one has $a_{j}(x) \geq 0$. Then $a_{j}(y) \geq 0$ for any $y \in \mathbb{R}^{d}$ such that $f(y) \leq f(x)$.
Proof. For any $\Delta y, y \in \mathbb{R}^{d}$ denote

$$
g_{j}(\Delta y, y)=\max _{(a, v) \in \underline{d f}(y)+z_{j}(y)}(a+\langle v, \Delta y\rangle)
$$

Applying (27) and (26) one gets that

$$
\begin{aligned}
& f(y+\Delta y)-f(y)=\max _{(a, v) \in \underline{d f(y)}}(a+\langle v, \Delta y\rangle)+\min _{(b, w) \in \bar{d} f(y)}(b+\langle w, \Delta y\rangle) \\
& \quad \leq \max _{(a, v) \in \underline{d} f(y)}(a+\langle v, \Delta y\rangle)+b_{j}-\bar{f}(y)+\left\langle w_{j}, y\right\rangle+\left\langle w_{j}, \Delta y\right\rangle=g_{j}(\Delta y, y)
\end{aligned}
$$

for any $\Delta y, y \in \mathbb{R}^{d}$. Hence taking into account the fact that $f$ is bounded below one obtains that the function $g(\cdot, y)$ is bounded below for any $y \in \mathbb{R}^{d}$. Furthermore, note that by the definition of $\underline{d} f(\cdot)$ and $z_{j}(\cdot)$ (see (26) and (38)) one has
$g_{j}(\Delta y, y)=\max _{i \in I}\left(a_{i}+\left\langle v_{i}, y\right\rangle-\underline{f}(y)+\left\langle v_{i}, \Delta y\right\rangle+b_{j}+\left\langle w_{j}, y\right\rangle-\bar{f}(y)+\left\langle w_{j}, \Delta y\right\rangle\right)$
for all $\Delta y, y \in \mathbb{R}^{d}$.
From Corollary 4.2 and the fact that $a_{j}(x) \geq 0$ it follows that the function $g(\cdot, x)$ is nonnegative. Hence with the use of (43) one obtains that for any $\Delta x \in \mathbb{R}^{d}$ there exists $i \in I$ such that

$$
a_{i}+\left\langle v_{i}, x\right\rangle-\underline{f}(x)+\left\langle v_{i}, \Delta x\right\rangle+b_{j}+\left\langle w_{j}, x\right\rangle-\bar{f}(x)+\left\langle w_{j}, \Delta x\right\rangle \geq 0
$$

```
Algorithm 3: The method of global codifferential descent (MGCD).
```

    Step 1 . Choose a starting point \(x_{0} \in \mathbb{R}^{d}\), and set \(M=J=\{1, \ldots, s\}\)
    and \(n:=0\).
    Step 2. Compute \(\underline{d} f\left(x_{n}\right)\) and \(z_{j}\left(x_{n}\right)\) for all \(j \in M\).
    Step 3. For any \(j \in M\) compute \(\left(a_{j}\left(x_{n}\right), v_{j}\left(x_{n}\right)\right) \in \mathbb{R}^{d+1}\) by solving
    $$
\min \|(a, v)\|^{2} \quad \text { s.t. }(a, v) \in \underline{d} f\left(x_{n}\right)+z_{j}\left(x_{n}\right) .
$$

If $a_{j}\left(x_{n}\right) \geq 0$, then $M:=M \backslash\{j\}$.
Step 4. If $M=\emptyset$, then stop. Otherwise, compute $j(n) \in M$ by solving

$$
\min _{j \in M} f\left(x_{n}+\frac{1}{a_{j}\left(x_{n}\right)} v_{j}\left(x_{n}\right)\right)
$$

Set $x_{n+1}=x_{n}+\left[a_{j(n)}\left(x_{n}\right)\right]^{-1} v_{j(n)}\left(x_{n}\right)$, and $n:=n+1$, and go to Step 2.

Setting $\Delta x=y-x+\Delta y$, and taking into account the fact that $f(x)=\underline{f}(x)+\bar{f}(x)$ by definition (see (24) and (25)) one gets that for any $\Delta y, y \in \mathbb{R}^{d}$ there exists $i \in I$ such that

$$
a_{i}+\left\langle v_{i}, y\right\rangle+\left\langle v_{i}, \Delta y\right\rangle+b_{j}+\left\langle w_{j}, y\right\rangle+\left\langle w_{j}, \Delta y\right\rangle \geq f(x)
$$

Subtracting $f(y)=\underline{f}(y)+\bar{f}(y)$ from both sides of this inequality one obtains that for any $y, \Delta y \in \overline{\mathbb{R}}^{d}$ there exists $i \in I$ such that

$$
a_{i}+\left\langle v_{i}, y\right\rangle-\underline{f}(y)+\left\langle v_{i}, \Delta y\right\rangle+b_{j}+\left\langle w_{j}, y\right\rangle-\bar{f}(y)+\left\langle w_{j}, \Delta y\right\rangle \geq f(x)-f(y) .
$$

Taking the maximum over all $i \in I$, and applying (43) one gets that $g_{j}(\Delta y, y) \geq$ $f(x)-f(y)$ for all $\Delta y, y \in \mathbb{R}^{d}$. Hence the function $g_{j}(\cdot, y)$ is nonnegative for any $y$ such that $f(y) \leq f(x)$, which by Corollary 4.2 and the definition of $a_{j}(y)$ (see (39)) implies that $a_{j}(y) \geq 0$ for all such $y$.

Now, we can introduce a modification of the method of codifferential descent for minimizing piecewise affine functions of the form (24), which we call the method of global codifferential descent (MGCD). The scheme of this method is given in Algorithm 3

Let a sequence $\left\{x_{n}\right\}$ be generated by the MGCD. Observe that from (40) it follows that for any $n \in \mathbb{N}$ either $f\left(x_{n+1}\right)<f\left(x_{n}\right)$ or $M=\emptyset$. Hence, in particular, if $a_{j}\left(x_{n}\right) \geq 0$ for some $j \in J$ and $n \in \mathbb{N}$, then

$$
\begin{equation*}
a_{j}\left(x_{k}\right) \geq 0 \quad \forall k \geq n \tag{44}
\end{equation*}
$$

by Lemma4.6. Therefore, if the MGCD terminates in an iteration $n$ (i.e. $M=\emptyset$ for some $n \in \mathbb{N}$ ), then $a_{j}\left(x_{n}\right) \geq 0$ for all $j \in J$, which by Theorem 4.5 implies that $x_{n}$ is a point of global minimum of the function $f$. Below, we prove that the MGCD always terminates in a finite number of steps, i.e. it finds a global minimizer of a nonconvex piecewise affine function in a finite number of steps.

At first, let us explain the idea behind the proof of this result, which also illuminates the way each step of the MGCD is performed. Suppose for the sake


Figure 1: The transformation of the global codifferential over one step of the MGCD: $\underline{d} f\left(x_{n}\right)$ (left figure) and $\underline{d} f\left(x_{n+1}\right)$ (right figure). Note that all points shift only horizontally, i.e. along the $a$-axis (see (26)).
of simplicity that the function $f$ is convex (i.e. $\bar{f}(x) \equiv\{0\}$, see (25)). The hypodifferential $\underline{d} f\left(x_{n}\right)$ is a convex polytope in $\mathbb{R}^{d+1}$. By (28) one has $a \leq 0$ for any $(a, v) \in \underline{d} f\left(x_{n}\right)$, and $\max _{(a, v) \in \underline{d} f\left(x_{n}\right)} a=0$. Thus, the set $\left\{(a, v) \in \mathbb{R}^{d+1} \mid\right.$ $a=0\} \cap \underline{d} f\left(x_{n}\right)$ is a nonempty face of $\underline{d} f\left(x_{n}\right)$ (by (41) this face is proper, i.e. it does not coincide with $\underline{d} f\left(x_{n}\right)$, since otherwise $f$ is unbounded below). We call it the active face of the polytope $\underline{d} f\left(x_{n}\right)$. It is easy to see that the subdifferential $\partial f\left(x_{n}\right)$ is exactly the set of those $v$ for which $(0, v)$ belongs to the active face of $\underline{d} f\left(x_{n}\right)$.

The point

$$
\begin{equation*}
\left\{\left(a_{n}, v_{n}\right)\right\}=\arg \min \left\{\|(a, v)\| \mid(a, v) \in \underline{d} f\left(x_{n}\right)\right\} \tag{45}
\end{equation*}
$$

lies on a face $F$ of $\underline{d} f\left(x_{n}\right)$, which is not active, since otherwise, $f$ is unbounded below by (41). When one performs one iteration of the MGCD, the polytope $\underline{d} f\left(x_{n}\right)$ transforms, and, as we will show in the proof below, the face $F$ becomes the active face of the polytope $\underline{d} f\left(x_{n+1}\right)$. Thus, the projection $\left(a_{n}, v_{n}\right)$ belongs to a face of the hypodifferential, which becomes active on the next iteration (see Fig. (1).

Bearing these observations in mind one can prove the finite convergence of the MGCD by showing that in a finite number of iterations the projection $\left(a_{n}, v_{n}\right)$ belongs to a face of $\underline{d} f\left(x_{n}\right)$ that intersects the axis $\left\{(a, 0) \in \mathbb{R}^{d+1} \mid\right.$ $a \in \mathbb{R}\}$. Then $0 \in \partial f\left(x_{n+1}\right)$, and the proof is complete. In the case, when the function $f$ is not convex, a similar argument allows one to prove that in a finite number of iterations an index $j(n)$ is discarded. Repeating the same argument $s$ times one can verify that in a finite number of iterations all indices are discarded, and the MGCD terminates.

Theorem 4.7. Let $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ be a bounded below piecewise affine function. Then $f$ attains a global minimum, and the MGCD finds a point of global minimum of this function in a finite number of steps.

Proof. Let $\left\{x_{n}\right\}$ be a possibly infinite sequence generated by the MGCD for the function $f$. Denote $a_{n}=a_{j(n)}\left(x_{n}\right)$ and $v_{n}=v_{j(n)}\left(x_{n}\right)$, where the index $j(n)$ is computed on Step 4 of the MGCD. Note that this definition of $\left(a_{n}, v_{n}\right)$ coincides with (45), if $\bar{f}(x) \equiv 0$, since in this case $z_{j}(x) \equiv 0$ for all $j$ (see Steps 3 and 4 of the MGCD, (25) and (38)).

From Theorem 4.5 it follows that if $x_{n}$ is not a global minimizer of $f$, then there exists $j \in J$ such that $a_{j}\left(x_{n}\right)<0$, and

$$
\begin{align*}
f\left(x_{n+1}\right) \leq f\left(x_{n}+\frac{1}{a_{j}\left(x_{n}\right)} v_{j}\left(x_{n}\right)\right) & \leq f\left(x_{n}\right)-\frac{1}{\left|a_{j}\left(x_{n}\right)\right|}\left\|\left(a_{j}\left(x_{n}\right), v_{j}\left(x_{n}\right)\right)\right\|^{2} \\
& =f\left(x_{n}\right)-\left|a_{j}\left(x_{n}\right)\right|-\frac{1}{\left|a_{j}\left(x_{n}\right)\right|}\left\|v_{j}\left(x_{n}\right)\right\|^{2} \tag{46}
\end{align*}
$$

(see (40) and Step 4 of the MGCD). Note that

$$
-\left|a_{j}\left(x_{n}\right)\right|-\frac{1}{\left|a_{j}\left(x_{n}\right)\right|}\left\|v_{j}\left(x_{n}\right)\right\|^{2} \leq \begin{cases}-1, & \text { if }\left|a_{j}\left(x_{n}\right)\right| \geq 1 \\ -\left\|v_{j}\left(x_{n}\right)\right\|^{2}, & \text { otherwise }\end{cases}
$$

Hence, if $x_{n}$ is a not a point of global minimum of $f$, then

$$
\begin{equation*}
f\left(x_{n+1}\right)-f\left(x_{0}\right) \leq-\sum_{k=0}^{n}\left(\left|a_{k}\right|+\frac{1}{\left|a_{k}\right|}\left\|v_{k}\right\|^{2}\right) \leq-\sum_{k=0}^{n} \min \left\{1,\left\|v_{k}\right\|^{2}\right\} \tag{47}
\end{equation*}
$$

Denote by $\mathcal{E}$ the family of all convex sets $C \subset \mathbb{R}^{d}$ such that $0 \notin C$, and

$$
C=\operatorname{co}\left\{v_{i_{1}}, \ldots, v_{i_{k}}\right\}+w_{j}
$$

for some $i_{1}, \ldots, i_{k} \in I, 1 \leq k \leq l$, and $j \in J$, where the vectors $v_{i}$ and $w_{j}$ are from the DC decomposition of the function $f$ (see (24)). Clearly, $\mathcal{E}$ is a finite family of compact convex sets, and $\theta=\min _{C \in \mathcal{E}} \min _{v \in C}\|v\|^{2}>0$.

Denote $f^{*}=\inf _{x \in \mathbb{R}^{d}} f(x)>-\infty$, and $n^{*}=\left\lfloor\left(f\left(x_{0}\right)-f^{*}\right) / \min \{\theta, 1\}\right\rfloor+1$ (here $\lfloor t\rfloor$ is the greatest integer less than or equial to $t \in \mathbb{R}$ ). From (47) it follows that there exists $n \leq n^{*}$ such that either the MGCD terminates at the step $n$ or $a_{n}<0$ and $\left\|v_{n}\right\|^{2}<\theta$.

Suppose that $x_{n}$ is not a global minimizer of $f$. By definition $\left(a_{n}, v_{n}\right)$ belongs to the convex polytope $\underline{d} f\left(x_{n}\right)+z_{j(n)}\left(x_{n}\right)$ (see Step 3 of the MGCD). Any convex polytope is equal to the disjoint union of the relative interiors of its faces, i.e. the relative interiors of all faces of a convex polytope are pairwise disjoint, and the polytope is equal to the union of these relative interiors (see [38], p. 61). Therefore, $\left(a_{n}, v_{n}\right)$ belongs to the relative interior relint $F$ of a face $F$ of $\underline{d} f\left(x_{n}\right)+z_{j(n)}\left(x_{n}\right)$.

With the use of the necessary and sufficient condition for a minimum of a convex function on a convex set [15, Proposition II.2.1] one obtains that

$$
\begin{equation*}
a_{n} a+\left\langle v_{n}, v\right\rangle \geq\left\|\left(a_{n}, v_{n}\right)\right\|^{2} \quad \forall(a, v) \in \underline{d} f\left(x_{n}\right)+z_{j(n)}\left(x_{n}\right), \tag{48}
\end{equation*}
$$

and this inequality turns into an equality when $(a, v)=\left(a_{n}, v_{n}\right)$. By 38, Prop. 2.3] the face $F$ is itself a polytope. Consequently, applying the characterization of relative interior points of a convex polytope [38, Lemma 2.9] and the fact that $\left(a_{n}, v_{n}\right) \in \operatorname{relint} F$ one gets that

$$
\begin{equation*}
a_{n} a+\left\langle v_{n}, v\right\rangle=\left\|\left(a_{n}, v_{n}\right)\right\|^{2} \quad \forall(a, v) \in F \tag{49}
\end{equation*}
$$

Note also that the face $F$ is a polytope whose vertices are vertices of $\underline{d} f\left(x_{n}\right)+$ $z_{j(n)}\left(x_{n}\right)$ as well [38, Prop. 2.3]. Therefore

$$
F=\operatorname{co}\left\{\left(a_{i_{r}}+\left\langle v_{i_{r}}, x_{n}\right\rangle-\underline{f}\left(x_{n}\right), v_{i_{r}}\right) \mid 1 \leq r \leq k\right\}+z_{j(n)}\left(x_{n}\right)
$$

for some $i_{1}, \ldots, i_{k} \in I$ and $1 \leq k \leq l$ (see (261). From the definition of $\theta$, and the facts that $\left(a_{n}, v_{n}\right) \in F$ and $\left\|v_{n}\right\|^{2}<\theta$ it follows that $F \cap(\mathbb{R} \times\{0\}) \neq \emptyset$.

Introduce the convex function

$$
\begin{equation*}
g_{n}(x)=\max _{(a, v) \in \underline{d} f\left(x_{n+1}\right)+z_{j(n)}\left(x_{n+1}\right)}(a+\langle v, x\rangle) \tag{50}
\end{equation*}
$$

Let us verify that $0 \in \partial g_{n}(0)$. Indeed, by the definition of $z_{j}(x)$ (see (38)) one has

$$
\begin{aligned}
z_{j(n)}\left(x_{n+1}\right) & =\left(b_{j(n)}-\bar{f}\left(x_{n+1}\right)+\left\langle w_{j(n)}, x_{n+1}\right\rangle, w_{j(n)}\right) \\
& =z_{j(n)}\left(x_{n}\right)+\left(\bar{f}\left(x_{n}\right)-\bar{f}\left(x_{n+1}\right)+\left\langle w_{j(n)}, x_{n+1}-x_{n}\right\rangle, 0\right)
\end{aligned}
$$

Similarly, by (35) one has

$$
\underline{d} f\left(x_{n+1}\right)=\left\{\left(a+\underline{f}\left(x_{n}\right)-\underline{f}\left(x_{n+1}\right)+\left\langle v, x_{n+1}-x_{n}\right\rangle, v\right) \in \mathbb{R}^{d+1} \mid(a, v) \in \underline{d} f\left(x_{n}\right)\right\}
$$

Consequently, applying the equality $f(x)=\underline{f}(x)+\bar{f}(x)$ (see (24) and (25)) one obtains that

$$
\begin{aligned}
& \underline{d} f\left(x_{n+1}\right)+z_{j(n)}\left(x_{n+1}\right) \\
= & \left\{\left(a+\left\langle v, x_{n+1}-x_{n}\right\rangle-f\left(x_{n+1}\right)+f\left(x_{n}\right), v\right) \mid(a, v) \in \underline{d} f\left(x_{n}\right)+z_{j(n)}\left(x_{n}\right)\right\} .
\end{aligned}
$$

Therefore

$$
g_{n}(0)=\max _{(a, v) \in \underline{d} f\left(x_{n}\right)+z_{j(n)}\left(x_{n}\right)}\left(a+\left\langle v, x_{n+1}-x_{n}\right\rangle\right)-f\left(x_{n+1}\right)+f\left(x_{n}\right)
$$

Hence taking into account (48), (49), and the facts that $x_{n+1}-x_{n}=a_{n}^{-1} v_{n}$ and $a_{n}<0$ one gets that

$$
\begin{equation*}
g_{n}(0)=\frac{1}{a_{n}}\left\|\left(a_{n}, v_{n}\right)\right\|^{2}-f\left(x_{n+1}\right)+f\left(x_{n}\right) \geq 0 \tag{51}
\end{equation*}
$$

where the last inequality follows from (46). Furthermore, the maximum in the definition of $g_{n}(0)$ is attained at the points $\left(a+\left\langle v, x_{n+1}-x_{n}\right\rangle-f\left(x_{n+1}\right)+\right.$ $f\left(x_{n}\right), v$ ) with $(a, v) \in F$. Consequently, one has $\{v \mid \exists(a, v) \in F\} \subseteq \partial g_{n}(0)$, which implies that $0 \in \partial g_{n}(0)$ (since $F \cap(\mathbb{R} \times\{0\}) \neq \emptyset$ ), i.e. 0 is the point of global minimum of the function $g_{n}(x)$. Hence and from (51) it follows that the function $g_{n}$ is nonnegative. Taking into account (50), and applying Corollary 4.2 one obtains that $a_{j(n)}\left(x_{n+1}\right) \geq 0$ (see Step 3 of the MGCD). Therefore the index
$j(n)$ is discarded by the MGCD, and by Lemma 4.6 one has $a_{j(n)}\left(x_{k}\right) \geq 0$ for all $k \geq n+1$.

Thus, there exists $n_{1} \leq n^{*}$ such that the MGCD discards the index $j\left(n_{1}\right)$ in the $\left(n_{1}+1\right)$ th iteration. Recall that $n^{*}=\left\lfloor\left(f\left(x_{0}\right)-f^{*}\right) / \min \{\theta, 1\}\right\rfloor+1$. Taking into account (47) one obtains that there exists $n_{2} \leq n_{1}+n^{*} \leq 2 n^{*}$ such that either the MGCD terminates at the $n_{2}$ th iteration or $a_{n_{2}}<0$, and $\left\|v_{n_{2}}\right\|^{2}<\theta$. Arguing in the same way as above one can easily verify that the MGCD discards the index $j\left(n_{2}\right)$ in the $\left(n_{2}+1\right)$ th iteration, and $a_{j\left(n_{2}\right)}\left(x_{k}\right) \geq 0$ for all $k \geq n_{2}+1$. Repeating the same argument $s$ times one obtains that the MGCD discards all indices from the set $M$ in at most $s n^{*}$ iterations, and, thus, terminates in a finite number of steps. Furthermore, if MGCD terminates at an $n$th iteration, then, as it was pointed out above (see (44)), by Lemma 4.6 one has $a_{j}\left(x_{n}\right) \geq 0$ for all $j \in J$, which with the use of Theorem 4.5 implies that $x_{n}$ is a point of global minimum of the function $f$, and the proof is complete.

Remark 4.3. Note that the theorem above is valid for any method generating a sequence $\left\{x_{n}\right\}$ such that for all $n \in \mathbb{N}$ one has $f\left(x_{n+1}\right) \leq f\left(x_{n}\right)$ and

$$
\begin{equation*}
f\left(x_{n+1}\right) \leq f\left(x_{n}+\frac{1}{a_{j}\left(x_{n}\right)} v_{j}\left(x_{n}\right)\right) \quad \forall j \in J: a_{j}\left(x_{n}\right)<0 \tag{52}
\end{equation*}
$$

Indeed, let

$$
j(n) \in \arg \min \left\{\left.f\left(x_{n}+\frac{1}{a_{j}\left(x_{n}\right)} v_{j}\left(x_{n}\right)\right) \right\rvert\, j \in J: a_{j}\left(x_{n}\right)<0\right\}
$$

and denote $\left(a_{n}, v_{n}\right)=\left(a_{j(n)}\left(x_{n}\right), v_{j(n)}\left(x_{n}\right)\right)$. Taking into account (46) it is easy to see that there exists $n \leq n^{*}$ such that either the method terminates at the $n$th iteration or $a_{n}<0$ and $\left\|v_{n}\right\|^{2}<\theta$, where $n^{*}$ and $\theta$ are defined in the proof of Theorem 4.7. Denote $y_{n}=x_{n}+a_{n}^{-1} v_{n}$. Arguing in the same way as in the proof of Theorem 4.7 one can check that $a_{j(n)}\left(y_{n}\right) \geq 0$, which with the use of Lemma 4.6 and inequality (52) implies that $a_{j(n)}\left(x_{n+1}\right) \geq 0$. Furthermore, from (46) and (52) it follows that $f\left(x_{k+1}\right)<f\left(x_{k}\right)$ for all $k \in \mathbb{N}$ such that $x_{k}$ is not a global minimizer of $f$. Therefore, applying Lemma 4.6 again one obtains that $a_{j(n)}\left(x_{k}\right) \geq 0$ for all $k \geq n+1$. Repeating the same argument $s$ times one can easily verify that there exists $n \leq s n^{*}$ such that $a_{j}\left(x_{k}\right) \geq 0$ for all $k \geq n+1$ and for all $j \in J$, which with the use of Theorem 4.5 implies the required result.

Observe that from (40) it follows that condition (52) is satisfied for the original version of method of codifferential descent with $\mu=+\infty$ and $\bar{d}_{\mu} f(x)=$ $\left\{z_{j}(x) \mid j \in J\right\}$, which implies that the MCD also finds a point of global minimum of a piecewise affine function in a finite number of steps.

Note that the MGCD discards those $\left(b_{j}, w_{j}\right)$ which no longer provide information about descent directions of the function $f$, while the MCD keeps using all points $\left(b_{j}, w_{j}\right)$. Sometimes directions $v_{j}\left(x_{n}\right)$ such that $a_{j}\left(x_{n}\right) \geq 0$ might provide some global information to the optimization method (i.e. $f\left(x_{n}\right)>$ $\min _{\alpha>0} f\left(x_{n}-\alpha v_{j}\left(x_{n}\right)\right)$; however, this effect seems to be purely random, and it is reasonable to discard those $j \in J$ for which $a_{j}\left(x_{n}\right) \geq 0$.

Let us finally note that it is unclear which version of the method of codifferential descent (the MCD or the MGCD) is better for minimizing piecewise affine functions in terms of overall performance. Further research and extensive numerical experiments are needed to answer this question. In particular, it is
interesting to find a sharp upper bound on the number of iterations of these methods. However, these questions lie outside the scope of this article, and we leave them as open problems for future research.

At the end of the paper, let us give a simple example demonstrating how one can compute a global codifferential mapping of a piecewise affine function with the use of Proposition 4.4, and how the MGCD can escape a local minimum, and find a point of global minimum in just one iteration.
Example 4.8. Let $d=2$, and

$$
f(x)=\min \left\{\max \left\{\left|x_{1}\right|,\left|x_{2}\right|\right\}, 1+\max \left\{2\left|x_{1}-2\right|,\left|x_{2}-2\right|\right\}\right\} .
$$

Set $x_{0}=(2,2)$. It is easily seen that $x_{0}$ is a point of local minimum of the function $f$, while a global minimum is attained at the point $x^{*}=(0,0)$. Our aim is to apply the MGCD with the starting point $x_{0}$ to the function $f$. Instead of computing a DC decomposition of the function $f$ of the form (24), and then applying (26) in order to find $D f\left(x_{0}\right)$, we will compute a global codifferential $D f\left(x_{0}\right)$ directly with the use of Proposition 4.4 (see Remark 4.2).

Let us compute $D f\left(x_{0}\right)$. Define

$$
g_{1}(x)=\max \left\{\left|x_{1}\right|,\left|x_{2}\right|\right\}, \quad g_{2}(x)=1+\max \left\{2\left|x_{1}-2\right|,\left|x_{2}-2\right|\right\}
$$

With the use of parts (11), (2) and (5) of Proposition 4.4 one obtains

$$
\begin{aligned}
& \underline{d} g_{1}\left(x_{0}\right)=\mathrm{co}\left\{\left(\begin{array}{l}
0 \\
1 \\
0
\end{array}\right),\left(\begin{array}{c}
-4 \\
-1 \\
0
\end{array}\right),\left(\begin{array}{l}
0 \\
0 \\
1
\end{array}\right),\left(\begin{array}{c}
-4 \\
0 \\
-1
\end{array}\right)\right\}, \quad \bar{d} g_{1}\left(x_{0}\right)=\{0\} \\
& \underline{d} g_{2}\left(x_{0}\right)=\mathrm{co}\left\{\left(\begin{array}{l}
0 \\
2 \\
0
\end{array}\right),\left(\begin{array}{c}
0 \\
-2 \\
0
\end{array}\right),\left(\begin{array}{l}
0 \\
0 \\
1
\end{array}\right),\left(\begin{array}{c}
0 \\
0 \\
-1
\end{array}\right)\right\}, \quad \bar{d} g_{2}\left(x_{0}\right)=\{0\}
\end{aligned}
$$

Taking into account the fact that $f(x)=\min \left\{g_{1}(x), g_{2}(x)\right\}$, and applying part (6) of Proposition 4.4 one gets that $\underline{d} f\left(x_{0}\right)=\underline{d} g_{1}\left(x_{0}\right)+\underline{d} g_{2}\left(x_{0}\right)$, i.e.

$$
\begin{gathered}
\underline{d} f\left(x_{0}\right)=\mathrm{co}\left\{\left(\begin{array}{l}
0 \\
3 \\
0
\end{array}\right),\left(\begin{array}{c}
-4 \\
1 \\
0
\end{array}\right),\left(\begin{array}{l}
0 \\
2 \\
1
\end{array}\right),\left(\begin{array}{c}
-4 \\
2 \\
-1
\end{array}\right),\left(\begin{array}{c}
0 \\
-1 \\
0
\end{array}\right),\left(\begin{array}{c}
-4 \\
-3 \\
0
\end{array}\right),\left(\begin{array}{c}
0 \\
-2 \\
1
\end{array}\right),\left(\begin{array}{c}
-4 \\
-2 \\
-1
\end{array}\right)\right. \\
\\
\left.\left(\begin{array}{l}
0 \\
1 \\
1
\end{array}\right),\left(\begin{array}{c}
-4 \\
-1 \\
1
\end{array}\right),\left(\begin{array}{l}
0 \\
0 \\
2
\end{array}\right),\left(\begin{array}{c}
-4 \\
0 \\
0
\end{array}\right),\left(\begin{array}{c}
0 \\
1 \\
-1
\end{array}\right),\left(\begin{array}{c}
-4 \\
-1 \\
-1
\end{array}\right),\left(\begin{array}{c}
0 \\
0 \\
0
\end{array}\right),\left(\begin{array}{c}
-4 \\
0 \\
-2
\end{array}\right)\right\} .
\end{gathered}
$$

One also has

$$
\begin{aligned}
\bar{d} f\left(x_{0}\right) & =\mathrm{co}\left\{\left(\begin{array}{l}
1 \\
0 \\
0
\end{array}\right)+\bar{d} g_{1}\left(x_{0}\right)-\underline{d} g_{2}\left(x_{0}\right), \bar{d} g_{2}\left(x_{0}\right)-\underline{d} g_{1}\left(x_{0}\right)\right\} \\
& =\operatorname{co}\left\{\left(\begin{array}{l}
1 \\
2 \\
0
\end{array}\right),\left(\begin{array}{c}
1 \\
-2 \\
0
\end{array}\right),\left(\begin{array}{l}
1 \\
0 \\
1
\end{array}\right),\left(\begin{array}{c}
1 \\
0 \\
-1
\end{array}\right),\left(\begin{array}{c}
0 \\
-1 \\
0
\end{array}\right),\left(\begin{array}{l}
4 \\
1 \\
0
\end{array}\right),\left(\begin{array}{c}
0 \\
0 \\
-1
\end{array}\right),\left(\begin{array}{l}
4 \\
0 \\
1
\end{array}\right)\right\} .
\end{aligned}
$$

Let us apply the MGCD. Solving the problem

$$
\min \|(a, v)\|^{2} \quad \text { subject to } \quad(a, v) \in \underline{d} f\left(x_{0}\right)+z_{i}\left(x_{0}\right)
$$

(Step 3 of the MGCD), one can check that for $z_{1}\left(x_{0}\right)=(1,2,0)^{T} \in \bar{d} f\left(x_{0}\right)$ one has

$$
\left(a_{1}\left(x_{0}\right), v_{1}\left(x_{0}\right)\right) \approx(-0.1111,0.2222,0.2222)
$$

Thus, $a_{1}\left(x_{0}\right)<0$, and $x_{0}$ is not a point of global minimum of $f$ by Theorem4.5. Furthermore, one has $x_{1}=x_{0}+\left[a_{1}\left(x_{0}\right)\right]^{-1} v_{1}\left(x_{0}\right)=(0,0)=x^{*}$, i.e. the MGCD finds a point of global minimum of the function $f$ in just one step.

## 5 Conclusions

In this paper we analysed the performance of the method of codifferential descent in the case when the objective function is either convex or piecewise affine. We proved that in the convex case this method has the iteration complexity bound $\mathcal{O}\left(\varepsilon^{-1}\right)$, provided the objective function satisfies some natural regularity assumptions, which in the smooth case are reduced to the Lipschitz continuity of the gradient. We also proposed a modification of the MCD for minimizing nonconvex piecewise affine function, and demonstrated that the modified method as well as the MCD itself find a global minimizer of a nonconvex piecewise affine function in a finite number of steps. The proof of this result is largely based on new global optimality conditions for piecewise affine functions obtained in this article.

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