Two-machine routing open shop on a tree: instance reduction and efficiently solvable subclass

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ABSTRACT

We consider two-machine routing open shop problem on a tree. In this problem a transportation network with a tree-like structure is given, and each node contains some jobs to be processed by two mobile machines. Machines are initially located in the predefined node called the depot, have to traverse the network to perform their operations on each job (and each job has to be processed by both machines in arbitrary order), and machines have to return to the depot after performing all the operations. The goal is to construct a feasible schedule for machines to process all the jobs and to return to the depot in shortest time possible. This problem is known to be NP-hard even in the case when the transportation network consists of just two nodes.

We propose an instance reduction procedure which allows to transform any instance of the problem to a simplified instance on a chain with limited number of jobs. The reduction considered preserves the standard lower bound on the optimum. We describe four possible outcomes of this procedure and prove that in three of them the initial instance can be solved to the optimum in linear time, thus introducing a wide polynomially solvable subclass of the problem considered. Our research can be used as a foundation to construct efficient approximation algorithms for the two-machine routing open shop on a tree.

KEYWORDS

Scheduling; open shop with delays; routing open shop; standard lower bound; instance reduction; polynomially solvable subclass; overloaded node; overloaded edge

1. Introduction

We consider the routing open shop problem, which is a natural combination of a well-known metric traveling salesman problem and a classical scheduling open shop problem. Metric TSP hardly needs an introduction. The open shop problem, introduced by Gonzalez and Sahni [12], can be described as follows. Sets $\mathcal{M} = \{M_1, \ldots, M_m\}$ of machines and $\mathcal{J} = \{J_1, \ldots, J_n\}$ of jobs are given and each machine M_i has to perform an operation on each job J_j , this operation O_{ji} requires $p_{ji} \geq 0$ time units to complete. Each machine has to process jobs in some sequence which is not given in advance but has to be chosen by a scheduler. Operations of the same job cannot be processed

simultaneously. The goal is to construct a schedule of processing of all jobs, *i.e.* to specify non-negative starting and completion times for each operation, such that the conditions above are satisfied and the maximum completion time (also referred to as the makespan) is minimized. Following the standard three-field notation for scheduling problems (see [17] for example) the open shop problem with m machines is denoted by $Om||C_{\max}$. Notation $O||C_{\max}$ is used when the number of machines is not bounded by any constant. The $Om||C_{\max}$ problem is known ([12]) to be polynomially solvable in the case of two machines and is NP-hard for $m \ge 3$. The $O||C_{\max}$ problem is strongly NP-hard. Moreover, unless P = NP, no ρ -approximation algorithm for $O||C_{\max}$ exists $\rho < \frac{5}{4}$ [23].

Several algorithms for solving the two-machine problem $O2||C_{\text{max}}$ to the optimum were proposed over the last decades, for example by Gonzalez and Sahni [12], Pinedo and Schrage [19] and de Werra [10]. All the known algorithms run in linear time and produce optimal schedules with different structures. An important property of the schedules produced by each of those algorithms (and therefore of the optimal schedule) is its so-called *normality*: the makespan of those schedules always coincides

with the standard lower bound
$$\bar{C} \doteq \max \left\{ \max_{i} \sum_{j=1}^{n} p_{ji}, \max_{j} \sum_{i=1}^{m} p_{ji} \right\}.$$

Most of the classical scheduling models (open shop included) share the following disadvantage. It is supposed that each machine is able to start a new operation at the same moment when it completes the previous one. In a real life environment that's not always possible. Usually jobs represent some material objects, therefore some delays between processing operations of two subsequent jobs may be unavoidable. Such delays can be machine-dependent, job- or sequence-dependent, and taking them into account can make the problem harder to investigate. Still, there is a number of papers considering scheduling problems with transportation delays (see [5, 13, 18, 20] for example). However, the problem we are considering in this paper uses a different approach to model transportation delays.

We consider the routing open shop problem [4] which can be described as the open shop meeting the metric traveling salesman problem (TSP). Let the input of the TSP be given by an edge-weighted graph G. Jobs from \mathcal{J} are distributed between the nodes of G, each node contains at least one job. Machines are mobile and are initially located at the predefined node referred to as the depot. Machines have to travel over the edges of G, weights of the edges represent travel times for each machine. Any number of machines can travel over the same edge at the same time. Each machine has to visit each node of G (not necessary once), perform all the respective operations (under the feasibility constraints from the open shop problem), and return back to the depot after processing all the jobs. One has to construct a schedule, specifying starting and completion times for each machine's activity, which is either a performing of some operation or traveling over some edge of G. Note that a machine has to arrive at some node to be able to process a job located at that node.

The makespan $R_{\rm max}$ of a schedule S for the routing open shop is the maximum completion time of machine's activity. (Note that $R_{\rm max} \geqslant C_{\rm max}$ and those to values are different in case the last activity is a traveling of machine to the depot.) The goal is to construct a feasible schedule minimizing the makespan. The routing open shop problem with m machines is denoted by $ROm||R_{\rm max}$, or $ROm|G = X|R_{\rm max}$ if we want to specify the structure X of the graph G. In the latter case we use either standard notation from graph theory, such as K_p for the complete graph with p nodes, or standard terms like tree or chain.

The $ROm||R_{max}$ problem has a certain similarity to the so-called open shop with batch setup times (see [13] for example). In the latter problem jobs a partitioned into several groups referred to as batches, and a machine has to spend a pre-defined setup time when switching from one batch to another. Batches can correspond to sets of jobs from the same node in the routing open shop, and setup times correspond to travel times. However, there are two significant differences. First, setup times usually only depend on the destination batch (in terms of the routing scheduling problem that would mean that the travel time between nodes u and v depend only on v); second, there is no initial state of machines which can correspond to the depot. Including the depot into the picture makes the problem to be a generalization of a metric TSP, and it worth mentioning that such combinations of hard discrete optimization problems get more attention during the last decades. For example, the routing scheduling model appeared independently while considering tasks arising both in production (see, e.g. [1, 2]), so in the service industry [9, 24].

The general routing open shop problem contains the metric TSP as a special case, moreover, the problem with a single machine is equivalent to the metric TSP and therefore is strongly NP-hard. On the other hand, the problem with zero travel times (or with $G=K_1$) is equivalent to the open shop problem and is NP-hard for $m \geq 3$. However it is known, that the routing open shop problem remains NP-hard even in the simplest case $RO2|G=K_2|R_{\rm max}$ with two machines and just two nodes of the transportation network [4]. On the other hand, FPTAS for such a case is described in [15].

Our research aims on the description of wide polynomially solvable cases of NP-hard problem $RO2||R_{\text{max}}$. A few such cases for the $RO2|G = K_2|R_{\text{max}}$ problem can be found in [8, 15], see Sections 2, 3 and 4 for details.

Although our research focuses on the two-machine version of the problem, the progress made in the study of the $ROm||R_{\max}$ problem should be mentioned as well. A series of approximation algorithms for the $ROm||R_{\max}$ problem was developed, starting with the $\frac{m+4}{2}$ -approximation [4]. The best known algorithm up to date has the approximation ration guarantee of $O(\log m)$ [16]. (An intriguing open question is whether an approximation algorithm with a constant approximation ratio exists for the $ROm||R_{\max}$ problem.) A number of papers is devoted to the research of a special case with unit processing times [11, 21, 22].

In this paper we consider the problem $RO2|G = tree|R_{\rm max}$, and describe several special cases which are solvable to the optimum in linear time, with the optimal makespan equal to the standard lower bound. The main special cases are formulated in terms of load distribution between the nodes, the formulation involves the definitions of an overloaded node and an overloaded edge (see Section 2), and is based on a special procedure of instance reduction and its properties (Section 3). For the sake of completeness we also provide a couple of additional special cases for $RO2|G = chain|R_{\rm max}$ problem formulated in terms of properties of the diagonal job, which plays important role in the Gonzalez-Sahni algorithm for the $O2||C_{\rm max}$ problem (Theorem 2.7). These cases are elementary extensions of known classes for $RO2|G = K_2|R_{\rm max}$ described by Kononov in [15] (see Theorem 2.5).

The structure of the remainder of the paper is as follows. Section 2 contains a detailed problem description, necessary notation and the formulation of known results we use. In Section 3, we describe the procedure of instance reduction, which is the main part of our algorithm. Polynomially solvable outcomes of the instance reduction procedure are described in Section 4, followed by the description of sufficient conditions

of polynomial solvability in terms of the properties of the initial instance in Section 5. Concluding remarks and some open questions are given in Section 6.

2. Preliminary notes

Let us give a formal description of the routing open shop problem.

A problem instance combines inputs from the metric TSP and the open shop problem in the following manner. A connected graph $G = \langle V, E \rangle$ is given, a non-negative weight function $\tau: E \to \mathbb{Z}_{\geq 0}$ is defined. One of the nodes $v_0 \in V$ is chosen to be the depot. Jobs from the set $\mathcal{J} = \{J_1, \ldots, J_n\}$ are distributed among the nodes from V. A set of jobs located at $v \in V$ is denoted by $\mathcal{J}(v)$ and is non-empty for any node with possible exclusion of the depot. Machines from the given set $\mathcal{M} = \{M_1, \dots, M_m\}$ are initially located at the depot and each machine can travel over the edges of G, travel time of each machine over an edge $e \in E$ equal to $\tau(e)$. Any number of machines can travel over the same edge in any direction at the same time. Machines are allowed to visit each node multiple times therefore we assume machines use the shortest paths while traveling from one location to another. Each machine M_i has to perform an operation O_{ji} on every job J_j . This operation takes $p_{ji} \in \mathbb{Z}_{\geqslant 0}$ time units and requires the machine to be at the location of J_i : while machine is in the node v, it can only process operations of jobs from $\mathcal{J}(v)$. Different operations of the same job cannot be processed simultaneously, and each machine can process at most one operation at a time. Machines have to return to the depot after processing all the operations. We use notation $p_{ii}(I)$, G(I), $\tau(I;e)$ and $\mathcal{J}(I;v)$, if we want to specify a problem instance I.

A schedule S can be described by specifying the starting time s_{ji} for each operation O_{ji} :

$$S = \{s_{ii} | i = 1, \dots, m, j = 1, \dots, n\}.$$

The completion time of operation O_{ji} in a schedule S is denoted by $c_{ji}(S) = s_{ji}(S) + p_{ji}$, notation S is omitted when not needed.

Let dist(v, u) denote the weighted distance between the nodes v and u (and vice versa), i.e. the minimal total weight of edges belonging to some chain connecting v and u. So dist(v, u) is the shortest time needed for a machine to reach u from v. We also use notation dist(I; v, u) for a specific instance I.

Definition 2.1. A schedule S for an instance I is referred to as *feasible* if it satisfies the following conditions:

1. If $i_1 = i_2$ or $j_1 = j_2$ (but not both) then

$$(s_{i_1i_1}, c_{i_1i_1}) \cap (s_{i_2i_2}, c_{i_2i_2}) = \emptyset.$$

2. If operation of job $J_i \in \mathcal{J}(v)$ is the first to start by machine M_i then

$$s_{ii} \geqslant \operatorname{dist}(I; v_0, v).$$

3. If machine M_i processes operation O_{ji} before the processing of an operation $O_{j'i}$, $J_j \in \mathcal{J}(v)$, and $J_{j'} \in \mathcal{J}(v')$, then

$$s_{j'i} \geqslant c_{ji} + \operatorname{dist}(I; v, v').$$

Condition 1 means that intervals of processing of dependent operations (i.e. operations of the same job or of the same machine) do not overlap. Conditions 2 and 3 mean that machine cannot start an operation before it reaches its location.

Suppose an operation of job $J_i \in \mathcal{J}(v)$ is the last to be processed by machine M_i in some schedule S. Then we define the release time of machine M_i as

$$R_i(S) \doteq c_{ii}(S) + \operatorname{dist}(v, v_0).$$

The makespan of schedule S is $R_{\max}(S) \doteq \max_{i} R_i(S)$. The goal is to find a feasible schedule minimizing the makespan.

For some problem instance I we use the following

Notation 1. • $\ell_i(I) \doteq \sum_{j=1}^n p_{ji}(I)$ — the load of machine M_i ; • $\ell_{\max}(I) \doteq \max_i \ell_i(I)$ — the maximal machine load;

- $d_j(I) \doteq \sum_{i=1}^m p_{ji}(I)$ the length of job J_j ; $d_{\max}(I;v) \doteq \max_{J_j \in \mathcal{J}(v)} d_j(I)$ the maximal job length at node v; $\Delta(I;v) \doteq \sum_{J_j \in \mathcal{J}(v)} d_j(I)$ the load of node v; $\Delta(I) \doteq \sum_{v \in V} \Delta(I;v)$ the $total\ load$ of instance I;

- $T^*(I)$ the optimum of the underlying TSP, i.e. the length of the shortest a cyclic route visiting each node at least once;
- $R_{\text{max}}^*(I)$ the optimal makespan.

We omit I from the notation in case when it does not lead to a confusion.

The following standard lower bound on the optimum for the routing open shop problem was introduced in [3]:

$$\bar{R}(I) \doteq \max \left\{ \ell_{\max}(I) + T^*(I), \max_{v \in V} \left(d_{\max}(I; v) + 2 \operatorname{dist}(I; v_0, v) \right) \right\}. \tag{1}$$

Note that \bar{R} coincides with \bar{C} in case when all edges have zero weight or $G = K_1$ (in this case our problem is reduced to the classical open shop problem).

Our study is focused on the case of two machines. In this case we use simplified notation for the operations of each job J_j : a_j and b_j instead of O_{j1} and O_{j2} , respectively. Moreover, we use the same notation $(a_i \text{ and } b_i)$ for operations' processing times whenever it does not lead to a confusion.

We use the following definitions inherited from [14].

Definition 2.2. A feasible schedule S for a problem instance I is referred to as normal if $R_{\max}(S) = R(I)$. Instance I is normal if it admits construction of a normal schedule.

A class of instances is normal if it consists of normal instances only. A normal class \mathcal{K} is referred to as efficiently normal if there exists a polynomial time algorithm for solving any instance from K to the optimum.

The goal of this paper is to describe wide efficiently normal classes for the RO2|G = $tree|R_{max}$ problem. Below we describe a few such classes known from previous research. The first efficiently normal class of instances of $RO2|G = K_2|R_{\text{max}}$ is due to Kononov [15] and its description is based on so-called diagonal job, which can be defined as follows.

Definition 2.3. The diagonal job of an instance of the $O2||C_{\text{max}}$ (or $RO2||R_{\text{max}}$) problem is such a job $J_r \in \mathcal{J}$ that

$$r = \arg\max_{j} \{\min\{a_j, b_j\}\}.$$

We also need the following

Definition 2.4. A feasible schedule S if referred to as *early* if no operation O_{ji} can start earlier than at $s_{ji}(S)$, providing that the sequences of operations of each job and each machine from S are preserved, without violating the feasibility.

Note that any early schedule is uniquely defined by sequences of operations of each job and each machine.

Theorem 2.5 (Kononov, [15]). A class of instances of the $RO2|G = K_2|R_{\text{max}}$ problem satisfying at least one of the following properties of the diagonal job J_r is solvable to the optimum in linear time:

- 1. $J_r \in \mathcal{J}(v_0)$,
- 2. $d_r \geqslant \ell_{\text{max}}$.

The proof of Theorem 2.5 is based on properties of the Gonzalez-Sahni algorithm for open shop problem $O2||C_{\text{max}}|$ [12]. This proof can be easily extended on a problem $RO2|G = chain|R_{\text{max}}$ under the following conditions:

- 1. the depot v_0 is one of the terminal nodes of chain G,
- 2. job J_r is located at some terminal node of G.

To present this proof, we use the following Gonzalez-Sahni formulation, similar to that described in [15].

Gonzalez-Sahni algorithm [12]

Input: An instance I of $O2||C_{\max}|$ problem.

Output: An optimal schedule for I.

Step 1. Partition the set of jobs \mathcal{J} into two subsets:

$$\mathcal{J}_A = \{J_i | a_i \leq b_i\}, \, \mathcal{J}_B = \{J_i | a_i > b_i\}.$$

- **Step 2.** Let J_r be a diagonal job. Without loss of generality assume $J_r \in \mathcal{J}_A$.
- Step 3. Choose enumerations of operations from $\mathcal{J} \setminus \{J_r\}$ (\mathfrak{A} and \mathfrak{B} for machines M_1 , M_2 respectively) in the following way:
 - 3.1. If $d_r < \ell_{\text{max}}$ then $\mathfrak{A} = \mathfrak{B}$ is a concatenation of arbitrary enumerations of $\mathcal{J}_A \setminus \{J_r\}$ and \mathcal{J}_B ,
 - 3.2. If $d_r \geqslant \ell_{\text{max}}$ then both \mathfrak{A} and \mathfrak{B} are arbitrary and independent.
- **Step 4.** Construct an early schedule in the following manner:
 - 4.1. Machine M_1 processes operation of jobs from $\mathcal{J} \setminus \{J_r\}$ according to \mathfrak{A} , then a_r :
 - 4.2. Machine M_2 processes b_r , then operations of jobs from $\mathcal{J} \setminus \{J_r\}$ according to \mathfrak{B} ;
 - 4.3. Operations of all jobs except J_r are processed first by M_1 , then by M_2 .

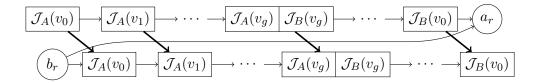


Figure 1. Structure of the schedule for Theorem 2.7, Case 1.

Lemma 2.6 ([12]). The Gonzalez-Sahni algorithm runs in O(n) time and obtains a normal schedule for any instance of $O(2)|C_{max}|$.

Theorem 2.7. Let I be an instance of the $RO2|G = chain|R_{max}$ problem with J_r being a diagonal job, G is a chain (v_0, \ldots, v_g) . Then any of the following conditions implies I is normal and an optimal schedule for I can be found in O(n):

- 1. $J_r \in \mathcal{J}(v_0)$, 2. $J_r \in \mathcal{J}(v)$ and $d_r \geqslant \ell_{\max}$.
- **Proof.** The algorithm for $RO2|G = chain|R_{max}$ is based on the Gonzalez-Sahni algorithm and its properties. The key fact is that operations of jobs from each of the sets \mathcal{J}_A and \mathcal{J}_B (Step 3) can be processed in an arbitrary order, and we show that is it possible to choose the order in such a way that each machine is guaranteed to take an optimal route. Thus the algorithm of building an optimal schedule in both cases has the following structure:
 - Choose a specific orders $\mathfrak A$ and $\mathfrak B$ for operations of non-diagonal jobs to apply at Step 3 of Gonzalez-Sahni algorithm.
 - Build a schedule S_{GS} ignoring travel times using Gonzalez-Sahni algorithm.
 - "Insert" travel times into S_{GS} to obtain a schedule for the initial $RO2||R_{\max}|$ problem instance.

Hereafter we assume without loss of generality that $J_r \in \mathcal{J}_A$, and use notation $\mathcal{J}_A(v) \doteq \mathcal{J}_A \cap \mathcal{J}(v) \setminus \{J_r\}$ $(\mathcal{J}_B(v) \doteq \mathcal{J}_B \cap \mathcal{J}(v))$ for each $v \in V$. Let us specify the orders \mathfrak{A} and \mathfrak{B} for both cases of the Theorem.

Case 1. Let $\mathfrak{A} = \mathfrak{B}$ be a concatenation of arbitrary enumerations of $\mathcal{J}_A(v_0), \mathcal{J}_A(v_1), \ldots, \mathcal{J}_A(v_g), \mathcal{J}_B(v_g), \ldots, \mathcal{J}_B(v_0)$. The structure of the resulting schedule is shown in Figure 1. Thick arcs represent multiple precedence constraints: not between the two blocks, but between respective operations of the same job.

Case 2. Let \mathfrak{A} be a concatenation of arbitrary enumerations of $\mathcal{J}(v_0), \mathcal{J}(v_1), \ldots, \mathcal{J}(v_g) \setminus \{J_r\}$, and \mathfrak{B} be a concatenation of arbitrary enumerations of $\mathcal{J}(v_g) \setminus \{J_r\}, \mathcal{J}(v_{g-1}), \ldots, \mathcal{J}(v_0)$ (see Figure 2). A thick dashed line represents the connection between operations of the diagonal job: $c_{r2} = s_{r1}$.

Note that in both cases the orders $\mathfrak A$ and $\mathfrak B$ comply with the Gonzalez-Sahni algorithm, hence the schedule built is normal. Now it is sufficient to observe that inserting travel times into those schedules does not introduce extra delay intervals into their structure.

The second normal class of instances of $RO2|G = K_2|R_{\text{max}}$ was introduced in [8] and is based on the following

Definition 2.8. A node v is referred to as *superoverloaded* if jobs from $\mathcal{J}(v)$ can be partitioned into three subsets $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ such that

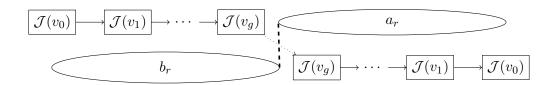


Figure 2. Structure of the schedule for Theorem 2.7, Case 2.

1.
$$\forall k \in \{1, 2, 3\} \sum_{J_j \in \mathcal{J}_k} d_j \leq \bar{R} - 2 \text{dist}(v_0, v),$$

2. $\forall k \neq l \in \{1, 2, 3\} \sum_{J_j \in \mathcal{J}_k \cup \mathcal{J}_l} d_j > \bar{R} - 2 \text{dist}(v_0, v).$

Such a partition is referred to as *irreducible* one.

It was proved in [8] that any instance of $RO2|G = K_2|R_{\text{max}}$ containing a superoverloaded node is normal, and the optimal schedule for such an instance can be built in linear time providing that an irreducible partition is known. We provide an elementary extension of this result on the special case with G = chain in Section 4.

Unfortunately the verification of existence of an irreducible partition is NP-complete [8] and therefore the problem of obtaining of such a partition is NP-hard. However, there is a description ([8]) of a sufficient condition for a node to be superoverloaded, together with a polynomial time procedure of obtaining of an irreducible partition.

Theorem 2.9 ([8]). Let $\Delta(v) > \frac{3}{2}(\bar{R} - 2\operatorname{dist}(v_0, v)) + d_{\max}(v)$. Then v is superoverloaded, and an irreducible partition can be obtained by the following procedure.

Procedure Partition.

Let
$$\mathcal{J}(v) = \{J_1, \dots, J_k\}$$
.

Step 1. Find minimal $x > 1$ such that $\sum_{j=1}^x d_j > \frac{1}{2}(\bar{R} - 2\mathrm{dist}(v_0, v))$.

Set $\mathcal{J}_1 = \{J_1, \dots, J_x\}$ and $X = \sum_{j=1}^x d_j$.

Step 2. Find minimal $y > x$ such that $\sum_{j=x+1}^y d_j > \bar{R} - 2\mathrm{dist}(v_0, v) - X$.

Set $\mathcal{J}_2 = \{J_{x+1}, \dots, J_y\}$.

Step 3. Set $\mathcal{J}_3 = \{J_{y+1}, \dots, J_k\}$.

This procedure runs correctly if for each job $J_j \in \mathcal{J}(v)$ its length $d_j \leqslant \frac{1}{2}(\bar{R} - 2\mathrm{dist}(v_0, v))$. The condition of Theorem 2.9 implies that inequality, and also guarantees the irreducibility of the partition obtained. In general case the procedure Partition still may be applied if a special treatment for "long" jobs (with $d_j > \frac{1}{2}(\bar{R} - 2\mathrm{dist}(v_0, v))$, if any) is provided. We describe the following version of the procedure guaranteed to run correctly in any case.

Procedure Partition 2.0.

Let
$$\mathcal{J}(v) = \{J_1, ..., J_k\}.$$

Step 0. If $\mathcal{J}(v)$ contains long jobs, rearrange the enumeration of jobs to comply with the following conditions:

0.1. J_1 is a long job,

0.2. J_2 is also a long job unless there are no more long jobs except for J_1 .

Step 1. Find minimal x > 1 such that $\sum_{j=1}^{x} d_j > \frac{1}{2}(\bar{R} - 2\operatorname{dist}(v_0, v) + d_{\max}(v))$.

If no such x exists, set x = k.

Set
$$\mathcal{J}_1 = \{J_1, \dots, J_x\}$$
 and $X = \sum_{j=1}^x d_j$.

If x = k, set $\mathcal{J}_2 = \mathcal{J}_3 = \emptyset$ and STOP.

Step 2. Find minimal y > x such that $\sum_{j=x+1}^{y} d_j > \bar{R} - 2\operatorname{dist}(v_0, v) + d_{\max}(v) - X$.

If no such y exists, set y = k.

Set $\mathcal{J}_2 = \{J_{x+1}, \dots, J_y\}$. If y = k, set $\mathcal{J}_3 = \emptyset$ and **STOP**.

Step 3. Set $\mathcal{J}_3 = \{J_{y+1}, \dots, J_k\}.$

Clearly, each step of the procedure requires O(n) time. However, without the conditions of Theorem 2.9 we cannot guarantee that the partition obtained by the procedure Partition 2.0 is irreducible. Note how we use this procedure in our algorithm in the next Section.

3. Instance reduction procedure

In this section we study some general properties of an instance of $RO2||R_{\text{max}}|$ and describe the reduction procedure which helps to reduce the number of jobs and to simplify the graph structure preserving the standard lower bound R. One of the important properties of the procedure is its reversibility: any feasible schedule for a reduced instance can be treated as a feasible schedule for the initial instance with the same makespan. In general case this procedure can increase the optimal makespan. However, in the next sections we prove that for our special cases of $RO2|G = tree|R_{max}$ the instance reduction procedure also preserves the optimum. Therefore, it can be used as a main part of an exact algorithm for solving the initial instance.

The procedure is based on two types of instance transformation: job aggregation and terminal edge contraction. The first one is described in detail in [7], while the second was used in [6] for a certain generalization of the routing open shop problem. We provide all the necessary details below.

Definition 3.1. Let I be an instance of the problem $ROm||R_{\text{max}}$ with graph G = $\langle V; E \rangle$, and $\mathcal{K} \subseteq \mathcal{J}(I; v)$ for some $v \in V$. Then we say that instance I' is obtained from I by aggregation of jobs from K if

$$\mathcal{J}(I';v) \doteq \mathcal{J}(I;v) \setminus \mathcal{K} \cup \{J_{j_{\mathcal{K}}}\}, \ \forall i = 1,\ldots, m \ p_{j_{\mathcal{K}}i}(I') \doteq \sum_{J_{j} \in \mathcal{K}} p_{ji}(I),$$

$$\forall u \neq v \ \mathcal{J}(I'; u) = \mathcal{J}(I; u).$$

(Here $j_{\mathcal{K}}$ is some new job index. A job $J_{j_{\mathcal{K}}}$ is to replace the set of jobs \mathcal{K} .) An instance \tilde{I} obtained from I by a series of job aggregations is referred to as an aggregation of I.

The idea behind the job aggregation is easy: to partition jobs into some number of groups, and treat each group as a new job with processing times equal to the total processing times of the jobs combined. Similar approach is used in de Werra's algorithm for the $O2||C_{\max}|$ problem [10].

It is easy to observe that any feasible schedule for any aggregation \tilde{I} can be treated as a feasible schedule for the initial instance I: one just needs to replace an aggregated operation with a sequence of operations of jobs from K to be processed in any order with no idle time. Therefore, $R_{\max}^*(\tilde{I}) \geq R_{\max}^*(I)$. Also as soon as we obtained a new job J_{j_K} in $\mathcal{J}(I';v)$, it is possible that $d_{j_K} > d_{\max}(I;v)$, so job aggregation can lead to the growth of the standard lower bound. Specifically, (1) implies

$$\bar{R}(I') > \bar{R}(I)$$
 if and only if $d_{j_{\kappa}} > \bar{R}(I) - 2\operatorname{dist}(v_0, v)$. (2)

We use job aggregation to simplify the instance preserving the standard lower bound. Such an aggregation is referred to as a *valid* one. An instance with no further legal job aggregation possible is called *irreducible*.

A natural question arises, if it is possible to perform a valid job aggregation of a whole set $\mathcal{J}(I;v)$ for some $v \in V$. To answer that question, we use the following definition from [7].

Definition 3.2. A node $v \in V$ of an instance I of the problem $ROm||R_{\max}$ is referred to as overloaded if

$$\Delta(I; v) > \bar{R}(I) - 2\operatorname{dist}(I; v_0, v).$$

Otherwise the node is called *underloaded*.

Note that Definition 2.8 implies $\Delta(v) > \frac{3}{2}(\bar{R} - 2\text{dist}(v_0, v))$, therefore any super-overloaded node is overloaded as well.

The job aggregation of the set $\mathcal{J}(I;v)$ is valid if and only if the node v is underloaded. Therefore, any node containing single job is an underloaded one.

By $L_V(I)$ we denote the number of overloaded nodes in an instance I. It was proved in [7] that for every instance I of the $RO2||R_{\max}|$ problem $L_V(I) \leq 1$. Further in this section we prove a more general result (Proposition 3.5).

Now let us describe the terminal edge contraction operation.

Definition 3.3. Let $v \in V \setminus \{v_0\}$ be some terminal node in graph G, containing a single job J_j in an instance I of the $ROm||R_{\max}$ problem. Let $e = [v, u] \in E$ be the edge incident to v. By the *contraction* of the edge e we understand the following instance transformation:

$$\mathcal{J}(I';u) \doteq \mathcal{J}(I;u) \cup \{J_j\}; \ p_{ji}(I') \doteq p_{ji}(I) + 2\tau(e); \ G(I') \doteq G(I) \setminus \{v\}.$$

In other words, job J_j is translated from v to u, while its operations' processing times increase by $2\tau(e)$ each. After that translation node v is obsolete (contains no jobs) and to be removed from G.

Consider an instance I' obtained from I by the contraction of edge e. Any feasible schedule for I' can be treated as a feasible schedule for the initial instance I. One just

needs to replace a scheduled interval of a new operation O_{ji} with three consecutive intervals: traveling of the machine M_i over the edge e to the node v, performing of the old operation O_{ji} , and traveling back to the node u.

Note that an edge contraction increases each machine load be $2\tau(e)$ while decreasing T^* by the same amount, therefore preserving the sum $\ell_{\max} + T^*$. However the length of J_j increases by $2m\tau(e)$ which might lead to the growth of the standard lower bound. We want to avoid that. Consider the two-machine case of our problem. The following definition describes the exact condition, under which an edge contraction increases \bar{R} .

Definition 3.4. Let $v \in V \setminus \{v_0\}$ be some terminal node in graph G, containing a single job J_j in instance I of the $RO2||R_{\max}|$ problem. Let $e = [v, u] \in E$ be the edge incident to v. The edge e is referred to as overloaded if

$$d_i(I) + 4\tau(e) > \bar{R}(I) - 2\operatorname{dist}(I; v_0, u),$$
 (3)

and underloaded otherwise.

(Note that we *could* perform a contraction of an overloaded edge — meaning that the edge is terminal, the respective terminal node is not the depot and contains a single job — but this would increase the standard lower bound. In the case the edge contraction cannot be performed, the edge is neither overloaded nor underloaded.)

For any problem instance I, we denote the number of overloaded edges by $L_E(I)$. The following property of any instance of $RO2||R_{\text{max}}$ is fundamental for the procedure of instance reduction.

Proposition 3.5. Let I be an instance of the problem $RO2||R_{\text{max}}|$. Then $L_V(I) + L_E(I) \leq 1$.

Proof. As proved in [7], any instance of $RO2||R_{\text{max}}$ contains at most one overloaded node, so $L_V(I) \leq 1$. Let us prove, that I contains at most one overloaded edge. Note that (1) implies

$$\Delta(I) = \ell_1(I) + \ell_2(I) \leqslant 2(\bar{R}(I) - T^*(I)). \tag{4}$$

Let v and v' be two different terminal nodes with single job in each, J_j and $J_{j'}$ respectively; e = [u, v] and e' = [u', v'] be the edges, incident to v and v', respectively, and both edges are overloaded. (Note that there is a possibility that u = u'.) Due to the metric property of distances we have

$$T^{\prime *} \geqslant \operatorname{dist}(v_0, u) + \operatorname{dist}(v_0, u^{\prime}). \tag{5}$$

From (3) we have

$$d_j + 4\tau(e) > \bar{R} - 2\operatorname{dist}(v_0, u); \ d_{j'} + 4\tau(e') > \bar{R} - 2\operatorname{dist}(v_0, u'),$$

and therefore

$$\Delta \geqslant d_j + d_{j'} > 2\bar{R} - 2\operatorname{dist}(v_0, u) - 2\operatorname{dist}(v_0, u') - 4\tau(e) - 4\tau(e'). \tag{6}$$

Consider a graph $G' = G \setminus \{v, v'\}$. Let T'^* be the optimum of the TSP on G'. Then

due to the fact that edges e and e' are terminal,

$$T^* = T'^* + 2\tau(e) + 2\tau(e'). \tag{7}$$

Indeed, in order to visit terminal nodes, one needs to travel twice over their respective incident edges. Combining (6), (5) and (7), we obtain the inequality

$$\Delta > 2\bar{R} - 2T'^* - 4\tau(e) - 4\tau(e') \geqslant 2\bar{R} - 2T^*.$$

By contradiction with (4) we have $L_E(I) \leq 1$.

Now suppose $L_V(I) = L_E(I) = 1$. Let e = [u, v] be the overloaded edge, $v \neq v_0$ is terminal node with single job J_j . Note that node v is underloaded as it contains a single job. Let $v' \neq v$ be the overloaded node. Then, by Definitions 3.2 and 3.4 we have

$$\Delta(v') > \bar{R} - 2\operatorname{dist}(v_0, v'), \tag{8}$$

$$d_j + 4\tau(e) > \bar{R} - 2\operatorname{dist}(v_0, u). \tag{9}$$

By using reasoning similar to that of (5) and (7), we deduce $T^* \ge \operatorname{dist}(v_0, v') + \operatorname{dist}(v_0, u) + 2\tau(e)$. Using this inequality, together with (8) and (9), we obtain

$$\Delta \geqslant \Delta(v') + d_j > 2\bar{R} - 2\operatorname{dist}(v_0, v') - 2\operatorname{dist}(v_0, u) - 4\tau(e) \geqslant 2\bar{R} - 2T^*,$$

contradicting (4). This concludes the proof of the Proposition.

The idea of the following instance reduction procedure is simple. First, we aggregate jobs in all the underloaded nodes to obtain single job in each, then contract all the underloaded edges, and repeat this step until there is no underloaded edge and each underloaded node contains exactly one job. Second, we deal with the only overloaded node (if any), using the Procedure Partition 2.0 and aggregation of the obtained job sets. This way we transform the initial instance, preserving the standard lower bound. The instance obtained is irreducible. Moreover, any further operation of terminal edge contraction would increase the standard lower bound.

Note that the reduction procedure is not used for solving a problem instance, but to simplify it and consequently verify it's properties. Depending on the outcome of the procedure (see Lemma 3.6) we further decide, whether the instance belong to the efficiently normal subclass we announced (see Theorem 4.3). The simplification procedure is described in detail in Table 1.

Let us illustrate this procedure on a small instance with transportation network from Figure 3 and job data from Table 2. Job data contains the description of jobs of type $J_j(a_j,b_j)$, with each node's load calculated in the last row. For this instance we have $T^*=28$ and $\ell_1=\ell_2=28$. As soon as each job length (and even node's load) is smaller than l_{max} , we have $\bar{R}=\ell_{\text{max}}+T^*=56$. We can also observe that for each node v we have $\Delta(v)+2\text{dist}(v_0,v)<\bar{R}$, therefore all the node are underloaded. After performing Step 1 of the procedure we will obtain an instance with a single job at each node (see Table 3).

Let us perform Step 2. Consider terminal node v_1 . The edge $[v_1, v_2]$ is underloaded as soon as $d_3 + 4\tau([v_1, v_2]) + 2\operatorname{dist}(v_2, v_0) = 5 + 4 + 2 = 11 < \bar{R}$. After the edge

Reduction

INPUT: An instance I of the problem $RO2||R_{\text{max}}|$.

OUTPUT: A simplified irreducible instance I.

Step 1. For each underloaded $v \in V$ perform the job aggregation of $\mathcal{J}(v)$.

Step 2. For each terminal node $v \neq v_0$:

2.1. e = [u, v] (e is incident to v, u is adjacent to v),

2.2. If e is underloaded then

2.2.1. Let J_i is the only job in $\mathcal{J}(v)$,

2.2.2. Perform the contraction of e,

2.2.3. If u is underloaded then perform the job aggregation of $\mathcal{J}(u)$.

Step 3. If some $v \in V$ is overloaded then

3.1. Obtain sets $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ by application of the Partition 2.0 to node v,

3.2. For each non-empty set \mathcal{J}_k perform the job aggregation of \mathcal{J}_k ,

3.3. If an aggregation of two obtained jobs in $\mathcal{J}(v)$ with the smallest length is valid **then** perform that aggregation.

Table 1. The simplification procedure Reduction.

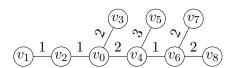


Figure 3. A sample transportation network.

contraction (Step 2.2.2) we have modified job $J_3(6,3)$ translated to node v_2 , so $\Delta(v_2)$ is now 15. The node v_2 is still underloaded, so we perform job aggregation (Step 2.2.3), see Table 4. Further contractions of terminal edges is show in Table 5. Terminal nodes are eliminated in order v_2 , v_3 , v_5 , v_7 , v_8 , v_6 . After the last contraction the node v_4 becomes overloaded, Step 3 does nothing and the procedure stops. The initial instance is reduced to a simple instance with two nodes v_0 and v_4 with three jobs.

Note that the procedure Reduction obtains a reversible instance in O(n) time. Indeed, Step 1 requires O(n) time. Step 2 is repeated once for each terminal node (and total number of nodes is not greater than n), and takes constant amount of time for a single edge contraction. Step 3 is also linear, as it's running time is majored by that of the procedure Partition 2.0.

The following Lemma describes all possible variants of the reduced instance for the problem $RO2|G=tree|R_{\rm max}$.

Lemma 3.6. Let I be an instance of $RO2|G = tree|R_{max}$ and \tilde{I} is obtained from I by the procedure Reduction. Then $\bar{R}(\tilde{I}) = \bar{R}(I)$ and the graph $G(\tilde{I})$ satisfies exactly one of the following conditions:

- 1. $G(\tilde{I})$ has a single node v_0 ;
- 2. $G(\tilde{I})$ is a chain connecting v_0 with an overloaded node v and each node contains

v:	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
Jobs:	$J_1(1,2)$	$J_3(4,1)$	$J_4(1,1)$	$J_7(2,1)$	$J_8(5,2)$	$J_{10}(2,1)$	$J_{11}(3,2)$	$J_{12}(1,4)$	$J_{13}(2,3)$
	$J_2(3,4)$		$J_5(1,1)$		$J_9(1,1)$				$J_{14}(1,5)$
			$J_6(1,1)$						
$\Delta(v)$:	10	5	6	3	9	3	5	5	11

Table 2. Job data for a sample instance.

Γ	v:	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
Г	Jobs:	$J_{1,2}(4,6)$	$J_3(4,1)$	$J_{4,5,6}(3,3)$	$J_7(2,1)$	$J_{8,9}(6,3)$	$J_{10}(2,1)$	$J_{11}(3,2)$	$J_{12}(1,4)$	$J_{13,14}(3,8)$
Γ	$\Delta(v)$:	10	5	6	3	9	3	5	5	11

Table 3. Job data for a sample instance after Step 1.

v:	v_0	v_2	v_3	v_4	v_5	v_6	v_7	v_8
Jobs:	$J_{1,2}(4,6)$	$J_3(6,3)$	$J_7(2,1)$	$J_{8,9}(6,3)$	$J_{10}(2,1)$	$J_{11}(3,2)$	$J_{12}(1,4)$	$J_{13,14}(3,8)$
$\Delta(v)$:	10	$J_{4,5,6}(3,3)$ 15	3	9	3	5	5	11
v:	v_0	v_2	v_3	v_4	v_5	v_6	v_7	v_8
Jobs:	$J_{1,2}(4,6)$	$J_{3,4,5,6}(9,6)$	$J_7(2,1)$	$J_{8,9}(6,3)$	$J_{10}(2,1)$	$J_{11}(3,2)$	$J_{12}(1,4)$	$J_{13,14}(3,8)$
$\Delta(v)$:	10	15	3	9	3	5	5	11

Table 4. Performing Steps 2.2.2 and 2.2.3 for v_1 .

only one job except v which contains two or three jobs;

3. $G(\tilde{I})$ is a chain connecting v_0 with a node v with single job at each node, and the edge incident to v is overloaded.

Proof. Each job aggregation used in the Procedure is valid and, therefore, does not grow the standard lower bound. Terminal edge contractions are applied only to underloaded edges, therefore, $\bar{R}(\tilde{I}) = \bar{R}(I)$.

Consider the case $G(\tilde{I}) \neq K_1$. Note that Steps 1 and 2.2.3 guarantee that each underloaded node in \tilde{I} contains exactly one job. Therefore, each terminal node in $G(\tilde{I})$ is either v_0 , or overloaded, or incident to an overloaded edge. By Proposition 3.5 the graph $G(\tilde{I})$ contains at most two terminal nodes (one of which is the depot) and hence is a chain. Step 2 of the procedure continues until we have no more underloaded terminal edges. Therefore, a terminal edge is contracted unless it is overloaded, or incident to the depot, or incident to an overloaded node. As soon as the first and the third options are mutually exclusive, the Lemma follows.

As soon as the procedure Reduction preserves \bar{R} , we have the following property: if the reduced instance \tilde{I} is normal, then the initial instance I is normal as well, and any normal (and hence optimal) schedule for \tilde{I} can be easily transformed into an optimal schedule of I. Obviously \tilde{I} is normal in case 1 of Lemma 3.6: the problem is reduced to a classical $O2||C_{\max}|$ and a normal schedule can be built by the Gonzalez-Sahni algorithm (Lemma 2.6). Therefore a class of instances of $RO2|G = tree|R_{\max}$, for which the procedure Reduction contracts the initial tree into a single node is efficiently normal and can be solved in three steps: Reduction, Gonzalez-Sahni algorithms and restoring a schedule for the initial instance. In the next Section we prove similar properties for case 3 and (under a certain condition) for case 2.

v:	v_0	v_3 v_4			v_5		v_6		v_7		v_8	
Jobs:	$J_{1,2,3,4,5,6}(15,14)$	$J_7(2,1)$ $J_{8,9}(6,1)$, 3)	$J_{10}(2,1)$		$J_{11}(3,2)$		$J_{12}(1,4)$	J_{13}	$J_{13,14}(3,8)$	
$\Delta(v)$:	29	3 9			3		5		5		11	
v:	v_0	v_4		v_5	v_5 v_6		v_7		v_8			
Jobs:	$J_{1,2,3,4,5,6,7}(21,19)$	$J_{8,9}(6,3)$ J_{10}		$(2,1)$ $J_{11}(3,1)$			$J_{12}(1,4)$		$J_{13,14}(3,8)$			
$\Delta(v)$:	40	9		3 5		5	5		11			
v:	v_0		v_4		v_6		v_7		v_8			
Jobs:	$J_{1,2,3,4,5,6,7}(21,19)$	$J_{8,9,10}(14,10)$							$_{,14}(3,8)$			
$\Delta(v)$:	40	24			5		5		11			
v:	v_0	v_4		v_6			v_8					
Jobs:	$J_{1,2,3,4,5,6,7}(21,19)$	$J_{8,9,10}(14,10)$		$J_{11,12}(8,10)$		0)	$J_{13,14}$					
$\Delta(v)$:	40	24		18			1.	1				
v:	v_0	v_4			v_6							
Jobs:	$J_{1,2,3,4,5,6,7}(21,19)$	$J_{8,9,10}(14,10)$		$J_{11,12,13,14}(15$			22)					
$\Delta(v)$:	40	24		37								
v:	v_0	v_4										
Jobs:	$J_{1,2,3,4,5,6,7}(21,19)$	$J_{8,9,10}$ (٥.4									
			$J_{11,12,13,14}(17,17)$									
$\Delta(v)$:	40	65										

Table 5. Performing Steps 2.2.2 and 2.2.3 for nodes v_2 , v_3 , v_5 , v_7 , v_8 , v_6 .

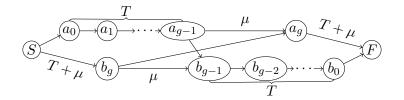


Figure 4. A scheme of an optimal schedule for an instance with overloaded edge.

4. Easy solvable cases on a chain

In this section we establish the normality of two special cases of $RO2|G = chain|R_{\text{max}}$. In both cases we assume that the instance is irreducible, the depot is one of the ends of G, while the other end is either incident to an overloaded edge (which corresponds to a case 3 of Lemma 3.6) or contains exactly three jobs (which is a special subcase of a case 2 of Lemma 3.6). A trivial corollary of those results is Theorem 4.3 providing a formulation of efficiently normal subcases for the $RO2|G = tree|R_{\text{max}}$ in terms of the outcome of the procedure Reduction applied to an instance of the problem.

In this Section we construct early schedules using partial orders of the operations, according to the remark to the Definition 2.4. Necessary partial orders are referred to as *schemes* and are described graphically. Auxiliary nodes S and F represent start and finish moments of a schedule.

Lemma 4.1. Let I be an instance of $RO2|G = chain|R_{max}$, with G(I) being a chain (v_0, \ldots, v_g) , $g \ge 1$, each node v_p contains a single job J_p and the edge $[v_{g-1}, v_g]$ is overloaded. Then one can build a normal schedule S for I in linear time.

Proof. Let $T \doteq \operatorname{dist}(v_0, v_{g-1})$ and $\mu \doteq \tau([v_{g-1}, v_g])$. Then $T^* = 2(T + \mu)$. As soon as the edge $[v_{g-1}, v_g]$ is overloaded, we have

$$d_q + 4\mu > \bar{R} - 2T.$$

Therefore, (4) implies

$$\sum_{j=0}^{g-1} d_j + 2T = \Delta - d_g + 2T < 2\bar{R} - 2T^* - \bar{R} + 2T + 4\mu + 2T = \bar{R}.$$
 (10)

Let S be the early schedule built according to the scheme from Figure 4. Following a well-known fact from project planning, the makespan of the schedule S coincides with the length of a critical path in graph from Figure 4:

$$R_{\max}(S) = \max \left\{ \ell_1 + T^*, \ell_2 + T^*, d_g + 2 \operatorname{dist}(v_0, v_g), \sum_{j=0}^{g-1} d_j + 2T \right\}.$$

From (1) and (10) we obtain $R_{\text{max}}(S) = \bar{R}(I)$, concluding the proof.

Lemma 4.2. Let I be an irreducible instance of $RO2|G = chain|R_{max}$, with G(I) being a chain (v_0, \ldots, v_g) , $g \ge 1$, and v_g contains three jobs, while each underloaded

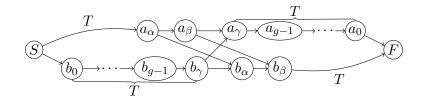


Figure 5. A scheme of the schedule S_1 .

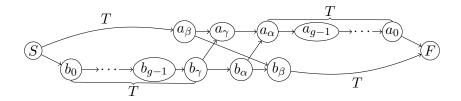


Figure 6. A scheme of the schedule S_2

node v_p contains a single job J_p , $p = 0, \ldots, g-1$. Then one can build a normal schedule S for I in linear time.

Proof. Let $\mathcal{J}(v_g) = \{J_\alpha, J_\beta, J_\gamma\}$. Let $T \doteq \operatorname{dist}(v_0, v_g)$, then we have $T^* = 2T$. Without loss of generality we may assume

$$a_{\alpha} \leqslant \min\{a_{\gamma}, b_{\gamma}, b_{\alpha}\} \tag{11}$$

(this can be achieved by renumeration of machines and/or jobs J_{α}, J_{γ}). As soon as I is irreducible we have

$$d_{\alpha} + d_{\beta} > \bar{R} - 2\operatorname{dist}(v_0, v_q) = \bar{R} - T^*. \tag{12}$$

Together (4) and (12) imply

$$\sum_{j=0}^{g-1} d_j + d_\gamma + T^* < \bar{R}. \tag{13}$$

Consider the early schedules S_1 and S_2 built according to the schemes from Figures 5 and 6, accordingly.

By the reasoning similar to that of the proof of Lemma 4.1, using (11) we have

$$R_{\max}(S_1) = \max \left\{ \ell_1 + T^*, \ell_2 + T^*, \sum_{j=0}^{g-1} d_j + d_\gamma + T^*, T^* + a_\alpha + a_\beta + b_\beta \right\}.$$

We prove that at least one of S_1 and S_2 is normal. Assume otherwise. Then $R_{\text{max}}(S_1) > \bar{R}$ together with (1) and (13) imply

$$R_{\max}(S_1) = T^* + a_\alpha + a_\beta + b_\beta,\tag{14}$$

and $R_{\max}(S_2) > \bar{R}$ implies

$$R_{\max}(S_2) = T^* + \sum_{j=0}^{g-1} d_j + b_\gamma + \max\{a_\gamma, b_\alpha\} + a_\alpha.$$

By the assumption (11) we have

$$R_{\max}(S_2) \leqslant T^* + \sum_{j=0}^{g-1} d_j + b_\gamma + a_\gamma + b_\alpha.$$
 (15)

Therefore, by (4)

$$R_{\max}(S_1) + R_{\max}(S_2) = 2T^* + \sum_{j=0}^{g-1} d_j + d_\alpha + d_\beta + d_\gamma = 2T^* + \Delta \leqslant 2\bar{R},$$

hence both S_1 and S_2 are normal. Lemma is proved by contradiction.

Now we are ready to declare the main result of this Section.

Theorem 4.3. Let I be an instance of the $RO2|G = tree|R_{max}$ problem, \tilde{I} is obtained from I by Reduction procedure and one of the following conditions is true:

- 1. $G(\tilde{I}) = K_1$,
- 2. $G(\tilde{I})$ contains an overloaded edge,
- 3. $G(\tilde{I})$ contains an superoverloaded node.

Then one can build a normal schedule S for I in linear time.

Theorem 4.3 can be seen as a description of efficiently normal class of instances of $RO2|G = tree|R_{max}$, formulated in terms of the outcome of the Procedure Reduction. In the next Section we suggest a formulation of sufficient conditions of efficient normality in terms of the properties of the initial instance (Theorem 5.3).

5. Sufficient conditions of polynomial solvability

Consider an instance I of the $RO2|G = tree|R_{max}$ problem. Let us introduce some notation and definitions convenient for the description of the further results.

Definition 5.1. Let $G' = \langle V'; E' \rangle$ be a subtree of G. We define the weight of G' as

$$W(G') \doteq \sum_{v \in V'} \Delta(v) + 4 \sum_{e \in E'} \tau(e).$$

It is easy to observe that W(G) is preserved by any operation of job aggregation and terminal edge contraction (and therefore by the Procedure Reduction). Moreover, if during the Reduction some subtree G' is completely contracted into a node v, then in the instance \tilde{I} obtained we have $\Delta(\tilde{I}; v) = W(G')$.

- **Notation 2.** Let $v \neq v_0$ and $e = [u, v] \in E$ the edge, incident to v in the chain connecting v_0 and v. Then G_v is the connected component of $G \setminus \{e\}$ containing v. In other words, G_v is a subtree of G induced by the set of all nodes u such that v belongs to a chain connecting v_0 and u. For the sake of completeness let $G_{v_0} = G$.
 - Let $e \in E$. Then by v(e) we denote the node incident to e such that e belongs to a chain connecting v_0 and v(e).
 - $B_{G'}(v)$ the set of all nodes of G', adjacent to v.

Proposition 5.2. Let $W(G_v) \leq \bar{R} - 2 \mathrm{dist}(v_0, v)$. Then during the Reduction procedure no node from G_v becomes overloaded.

Proof. It is sufficient to prove that any tree G_v with such a property cannot contain an overloaded node. Assume otherwise, let some $u \in G_v$ be overloaded. Then by Definition 3.2

$$\Delta(u) > \bar{R} - 2\operatorname{dist}(v_0, u) \geqslant \bar{R} - 2\operatorname{dist}(v_0, v) \geqslant W(G_v),$$

which contradicts with Definition 5.1.

The next Theorem describes an efficiently normal class of instances of $RO2|G = tree|R_{\text{max}}$.

Theorem 5.3. Suppose an instance I of the $RO2|G = tree|R_{max}$ problem satisfies at least one of the following properties:

- 1. The depot v_0 is overloaded.
- 2. $\forall v \in B_{G(I)}(v_0) \ W(v) \leqslant \bar{R} 2\tau([v_0, v]).$
- 3. There exists $e \in E$ such that

$$W\left(G_{v(e)}\right) \in \left(\bar{R} - 2\operatorname{dist}(v_0, v) - 2\tau(e), \bar{R} - 2\operatorname{dist}(v_0, v)\right]. \tag{16}$$

4. There exists $v \neq v_0$ such that 4.1. $\forall u \in B_{G_v}(v) \ W(G_u) \leq \bar{R} - 2 \text{dist}(v_0, u), \text{ and}$ 4.2. $W(G_v) > \frac{3}{2}(\bar{R} - 2 \text{dist}(v_0, v)) + M, \text{ there}$

$$M = \max \left\{ d_{\max}(v), \max_{u \in B_{G_v}(v)} \left(W(G_u) + 4\tau([v, u]) \right) \right\}.$$

Then a normal schedule for I can be built in linear time.

Proof. It is sufficient to prove that such an instance I satisfy the conditions of Theorem 4.3.

- 1. By Lemma 3.6 the only possible outcome of the Reduction procedure is that G is contracted into v_0 , and we have condition 1 of Theorem 4.3.
- 2. Let us prove that in this case the initial tree is contracted by the Reduction procedure either into v_0 or into a chain containing an overloaded edge. By Proposition 5.2, in this case each of the subtrees G_v , $v \in B_G(v_0)$ is either contracted into v or there occurs an overloaded edge. In the later case we have condition 2 of Theorem 4.3, otherwise each of G_v is contracted into v, and after that v is underloaded. Any further reduction can only end up either with v_0 or a link with

- overloaded edge (if any of the edges incident to v_0 become overloaded). Either way we have one of the conditions 1, 2 of Theorem 4.3.
- 3. Let us prove that condition (16) guarantees that the Reduction procedure will end up with a chain containing an overloaded edge. Indeed, by (16) and Proposition 5.2 the tree $G_{v(e)}$ is either contracted into v(e) or an overloaded edge occurs during the process and we have the claim. In the first case we obtain $\Delta(v(e)) = W(G_{v(e)})$, by (16) v(e) is underloaded, and hence all the jobs from $\mathcal{J}(v(e))$ are aggregated into a single job J_x of length $d_x = \Delta(v(e))$ during the Reduction. Let e = [u, v]. From (16) we have

$$d_x > \bar{R} - 2\operatorname{dist}(v_0, v) - 2\tau(e) = \bar{R} - 2\operatorname{dist}(v_0, u) - 4\tau(e),$$

and by Definition 3.4 the edge e is overloaded.

4. Suppose that no overloaded edge occurs during the Reduction of subtree v. By condition 4.1 and Proposition 5.2 G_v is contracted into overloaded node v. The value M equals the maximal job length in $\mathcal{J}(v)$ after the reduction prior to Step 3 of the Reduction procedure: indeed, for each $u \in B_{G_v}(v)$ all jobs from G_u are reduced into a single job of length $W(G_u)$, which is further translated into v while its length becomes $W(G_u) + 4\tau([v, u])$ by the contraction of edge [v, u]. Now by condition 4.2 set of jobs from $\mathcal{J}(v)$ satisfy the Theorem 2.9, an as soon as Step 3 of the Reduction procedure is an application of the Procedure Partition 2.0 (see Section 2), all the jobs from $\mathcal{J}(v)$ are aggregated into exactly three jobs, and v is superoverloaded. Hence the reduced instance satisfies condition 3 of Theorem 4.3.

6. Conclusion

We described the instance reduction procedure, and proved that any instance with G=tree can reduced to a chain preserving the standard lower bound. We may distinguish four outcomes of that procedure: Lemma 3.6 describes three, but the second one actually consists of two: with 2 and 3 jobs at the overloaded node. We cannot guarantee the normality of the initial instance only in one of that four outcomes (with 2 jobs in the overloaded node). However it would be of interest to find out, how often does such an abnormal outcome occur, and therefore, how justified the use of term "normal" in this context is. This might be a subject for an experimental study.

On top of that we suggest the following directions for future investigation.

Problem 1. Find new normal (or efficiently normal) classes of instances of the $RO2|G = cycle|R_{max}$ problem.

Problem 2. Determine whether $RO2|G = tree|R_{\text{max}}$ is strongly NP-hard.

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