# On Basic Operations Related to Network Induction of Discrete Convex Functions 

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#### Abstract

Discrete convex functions are used in many areas, including operations research, discrete-event systems, game theory, and economics. The objective of this paper is to investigate basic operations such as direct sum, splitting, and aggregation that are related to network induction of discrete convex functions as well as discrete convex sets. Various kinds of discrete convex functions in discrete convex analysis are considered such as integrally convex functions, L-convex functions, M-convex functions, multimodular functions, and discrete midpoint convex functions.


Keywords: Discrete convex analysis, Integrally convex function, Multimodular function, Splitting, Aggregation, Network induction

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Figure 1: Bipartite graphs for operations on discrete structures

## 1 Introduction

In matroid theory it is well known that a matroid is induced or transformed by bipartite graphs through matchings ([46, Section 11.2], [58, Section 8.2]). Let $G$ be a bipartite graph with vertex bipartition consisting of $N=\{1,2, \ldots, n\}$ and $M=\{1,2, \ldots, m\}$ as in Fig. 1 (a). When a matroid $(N, \mathcal{I})$ is given on $N$ in terms of the family $\mathcal{I}$ of independent sets, let $\mathcal{J}$ denote the collection of subsets of $M$ which can be matched in $G$ with an independent subset of $N$. Then $(M, \mathcal{J})$ is a matroid, which is referred to as the matroid induced from $(N, \mathcal{I})$ by $G$. We may regard this construction as a transformation of a matroid to another matroid. If a free matroid is given on $N$, for example, the matroid induced on $M$ is a transversal matroid. In particular, a free matroid on $N$ is transformed to a partition matroid on $M$, if the graph $G$ has a special structure like Fig. [1(b), where each vertex of $M$ has exactly one incident arc. The union (or sum) operation for matroids can also be understood as a transformation of this kind. Given two matroids on $\{1,2, \ldots, n\}$ we consider a bipartite graph of the form of Fig. 1 (d), in which the direct sum of the given matroids is associated with the left vertex set $\{1,2, \ldots, n\} \cup\left\{1^{\prime}, 2^{\prime}, \ldots, n^{\prime}\right\}$, and the induced matroid coincides with the union of the given matroids. It is possible to generalize the above construction by replacing a bipartite graph with a general directed graph and matchings with linkings; see [46, Section 11.2] and [58, Section 13.3].

In discrete convex analysis [11, 32, 34, 37, 38], the transformation of matroids described above is generalized to a transformation of discrete convex sets and functions by capacitated networks, which is called the network transformation. The objective of this paper is to systematically investigate the network transformation, together with the related basic operations, for discrete convex sets and functions. The network transformation in discrete convex analysis is more general than the transformation of matroids in the following two respects:

- From $\{0,1\}$ to $\mathbb{Z}$ : A set family on the ground set $\{1,2, \ldots, n\}$ can be identified with a subset of $\{0,1\}^{n}$, and hence the transformation of a matroid can be regarded as a transformation of a subset of $\{0,1\}^{n}$ to a subset of $\{0,1\}^{m}$. A discrete convex set is a subset of $\mathbb{Z}^{n}$ that has some defining properties, and the network transformation of a discrete convex set amounts to a transformation of a subset $S$ of $\mathbb{Z}^{n}$ to a subset $T$ of $\mathbb{Z}^{m}$ via integral flows in an arc-capacitated network. We are naturally interested in whether the resulting set $T$ is a discrete convex set of the same kind.
- From sets to functions: A discrete convex set, which is a subset of $\mathbb{Z}^{n}$, can be identified with its indicator function, which is equal to 0 on that set and $+\infty$ elsewhere.

We generalize this by considering functions $f: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ that have certain discrete convexity properties. The network transformation of a discrete convex function is defined via integral flows in a network with arc costs. We are naturally interested in whether the resulting function $g: \mathbb{Z}^{m} \rightarrow \mathbb{R} \cup\{+\infty\}$ is a discrete convex function of the same kind.

The network transformation of a subset of $\mathbb{Z}^{n}$ is defined (roughly) as follows. For simplicity of presentation, we restrict ourselves to a bipartite network. Let $G$ be a bipartite graph with vertex bipartition consisting of $N=\{1,2, \ldots, n\}$ and $M=\{1,2, \ldots, m\}$ as in Fig. $\mathbb{1}$ (a), and suppose that a nonnegative integer (upper) capacity is specified for each arc, where the lower capacity is assumed to be zero. When a set $S \subseteq \mathbb{Z}^{n}$ is given, let $T$ denote the collection of vectors $y \in \mathbb{Z}^{m}$ which can be linked from some $x \in S$ via an integer-valued flow meeting the capacity constraint. If we interpret $S$ as a set of feasible supply vectors, then the resulting set $T$ represents the set of demand vectors that can be realized by a feasible transportation scheme. The transformation (or induction) by $G$ will mean the operation of obtaining $T$ from $S$.

It turns out to be convenient to single out two special types of bipartite graphs, which are depicted in Fig. [1(b) and (c). In the graph in (b), each vertex of $M$ has exactly one incident arc, and the transformation represented by such a graph will be called a splitting. In the graph in (c), in contrast, each vertex of $N$ has exactly one incident arc, and the transformation by such a graph will be called an aggregation. While splitting and aggregation are special cases of the transformation by bipartite networks, they are general enough in the sense that the transformation by an arbitrary bipartite graph $G$ can be represented as a composition of the transformation by a graph $G_{1}$ of type (b) followed by the transformation by a graph $G_{2}$ of type (c), where $G_{1}$ and $G_{2}$ are obtained from $G$ (drawn as in Fig. (1) by "vertically cutting $G$ into left and right parts." The Minkowski sum $S_{1}+S_{2}=\left\{y \in \mathbb{Z}^{n} \mid y=x+x^{\prime}, x \in S_{1}, x^{\prime} \in S_{2}\right\}$ of sets $S_{1}, S_{2} \subseteq \mathbb{Z}^{n}$ is represented by the graph in Fig. [(d), that is, the Minkowski sum can be represented as a combination of the direct sum and aggregation operations. Furthermore, it is known [20] that the transformation by a general capacitated network (to be defined in Section 3.4) can be realized by a combination of splitting, aggregation, and other basic operations.

The network transformation of a function on $\mathbb{Z}^{n}$ is defined (roughly) as follows. We continue to refer to a bipartite graph $G$ in Fig. 1 (a), but we now suppose that each arc is associated with a (convex) function to represent the cost of an integral flow in the arc. When a function $f$ on $\mathbb{Z}^{n}$ is given, we interpret $f(x)$ as the production cost of $x \in \mathbb{Z}^{n}$. For $y \in \mathbb{Z}^{m}$, interpreted as a demand, let $g(y)$ denote the minimum cost of an integral flow that meets the demand $y$ by an appropriate choice of production $x$ and transportation scheme using an integer-valued flow. The transformation (or induction) by $G$ will mean the operation of obtaining $g$ from $f$.

The special types of bipartite graphs in Fig. (1)(b) and (c) continue to play the key role also for operations on functions. The transformations of a function by the graphs in (b) and (c) are called a splitting and an aggregation of the function, respectively. As with the transformation of a discrete convex set, the transformation of a function by an arbitrary bipartite graph can be represented as a composition of the transformation by a graph of type (b) followed by the transformation by a graph of type (c). For functions $f_{1}$ and $f_{2}$ on $\mathbb{Z}^{n}$, their convolution

$$
\left(f_{1} \square f_{2}\right)(y)=\inf \left\{f_{1}(x)+f_{2}\left(x^{\prime}\right) \mid y=x+x^{\prime}\right\} \quad\left(y \in \mathbb{Z}^{n}\right)
$$

is represented by the graph in Fig. $\mathbb{1}$ (d), that is, the convolution can be represented as a combination of the direct sum and aggregation operations. Furthermore, it is known [20] that
the transformation of a function by a general network (to be defined in Section 4.4) can be realized by a combination of splitting, aggregation, and other basic operations.

Discrete convex functions treated in this paper include integrally convex functions [8], Land $L^{\natural}$-convex functions [12, 32], M- and $\mathrm{M}^{\natural}$-convex functions [30, 32, 41], multimodular functions [14], globally and locally discrete midpoint convex functions [29], and M- and $M^{\natural}$-convex functions on jump systems [36, 40]. It is noted that "L"" and " $M^{\natural}$ " should be pronounced as "ell natural" and "em natural," respectively. L- and $L^{\natural}$-convex functions have applications in several different fields including image processing, auction theory, inventory theory, and scheduling [6, 38, 50, 52]. M - and $\mathrm{M}^{\natural}$-convex functions find applications in game theory and economics [34, 38, 44, 51] as well as in matrix theory [33, Chapter 5]. Multimodular functions have been used as a fundamental tool in the literature of queueing theory, discrete-event systems, and operations research [1, 2, 10, 13, 14, 23, 25, 53, 56, 57, 62]. Jump M - and $\mathrm{M}^{\natural}$-convex functions find applications in several fields including matching theory [3, 21, 22, 54] and algebra [5]. Integrally convex functions are used in formulating discrete fixed point theorems [15, 16, 61], and designing solution algorithms for discrete systems of nonlinear equations [24, 60]. In game theory the integral concavity of payoff functions guarantees the existence of a pure strategy equilibrium in finite symmetric games [17].

This paper is intended to be a continuation of the recent paper [39], which is the first systematic study of fundamental operations for various kinds of discrete convex functions including multimodular functions and discrete midpoint convex functions. While the paper [39] dealt with basic operations such as restriction, projection, scaling, and convolution, this paper focuses on operations related to the network transformation including direct sum, splitting, and aggregation. We mention that a systematic study of fundamental operations for discrete convex functions, though not covering multimodular functions and discrete midpoint convex functions, was conducted in [42] at the early stage of discrete convex analysis.

Table $\prod$ is a summary of the behavior of discrete convex sets with respect to the operations of direct sum, splitting, aggregation, and network transformation discussed in this paper. In the table, " Y " means that the set class is closed under the operation and " $N$ " means it is not, where we use different fonts for easier distinction. For the results obtained in the paper, specific references are made to the corresponding propositions (Propositions 3.2, 3.4 and 3.5) and counterexamples. The results about M - and $\mathrm{M}^{\natural}$-convex sets are not particularly new, as they are no more than restatements of well known facts in the literature of polymatroids and submodular functions [9, 11]. These operations for jump systems are considered by Bouchet and Cunningham [4] and Kabadi and Sridhar [18]. Table 2] offers a similar summary for operations on functions, with pointers to the major propositions (Propositions 4.2, 4.4, and 4.5) as well as to counterexamples in this paper. Network induction for M-convex functions originates in [30], and that for jump M-convex functions is due to [20]. The reader is referred to Tables 3 to 6 in [39] for summaries about other operations such as restriction, projection, scaling, and convolution.

This paper is organized as follows. Section 2 is a brief summary of the definitions of discrete convex sets and functions, including new observations (Theorems 2.3 and 2.5, Example 2.2). Section 3 treats operations on discrete convex sets such as direct sum, splitting, aggregation, and network transformation. Section 4 treats the corresponding operations on discrete convex functions. Section 5 gives the proofs.

Table 1: Operations on discrete convex sets

| Discrete convex set | Direct <br> sum | Splitting | $\begin{aligned} & \hline \text { Aggrega- } \\ & \text { tion } \end{aligned}$ | Network induction | Reference |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Integer box | Y | $\begin{gathered} \hline \boldsymbol{N} \\ \operatorname{Ex}, \sqrt{3.2} \end{gathered}$ | Y | $\begin{gathered} \hline \boldsymbol{N} \\ \operatorname{Ex}, \sqrt{3.2} \end{gathered}$ | (this paper) |
| Integrally convex | Y | $\begin{gathered} \mathrm{Y} \\ \operatorname{Prop}, 3.4 \end{gathered}$ | $\begin{gathered} N \\ \operatorname{Ex} \sqrt{3.4} \end{gathered}$ | $\begin{gathered} \frac{N}{\operatorname{Ex}, \sqrt{3.4}} \end{gathered}$ | (this paper) |
| L ${ }^{\text {b }}$-convex | Y | $\begin{gathered} N \\ \text { Ex. } 3.2 \end{gathered}$ | $\begin{gathered} \boldsymbol{N} \\ \operatorname{Ex} .3 .5 \end{gathered}$ | $\begin{gathered} \boldsymbol{N} \\ \operatorname{Ex} 3.2,3.5 \end{gathered}$ | (this paper) |
| L-convex | Y | $\begin{gathered} \bar{N} \\ \operatorname{Ex} .3 .3 \end{gathered}$ | $\begin{gathered} \hline \boldsymbol{N} \\ \operatorname{Ex}, 3.6 \end{gathered}$ | $\begin{gathered} N \\ \operatorname{Ex} 3.3,3.6 \end{gathered}$ | (this paper) |
| $\mathrm{M}^{\mathrm{b}}$-convex | Y | Y | Y | Y | [9, 34, 41] |
| M-convex | Y | Y | Y | Y | [11, 34] |
| Multimodular | $\begin{gathered} \hline \mathrm{Y} \\ \text { Prop } 3.2 \end{gathered}$ | $\begin{gathered} \mathrm{Y} \\ \text { Prop } 3.5 \end{gathered}$ | $\begin{gathered} \hline \boldsymbol{N} \\ \operatorname{Ex}, 3.7 \end{gathered}$ | $\begin{gathered} \hline N \\ \operatorname{Ex} \sqrt{3.7} \end{gathered}$ | (this paper) |
| Disc. midpt convex | $\begin{gathered} \bar{N} \\ \operatorname{Ex} 3.1 \end{gathered}$ | $\begin{gathered} \bar{N} \\ \operatorname{Ex}, 3.2 \end{gathered}$ | $\begin{gathered} \bar{N} \\ \operatorname{Ex} \widehat{3.4} \end{gathered}$ | $\begin{gathered} \bar{N} \\ \operatorname{Ex} \sqrt{3.4} \end{gathered}$ | (this paper) |
| Simul. exch. jump | Y | Y | Y | Y | [18, 40] |
| Const-parity jump | Y | Y | Y | Y | [4, 18] |

"Y" means "Yes, this set class is closed under this operation."
" $N$ " means "No, this set class is not closed under this operation."

Table 2: Operations on discrete convex functions

| Discrete convex function | Direct sum | Splitting | $\begin{gathered} \text { Aggrega- } \\ \text { tion } \end{gathered}$ | Network induction | Reference |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Separable convex | Y | $\begin{gathered} \hline N \\ \operatorname{Ex} 3.2 \end{gathered}$ | Y | $\begin{gathered} \hline \hline N \\ \operatorname{Ex} .3 .2 \\ \hline \end{gathered}$ | (this paper) |
| Integrally convex | Y | $\begin{gathered} \mathrm{Y} \\ \text { Prop. } 4.4 \end{gathered}$ | $\begin{gathered} N \\ \operatorname{Ex} .3 .4 \end{gathered}$ | $\begin{gathered} \bar{N} \\ \operatorname{Ex} \widehat{3.4} \end{gathered}$ | (this paper) |
| $L^{\text {b }}$-convex | Y | $\begin{gathered} N \\ \text { Ex } 3.2 \\ \hline \end{gathered}$ | $\begin{gathered} \bar{N} \\ \operatorname{Ex} .3 .5 \\ \hline \end{gathered}$ | $\begin{gathered} \bar{N} \\ \operatorname{Ex} 3.2,3.5 \end{gathered}$ | (this paper) |
| L-convex | Y | $\begin{gathered} N \\ \operatorname{Ex}, 3.3 \end{gathered}$ | $\begin{gathered} N \\ \operatorname{Ex} .3 .6 \end{gathered}$ | $\begin{gathered} N \\ \operatorname{Ex}, 3.3,3.6 \end{gathered}$ | (this paper) |
| $\mathrm{M}^{\text {b }}$-convex | Y | Y | Y | Y | [34] |
| M-convex | Y | Y | Y | Y | [30, 34] |
| Multimodular | $\begin{gathered} \text { Y } \\ \text { Prop. } 4.2 \end{gathered}$ | $\begin{gathered} \hline \mathrm{Y} \\ \text { Prop, } 4.5 \end{gathered}$ | $\begin{gathered} \hline N \\ \operatorname{Ex} 3.7 \end{gathered}$ | $\begin{gathered} \hline \boldsymbol{N} \\ \operatorname{Ex} .3 .7 \end{gathered}$ | (this paper) |
| Globally d.m.c. | $\frac{N}{\operatorname{Ex} \sqrt{3.1}, 4.1}$ | $\begin{gathered} \overline{\boldsymbol{N}} \\ \operatorname{Ex} .3 .2 \end{gathered}$ | $\begin{gathered} \bar{N} \\ \operatorname{Ex} .3 .4 \end{gathered}$ | $\begin{gathered} \bar{N} \\ \operatorname{Ex} .3 .4 \end{gathered}$ | (this paper) |
| Locally d.m.c. | $\frac{N}{\operatorname{Ex} .3 .1,4.1}$ | $\begin{gathered} \bar{N} \\ \operatorname{Ex} 3.2 \end{gathered}$ | $\begin{gathered} N \\ \operatorname{Ex} .3 .4 \end{gathered}$ | $\begin{gathered} \boldsymbol{N} \\ \operatorname{Ex} .3 .4 \end{gathered}$ | (this paper) |
| Jump M ${ }^{\natural}$-convex | Y | Y | Y | Y | [40] |
| Jump M-convex | Y | Y | Y | Y | [20] |

"Y" means "Yes, this function class is closed under this operation."
" $N$ " means "No, this function class is not closed under this operation."

## 2 Definitions of Discrete Convex Sets and Functions

In this section we provide a minimum account of definitions of discrete convex sets $S \subseteq \mathbb{Z}^{n}$ and functions $f: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$. Let $N=\{1,2, \ldots, n\}$.

For $i \in\{1,2, \ldots, n\}$, the $i$ th unit vector is denoted by $\mathbf{1}^{i}$. We define $\mathbf{1}^{0}=\mathbf{0}$ where $\mathbf{0}=$ $(0,0, \ldots, 0)$. We also define $\mathbf{1}=(1,1, \ldots, 1)$.

For a vector $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ and a subset $A \subseteq\{1,2, \ldots, n\}, x(A)$ denotes the component sum within $A$, i.e., $x(A)=\sum\left\{x_{i} \mid i \in A\right\}$. The positive and negative supports of $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ are defined as

$$
\begin{equation*}
\operatorname{supp}^{+}(x)=\left\{i \mid x_{i}>0\right\}, \quad \operatorname{supp}^{-}(x)=\left\{i \mid x_{i}<0\right\} . \tag{2.1}
\end{equation*}
$$

The indicator function of a set $S \subseteq \mathbb{Z}^{n}$ is the function $\delta_{S}: \mathbb{Z}^{n} \rightarrow\{0,+\infty\}$ defined by

$$
\delta_{S}(x)= \begin{cases}0 & (x \in S),  \tag{2.2}\\ +\infty & (x \notin S) .\end{cases}
$$

The convex hull of a set $S$ is denoted by $\bar{S}$. The effective domain of a function $f$ means the set of $x$ with $f(x)<+\infty$ and is denoted by $\operatorname{dom} f=\left\{x \in \mathbb{Z}^{n} \mid f(x)<+\infty\right\}$. We always assume that $\operatorname{dom} f$ is nonempty.

### 2.1 Separable convexity

For integer vectors $a \in(\mathbb{Z} \cup\{-\infty\})^{n}$ and $b \in(\mathbb{Z} \cup\{+\infty\})^{n}$ with $a \leq b,[a, b]_{\mathbb{Z}}$ denotes the integer box (discrete rectangle, integer interval) between $a$ and $b$. A function $f: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ in $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in \mathbb{Z}^{n}$ is called separable convex if it can be represented as

$$
\begin{equation*}
f(x)=\varphi_{1}\left(x_{1}\right)+\varphi_{2}\left(x_{2}\right)+\cdots+\varphi_{n}\left(x_{n}\right) \tag{2.3}
\end{equation*}
$$

with univariate discrete convex functions $\varphi_{i}: \mathbb{Z} \rightarrow \mathbb{R} \cup\{+\infty\}$, which, by definition, satisfy

$$
\begin{equation*}
\varphi_{i}(t-1)+\varphi_{i}(t+1) \geq 2 \varphi_{i}(t) \quad(t \in \mathbb{Z}) . \tag{2.4}
\end{equation*}
$$

### 2.2 Integral convexity

For $x \in \mathbb{R}^{n}$ the integral neighborhood of $x$ is defined in [8] as

$$
\begin{equation*}
N(x)=\left\{z \in \mathbb{Z}^{n}| | x_{i}-z_{i} \mid<1(i=1,2, \ldots, n)\right\} . \tag{2.5}
\end{equation*}
$$

It is noted that strict inequality " $<$ " is used in this definition and hence $N(x)$ admits an alternative expression

$$
\begin{equation*}
N(x)=\left\{z \in \mathbb{Z}^{n} \mid\left\lfloor x_{i}\right\rfloor \leq z_{i} \leq\left\lceil x_{i}\right\rceil(i=1,2, \ldots, n)\right\}, \tag{2.6}
\end{equation*}
$$

where, for $t \in \mathbb{R}$ in general, $\lceil t\rceil$ denotes the smallest integer not smaller than $t$ (rounding-up to the nearest integer) and $\lfloor t\rfloor$ the largest integer not larger than $t$ (rounding-down to the nearest integer). For a set $S \subseteq \mathbb{Z}^{n}$ and $x \in \mathbb{R}^{n}$ we call the convex hull of $S \cap N(x)$ the local convex hull of $S$ at $x$. A nonempty set $S \subseteq \mathbb{Z}^{n}$ is said to be integrally convex if the union of the local convex hulls $\overline{S \cap N(x)}$ over $x \in \mathbb{R}^{n}$ is convex [34]. This is equivalent to saying that, for any $x \in \mathbb{R}^{n}, x \in \bar{S}$ implies $x \in \overline{S \cap N(x)}$.

It is recognized only recently that the concept of integrally convex sets is closely related (or essentially equivalent) to the concept of box-integer polyhedra. Recall from [47, Section 5.15] that a polyhedron $P \subseteq \mathbb{R}^{n}$ is called box-integer if $P \cap\left\{x \in \mathbb{R}^{n} \mid a \leq x \leq b\right\}$ is an integer polyhedron for each choice of integer vectors $\frac{a}{}$ and $b$. Then it is easy to see that if a set $S \subseteq \mathbb{Z}^{n}$ is integrally convex, then its convex hull $\bar{S}$ is a box-integer polyhedron, and conversely, if $P$ is a box-integer polyhedron, then $S=P \cap \mathbb{Z}^{n}$ is an integrally convex set.

For a function $f: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ the local convex extension $\tilde{f}: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ of $f$ is defined as the union of all convex envelopes of $f$ on $N(x)$. That is,

$$
\begin{equation*}
\tilde{f}(x)=\min \left\{\sum_{y \in N(x)} \lambda_{y} f(y) \mid \sum_{y \in N(x)} \lambda_{y} y=x,\left(\lambda_{y}\right) \in \Lambda(x)\right\} \quad\left(x \in \mathbb{R}^{n}\right), \tag{2.7}
\end{equation*}
$$

where $\Lambda(x)$ denotes the set of coefficients for convex combinations indexed by $N(x)$ :

$$
\Lambda(x)=\left\{\left(\lambda_{y} \mid y \in N(x)\right) \mid \sum_{y \in N(x)} \lambda_{y}=1, \lambda_{y} \geq 0 \text { for all } y \in N(x)\right\} .
$$

If $\tilde{f}$ is convex on $\mathbb{R}^{n}$, then $f$ is said to be integrally convex [8]. The effective domain of an integrally convex function is an integrally convex set. A set $S \subseteq \mathbb{Z}^{n}$ is integrally convex if and only if its indicator function $\delta_{S}: \mathbb{Z}^{n} \rightarrow\{0,+\infty\}$ is an integrally convex function.

Integral convexity of a function can be characterized as follows.
Theorem 2.1 ([29, Theorem A.1]). A function $f: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ with $\operatorname{dom} f \neq \emptyset$ is integrally convex if and only if, for every $x, y \in \mathbb{Z}^{n}$ we have

$$
\tilde{f}\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(f(x)+f(y)),
$$

where $\tilde{f}$ is the local convex extension of $f$ defined by (2.7).
The reader is referred to [26, 28, 45] for recent developments in the theory of integral convexity.

### 2.3 L-convexity and discrete midpoint convexity

### 2.3.1 L-convex sets and functions

A nonempty set $S \subseteq \mathbb{Z}^{n}$ is called $L^{\natural}$-convex if

$$
\begin{equation*}
x, y \in S \Longrightarrow\left\lceil\frac{x+y}{2}\right\rceil,\left\lfloor\frac{x+y}{2}\right\rfloor \in S, \tag{2.8}
\end{equation*}
$$

where $\lceil z\rceil=\left(\left\lceil z_{1}\right\rceil,\left\lceil z_{2}\right\rceil, \ldots,\left\lceil z_{n}\right\rceil\right)$ and $\lfloor z\rfloor=\left(\left\lfloor z_{1}\right\rfloor,\left\lfloor z_{2}\right\rfloor, \ldots,\left\lfloor z_{n}\right\rfloor\right)$ for $z=\left(z_{1}, z_{2}, \ldots, z_{n}\right) \in \mathbb{R}^{n}$. The property (2.8) is called discrete midpoint convexity.

A function $f: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ with $\operatorname{dom} f \neq \emptyset$ is said to be $L^{\natural}$-convex if it satisfies a quantitative version of discrete midpoint convexity, i.e., if

$$
\begin{equation*}
f(x)+f(y) \geq f\left(\left\lceil\frac{x+y}{2}\right\rceil\right)+f\left(\left\lfloor\frac{x+y}{2}\right\rfloor\right) \tag{2.9}
\end{equation*}
$$

holds for all $x, y \in \mathbb{Z}^{n}$. The effective domain of an $L^{\natural}$-convex function is an $L^{\natural}$-convex set. A set $S$ is $L^{\natural}$-convex if and only if its indicator function $\delta_{S}$ is an $\mathrm{L}^{\natural}$-convex function. It is
known [34, Section 7.1] that $L^{\text {h }}$-convex functions can be characterized by several different conditions.

For example, $f\left(x_{1}, x_{2}, x_{3}\right)=\max \left\{x_{1}, x_{2}, x_{3}\right\}$ is an $\mathrm{L}^{\natural}$-convex function. Another function $f\left(x_{1}, x_{2}, x_{3}\right)=x_{1}^{2}+\left|x_{1}-x_{2}\right|+\left(x_{2}-x_{3}\right)^{2}$ is also L ${ }^{\natural}$-convex. More generally [34, Section 7.3],

$$
\begin{equation*}
f(x)=\sum_{i=1}^{n} \varphi_{i}\left(x_{i}\right)+\sum_{i \neq j} \varphi_{i j}\left(x_{i}-x_{j}\right) \tag{2.10}
\end{equation*}
$$

with univariate convex functions $\varphi_{i}(i=1,2, \ldots, n)$ and $\varphi_{i j}(i, j=1,2, \ldots, n ; i \neq j)$ is $\mathrm{L}^{\natural}$-convex. A function of the form of (2.10) is sometimes called a 2 -separable diff-convex function.

A function $f\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ is said to be submodular if

$$
\begin{equation*}
f(x)+f(y) \geq f(x \vee y)+f(x \wedge y) \tag{2.11}
\end{equation*}
$$

holds for all $x, y \in \mathbb{Z}^{n}$, where $x \vee y$ and $x \wedge y$ denote, respectively, the vectors of componentwise maximum and minimum of $x$ and $y$, i.e.,

$$
\begin{equation*}
(x \vee y)_{i}=\max \left(x_{i}, y_{i}\right), \quad(x \wedge y)_{i}=\min \left(x_{i}, y_{i}\right) \quad(i=1,2, \ldots, n) . \tag{2.12}
\end{equation*}
$$

A function $f\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ with $\operatorname{dom} f \neq \emptyset$ is called L-convex if it is submodular and there exists $r \in \mathbb{R}$ such that

$$
\begin{equation*}
f(x+\mathbf{1})=f(x)+r \tag{2.13}
\end{equation*}
$$

for all $x \in \mathbb{Z}^{n}$. If $f$ is L-convex, the function $g\left(x_{2}, \ldots, x_{n}\right):=f\left(0, x_{2}, \ldots, x_{n}\right)$ is an $\mathrm{L}^{\mathrm{q}}$ convex function, and every $\mathrm{L}^{\mathrm{h}}$-convex function arises in this way. For example, $f\left(x_{1}, x_{2}, x_{3}\right)=$ $\max \left\{x_{1}, x_{2}, x_{3}\right\}$, mentioned above as an $L^{\natural}$-convex function, is actually L-convex. A 2-separable diff-convex function in (2.10) is L-convex if $\varphi_{i}=0$ for $i=1,2, \ldots, n$.

A nonempty set $S$ is called $L$-convex if its indicator function $\delta_{S}$ is an L-convex function. The effective domain of an L-convex function is an L-convex set.

### 2.3.2 Discrete midpoint convex sets and functions

A nonempty set $S \subseteq \mathbb{Z}^{n}$ is said to be discrete midpoint convex [29] if

$$
\begin{equation*}
x, y \in S,\|x-y\|_{\infty} \geq 2 \Longrightarrow\left\lceil\frac{x+y}{2}\right\rceil,\left\lfloor\frac{x+y}{2}\right\rfloor \in S . \tag{2.14}
\end{equation*}
$$

This condition is weaker than the defining condition (2.8) for an $L^{\natural}$-convex set, and hence every $L^{\natural}$-convex set is a discrete midpoint convex set.

A function $f: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ with $\operatorname{dom} f \neq \emptyset$ is called globally discrete midpoint convex if the discrete midpoint convexity (2.9) is satisfied by every pair $(x, y) \in \mathbb{Z}^{n} \times \mathbb{Z}^{n}$ with $\|x-y\|_{\infty} \geq 2$. The effective domain of a globally discrete midpoint convex function is necessarily a discrete midpoint convex set. A function $f: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ with $\operatorname{dom} f \neq \emptyset$ is called locally discrete midpoint convex if $\operatorname{dom} f$ is a discrete midpoint convex set and the discrete midpoint convexity (2.9) is satisfied by every pair $(x, y) \in \mathbb{Z}^{n} \times \mathbb{Z}^{n}$ with $\|x-y\|_{\infty}=$ 2 (exactly equal to 2 ). Obviously, every $L^{\natural}$-convex function is globally discrete midpoint convex, and every globally discrete midpoint convex function is locally discrete midpoint convex. We sometimes abbreviate "discrete midpoint convex(ity)" to "d.m.c."

The inclusion relations for sets and functions equipped with (variants of) L-convexity are summarized as follows:
$\{$ L-convex sets $\} \varsubsetneqq\left\{L^{\text {b}}\right.$-convex sets $\} \varsubsetneqq\{$ discrete midpoint convex sets $\}$, $\{$ L-convex fns $\} \varsubsetneqq\left\{L^{\natural}\right.$-convex fns $\} \varsubsetneqq\{$ globally d.m.c. fns $\} \varsubsetneqq\{$ locally d.m.c. fns $\}$.

### 2.4 M-convexity and jump M-convexity

### 2.4.1 M-convex sets and functions

A nonempty set $S \subseteq \mathbb{Z}^{n}$ is called an $M^{\natural}$-convex set if it satisfies the following exchange property:
( $\mathbf{B}^{\natural}$-EXC) For any $x, y \in S$ and $i \in \operatorname{supp}^{+}(x-y)$, we have (i) $x-\mathbf{1}^{i} \in S$ and $y+\mathbf{1}^{i} \in S$ or
(ii) there exists some $j \in \operatorname{supp}^{-}(x-y)$ such that $x-\mathbf{1}^{i}+\mathbf{1}^{j} \in S$ and $y+\mathbf{1}^{i}-\mathbf{1}^{j} \in S$.
$\mathrm{M}^{\natural}$-convex set is an alias for the set of integer points in an integral generalized polymatroid. In particular, the family of independent sets of a matroid can be regarded as an $\mathrm{M}^{\natural}$-convex set consisting of $\{0,1\}$-vectors.

A function $f: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ with $\operatorname{dom} f \neq \emptyset$ is called $M^{\natural}$-convex, if, for any $x, y \in \operatorname{dom} f$ and $i \in \operatorname{supp}^{+}(x-y)$, we have (i)

$$
\begin{equation*}
f(x)+f(y) \geq f\left(x-\mathbf{1}^{i}\right)+f\left(y+\mathbf{1}^{i}\right) \tag{2.15}
\end{equation*}
$$

or (ii) there exists some $j \in \operatorname{supp}^{-}(x-y)$ such that

$$
\begin{equation*}
f(x)+f(y) \geq f\left(x-\mathbf{1}^{i}+\mathbf{1}^{j}\right)+f\left(y+\mathbf{1}^{i}-\mathbf{1}^{j}\right) . \tag{2.16}
\end{equation*}
$$

This property is referred to as the exchange property. A more compact expression of this exchange property is as follows:
$\left(\mathbf{M}^{\natural}-\mathbf{E X C}\right)$ For any $x, y \in \operatorname{dom} f$ and $i \in \operatorname{supp}^{+}(x-y)$, we have

$$
\begin{equation*}
f(x)+f(y) \geq \min _{j \in \operatorname{supp}^{-}(x-y) \cup\{0\}}\left\{f\left(x-\mathbf{1}^{i}+\mathbf{1}^{j}\right)+f\left(y+\mathbf{1}^{i}-\mathbf{1}^{j}\right)\right\}, \tag{2.17}
\end{equation*}
$$

where $\mathbf{1}^{0}=\mathbf{0}$ (zero vector).
For example, $f\left(x_{1}, x_{2}, x_{3}\right)=\left|x_{1}+x_{2}+x_{3}\right|+\left(x_{1}+x_{2}\right)^{2}+x_{3}{ }^{2}$ is an $\mathrm{M}^{\natural}$-convex function. More generally [34, Section 6.3], a laminar convex function is $\mathrm{M}^{\natural}$-convex, where a function $f$ is called laminar convex if it can be represented as

$$
\begin{equation*}
f(x)=\sum_{A \in \mathcal{T}} \varphi_{A}(x(A)) \tag{2.18}
\end{equation*}
$$

for a laminar family $\mathcal{T} \subseteq 2^{N}$ (i.e., $A \cap B=\emptyset, A \subseteq B$, or $A \supseteq B$ for any $A, B \in \mathcal{T}$ ) and a family of univariate discrete convex functions $\varphi_{A}: \mathbb{Z} \rightarrow \mathbb{R} \cup\{+\infty\}$ indexed by $A \in \mathcal{T}$.
$\mathrm{M}^{\natural}$-convex functions can be characterized by a number of different exchange properties including a local exchange property under the assumption that function $f$ is (effectively) defined on an $\mathrm{M}^{\natural}$-convex set. See [43] as well as [38, Theorem 4.2] and [51, Theorem 6.8].

If a set $S \subseteq \mathbb{Z}^{n}$ lies on a hyperplane with a constant component sum (i.e., $x(N)=y(N)$ for all $x, y \in S$ ), the exchange property ( $\mathrm{B}^{\natural}-\mathrm{EXC}$ ) takes a simpler form (without the possibility of the first case (i)):
(B-EXC) For any $x, y \in S$ and $i \in \operatorname{supp}^{+}(x-y)$, there exists some $j \in \operatorname{supp}^{-}(x-y)$ such that $x-\mathbf{1}^{i}+\mathbf{1}^{j} \in S$ and $y+\mathbf{1}^{i}-\mathbf{1}^{j} \in S$.
A nonempty set $S \subseteq \mathbb{Z}^{n}$ having this exchange property is called an $M$-convex set, which is an alias for the set of integer points in an integral base polyhedron. In particular, the basis family of a matroid can be identified precisely with an M-convex set consisting of $\{0,1\}$-vectors.

An $\mathrm{M}^{\natural}$-convex function whose effective domain is an M -convex set is called an $M$-convex function [30, 32, 34]. In other words, a function $f: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ with $\operatorname{dom} f \neq \emptyset$ is M-convex if and only if it satisfies the exchange property:
(M-EXC) For any $x, y \in \operatorname{dom} f$ and $i \in \operatorname{supp}^{+}(x-y)$, there exists $j \in \operatorname{supp}^{-}(x-y)$ such that (2.16) holds.

M-convex functions can be characterized by a local exchange property under the assumption that function $f$ is (effectively) defined on an M-convex set. See [34, Section 6.2].
$M$-convex functions and $M^{\natural}$-convex functions are equivalent concepts, in that $M^{\natural}$-convex functions in $n$ variables can be obtained as projections of M-convex functions in $n+1$ variables. More formally, a function $f: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ is $\mathrm{M}^{\natural}$-convex if and only if the function $\tilde{f}: \mathbb{Z}^{n+1} \rightarrow \mathbb{R} \cup\{+\infty\}$ defined by

$$
\tilde{f}\left(x_{0}, x\right)=\left\{\begin{array}{ll}
f(x) & \text { if } x_{0}=-x(N)  \tag{2.19}\\
+\infty & \text { otherwise }
\end{array} \quad\left(x_{0} \in \mathbb{Z}, x \in \mathbb{Z}^{n}\right)\right.
$$

is an M-convex function.

### 2.4.2 Jump systems and jump M-convex functions

Let $x$ and $y$ be integer vectors. The smallest integer box containing $x$ and $y$ is given by $[x \wedge y, x \vee y]_{Z}$. A vector $s \in \mathbb{Z}^{n}$ is called an ( $x, y$ )-increment if $s=\mathbf{1}^{i}$ or $s=-\mathbf{1}^{i}$ for some $i \in N$ and $x+s \in[x \wedge y, x \vee y]_{z}$.

A nonempty set $S \subseteq \mathbb{Z}^{n}$ is said to be a jump system [4] if satisfies an exchange axiom, called the 2 -step axiom:
(2-step axiom) For any $x, y \in S$ and any ( $x, y$ )-increment $s$ with $x+s \notin S$, there exists an $(x+s, y)$-increment $t$ such that $x+s+t \in S$.

Note that we have the possibility of $s=t$ in the 2-step axiom.
A set $S \subseteq \mathbb{Z}^{n}$ is called a constant-sum system if $x(N)=y(N)$ for any $x, y \in S$. A constantsum jump system is nothing but an M-convex set.

A set $S \subseteq \mathbb{Z}^{n}$ is called a constant-parity system if $x(N)-y(N)$ is even for any $x, y \in S$. It is known [36] that a constant-parity jump system (or c.p. jump system) is characterized by
(J-EXC) For any $x, y \in S$ and any ( $x, y$ )-increment $s$, there exists an $(x+s, y)$-increment $t$ such that $x+s+t \in S$ and $y-s-t \in S$.

A function $f: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ with $\operatorname{dom} f \neq \emptyset$ is called ${ }^{1}$ jump $M$-convex if it satisfies the following exchange axiom:
(JM-EXC) For any $x, y \in \operatorname{dom} f$ and any $(x, y)$-increment $s$, there exists an $(x+s, y)$ increment $t$ such that $x+s+t \in \operatorname{dom} f, y-s-t \in \operatorname{dom} f$, and

$$
\begin{equation*}
f(x)+f(y) \geq f(x+s+t)+f(y-s-t) . \tag{2.20}
\end{equation*}
$$

The effective domain of a jump M-convex function is a constant-parity jump system.
A jump system is called a simultaneous exchange jump system (or s.e. jump system) [40] if it satisfies the following exchange axiom
( $\left.\mathbf{J}^{\natural}-\mathbf{E X C}\right)$ For any $x, y \in S$ and any ( $x, y$ )-increment $s$, we have (i) $x+s \in S$ and $y-s \in S$, or (ii) there exists an $(x+s, y)$-increment $t$ such that $x+s+t \in S$ and $y-s-t \in S$.

[^1]Every constant-parity jump system is a simultaneous exchange jump system, since the condition (J-EXC) implies ( $\mathrm{J}^{\mathrm{h}}$-EXC). Not every jump system is a simultaneous exchange jump system, as is shown in [39, Examples 2.2 and 2.3].

A function $f: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ with $\operatorname{dom} f \neq \emptyset$ is called jump $M^{\natural}$-convex [40] if it satisfies the following exchange axiom
(JM ${ }^{\natural}$-EXC) For any $x, y \in \operatorname{dom} f$ and any $(x, y)$-increment $s$, we have
(i) $x+s \in \operatorname{dom} f, y-s \in \operatorname{dom} f$, and

$$
\begin{equation*}
f(x)+f(y) \geq f(x+s)+f(y-s), \tag{2.21}
\end{equation*}
$$

or (ii) there exists an $(x+s, y)$-increment $t$ such that $x+s+t \in \operatorname{dom} f, y-s-t \in \operatorname{dom} f$, and (2.20) holds.

The condition ( $\mathrm{JM}^{\natural}$-EXC) is weaker than (JM-EXC), and hence every jump M-convex function is a jump $\mathrm{M}^{\natural}$-convex function. However, the concepts of jump M-convexity and jump $\mathrm{M}^{\natural}$-convexity are in fact equivalent to each other in the sense that jump $\mathrm{M}^{\natural}$-convex functions in $n$ variables can be identified with jump M-convex functions in $n+1$ variables. More specifically, for any integer vector $x \in \mathbb{Z}^{n}$ we define $\pi(x)=0$ if its component sum $x(N)$ is even, and $\pi(x)=1$ if $x(N)$ is odd. It is known [40] that a function $f: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ is jump $\mathrm{M}^{\natural}$-convex if and only if the function $\tilde{f}: \mathbb{Z}^{n+1} \rightarrow \mathbb{R} \cup\{+\infty\}$ defined by

$$
\tilde{f}\left(x_{0}, x\right)=\left\{\begin{array}{ll}
f(x) & \left(x_{0}=\pi(x)\right)  \tag{2.22}\\
+\infty & (\text { otherwise })
\end{array} \quad\left(x_{0} \in \mathbb{Z}, x \in \mathbb{Z}^{n}\right)\right.
$$

is a jump M-convex function.
The inclusion relations for sets and functions equipped with (variants of) M-convexity is summarized as follows:

$$
\begin{aligned}
& \left.\left.\{\text { M-convex sets }\} \varsubsetneqq\left\{\begin{array}{l}
\left\{\mathrm{M}^{\natural} \text {-convex sets }\right\} \\
\{\text { c.p. jump systems }\}
\end{array}\right\} \varsubsetneqq \text { \{s.e. jump systems }\right\} \varsubsetneqq \text { \{jump systems }\right\}, \\
& \{\text { M-convex fns }\} \varsubsetneqq\left\{\begin{array}{l}
\left\{\mathrm{M}^{\natural} \text {-convex fns }\right\} \\
\text { \{jump M-convex fns }\}
\end{array}\right\} \varsubsetneqq\left\{\text { jump } \mathrm{M}^{\natural} \text {-convex fns }\right\} .
\end{aligned}
$$

It is noted that no convexity class is introduced for functions defined on general jump systems.
Finally we mention an example to show that a jump M-convex function may not look like a convex function in the intuitive sense. Nevertheless, jump $M$ - and $M^{\natural}$-convex functions find applications in several fields including matching theory [3, 21, 22, 54] and algebra [5].

Example 2.1. Let $S$ be a subset of $\mathbb{Z}^{2}$ defined by

$$
S=\left\{\left(x_{1}, x_{2}\right) \in \mathbb{Z}^{2} \mid 0 \leq x_{1} \leq 3,0 \leq x_{2} \leq 3, x_{1}+x_{2}: \text { even }\right\} .
$$

This set is a constant-parity jump system. Consider $f: S \rightarrow \mathbb{R}$ defined by

$$
f\left(x_{1}, x_{2}\right)= \begin{cases}0 & \left(x_{1}, x_{2} \in\{0,2\}\right),  \tag{2.23}\\ 1 & \left(x_{1}, x_{2} \in\{1,3\}\right),\end{cases}
$$

which may be shown as

$$
f\left(x_{1}, x_{2}\right)=\begin{array}{|cccc}
- & 1 & - & 1 \\
0 & - & 0 & - \\
- & 1 & - & 1 \\
0 & - & 0 & - \\
\hline
\end{array} .
$$

This function is jump M-convex. Indeed, for $x=(0,0), y=(2,2)$, and $s=(1,0)$, for example, we can take $t=(1,0)$, for which $x+s+t=(2,0), y-s-t=(0,2)$, and $f(x)+f(y)=0+0=$ $f(x+s+t)+f(y-s-t)$ in (2.20). For $x=(0,0), y=(3,3)$, and $s=(1,0)$, we can choose $t=(1,0)$ or $t=(0,1)$. For either choice we have $f(x+s+t)+f(y-s-t)=1=f(x)+f(y)$.

It is noted that the function $f$ above arises from the degree sequences of a graph as in [36, Example 2.2]. Let $G=(V, E)$ be an undirected graph with vertex set $V=\left\{v_{1}, v_{2}\right\}$ and edge set $E$ consisting of three edges, $E=\left\{\left(v_{1}, v_{2}\right),\left(v_{1}, v_{1}\right),\left(v_{2}, v_{2}\right)\right\}$, where $\left(v_{i}, v_{i}\right)$ denotes a self-loop at $v_{i}$ for $i=1,2$. The set $S$ above is the degree system (the set of the degree sequences of a subgraph) of this graph $G$, and $f(x)$ coincides with the minimum weight of a subgraph with degree sequence $x$ when the $\left(v_{1}, v_{2}\right)$ has weight 1 and the self-loops have weight 0 .

### 2.5 Multimodularity

Recall that $\mathbf{1}^{i}$ denotes the $i$ th unit vector for $i=1,2, \ldots, n$, and $\mathcal{F} \subseteq \mathbb{Z}^{n}$ be the set of vectors defined by

$$
\begin{equation*}
\mathcal{F}=\left\{-\mathbf{1}^{1}, \mathbf{1}^{1}-\mathbf{1}^{2}, \mathbf{1}^{2}-\mathbf{1}^{3}, \ldots, \mathbf{1}^{n-1}-\mathbf{1}^{n}, \mathbf{1}^{n}\right\} . \tag{2.24}
\end{equation*}
$$

A finite-valued function $f: \mathbb{Z}^{n} \rightarrow \mathbb{R}$ is said to be multimodular [14] if it satisfies

$$
\begin{equation*}
f(z+d)+f\left(z+d^{\prime}\right) \geq f(z)+f\left(z+d+d^{\prime}\right) \tag{2.25}
\end{equation*}
$$

for all $z \in \mathbb{Z}^{n}$ and all distinct $d, d^{\prime} \in \mathcal{F}$. It is known [14, Proposition 2.2] that $f: \mathbb{Z}^{n} \rightarrow \mathbb{R}$ is multimodular if and only if the function $\tilde{f}: \mathbb{Z}^{n+1} \rightarrow \mathbb{R}$ defined by

$$
\begin{equation*}
\tilde{f}\left(x_{0}, x\right)=f\left(x_{1}-x_{0}, x_{2}-x_{1}, \ldots, x_{n}-x_{n-1}\right) \quad\left(x_{0} \in \mathbb{Z}, x \in \mathbb{Z}^{n}\right) \tag{2.26}
\end{equation*}
$$

is submodular in $n+1$ variables. This characterization enables us to define multimodularity for a function that may take the infinite value $+\infty$. That is, we say [27, 35] that a function $f: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ with dom $f \neq \emptyset$ is multimodular if the function $\tilde{f}: \mathbb{Z}^{n+1} \rightarrow \mathbb{R} \cup\{+\infty\}$ associated with $f$ by (2.26) is submodular.

Multimodularity and $\mathrm{L}^{\mathrm{h}}$-convexity have the following close relationship.
Theorem 2.2 ([35]). A function $f: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ is multimodular if and only if the function $g: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ defined by

$$
\begin{equation*}
g(p)=f\left(p_{1}, p_{2}-p_{1}, p_{3}-p_{2}, \ldots, p_{n}-p_{n-1}\right) \quad\left(p \in \mathbb{Z}^{n}\right) \tag{2.27}
\end{equation*}
$$

is $L^{\natural}$-convex.
Note that the relation (2.27) between $f$ and $g$ can be rewritten as

$$
\begin{equation*}
f(x)=g\left(x_{1}, x_{1}+x_{2}, x_{1}+x_{2}+x_{3}, \ldots, x_{1}+\cdots+x_{n}\right) \quad\left(x \in \mathbb{Z}^{n}\right) \tag{2.28}
\end{equation*}
$$

Using a bidiagonal matrix $D=\left(d_{i j} \mid 1 \leq i, j \leq n\right)$ defined by

$$
\begin{equation*}
d_{i i}=1 \quad(i=1,2, \ldots, n), \quad d_{i+1, i}=-1 \quad(i=1,2, \ldots, n-1), \tag{2.29}
\end{equation*}
$$

we can express (2.27) and (2.28) more compactly as $g(p)=f(D p)$ and $f(x)=g\left(D^{-1} x\right)$, respectively. The matrix $D$ is unimodular, and its inverse $D^{-1}$ is a lower triangular integer matrix whose $(i, j)$ entry is given by

$$
\left(D^{-1}\right)_{i j}= \begin{cases}1 & (i \geq j)  \tag{2.30}\\ 0 & (i<j)\end{cases}
$$

For $n=5$, for example, we have

$$
D=\left[\begin{array}{rrrrr}
1 & 0 & 0 & 0 & 0  \tag{2.31}\\
-1 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & -1 & 1
\end{array}\right], \quad D^{-1}=\left[\begin{array}{lllll}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1
\end{array}\right] .
$$

A nonempty set $S$ is called multimodular if its indicator function $\delta_{S}$ is multimodular. A multimodular set $S$ can be represented as $S=\{x=D p \mid p \in T\}$ for some $L^{\natural}$-convex set $T$, where $T$ is uniquely determined from $S$ as $T=\left\{p=D^{-1} x \mid x \in S\right\}$. It follows from (2.27) that the effective domain of a multimodular function is a multimodular set.

A polyhedral description of a multimodular set is given as follows. A subset of the index set $N=\{1,2, \ldots, n\}$ is said to be consecutive if it consists of consecutive numbers, that is, it is a set of the form $\{k, k+1, \ldots, l-1, l\}$ for some $k \leq l$.

Theorem 2.3. A set $S \subseteq \mathbb{Z}^{n}$ is multimodular if and only if

$$
S=\left\{x \in \mathbb{Z}^{n} \mid a_{I} \leq x(I) \leq b_{I}(I: \text { consecutive interval in } N)\right\}
$$

for some $a_{I} \in \mathbb{Z} \cup\{-\infty\}$ and $b_{I} \in \mathbb{Z} \cup\{+\infty\}$ indexed by consecutive intervals $I \subseteq N$.
Proof. As is well known ([34, Section 5.5]), an $\mathrm{L}^{\natural}$-convex set can be described by a system of inequalities of the form $p_{i}-p_{j} \leq d_{i j}$ and $a_{i} \leq p_{i} \leq b_{i}$. On substituting $p_{i}=x_{1}+x_{2}+\cdots+x_{i}$ ( $i=1,2, \ldots, n$ ) into these inequalities, we obtain the claim.

### 2.6 Discrete convexity of functions in terms of the minimizers

In this section we discuss how discrete convexity of functions can be characterized in terms of the discrete convexity of the minimizer sets.

For a function $f: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ and a vector $c \in \mathbb{R}^{n}, f[-c]$ will denote the function defined by

$$
f[-c](x)=f(x)-\sum_{i=1}^{n} c_{i} x_{i} \quad\left(x \in \mathbb{Z}^{n}\right) .
$$

It is often the case that $f$ is equipped with some kind of discrete convexity if and only if, for every $c \in \mathbb{R}^{n}$, the set of the minimizers of $f[-c]$, i.e.,

$$
\arg \min f[-c]=\left\{x \in \mathbb{Z}^{n} \mid f[-c](x) \leq f[-c](y) \text { for all } y \in \mathbb{Z}^{n}\right\}
$$

is equipped with the discrete convexity of the same kind. This implies that the concept of discrete convex functions can also be defined from that of discrete convex sets.

Indeed the following facts are known.
Theorem 2.4. Let $f: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be a function that is convex-extensible or has a bounded nonempty effective domain. 2
(1) $f$ is separable convex if and only if $\arg \min f[-c]$ is an integer box for each $c \in \mathbb{R}^{n}$.

[^2](2) $f$ is integrally convex if and only if $\arg \min f[-c]$ is an integrally convex set for each $c \in \mathbb{R}^{n}$.
(3) $f$ is $L^{\natural}$-convex if and only if $\arg \min f[-c]$ is an $L^{\natural}$-convex set for each $c \in \mathbb{R}^{n}$.
(4) $f$ is $L$-convex if and only if $\arg \min f[-c]$ is an $L$-convex set for each $c \in \mathbb{R}^{n}$.
(5) $f$ is $M^{\natural}$-convex if and only if arg $\min f[-c]$ is an $M^{\natural}$-convex set for each $c \in \mathbb{R}^{n}$.
(6) $f$ is $M$-convex if and only if $\arg \min f[-c]$ is an $M$-convex set for each $c \in \mathbb{R}^{n}$.

Proof. It follows easily from the definitions that the only-if parts in all cases (1)-(6) hold without the assumption of convex-extensibility or boundedness of $\operatorname{dom} f$.

The if-parts under the assumption of bounded $\operatorname{dom} f$ are known in the literature. Part (1) for separable convexity is obvious. Part (2) for integral convexity is given in [34, Theorem 3.29]. Parts (3) and (4) for $L^{4}$ - and L-convexity are given in [34, Theorem 7.17]. Parts (5) and (6) for $\mathrm{M}^{\natural}$ - and M -convexity are given in [34, Theorem 6.30].

The proof of the if-part under the assumption of convex-extensibility of $f$ can be reduced to the case of a bounded effective domain. We demonstrate this reduction for $L^{\natural}$-convex functions. A function $f$ is $\mathrm{L}^{\mathrm{h}}$-convex if and only if its restriction to every finite box is $\mathrm{L}^{\mathrm{h}}$ convex. Let $f_{[a, b]}$ denote the restriction of $f$ to a finite integer box $[a, b]=[a, b]_{\mathbb{Z}}$, and note that $(f[-c])_{[a, b]}=f_{[a, b]}[-c]$. When $f$ is convex-extensible, we have the relation

$$
\begin{equation*}
(\arg \min f[-c]) \cap[a, b]=\arg \min \left(f_{[a, b]}[-c]\right) . \tag{2.32}
\end{equation*}
$$

By the assumption, $\arg \min f[-c]$ is an $\mathrm{L}^{\natural}$-convex set, from which follows that its intersection with the box $[a, b]$, i.e., $(\arg \min f[-c]) \cap[a, b]$, is also an $L^{\text {h}}$-convex set. Hence, by (2.32), $\arg \min \left(f_{[a, b]}[-c]\right)$ is an L ${ }^{\natural}$-convex set for every $c$. Since $\operatorname{dom} f_{[a, b]}$ is bounded, $f_{[a, b]}$ is an Lconvex function. Therefore, $f$ is $\mathrm{L}^{\text {h}}$-convex. The same argument is valid for other kinds of discrete convex functions.

Moreover, we can show a similar statement for multimodularity, which does not seem to have been made in the literature.

Theorem 2.5. Let $f: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be a function that is convex-extensible or has a bounded nonempty effective domain. Then $f$ is multimodular if and only if $\arg \min f[-c]$ is a multimodular set for each $c \in \mathbb{R}^{n}$.
Proof. This is a straightforward translation of Theorem 2.4 (3) for an $L^{\text {h }}$-convex function. Let $g(p)=f(D p)$. By Theorem 2.2, $f$ is multimodular if and only if $g$ is $L^{\natural}$-convex, whereas the relation

$$
\arg \min f[-c]=\left\{x=D p \mid p \in \arg \min g\left[-D^{\top} c\right]\right\}
$$

shows that $\arg \min f[-c]$ is multimodular if and only if $\arg \min g\left[-c^{\prime}\right]$ is $L^{\natural}$-convex for $c^{\prime}=$ $D^{\top} c$.

Jump M-convexity as well as jump $\mathrm{M}^{\natural}$-convexity does not admit such characterization. This is demonstrated by the following example.
Example 2.2. Let $S$ be a subset of $\mathbb{Z}^{2}$ defined by

$$
S=\left\{\left(x_{1}, x_{2}\right) \in \mathbb{Z}^{2} \mid 0 \leq x_{1} \leq 4,0 \leq x_{2} \leq 4, x_{1}+x_{2}: \text { even }\right\} .
$$

This set is a constant-parity jump system. Consider $f: S \rightarrow \mathbb{R}$ defined by

$$
f\left(x_{1}, x_{2}\right)= \begin{cases}0 & \left(x_{1}, x_{2} \in\{0,2,4\}\right)  \tag{2.33}\\ \alpha & \left(\left(x_{1}, x_{2}\right)=(1,1),(1,3)\right), \\ \beta & \left(\left(x_{1}, x_{2}\right)=(3,1),(3,3)\right)\end{cases}
$$

with parameters $\alpha$ and $\beta$, which may be shown as

$$
f\left(x_{1}, x_{2}\right)=\begin{array}{|ccccc}
0 & - & 0 & - & 0 \\
- & \alpha & - & \beta & - \\
0 & - & 0 & - & 0 \\
- & \alpha & - & \beta & - \\
0 & - & 0 & - & 0 \\
\hline
\end{array} .
$$

This function is jump M-convex if and only if $\alpha=\beta$. Indeed, for $x=(0,0), y=(2,2)$, and $s=(1,0)$, for example, we can take $t=(1,0)$, for which $x+s+t=(2,0), y-s-t=(0,2)$, and $f(x)+f(y)=0+0=f(x+s+t)+f(y-s-t)$ in (2.20). For $x=(0,0), y=(3,3)$, and $s=(1,0)$, we can choose $t=(1,0)$ or $t=(0,1)$. For either choice we have $f(x+s+t)+f(y-s-t)=\alpha$, while $f(x)+f(y)=\beta$. Therefore, the inequality (2.20) is satisfied if and only if $\alpha \leq \beta$. By considering $x=(4,4), y=(1,1)$, and $s=(-1,0)$, we obtain $\alpha \geq \beta$. From this argument and symmetry, we can conclude that $f$ is jump M-convex if and only if $\alpha=\beta$.

Now suppose that $0<\alpha<\beta$. Then $f$ is not jump M-convex. However, $\arg \min f[-c]$ is a constant-parity jump system for each $c \in \mathbb{R}^{2}$. Indeed, $\arg \min f[-c]=S \cap(2 \mathbb{Z})^{2}$ for $c=(0,0)$, and for $c \neq(0,0)$, arg $\min f[-c]$ is equal to a singleton or a set of three points like $\{(0,0),(2,0),(4,0)\}$ lying on a horizontal or vertical line.

However, such characterization is valid for jump M-convex functions if $\operatorname{dom} f \subseteq\{0,1\}^{n}$.
Theorem 2.6. Assume that dom $f$ is a constant-parity jump system contained in $\{0,1\}^{n}$. Then $f$ is jump $M$-convex if and only if $\arg \min f[-c]$ is a constant-parity jump system for each $c \in \mathbb{R}^{n}$.

Proof. A constant-parity jump system contained in $\{0,1\}^{n}$ can be identified with an even delta-matroid, and a function $f$ with $\operatorname{dom} f \subseteq\{0,1\}^{n}$ is jump M-convex if and only if $-f$ is a valuated delta-matroid [7, 59]. With this correspondence, Theorem 2.2 of [31] for valuated delta-matroids is translated into this theorem.

Such characterization fails for (global and local) discrete midpoint convexity, as pointed out by [55] only recently (after the submission of this paper). That is, there is a function $f$ which is not discrete midpoint convex but for which $\arg \min f[-c]$ is discrete midpoint convex for every $c$.

## 3 Operations on Discrete Convex Sets

In this section we consider operations on discrete convex sets. The behavior of discrete convex sets with respect to the operations discussed below is summarized in Table 1 in Introduction.

### 3.1 Direct sum

For two sets $S_{1} \subseteq \mathbb{Z}^{n_{1}}$ and $S_{2} \subseteq \mathbb{Z}^{n_{2}}$, their direct sum is defined as

$$
\begin{equation*}
S_{1} \oplus S_{2}=\left\{(x, y) \mid x \in S_{1}, y \in S_{2}\right\}, \tag{3.1}
\end{equation*}
$$

which is a subset of $\mathbb{Z}^{n_{1}+n_{2}}$.
In most cases it is obvious that the direct sum operation preserves the discrete convexity in question. However, this is not the case with multimodularity and discrete midpoint convexity. We have the following proposition for the obvious cases.

Proposition 3.1. The direct sum of two integrally convex sets is an integrally convex set. Similarly for $L^{\natural}$-convex sets, $L$-convex sets, $M^{\natural}$-convex sets, $M$-convex sets, simultaneous exchange jump systems, and constant-parity jump systems.

A multimodular set is defined with reference to an ordering of the underlying set. When we consider the direct sum of two multimodular sets $S_{1} \subseteq \mathbb{Z}^{n_{1}}$ and $S_{2} \subseteq \mathbb{Z}^{n_{2}}$, we assume that the components of $(x, y)$ are ordered naturally with $x_{1}, x_{2}, \ldots, x_{n_{1}}$ followed by $y_{1}, y_{2}, \ldots, y_{n_{2}}$. In this sense, it is more appropriate to regard an element $(x, y)$ of $S_{1} \oplus S_{2}$ as a concatenation of $x \in S_{1}$ and $y \in S_{2}$.

Proposition 3.2 below states that the direct sum $S_{1} \oplus S_{2}$ is also multimodular. It is noted that this is a nontrivial statement, since the definition of the multimodularity of $S_{1} \oplus S_{2}$ involves the vector $\mathbf{1}^{i}-\mathbf{1}^{i+1}$ for $i=n_{1}$ in (2.24), which does not appear in the definitions of the multimodularity of $S_{1}$ and $S_{2}$.

Proposition 3.2. The direct sum of two multimodular sets is multimodular.
Proof. The proof is given in Section 5.3 ,
In contrast, the direct sum of discrete midpoint convex sets is not necessarily discrete midpoint convex, as shown in Example 3.1 below.

Example 3.1. Let

$$
S_{1}=\{(1,0),(0,1)\}, \quad S_{2}=\mathbb{Z},
$$

for which $S_{1} \oplus S_{2}=\{(1,0, t),(0,1, t) \mid t \in \mathbb{Z}\}$. The sets $S_{1}$ and $S_{2}$ are both discrete midpoint convex, whereas $S_{1} \oplus S_{2}$ is not. Indeed, for $x=(1,0,2)$ and $y=(0,1,0)$ in $S_{1} \oplus S_{2}$, we have $\|x-y\|_{\infty}=2,(x+y) / 2=(1 / 2,1 / 2,1)$, for which $\lceil(x+y) / 2\rceil=(1,1,1) \notin S_{1} \oplus S_{2}$, and $\lfloor(x+y) / 2\rfloor=(0,0,1) \notin S_{1} \oplus S_{2}$.

### 3.2 Splitting

Suppose that we are given a family $\left\{U_{1}, U_{2}, \ldots, U_{n}\right\}$ of disjoint nonempty sets indexed by $N=\{1,2, \ldots, n\}$. Let $m_{i}=\left|U_{i}\right|$ for $i=1,2, \ldots, n$ and define $m=\sum_{i=1}^{n} m_{i}$, where $m \geq n$. For a set $S \subseteq \mathbb{Z}^{n}$, the subset of $\mathbb{Z}^{m}$ defined by

$$
\begin{equation*}
T=\left\{\left(y_{1}, y_{2}, \ldots, y_{n}\right) \in \mathbb{Z}^{m} \mid y_{i} \in \mathbb{Z}^{m_{i}}, x_{i}=y_{i}\left(U_{i}\right)(i \in N), x \in S\right\} \tag{3.2}
\end{equation*}
$$

is called the splitting of $S$ by $\left\{U_{1}, U_{2}, \ldots, U_{n}\right\}$. A splitting is called an elementary splitting if $\left|U_{k}\right|=2$ for some $k$ and $\left|U_{i}\right|=1$ for other $i \neq k$. For example, $T=\left\{\left(y_{1}, y_{2}, y_{3}\right) \in \mathbb{Z}^{3} \mid\right.$ $\left.\left(y_{1}, y_{2}+y_{3}\right) \in S\right\}$ is an elementary splitting of $S \subseteq \mathbb{Z}^{2}$. Any (general) splitting can be obtained by repeated applications of elementary splittings. It should be clear that the definition of splitting by (3.2) is consistent with the definition, given in Introduction, in terms of the graph in Fig. [(b).

M-convexity and its relatives are well-behaved with respect to the splitting operation, which is easy to see.

## Proposition 3.3.

(1) The splitting of an $M^{\natural}$-convex set is $M^{\natural}$-convex.
(2) The splitting of an $M$-convex set is $M$-convex.
(3) The splitting of a simultaneous exchange jump system is a simultaneous exchange jump system.
(4) The splitting of a constant-parity jump system is a constant-parity jump system.

The splitting operation has never been investigated for integrally convex sets and multimodular sets. For integrally convex sets we can show the following.

Proposition 3.4. The splitting of an integrally convex set is integrally convex.
Proof. The proof is given in Section5.1.
In the definition of multimodularity, the ordering of the components of a vector is crucial. Accordingly, in defining the splitting operation for multimodular sets, we assume that the components of vector $y \in \mathbb{Z}^{m}$ are ordered naturally, first the $m_{1}$ components of $y_{1}$, then the $m_{2}$ components of $y_{2}$, etc., and finally the $m_{n}$ components of $y_{n}$.

Proposition 3.5. The splitting of a multimodular set is multimodular (under the natural ordering of the elements).

Proof. The proof is given in Section 5.4,
Other kinds of discrete convexity are not compatible with the splitting operation. The splitting of an integer box is not necessarily an integer box. Similarly, the splitting of an $L^{\natural}$-convex (resp., L-convex, discrete midpoint convex) set is not necessarily $L^{\natural}$-convex (resp., L-convex, discrete midpoint convex). See Examples 3.2 and 3.3 .

Example 3.2. The elementary splitting of a singleton set $S=\{0\}$ is given by $T=\{(t,-t) \mid t \in$ $\mathbb{Z}$ \}. The set $S$ is an integer box but $T$ is not. Also $S$ is an $\mathrm{L}^{\natural}$-convex set but $T$ is not.

Example 3.3. The set $S=\left\{x \in \mathbb{Z}^{2} \mid x_{1}=x_{2}\right\}$ is an L-convex set. The elementary splitting of $S$ at the second component is given by $T=\left\{y \in \mathbb{Z}^{3} \mid y_{1}=y_{2}+y_{3}\right\}$. This set is not L-convex since the vector $y+\mathbf{1}$ does not belong to $T$ for $y \in T$.

### 3.3 Aggregation

Let $\mathcal{P}=\left\{N_{1}, N_{2}, \ldots, N_{m}\right\}$ be a partition of $N=\{1,2, \ldots, n\}$ into disjoint nonempty subsets, i.e., $N=N_{1} \cup N_{2} \cup \cdots \cup N_{m}$ and $N_{i} \cap N_{j}=\emptyset$ for $i \neq j$. We have $m \leq n$. For a set $S \subseteq \mathbb{Z}^{n}$ the subset of $\mathbb{Z}^{m}$ defined by

$$
\begin{equation*}
T=\left\{\left(y_{1}, y_{2}, \ldots, y_{m}\right) \in \mathbb{Z}^{m} \mid y_{j}=x\left(N_{j}\right)(j=1,2, \ldots, m), x \in S\right\} \tag{3.3}
\end{equation*}
$$

is called the aggregation of $S$ by $\mathcal{P}$. An aggregation with $m=n-1$ is called an elementary aggregation, in which $\left|N_{k}\right|=2$ for some $k$ and $\left|N_{j}\right|=1$ for other $j \neq k$. For example, $T=$ $\left\{\left(y_{1}, y_{2}\right) \in \mathbb{Z}^{2} \mid y_{1}=x_{1}, y_{2}=x_{2}+x_{3}\right.$ for some $\left.\left(x_{1}, x_{2}, x_{3}\right) \in S\right\}$ is an elementary aggregation of $S \subseteq \mathbb{Z}^{3}$. Any (general) aggregation can be obtained by repeated applications of elementary aggregations. It should be clear that the definition of aggregation by (3.3) is consistent with the definition, given in Introduction, in terms of the graph in Fig. 1 (c).

It is known that M -convexity and its relatives are well-behaved with respect to the aggregation operation.

## Proposition 3.6.

(1) The aggregation of an integer box is an integer box.
(2) The aggregation of an $M^{\natural}$-convex set is $M^{\natural}$-convex.
(3) The aggregation of an $M$-convex set is $M$-convex.
(4) The aggregation of a simultaneous exchange jump system is a simultaneous exchange jump system.
(5) The aggregation of a constant-parity jump system is a constant-parity jump system.

Proof. (1) The aggregation of an integer box $\left\{x \in \mathbb{Z}^{n} \mid a_{i} \leq x_{i} \leq b_{i}(i=1,2, \ldots, n)\right\}$ is given by $\left\{y \in \mathbb{Z}^{m} \mid a\left(N_{j}\right) \leq y_{j} \leq b\left(N_{j}\right)(j=1,2, \ldots, m)\right\}$, which is an integer box. The aggregation operations for M-convex and $\mathrm{M}^{\natural}$-convex sets in Parts (2) and (3) are well known in polymatroid/submodular function theory (see, e.g., [11, Section 3.1(d)]). The aggregation operation for (general) jump systems was considered by Kabadi and Sridhar [18]. Part (5) for constant-parity jump systems follows from this, since $\sum_{j=1}^{m} y_{j}=\sum_{i=1}^{n} x_{i}$ if $y_{j}=x\left(N_{j}\right)$ ( $j=1,2, \ldots, m$ ). Part (4) for simultaneous exchange jump systems can be derived from Part (5) for constant-parity jump systems on the basis of their relation (2.22) in Section 2.4.2 by specializing the proof of [40, Lemma 4.5] to indicator functions.

We point out here that other kinds of discrete convexity are not compatible with the aggregation operation by presenting counterexamples, as follows.

- The aggregation of an integrally convex set is not necessarily integrally convex (Example 3.4).
- The aggregation of an $L^{\natural}$-convex set is not necessarily $L^{\natural}$-convex (Example 3.5).
- The aggregation of an L-convex set is not necessarily L-convex (Example 3.6).
- The aggregation of a multimodular set is not necessarily multimodular (Example 3.7).
- The aggregation of a discrete midpoint convex set is not necessarily discrete midpoint convex (Example 3.4).

Example 3.4. The set

$$
S=\{(0,0,1,0),(0,0,0,1),(1,1,1,0),(1,1,0,1)\}
$$

is an integrally convex set. For the partition of $N=\{1,2,3,4\}$ into $N_{1}=\{1,3\}$ and $N_{2}=\{2,4\}$, the aggregation of $S$ by $\left\{N_{1}, N_{2}\right\}$ is given by

$$
T=\{(1,0),(0,1),(2,1),(1,2)\},
$$

which is not integrally convex. The set $S$ is also discrete midpoint convex, but $T$ is not.
Example 3.5. The set

$$
\begin{equation*}
S=\{(0,0,0,0,0,0),(0,0,0,0,1,1),(1,1,0,0,0,0),(1,1,0,0,1,1)\} \tag{3.4}
\end{equation*}
$$

is an $\mathrm{L}^{\mathrm{h}}$-convex set. For the partition of $N=\{1,2, \ldots, 6\}$ into three pairs $N_{1}=\{1,4\}, N_{2}=$ $\{2,5\}$, and $N_{3}=\{3,6\}$, the aggregation of $S$ by $\left\{N_{1}, N_{2}, N_{3}\right\}$ is given by

$$
T=\{(0,0,0),(0,1,1),(1,1,0),(1,2,1)\},
$$

which is not $\mathrm{L}^{\mathrm{h}}$-convex. Indeed, for $x=(0,1,1)$ and $y=(1,1,0)$ in $T$, we have $(x+y) / 2=$ $(1 / 2,1,1 / 2)$, for which $\lceil(x+y) / 2\rceil=(1,1,1) \notin T$, and $\lfloor(x+y) / 2\rfloor=(0,1,0) \notin T$. Therefore, $T$ is not $\mathrm{L}^{\natural}$-convex.

Example 3.6. (This is an adaptation of Example 3.5to L-convex sets.) Let $S_{1}=\{(0,0,0,0)+$ $\alpha \mathbf{1},(1,1,0,0)+\alpha \mathbf{1} \mid \alpha \in \mathbb{Z}\}$ and $S_{2}=\{(0,0,0,0)+\alpha \mathbf{1},(0,1,1,0)+\alpha \mathbf{1} \mid \alpha \in \mathbb{Z}\}$ with $\mathbf{1}=(1,1,1,1)$, and define $S=S_{1} \oplus S_{2} \subseteq \mathbb{Z}^{8}$. This set $S$ is L-convex. For the partition of $N=\{1,2, \ldots, 8\}$ into four pairs $N_{j}=\{j, j+4\}(j=1,2,3,4)$, the aggregation of $S$ is given by

$$
T=\{(0,0,0,0)+\alpha \mathbf{1},(0,1,1,0)+\alpha \mathbf{1},(1,1,0,0)+\alpha \mathbf{1},(1,2,1,0)+\alpha \mathbf{1} \mid \alpha \in \mathbb{Z}\}
$$

which is not L-convex, since for the elements $x=(0,1,1,0)$ and $y=(1,1,0,0)$ of $T$, we have $\lceil(x+y) / 2\rceil=(1,1,1,0) \notin T$ and $\lfloor(x+y) / 2\rfloor=(0,1,0,0) \notin T$.

Example 3.7. Here is an example of the aggregation of multimodular sets. For the $L^{\natural}$-convex set $S$ in (3.4) (Example 3.5), let $\tilde{S}=\{D x \mid x \in S\}$ be the multimodular set corresponding to $S$, where $D$ is the matrix defined in (2.29) in Section 2.5. That is,

$$
\tilde{S}=\{(0,0,0,0,0,0),(0,0,0,0,1,0),(1,0,-1,0,0,0),(1,0,-1,0,1,0)\} .
$$

For the partition of $N=\{1,2, \ldots, 6\}$ into three pairs $N_{1}=\{1,4\}, N_{2}=\{2,5\}$, and $N_{3}=\{3,6\}$, the aggregation of $\tilde{S}$ is given by

$$
\tilde{T}=\{(0,0,0),(0,1,0),(1,0,-1),(1,1,-1)\} .
$$

This set $\tilde{T}$ is not multimodular. We can check this directly or by detecting that the transformed set

$$
T=\left\{D^{-1} x \mid x \in \tilde{T}\right\}=\{(0,0,0),(0,1,1),(1,1,0),(1,2,1)\}
$$

is not $\mathrm{L}^{\natural}$-convex. Indeed, $x=(0,1,1)$ and $y=(1,1,0)$ in $T$, we have $\lceil(x+y) / 2\rceil=(1,1,1) \notin$ $T$ and $\lfloor(x+y) / 2\rfloor=(0,1,0) \notin T$.

Remark 3.1. The Minkowski sum of two sets $S_{1}, S_{2} \subseteq \mathbb{Z}^{n}$ means the subset of $\mathbb{Z}^{n}$ defined by

$$
\begin{equation*}
S_{1}+S_{2}=\left\{x+y \mid x \in S_{1}, y \in S_{2}\right\}, \tag{3.5}
\end{equation*}
$$

which is useful and important in applications. The Minkowski sum can be realized through a combination of direct sum and aggregation operations. We first form their direct sum $S=S_{1} \oplus S_{2} \subseteq \mathbb{Z}^{2 n}$. The underlying set of $S$ is the union of two disjoint copies of $\{1,2, \ldots, n\}$, which we denote by $\left\{\psi_{1}(i) \mid i=1,2, \ldots, n\right\} \cup\left\{\psi_{2}(i) \mid i=1,2, \ldots, n\right\}$. Consider the partition of this underlying set into the pairs $\left\{\psi_{1}(i), \psi_{2}(i)\right\}$ of corresponding elements. Then the aggregation of $S$ coincides with the Minkowski sum $S_{1}+S_{2}$.

### 3.4 Transformation by networks

In this section, we consider the transformation of a discrete (convex) set through a network. Let $G=(V, A ; U, W)$ be a directed graph with vertex set $V$, arc set $A$, entrance set $U$, and exit set $W$, where $U$ and $W$ are disjoint subsets of $V$ (cf., Fig. 2]). For each arc $a \in A$, an integer interval $[\ell(a), u(a)]_{\mathbb{Z}}$ is given as the capacity constraint, where $\ell(a) \in \mathbb{Z} \cup\{-\infty\}$ and $u(a) \in \mathbb{Z} \cup\{+\infty\}$.

We consider an integral flow $\xi: A \rightarrow \mathbb{Z}$ that satisfies the capacity constraint on arcs:

$$
\begin{equation*}
\ell(a) \leq \xi(a) \leq u(a) \quad(a \in A) \tag{3.6}
\end{equation*}
$$



Figure 2: Transformation of a discrete convex set by a network
and the flow-conservation at internal vertices:

$$
\begin{equation*}
\sum_{a: a \text { leaves } v} \xi(a)-\sum_{a: a \text { enters } v} \xi(a)=0 \quad(v \in V \backslash(U \cup W)) . \tag{3.7}
\end{equation*}
$$

For $v \in V$ we use notation

$$
\begin{equation*}
\partial \xi(v)=\sum_{a: a \text { leaves } v} \xi(a)-\sum_{a: a \text { enters } v} \xi(a), \tag{3.8}
\end{equation*}
$$

which means the net flow-supply from outside of the network at vertex $v$. Accordingly, $\partial \xi \in$ $\mathbb{Z}^{V}$ is the vector of net supplies. The restriction of $\partial \xi$ to $U$ is denoted by $\partial \xi \mid U$, that is, $x=\partial \xi \mid U$ is a vector with components indexed by $U$ such that $x(v)=\partial \xi(v)$ for $v \in U$. Similarly we define $\partial \xi \mid W \in \mathbb{Z}^{W}$.

Given a set $S \subseteq \mathbb{Z}^{U}$ of integer vectors on the entrance set $U$, we consider the set $T \subseteq \mathbb{Z}^{W}$ of integer vectors $y$ on the exit set $W$ for which there is a feasible flow $\xi$ such that the net supply vector on $U$ belongs to the given set $S$ (i.e., $\partial \xi \mid U \in S$ ) and the net supply vector on $W$ coincides with $-y$ (i.e., $\partial \xi \mid W=-y$ ). That is,

$$
\begin{gather*}
T=\left\{y \in \mathbb{Z}^{W} \mid \text { there exists } \xi \in \mathbb{Z}^{A}\right. \text { satisfying (3.6), (3.7), } \\
\partial \xi \mid U \in S, \text { and } \partial \xi \mid W=-y\} . \tag{3.9}
\end{gather*}
$$

We regard $T$ as a result of transformation (or induction) of $S$ by the network. It is assumed that $T$ is nonempty.

It is known that M-convexity and its relatives are well-behaved with respect to the network induction.

## Theorem 3.7.

(1) The network induction of an $M^{\natural}$-convex set is $M^{\natural}$-convex.
(2) The network induction of an M-convex set is M-convex.
(3) The network induction of a simultaneous exchange jump system is a simultaneous exchange jump system.
(4) The network induction of a constant-parity jump system is a constant-parity jump system.

Remark 3.2. Here is a supplement to Theorem 3.7. These statements are reformulations of known facts in matroid/polymatroid/submodular function theory (see, e.g., [4, 11, 18, 47]). Parts (1) and (2) for $\mathrm{M}^{\mathrm{4}}$-convex and M -convex sets are variants of the statement that an integral polymatroid (defined in terms of independent sets or bases) is transformed to another integral polymatroid through Menger-type linkings in a given directed graph. Part (3) for simultaneous exchange jump systems is a special case of [40, Theorem 4.12]. Part (4) for constant-parity jump systems is a special case of [20, Theorem 14].

In contrast, other kinds of discrete convexity are not compatible with the network induction. The network induction of an integer box is not necessarily an integer box. Similarly, the network induction of an integrally convex (resp., Lh-convex, L-convex, multimodular, discrete midpoint convex) set is not necessarily integrally convex (resp., $L^{\natural}$-convex, L-convex, multimodular, discrete midpoint convex). Note that these statements are immediate from the corresponding statements for splitting and aggregation in Sections 3.2 and 3.3, since the network induction is more general than those operations.

## 4 Operations on Discrete Convex Functions

In this section we consider operations on discrete convex functions. The behavior of discrete convex functions with respect to the operations discussed below is summarized in Table 2 in Introduction.

### 4.1 Direct sum

The direct sum of two functions $f_{1}: \mathbb{Z}^{n_{1}} \rightarrow \mathbb{R} \cup\{+\infty\}$ and $f_{2}: \mathbb{Z}^{n_{2}} \rightarrow \mathbb{R} \cup\{+\infty\}$ is a function $f_{1} \oplus f_{2}: \mathbb{Z}^{n_{1}+n_{2}} \rightarrow \mathbb{R} \cup\{+\infty\}$ defined as

$$
\begin{equation*}
\left(f_{1} \oplus f_{2}\right)(x, y)=f_{1}(x)+f_{2}(y) \quad\left(x \in \mathbb{Z}^{n_{1}}, y \in \mathbb{Z}^{n_{2}}\right) \tag{4.1}
\end{equation*}
$$

The effective domain of the direct sum is equal to the direct sum of the effective domains of the given functions, that is,

$$
\begin{equation*}
\operatorname{dom}\left(f_{1} \oplus f_{2}\right)=\operatorname{dom} f_{1} \oplus \operatorname{dom} f_{2} . \tag{4.2}
\end{equation*}
$$

For two sets $S_{1} \subseteq \mathbb{Z}^{n_{1}}$ and $S_{2} \subseteq \mathbb{Z}^{n_{2}}$, the direct sum of their indicator functions $\delta_{S_{1}}$ and $\delta_{S_{2}}$ coincides with the indicator function of their direct sum $S_{1} \oplus S_{2}$, that is,

$$
\delta_{S_{1}} \oplus \delta_{S_{2}}=\delta_{S_{1} \oplus S_{2}} .
$$

In most cases it is obvious that the direct sum operation preserves the discrete convexity in question. However, this is not the case with multimodularity and discrete midpoint convexity. We have the following proposition for the obvious cases.

Proposition 4.1. The direct sum operation (4.1) for functions preserves separable convexity, integral convexity, $L^{\natural}$-convexity, L-convexity, $M^{\natural}$-convexity, $M$-convexity, and jump $M^{\natural}$ convexity, and jump M-convexity.

Proposition 4.2 below states that the direct sum $f_{1} \oplus f_{2}$ of two multimodular functions $f_{1}$ and $f_{2}$ is also multimodular. It is noted that this is a nontrivial statement, since the definition of the multimodularity of $f_{1} \oplus f_{2}$ involves the vector $\mathbf{1}^{i}-\mathbf{1}^{i+1}$ for $i=n_{1}$ in (2.24), which does not appear in the definitions of the multimodularity of $f_{1}$ and $f_{2}$. Just as for the direct sum of multimodular sets, it is assumed that the components of $(x, y)$ in the definition (4.1) of $f_{1} \oplus f_{2}: \mathbb{Z}^{n_{1}+n_{2}} \rightarrow \mathbb{R} \cup\{+\infty\}$ are ordered naturally with $x_{1}, x_{2}, \ldots, x_{n_{1}}$ followed by $y_{1}, y_{2}, \ldots, y_{n_{2}}$.

Proposition 4.2. The direct sum of two multimodular functions is multimodular.
Proof. The proof is given in Section5.3,

In contrast, the direct sum of globally (resp., locally) discrete midpoint convex functions is not necessarily globally (resp., locally) discrete midpoint convex. This is shown already by Example 3.1, and the following example gives $f_{1}$ and $f_{2}$ that are finite-valued at every integer point.

Example 4.1. Let $f_{1}: \mathbb{Z}^{2} \rightarrow \mathbb{R}$ and $f_{2}: \mathbb{Z} \rightarrow \mathbb{R}$ be defined by

$$
f_{1}\left(x_{1}, x_{2}\right)=x_{1}^{2}+x_{1} x_{2}+x_{2}^{2}, \quad f_{2}\left(x_{3}\right)=0 .
$$

While $f_{1}$ and $f_{2}$ are (globally and locally) discrete midpoint convex, their direct sum

$$
g\left(x_{1}, x_{2}, x_{3}\right)=\left(f_{1} \oplus f_{2}\right)\left(x_{1}, x_{2}, x_{3}\right)=x_{1}^{2}+x_{1} x_{2}+x_{2}^{2}
$$

is not (globally and locally) discrete midpoint convex. Indeed, for $x=(1,0,0), y=(0,1,2)$, we have $\|x-y\|_{\infty}=2, u=\left\lceil\frac{x+y}{2}\right\rceil=(1,1,1), v=\left\lfloor\frac{x+y}{2}\right\rfloor=(0,0,1)$, and $g(x)+g(y)=1+1<$ $g(u)+g(v)=3+0$.

### 4.2 Splitting

Suppose that we are given a family $\left\{U_{1}, U_{2}, \ldots, U_{n}\right\}$ of disjoint nonempty sets indexed by $N=\{1,2, \ldots, n\}$. Let $m_{i}=\left|U_{i}\right|$ for $i=1,2, \ldots, n$ and define $m=\sum_{i=1}^{n} m_{i}$, where $m \geq n$. For a function $f: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$, the splitting of $f$ by $\left\{U_{1}, U_{2}, \ldots, U_{n}\right\}$ is defined as a function $g: \mathbb{Z}^{m} \rightarrow \mathbb{R} \cup\{+\infty\}$ given by

$$
\begin{equation*}
g\left(y_{1}, y_{2}, \ldots, y_{n}\right)=f\left(y_{1}\left(U_{1}\right), y_{2}\left(U_{2}\right), \ldots, y_{n}\left(U_{n}\right)\right), \tag{4.3}
\end{equation*}
$$

where, for each $i \in N, y_{i}=\left(y_{i j} \mid j \in U_{i}\right)$ is an integer vector of dimension $m_{i}$ and $y_{i}\left(U_{i}\right)=$ $\sum\left\{y_{i j} \mid j \in U_{i}\right\}$ is the component sum of vector $y_{i} \in \mathbb{Z}^{m_{i}}$. If $m=n+1$ (in which case we have $\left|U_{k}\right|=2$ for some $k$ and $\left|U_{i}\right|=1$ for other $i \neq k$ ), this is called an elementary splitting. For example, $g\left(y_{1}, y_{2}, y_{3}\right)=f\left(y_{1}, y_{2}+y_{3}\right)$ is an elementary splitting of $f$. Any (general) splitting can be obtained by repeated applications of elementary splittings. It should be clear that the definition of splitting by (4.3) is consistent with the definition, given in Introduction, in terms of the graph in Fig. [1(b).

It is known that M-convexity and its relatives are well-behaved with respect to the splitting operation.

## Proposition 4.3.

(1) The splitting of an $M^{\natural}$-convex function is $M^{\natural}$-convex.
(2) The splitting of an $M$-convex function is $M$-convex.
(3) The splitting of a jump $M^{\natural}$-convex function is jump $M^{\natural}$-convex.
(4) The splitting of a jump M-convex function is jump M-convex.

Remark 4.1. Here is a supplement to Proposition 4.3. The splitting operation for discrete convex functions is considered explicitly in [20] for jump M-convex functions. This is given in Part (4). As the splitting operation is a special case of the transformation by a bipartite network (cf., Remark 4.4), Parts (1) and (2) for $\mathrm{M}^{\natural}$-convex and M -convex functions follow from the previous results on the network induction for $\mathrm{M}^{\natural}$-convex and M -convex functions stated in [34, Theorem 9.26]. Part (3) for jump $\mathrm{M}^{\natural}$-convex functions is derived in [40] from (4) for jump M-convex functions.

The splitting operation has never been investigated for integrally convex functions and multimodular functions. For integrally convex functions we can show the following.

Proposition 4.4. The splitting of an integrally convex function is integrally convex.
Proof. The proof is given in Section 5.2,
In the definition of multimodularity, the ordering of the components of a vector is crucial. Accordingly, in defining the splitting operation for multimodular functions, we assume that the components of vector $y \in \mathbb{Z}^{m}$ are ordered naturally, first the $m_{1}$ components of $y_{1}$, then the $m_{2}$ components of $y_{2}$, etc., and finally the $m_{n}$ components of $y_{n}$.
Proposition 4.5. The splitting of a multimodular function is multimodular (under the natural ordering of the elements).

Proof. The proof is given in Section 5.4,
In contrast, L-convexity and its relatives are not compatible with the splitting operation. That is, the splitting operation does not preserve separable convexity, $\mathrm{L}^{\natural}$-convexity, L-convexity, and (global, local) discrete midpoint convexity. This is immediate from the corresponding statements for the splitting of sets in Section 3.2.

### 4.3 Aggregation

Let $\mathcal{P}=\left\{N_{1}, N_{2}, \ldots, N_{m}\right\}$ be a partition of $N=\{1,2, \ldots, n\}$ into disjoint (nonempty) subsets, i.e., $N=N_{1} \cup N_{2} \cup \cdots \cup N_{m}$ and $N_{i} \cap N_{j}=\emptyset$ for $i \neq j$. We have $m \leq n$. For a function $f: \mathbb{Z}^{n} \rightarrow$ $\mathbb{R} \cup\{+\infty\}$, the aggregation of $f$ with respect to $\mathcal{P}$ is the function $g: \mathbb{Z}^{m} \rightarrow \mathbb{R} \cup\{+\infty,-\infty\}$ defined by

$$
\begin{equation*}
g\left(y_{1}, y_{2}, \ldots, y_{m}\right)=\inf \left\{f(x) \mid x\left(N_{j}\right)=y_{j}(j=1,2, \ldots, m)\right\}, \tag{4.4}
\end{equation*}
$$

where $y_{j} \in \mathbb{Z}$ for $j=1,2, \ldots, m$. If $m=n-1$ (in which case we have $\left|N_{k}\right|=2$ for some $k$ and $\left|N_{j}\right|=1$ for other $j \neq k$ ), this is called an elementary aggregation. For example, $g\left(y_{1}, y_{2}\right)=\inf \left\{f\left(x_{1}, x_{2}, x_{3}\right) \mid x_{1}=y_{1}, x_{2}+x_{3}=y_{2}\right\}$ is an elementary aggregation of $f$. Any (general) aggregation can be obtained by repeated applications of elementary aggregations. It should be clear that the definition of aggregation by (4.4) is consistent with the definition, given in Introduction, in terms of the graph in Fig. [1(c).

It is known that M-convexity and its relatives are well-behaved with respect to the aggregation operation.

## Proposition 4.6.

(1) The aggregation of a separable convex function is separable convex.
(2) The aggregation of an $M^{\natural}$-convex function is $M^{\natural}$-convex.
(3) The aggregation of an M-convex function is M-convex.
(4) The aggregation of a jump $M^{\natural}$-convex function is jump $M^{\natural}$-convex.
(5) The aggregation of a jump $M$-convex function is jump $M$-convex.

Remark 4.2. Here is a supplement to Proposition 4.6. The aggregation of a separable convex function $\sum_{i=1}^{n} \varphi_{i}\left(x_{i}\right)$ is given by a separable convex function $\sum_{j=1}^{m} \psi_{j}\left(y_{j}\right)$ with $\psi_{j}\left(y_{j}\right)=$ $\inf \left\{\sum_{i \in N_{j}} \varphi_{i}\left(x_{i}\right) \mid x\left(N_{j}\right)=y_{j}\right\}$, where $y_{j} \in \mathbb{Z}$ for $j=1,2, \ldots, m$. The aggregation operations for M-convex and $\mathrm{M}^{\natural}$-convex functions in (2) and (3) are given in [34, Theorem 6.13] and [34, Theorem 6.15], respectively. Part (5) for jump M-convex functions is established in [20] by a long proof. Part (4) for jump $\mathrm{M}^{\natural}$-convex functions is derived in [40] from (5) for jump M -convex functions.

In contrast, other kinds of discrete convexity are not compatible with the aggregation operation. That is, the aggregation operation does not preserve integral convexity, $\mathrm{L}^{\natural}$-convexity, L-convexity, multimodularity, and (global, local) discrete midpoint convexity. This is immediate from the corresponding statements for the aggregation of sets in Section 3.3.

Remark 4.3. The convolution of two functions can be realized through a combination of direct sum and aggregation operations. We recall that the (infimal) convolution of two functions $f_{1}, f_{2}: \mathbb{Z}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ is defined by

$$
\begin{equation*}
\left(f_{1} \square f_{2}\right)(x)=\inf \left\{f_{1}(y)+f_{2}(z) \mid x=y+z, y, z \in \mathbb{Z}^{n}\right\} \quad\left(x \in \mathbb{Z}^{n}\right), \tag{4.5}
\end{equation*}
$$

where it is assumed that the infimum is bounded from below (i.e., $\left(f_{1} \square f_{2}\right)(x)>-\infty$ for every $x \in \mathbb{Z}^{n}$ ). For the given functions $f_{1}$ and $f_{2}$ we first form their direct sum

$$
f\left(x_{1}, x_{2}\right)=f_{1}\left(x_{1}\right)+f_{2}\left(x_{2}\right),
$$

where $x_{1}, x_{2} \in \mathbb{Z}^{n}$. The underlying set of $f$ is the union of two disjoint copies of $\{1,2, \ldots, n\}$, which we denote by $\left\{\psi_{1}(i) \mid i=1,2, \ldots, n\right\} \cup\left\{\psi_{2}(i) \mid i=1,2, \ldots, n\right\}$. Consider the partition of this underlying set into the pairs $\left\{\psi_{1}(i), \psi_{2}(i)\right\}$ of corresponding elements. Then the aggregation of $f$ coincides with the convolution $f_{1} \square f_{2}$.

### 4.4 Transformation by networks

In this section, we consider the transformation of a discrete (convex) function through a network. As in Section 3.4 let $G=(V, A ; U, W)$ be a directed graph with vertex set $V$, arc set $A$, entrance set $U$, and exit set $W$, where $U$ and $W$ are disjoint subsets of $V$ (cf., Fig. (2). For each arc $a \in A$, an integer interval $[\ell(a), u(a)]_{Z}$ is given as the capacity constraint, where $\ell(a) \in \mathbb{Z} \cup\{-\infty\}$ and $u(a) \in \mathbb{Z} \cup\{+\infty\}$. We consider an integral flow $\xi: A \rightarrow \mathbb{Z}$ that satisfies the capacity constraint (3.6) on arcs and the flow-conservation (3.7) at internal vertices. Recall notations $\partial \xi \in \mathbb{Z}^{V}, \partial \xi \mid U \in \mathbb{Z}^{U}$, and $\partial \xi \mid W \in \mathbb{Z}^{W}$.

In addition, we assume that the cost of integer-flow $\xi$ is measured in each $\operatorname{arc} a \in A$ in terms of a function $\varphi_{a}: \mathbb{Z} \rightarrow \mathbb{R} \cup\{+\infty\}$, where $\operatorname{dom} \varphi_{a}=[\ell(a), u(a)]_{\mathbb{Z}}$ and $\varphi_{a}$ is (discrete) convex in the sense that

$$
\begin{equation*}
\varphi_{a}(t-1)+\varphi_{a}(t+1) \geq 2 \varphi_{a}(t) \quad(t \in \mathbb{Z}) \tag{4.6}
\end{equation*}
$$

Suppose we are given a function $f: \mathbb{Z}^{U} \rightarrow \mathbb{R} \cup\{+\infty\}$ associated with the entrance set $U$. For each vector $y \in \mathbb{Z}^{W}$ on the exit set $W$, we define a function $g(y)$ as the minimum cost of a flow $\xi$ to meet the demand specification $\partial \xi \mid W=-y$ at the exit, where the cost of flow $\xi$ consists of two parts, the production cost $f(x)$ of $x=\partial \xi \mid U$ at the entrance and the transportation cost $\sum_{a \in A} \varphi_{a}(\xi(a))$ at arcs; the sum of these is to be minimized over varying supply $x$ and flow $\xi$ subject to the supply-demand constraints $\partial \xi \mid U=x$ and $\partial \xi \mid W=-y$ as well as the flow conservation constraint (3.7) at internal vertices. That is, $g: \mathbb{Z}^{W} \rightarrow \mathbb{R} \cup\{+\infty,-\infty\}$ is defined as

$$
\begin{gather*}
g(y)=\inf _{x, \xi}\left\{f(x)+\sum_{a \in A} \varphi_{a}(\xi(a)) \mid x \in \mathbb{Z}^{U} \text { and } \xi \in \mathbb{Z}^{A}\right. \text { satisfy (3.6), (3.7), } \\
\partial \xi \mid U=x, \text { and } \partial \xi \mid W=-y\} \quad\left(y \in \mathbb{Z}^{W}\right), \tag{4.7}
\end{gather*}
$$

where $g(y)=+\infty$ if no such $(x, \xi)$ exists. It is assumed that the effective domain dom $g$ is nonempty and that the infimum is bounded from below (i.e., $g(y)>-\infty$ for every $y \in \mathbb{Z}^{W}$ ). We regard $g$ as a result of transformation (or induction) of $f$ by the network.

Remark 4.4. Splitting, aggregation, and convolution can be regarded as special cases of the transformation by means of bipartite networks (cf., Fig. (1). For the convolution we use the bipartite graph (d) in Fig. (1)

It is known that M-convexity and its relatives are well-behaved with respect to the network induction.

## Theorem 4.7.

(1) The network induction of an $M^{\natural}$-convex function is $M^{\natural}$-convex.
(2) The network induction of an M-convex function is $M$-convex.
(3) The network induction of a jump $M^{\natural}$-convex function is jump $M^{\natural}$-convex.
(4) The network induction of a jump $M$-convex function is jump $M$-convex.

Proof. (4) The proof for jump M-convex functions, given in [20], is based on splitting and aggregation (Propositions 4.3 and 4.6), and other simple operations such as independent coordinate inversion, restriction, and addition of a separable convex function treated in [39, Propositions 4.3, 4.9, 4.14].
(3) The proof for jump $\mathrm{M}^{\natural}$-convex functions can be obtained as an adaptation of the proof for jump M-convex functions, as pointed out in [40]. This is possible since splitting and aggregation are allowed also for jump $\mathrm{M}^{\natural}$-convex functions by Propositions 4.3 and 4.6 , as well as independent coordinate inversion, restriction, and addition of a separable convex function (cf., [39, Propositions 4.3, 4.9, 4.14]).
(2) Two kinds of proofs are known for M-convex functions. The first proof [30] uses a dual variable and a characterization of M-convexity of a function in terms of its minimizers. The second proof [48, 49] is an algorithmic proof, which is described in [34, Section 9.6.2]. Yet another proof is possible, which derives this as a corollary of Part (4) for jump M-convex functions. Recall that an M-convex function is characterized as a jump M-convex function that has a constant-sum effective domain. If the given function $f$ is M-convex, then it is jump M-convex, and therefore, $g$ is jump M-convex by Part (4). In addition, dom $g$ is a constant-sum system, since $\operatorname{dom} f$ is a constant-sum system and $\partial \xi(U)+\partial \xi(W)=0$ by (3.7). Therefore, $g$ is M-convex.
(1) The proof for $\mathrm{M}^{\natural}$-convex functions can be obtained from Part (2) for M-convex functions as follows. Let $f$ be an $\mathrm{M}^{\natural}$-convex function given on $U$. Consider two new vertices $u_{0}$ and $w_{0}$ and an arc $\left(u_{0}, w_{0}\right)$, and let $\tilde{U}=U \cup\left\{u_{0}\right\}, \tilde{W}=W \cup\left\{w_{0}\right\}, \tilde{V}=V \cup\left\{u_{0}, w_{0}\right\}$, $\tilde{A}=A \cup\left\{\left(u_{0}, w_{0}\right)\right\}$, and $\tilde{G}=(\tilde{V}, \tilde{A} ; \tilde{U}, \tilde{W})$. For $a=\left(u_{0}, w_{0}\right)$ we define $\ell(a)=-\infty, u(a)=+\infty$, and $\varphi_{a} \equiv 0$. Let $\tilde{f}$ and $\tilde{g}$ be the functions associated, respectively, with $f$ and $g$ as in (2.19), where $\operatorname{dom} \tilde{f} \subseteq\left\{x \in \mathbb{Z}^{\tilde{U}} \mid x(\tilde{U})=0\right\}$ and $\operatorname{dom} \tilde{g} \subseteq\left\{y \in \mathbb{Z}^{\tilde{W}} \mid y(\tilde{W})=0\right\}$. If the function $g$ is induced from $f$ by $G$, then $\tilde{g}$ coincides with the function induced from $\tilde{f}$ by $\tilde{G}$. Since $f$ is $\mathrm{M}^{\natural}$-convex, $\tilde{f}$ is M-convex, and hence $\tilde{g}$ is M-convex by Part (2). This implies that $g$ is $\mathrm{M}^{\natural}$-convex. It is also possible to adapt the first and second proofs for M-convex functions to $\mathrm{M}^{\natural}$-convex functions.

Remark 4.5. Here is a supplement to Theorem4.7, The network induction for discrete convex functions is considered first by Murota [30] for M-convex functions, and stated also in [34, Theorem 9.26]. Part (1) for $\mathrm{M}^{\natural}$-convex functions is a variant thereof, and stated in [34, Theorem 9.26]. Part (4) for jump M-convex function is established in [20] and Part (3) for jump $\mathrm{M}^{\natural}$-convex functions is derived therefrom in [40]. Theorem4.7 here is a generalization of Theorem 3.7 for discrete convex sets. The transformation by networks can be generalized by replacing networks with poly-linking systems, and it is shown in [19] that the transformation by valuated integral poly-linking systems preserves M-convexity and jump M-convexity.


Figure 3: Network induction for a laminar convex function

Example 4.2 ([34, Note 9.31]). A laminar convex function introduced in Section 2.4.1 can be constructed by means of the network induction. As a concrete example, consider

$$
g\left(y_{1}, y_{2}, y_{3}\right)=\left|y_{1}+y_{2}+y_{3}\right|+\left(y_{1}+y_{2}\right)^{2}+y_{3}^{2},
$$

which is a laminar convex function of the form of (2.18) with a laminar family $\mathcal{T}=\{\{1,2,3\}$, $\{1,2\},\{1\},\{2\},\{3\}\}$ and univariate convex functions $\varphi_{123}(t)=|t|, \varphi_{12}(t)=\varphi_{3}(t)=t^{2}$, and $\varphi_{1}(t)=\varphi_{2}(t)=0$. For this function we consider the graph $G$, a rooted directed tree, depicted in Fig. 3, Each vertex other than the root $u$ corresponds to a member of $\mathcal{T}$. The entrance set $U$ is the singleton set of the root, i.e., $U=\{u\}$, and the exit set $W$ is the set of the leaves, i.e., $W=\left\{v_{1}, v_{2}, v_{3}\right\}$. The cost function $\varphi_{a}$ on each arc is determined by the head of the arc; for example, we have $\varphi_{123}(t)=|t|$ for $\operatorname{arc}\left(u, v_{123}\right)$ and $\varphi_{12}(t)=t^{2}$ for $\operatorname{arc}\left(v_{123}, v_{12}\right)$. Assume that the identically zero function $f \equiv 0$ is defined on the entrance set $U$. Then the function induced from $f$ by $G$ coincides with the function $g\left(y_{1}, y_{2}, y_{3}\right)=g\left(-y_{1},-y_{2},-y_{3}\right)$. Since $f \equiv 0$ is $\mathrm{M}^{\natural}$-convex, Theorem 4.7(1) shows that $g\left(y_{1}, y_{2}, y_{3}\right)$ is $\mathrm{M}^{\natural}$-convex.

In contrast, other kinds of discrete convexity are not compatible with the network induction. That is, the network induction does not preserve separable convexity, integral convexity, $\mathrm{L}^{\natural}$-convexity, L-convexity, multimodularity, and (global, local) discrete midpoint convexity. Note that these statements are immediate from the corresponding statements for splitting and aggregation in Sections 4.2 and 4.3 since the network induction is more general than those operations.

## 5 Proofs

In this section we give proofs for Propositions 3.2, 3.4, 3.5, 4.2, 4.4, and 4.5, We deal with propositions concerning integral convexity in the first two subsections, and then those concerning multimodularity in the following subsections, as follows:

- Section 5.1: Proposition 3.4 for splitting of integrally convex sets,
- Section 5.2. Proposition 4.4 for splitting of integrally convex functions,
- Section 5.3. Propositions 3.2 and 4.2 for direct sum of multimodular sets and functions,
- Section 5.4, Propositions 3.5 and 4.5 for splitting of multimodular sets and functions.


### 5.1 Proof for the splitting of integrally convex sets

Here is a proof of Proposition 3.4 concerning the splitting of an integrally convex set $S$. It suffices to consider an elementary splitting. Specifically we consider the splitting of the first variable $x_{1}$ of $\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in S$ into two variables $\left(y_{0}, y_{1}\right)$ satisfying $x_{1}=y_{0}+y_{1}$, that is, $U_{1}=\{0,1\}$ and $U_{i}=\{i\}$ for $i=2, \ldots, n$ in the notation of Section 3.2. The resulting set $T$ is given by

$$
T=\left\{\left(y_{0}, y_{1}, y_{2}, \ldots, y_{n}\right) \in \mathbb{Z}^{n+1} \mid y_{0}+y_{1}=x_{1}, y_{i}=x_{i}(i=2, \ldots, n),\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in S\right\} .
$$

To show the integral convexity of $T$, take any $y \in \bar{T} \subseteq \mathbb{R}^{n+1}$. A crucial step of the proof is to find a set of vectors $v^{\ell} \in T \cap N(y)$ to represent $y$ as their convex combination:

$$
\begin{equation*}
y=\sum_{\ell} \mu_{\ell} v^{\ell} \tag{5.1}
\end{equation*}
$$

where $\mu_{\ell} \geq 0$ and $\sum_{\ell} \mu_{\ell}=1$. This means, in particular, that each $v^{\ell} \in \mathbb{Z}^{n+1}$ must satisfy the condition $\lfloor y\rfloor \leq v^{\ell} \leq\lceil y\rceil$, since (cf., (2.6))

$$
N(y)=\left\{z \in \mathbb{Z}^{n+1} \mid\left\lfloor y_{i}\right\rfloor \leq z_{i} \leq\left\lceil y_{i}\right\rceil(i=0,1, \ldots, n)\right\} .
$$

We introduce notation $\hat{y}=\left(y_{2}, \ldots, y_{n}\right) \in \mathbb{R}^{n-1}$. Then $y=\left(y_{0}, y_{1}, \hat{y}\right)$. Let

$$
x=\left(y_{0}+y_{1}, \hat{y}\right)=\left(y_{0}+y_{1}, y_{2}, \ldots, y_{n}\right) \in \mathbb{R}^{n} .
$$

We have $x \in \bar{S}$. By the integral convexity of $S$, we can represent $x$ as a convex combination of some $u^{k} \in S \cap N(x)(k=1,2, \ldots, m)$, that is,

$$
\begin{equation*}
x=\sum_{k=1}^{m} \lambda_{k} u^{k} \tag{5.2}
\end{equation*}
$$

with $\lambda_{k} \geq 0$ and $\sum_{k} \lambda_{k}=1$, where $u^{k} \in S\left(\subseteq \mathbb{Z}^{n}\right)$ and $\lfloor x\rfloor \leq u^{k} \leq\lceil x\rceil$ for $k=1,2, \ldots, m$. The equation (5.2) shows

$$
\begin{equation*}
y_{0}+y_{1}=\sum_{k=1}^{m} \lambda_{k} u_{1}^{k}, \quad \hat{y}=\sum_{k=1}^{m} \lambda_{k} \hat{u}^{k}, \tag{5.3}
\end{equation*}
$$

where $u^{k}=\left(u_{1}^{k}, \hat{u}^{k}\right)$ with $u_{1}^{k} \in \mathbb{Z}$ and $\hat{u}^{k} \in \mathbb{Z}^{n-1}$. The condition $\lfloor x\rfloor \leq u^{k} \leq\lceil x\rceil$ is equivalent to

$$
\begin{equation*}
\left\lfloor y_{0}+y_{1}\right\rfloor \leq u_{1}^{k} \leq\left\lceil y_{0}+y_{1}\right\rceil, \quad\lfloor\hat{y}\rfloor \leq \hat{u}^{k} \leq\lceil\hat{y}\rceil . \tag{5.4}
\end{equation*}
$$

Let

$$
\begin{equation*}
K_{0}=\left\{k \mid u_{1}^{k}=\left\lfloor y_{0}+y_{1}\right\rfloor\right\}, \quad K_{1}=\left\{k \mid u_{1}^{k}=\left\lfloor y_{0}+y_{1}\right\rfloor+1\right\} . \tag{5.5}
\end{equation*}
$$

Denote the fractional parts of $y_{0}$ and $y_{1}$ by

$$
\begin{equation*}
\eta_{0}=y_{0}-\left\lfloor y_{0}\right\rfloor, \quad \eta_{1}=y_{1}-\left\lfloor y_{1}\right\rfloor . \tag{5.6}
\end{equation*}
$$

We have $0 \leq \eta_{0}<1$ and $0 \leq \eta_{1}<1$, from which follows $0 \leq \eta_{0}+\eta_{1}<2$. We distinguish the following cases:

Case 1: $0<\eta_{0}<1,0<\eta_{1}<1, \eta_{0}+\eta_{1}<1 \quad$ (This is the essential case);
Case 2: $0<\eta_{0}<1,0<\eta_{1}<1, \eta_{0}+\eta_{1}>1$;
Case 3: $\eta_{0}=0$ or $\eta_{1}=0$ or $\eta_{0}+\eta_{1}=1$ (in addition to $0 \leq \eta_{0}<1$ and $0 \leq \eta_{1}<1$ ).

### 5.1.1 Case 1: $0<\eta_{0}<1,0<\eta_{1}<1, \eta_{0}+\eta_{1}<1$

In this case we have

$$
\begin{equation*}
\left\lfloor y_{0}\right\rfloor+1=\left\lceil y_{0}\right\rceil, \quad\left\lfloor y_{1}\right\rfloor+1=\left\lceil y_{1}\right\rceil, \quad\left\lfloor y_{0}\right\rfloor+\left\lfloor y_{1}\right\rfloor=\left\lfloor y_{0}+y_{1}\right\rfloor . \tag{5.7}
\end{equation*}
$$

For $k=1,2, \ldots, m$, we define $(n+1)$-dimensional integer vectors $v^{k}$ or $\left\{v^{k 0}, v^{k 1}\right\}$ from the vectors $u^{k}=\left(u_{1}^{k}, \hat{u}^{k}\right)$ in (5.2). Define

$$
\begin{array}{rlrl}
v^{k} & :=\left(\left\lfloor y_{0}\right\rfloor,\right. & \left\lfloor y_{1}\right\rfloor, & \left.\hat{u}^{k}\right) \\
v^{k 0} & :=\left(\begin{array}{ll}
\text { for } \left.k \in K_{0}\right\rfloor \\
\left.v_{0}\right\rfloor+1,\left\lfloor y_{1}\right\rfloor, & \hat{u}^{k}
\end{array}\right) & \text { for } k \in K_{1}, \\
& =\left(\left\lfloor y_{0}\right\rfloor,\right. & \left.\left\lfloor y_{1}\right\rfloor+1, \hat{u}^{k}\right) &  \tag{5.10}\\
\text { for } k \in K_{1} .
\end{array}
$$

We use notations $\nu^{k}=\left(v_{0}^{k}, v_{1}^{k}, \hat{\nu}^{k}\right), \nu^{k 0}=\left(v_{0}^{k 0}, v_{1}^{k 0}, \hat{\nu}^{k 0}\right)$, and $\nu^{k 1}=\left(v_{0}^{k 1}, v_{1}^{k 1}, \hat{\nu}^{k 1}\right)$.
Claim 1: (i) $v^{k} \in T \cap N(y)$ for $k \in K_{0}$, and (ii) $v^{k 0}, v^{k 1} \in T \cap N(y)$ for $k \in K_{1}$.
Proof of Claim 1. (i) Let $k \in K_{0}$. We have $v^{k} \in T$ since

$$
v_{0}^{k}+v_{1}^{k}=\left\lfloor y_{0}\right\rfloor+\left\lfloor y_{1}\right\rfloor=\left\lfloor y_{0}+y_{1}\right\rfloor=u_{1}^{k} .
$$

We have $v^{k} \in N(y)$ since $v_{i}^{k}=\left\lfloor y_{i}\right\rfloor$ for $i=0,1$ and $\lfloor\hat{y}\rfloor \leq \hat{v}^{k}=\hat{u}^{k} \leq\lceil\hat{y}\rceil$ by (5.4).
(ii) Let $k \in K_{1}$. We have $v^{k 0}, \nu^{k 1} \in T$ since

$$
v_{0}^{k j}+v_{1}^{k j}=\left\lfloor y_{0}\right\rfloor+\left\lfloor y_{1}\right\rfloor+1=\left\lfloor y_{0}+y_{1}\right\rfloor+1=u_{1}^{k}
$$

for $j=0,1$. We have $v^{k 0} \in N(y)$ since

$$
v_{0}^{k 0}=\left\lfloor y_{0}\right\rfloor+1=\left\lceil y_{0}\right\rceil, \quad v_{1}^{k 0}=\left\lfloor y_{1}\right\rfloor, \quad\lfloor\hat{y}\rfloor \leq \hat{v}^{k 0}=\hat{u}^{k} \leq\lceil\hat{y}\rceil
$$

by (5.4). Similarly, we have $v^{k 1} \in N(y)$.
We will show that we can represent $y$ as a convex combination of the vectors in (5.8)(5.10), that is,

$$
\begin{equation*}
y=\sum_{k \in K_{0}} \mu_{k} v^{k}+\sum_{k \in K_{1}}\left(\mu_{k 0} v^{k 0}+\mu_{k 1} v^{k 1}\right) \tag{5.11}
\end{equation*}
$$

for some $\mu_{k}, \mu_{k 0}, \mu_{k 1} \geq 0$ with $\sum_{k \in K_{0}} \mu_{k}+\sum_{k \in K_{1}}\left(\mu_{k 0}+\mu_{k 1}\right)=1$. For the coefficients for $k \in K_{0}$ we take

$$
\begin{equation*}
\mu_{k}=\lambda_{k} \quad\left(k \in K_{0}\right) . \tag{5.12}
\end{equation*}
$$

For the coefficients for $k \in K_{1}$ we have the following.
Claim 2: There exist nonnegative $\mu_{k 0}, \mu_{k 1}\left(k \in K_{1}\right)$ satisfying

$$
\begin{align*}
\sum_{k \in K_{1}} \mu_{k 0} & =\eta_{0},  \tag{5.13}\\
\sum_{k \in K_{1}} \mu_{k 1} & =\eta_{1},  \tag{5.14}\\
\mu_{k 0}+\mu_{k 1} & =\lambda_{k} \quad \text { for each } k \in K_{1} . \tag{5.15}
\end{align*}
$$

Proof of Claim 2. Consider a $2 \times\left|K_{1}\right|$ matrix (array), say, $M$ in which the first row is $\left(\mu_{k 0} \mid\right.$ $k \in K_{1}$ ) and the second row is ( $\mu_{k 1} \mid k \in K_{1}$ ). The conditions above say that the first row-sum of $M$ is equal to $\eta_{0}$, the second row-sum is equal to $\eta_{1}$, and the $k$-th column-sum is equal to $\lambda_{k}$. Note that the sum of the row-sums is equal to the sum of the column-sums, that is,

$$
\eta_{0}+\eta_{1}=\sum_{k \in K_{1}} \lambda_{k},
$$

since (5.6) and (5.7) imply

$$
\eta_{0}+\eta_{1}=y_{0}+y_{1}-\left(\left\lfloor y_{0}\right\rfloor+\left\lfloor y_{1}\right\rfloor\right)=y_{0}+y_{1}-\left\lfloor y_{0}+y_{1}\right\rfloor,
$$

whereas (5.3) implies

$$
y_{0}+y_{1}=\left\lfloor y_{0}+y_{1}\right\rfloor+\sum_{k \in K_{1}} \lambda_{k} .
$$

Thus the proof of Claim 2 is reduced to showing the existence of a feasible (nonnegative) solution to a transportation problem. As is well known, a feasible solution always exists and it can be constructed by the so-called north-west corner method (or north-west rule [47]).

The coefficients $\mu_{k}, \mu_{k 0}, \mu_{k 1}$ constructed above have the desired properties. Indeed, we have the following:

- By (5.12) and (5.15), they are nonnegative numbers adding up to one:

$$
\sum_{k \in K_{0}} \mu_{k}+\sum_{k \in K_{1}}\left(\mu_{k 0}+\mu_{k 1}\right)=\sum_{k \in K_{0}} \lambda_{k}+\sum_{k \in K_{1}} \lambda_{k}=1 .
$$

- By (5.13), the first ( $0-\mathrm{th}$ ) component of the right-hand side of (5.11) is equal to

$$
\left\lfloor y_{0}\right\rfloor+\sum_{k \in K_{1}} \mu_{k 0}=\left\lfloor y_{0}\right\rfloor+\eta_{0}=y_{0} .
$$

- By (5.14), the second component of the right-hand side of (5.11) is equal to

$$
\left\lfloor y_{1}\right\rfloor+\sum_{k \in K_{1}} \mu_{k 1}=\left\lfloor y_{1}\right\rfloor+\eta_{1}=y_{1} .
$$

- By (5.12), (5.15), and (5.3), the remaining part is equal to

$$
\sum_{k \in K_{0}} \mu_{k} \hat{v}^{k}+\sum_{k \in K_{1}}\left(\mu_{k} \hat{v}^{k 0}+\mu_{k 1} \hat{v}^{k 1}\right)=\sum_{k \in K_{0}} \mu_{k} \hat{u}^{k}+\sum_{k \in K_{1}}\left(\mu_{k 0}+\mu_{k 1}\right) \hat{u}^{k}=\sum_{k \in K_{0} \cup K_{1}} \lambda_{k} \hat{u}^{k}=\hat{y} .
$$

The above argument shows the following lemma, which will be used in the proof of the splitting of integrally convex functions in Section 5.2,
Lemma 5.1. Let $y \in \bar{T}$ and $x=\left(y_{0}+y_{1}, \hat{y}\right)=\left(y_{0}+y_{1}, y_{2}, \ldots, y_{n}\right)$, and consider an arbitrary representation $x=\sum_{k=1}^{m} \lambda_{k} u^{k}$ of $x$ as a convex combination of $u^{k} \in S \cap N(x)(k=1,2, \ldots, m)$, where $y \in \mathbb{R}^{n+1}$ and $x, u^{1}, \ldots, u^{m} \in \mathbb{R}^{n}$. Assuming Case 1 , the vectors $v^{k}$, $v^{k 0}$, $v^{k 1}$ defined by (5.8), (5.9), (5.10) all belong to $T \cap N(y)$, and $y$ can be represented as their convex combination as

$$
\begin{equation*}
y=\sum_{k \in K_{0}} \mu_{k} v^{k}+\sum_{k \in K_{1}}\left(\mu_{k} v^{k 0}+\mu_{k 1} v^{k 1}\right), \tag{5.16}
\end{equation*}
$$

where $\lambda_{k}=\mu_{k}$ for $k \in K_{0}$ and $\lambda_{k}=\mu_{k 0}+\mu_{k 1}$ for $k \in K_{1}$.

### 5.1.2 Case 2: $0<\eta_{0}<1,0<\eta_{1}<1, \eta_{0}+\eta_{1}>1$

By coordinate inversion we can reduce this case to Case 1. Let

$$
\check{S}=-S, \quad \check{T}=-T, \quad \check{y}=-y, \quad \check{x}=-x .
$$

Then $\check{S}$ is integrally convex and $\check{T}$ is an elementary splitting of $\check{S}$.
Denote the fractional parts of $\check{y}_{0}$ and $\check{y}_{1}$ by

$$
\check{\eta}_{0}=\check{y}_{0}-\left\lfloor\check{y}_{0}\right\rfloor, \quad \check{\eta}_{1}=\check{y}_{1}-\left\lfloor\check{y}_{1}\right\rfloor .
$$

For $i=0,1$ we have

$$
\check{\eta}_{i}=-y_{i}-\left\lfloor-y_{i}\right\rfloor=-y_{i}+\left\lceil y_{i}\right\rceil=-y_{i}+\left(\left\lfloor y_{i}\right\rfloor+1\right)=1-\eta_{i},
$$

and therefore, $0<\check{\eta}_{0}<1,0<\check{\eta}_{1}<1, \check{\eta}_{0}+\check{\eta}_{1}<1$. By the argument for Case 1, we have $\check{y} \in \bar{T} \cap N(\breve{y})$, which is equivalent to $y \in \overline{T \cap N(y)}$.

### 5.1.3 Case 3: $\eta_{0}=0$ or $\eta_{1}=0$ or $\eta_{0}+\eta_{1}=1$

In this case, $y$ lies on the boundary of the region of Case 1 . We consider a perturbation of $y$ in the first two components $y_{0}$ and $y_{1}$ For an arbitrary $\varepsilon>0$, take $y^{\varepsilon}=\left(y_{0}^{\varepsilon}, y_{1}^{\varepsilon}, y_{2}, \ldots, y_{n}\right) \in \bar{T}$ with $\left|y_{i}^{\varepsilon}-y_{i}\right| \leq \varepsilon(i=0,1)$ such that $\eta_{0}^{\varepsilon}=y_{0}^{\varepsilon}-\left\lfloor y_{0}^{\varepsilon}\right\rfloor$ and $\eta_{1}^{\varepsilon}=y_{1}^{\varepsilon}-\left\lfloor y_{1}^{\varepsilon}\right\rfloor$ satisfy $0<\eta_{0}^{\varepsilon}<1$, $0<\eta_{1}^{\varepsilon}<1$, and $\eta_{0}^{\varepsilon}+\eta_{1}^{\varepsilon}<1$. Then we have

$$
N\left(y^{\varepsilon}\right)=\left\{z \in \mathbb{Z}^{n+1} \mid\left\lfloor y_{i}\right\rfloor \leq z_{i} \leq\left\lfloor y_{i}\right\rfloor+1(i=0,1),\left\lfloor y_{i}\right\rfloor \leq z_{i} \leq\left\lceil y_{i}\right\rceil(i=2, \ldots, n)\right\},
$$

which we denote by $N\left(y_{*}\right)$ since it does not depend on $\varepsilon$. Note that $N\left(y_{*}\right)$ is strictly larger than $N(y)$. By the argument of Case 1 , we have $y^{\varepsilon} \in \overline{T \cap N\left(y_{*}\right)}$. By letting $\varepsilon \rightarrow 0$, we obtain $y \in \overline{T \cap N\left(y_{*}\right)}$ since the convex hull of $T \cap N\left(y_{*}\right)$ is a closed set. Furthermore, $y \in \overline{T \cap N\left(y_{*}\right)}$ implies $y \in \overline{T \cap N(y)}$ in spite of the proper inclusion $N\left(y_{*}\right) \supset N(y)$.

We have completed the proof of Proposition 3.4.

### 5.2 Proof for the splitting of integrally convex functions

Here is a proof of Proposition 4.4 concerning the splitting of an integrally convex function. Let $g$ be an elementary splitting of an integrally convex function $f$ :

$$
\begin{equation*}
g\left(y_{0}, y_{1}, y_{2}, \ldots, y_{n}\right)=f\left(y_{0}+y_{1}, y_{2}, \ldots, y_{n}\right) \quad\left(y \in \mathbb{Z}^{n+1}\right) . \tag{5.17}
\end{equation*}
$$

The effective domain $T=\operatorname{dom} g$ is an elementary splitting of $S=\operatorname{dom} f$.
To prove the integral convexity of $g$, it suffices, by Theorem 2.1 (if part), to show that the local convex extension $\tilde{g}$ of $g$ satisfies the inequality

$$
\begin{equation*}
\tilde{g}\left(\frac{z+w}{2}\right) \leq \frac{1}{2}(g(z)+g(w)) \tag{5.18}
\end{equation*}
$$

for all $z, w \in \operatorname{dom} g$. Let $\check{z}=\left(z_{0}+z_{1}, z_{2}, \ldots, z_{n}\right)$ and $\check{w}=\left(w_{0}+w_{1}, w_{2}, \ldots, w_{n}\right)$. By Theorem 2.1 (only-if part), the local convex extension $\tilde{f}$ of $f$ satisfies the inequality

$$
\tilde{f}\left(\frac{\check{z}+\check{w}}{2}\right) \leq \frac{1}{2}(f(\check{z})+f(\check{w})),
$$

whereas

$$
\frac{1}{2}(f(\check{z})+f(\check{w}))=\frac{1}{2}(g(z)+g(w))
$$

from (5.17). Therefore, the desired inequality (5.18) follows from Lemma 5.2 below, where the technical result stated in Lemma 5.1] plays the crucial role in the proof.

## Lemma 5.2.

$$
\begin{equation*}
\tilde{g}\left(\frac{z+w}{2}\right) \leq \tilde{f}\left(\frac{\check{z}+\check{w}}{2}\right) . \tag{5.19}
\end{equation*}
$$

Proof. Let $y=(z+w) / 2$. We have $y=\left(y_{0}, y_{1}, y_{2}, \ldots, y_{n}\right) \in \bar{T}$. Depending on the fractional parts $\eta_{0}=y_{0}-\left\lfloor y_{0}\right\rfloor$ and $\eta_{1}=y_{1}-\left\lfloor y_{1}\right\rfloor$ of $y_{0}$ and $y_{1}$, we have three cases as in Section 5.1. Here we assume Case $1\left(0<\eta_{0}<1,0<\eta_{1}<1, \eta_{0}+\eta_{1}<1\right)$, which is the essential case.

Let $x=\left(y_{0}+y_{1}, y_{2}, \ldots, y_{n}\right)=(\check{z}+\check{w}) / 2$. By the definition of the local convex extension $\tilde{f}$, there exist some $u^{k} \in S \cap N(x)(k=1,2, \ldots, m)$ such that

$$
x=\sum_{k=1}^{m} \lambda_{k} u^{k}, \quad \tilde{f}(x)=\sum_{k=1}^{m} \lambda_{k} f\left(u^{k}\right),
$$

where $\lambda_{k} \geq 0$ and $\sum_{k} \lambda_{k}=1$. We now apply Lemma5.1 in Section 5.1.1 to obtain

$$
y=\sum_{k \in K_{0}} \mu_{k} v^{k}+\sum_{k \in K_{1}}\left(\mu_{k 0} v^{k 0}+\mu_{k 1} v^{k 1}\right)
$$

in (5.16). It follows from this and the definition of the local convex extension $\tilde{g}$ that

$$
\tilde{g}(y) \leq \sum_{k \in K_{0}} \mu_{k} g\left(v^{k}\right)+\sum_{k \in K_{1}}\left(\mu_{k 0} g\left(v^{k 0}\right)+\mu_{k 1} g\left(v^{k 1}\right)\right) .
$$

On the right-hand side we have

$$
\begin{aligned}
& g\left(v^{k}\right)=f\left(u^{k}\right) \quad \text { for } k \in K_{0}, \\
& g\left(v^{k 0}\right)=g\left(v^{k 1}\right)=f\left(u^{k}\right) \quad \text { for } k \in K_{1}
\end{aligned}
$$

by (5.8), (5.9), and (5.10). We also have $\lambda_{k}=\mu_{k}\left(k \in K_{0}\right)$ and $\lambda_{k}=\mu_{k 0}+\mu_{k 1}\left(k \in K_{1}\right)$. Therefore, we have

$$
\sum_{k \in K_{0}} \mu_{k} g\left(v^{k}\right)+\sum_{k \in K_{1}}\left(\mu_{k 0} g\left(v^{k 0}\right)+\mu_{k 1} g\left(v^{k 1}\right)\right)=\sum_{k=1}^{m} \lambda_{k} f\left(u^{k}\right) .
$$

From the above argument we obtain

$$
\begin{aligned}
\tilde{g}\left(\frac{z+w}{2}\right)=\tilde{g}(y) & \leq \sum_{k \in K_{0}} \mu_{k} g\left(v^{k}\right)+\sum_{k \in K_{1}}\left(\mu_{k 0} g\left(v^{k 0}\right)+\mu_{k 1} g\left(v^{k 1}\right)\right) \\
& =\sum_{k=1}^{m} \lambda_{k} f\left(u^{k}\right)=\tilde{f}(x)=\tilde{f}\left(\frac{\check{z}+\check{w}}{2}\right),
\end{aligned}
$$

which shows (5.19).
This completes the proof of Proposition 4.4,

### 5.3 Proof for the direct sum of multimodular sets and functions

In Section 5.3.1 we give a proof of Proposition 4.2 concerning the direct sum of multimodular functions. Proposition 3.2 for multimodular sets follows from this as a special case for the indicator functions of sets. In Section 5.3.2 we give an alternative proof of Proposition 3.2 for multimodular sets based on the polyhedral description of a multimodular set.

### 5.3.1 Proof via discrete midpoint convexity

We make use of Theorem 2.2 to reduce the argument for multimodular functions to that for $\mathrm{L}^{\text {¢ }}$ convex functions. The direct sum operation for multimodular functions does not correspond to the direct sum of the corresponding $L^{\text {h}}$-convex functions, but to a certain new operation on variables of the $L^{\mathrm{h}}$-convex functions (cf., Lemma 5.4). By investigating discrete midpoint convexity we shall show that this new operation preserves $L^{\mathrm{h}}$-convexity.

First we note a simple fact about integers.
Lemma 5.3. For $a, b \in \mathbb{Z}$ we have

$$
\begin{align*}
& \left\lceil\frac{a+b}{2}\right\rceil= \begin{cases}\lceil a / 2\rceil+\lceil b / 2\rceil & \text { (if } a \text { is even), } \\
\lceil a / 2\rceil+\lfloor b / 2\rfloor & \text { (if } a \text { is odd), }\end{cases}  \tag{5.20}\\
& \left\lfloor\frac{a+b}{2}\right\rfloor= \begin{cases}\lfloor a / 2\rfloor+\lfloor b / 2\rfloor & \text { (if } a \text { is even), } \\
\lfloor a / 2\rfloor+\lceil b / 2\rceil & \text { (if } a \text { is odd). }\end{cases} \tag{5.21}
\end{align*}
$$

We use variables $x \in \mathbb{Z}^{n_{1}}$ and $y \in \mathbb{Z}^{n_{2}}$ for multimodular functions $f_{1}$ and $f_{2}$, respectively. To reduce the argument to $L^{\natural}$-convex functions, we transform the variables $x$ and $y$ for multimodular functions to variables $p$ and $q$ for $\mathrm{L}^{\natural}$-convex functions through the relations $x=D_{1} p$ and $y=D_{2} q$ using matrices $D_{1}$ and $D_{2}$ of the form of (2.29) of sizes $n_{1}$ and $n_{2}$. We also transform the variable $(x, y)$ for $f_{1} \oplus f_{2}$ to a variable $r \in \mathbb{Z}^{n_{1}+n_{2}}$ in a similar manner. The following lemma reveals that $r$ is not equal to ( $p, q$ ), but is equal to ( $p, p_{*} \mathbf{1}+q$ ), where $p_{*}$ denotes the last component of $p$.

Lemma 5.4. If $z=(x, y)$ with $x=D_{1} p, y=D_{2} q$, and $z=\tilde{D} r$, then

$$
\begin{equation*}
r=\left(p, p_{*} \mathbf{1}+q\right), \tag{5.22}
\end{equation*}
$$

where $p_{*}$ denotes the last component of $p$, i.e., $p_{*}=p_{n_{1}}$.
Proof. The inverse of a matrix of the form (2.29) is the lower triangular matrix in (2.30), which implies

$$
\tilde{D}^{-1}=\left[\begin{array}{cc}
D_{1}^{-1} & O \\
\mathbf{1 1}^{\top} & D_{2}^{-1}
\end{array}\right],
$$

where $11^{\top}$ is an $n_{2} \times n_{1}$ matrix. Since $z=\tilde{D} r, x=D_{1} p$, and $y=D_{2} q$, we obtain

$$
r=\tilde{D}^{-1} z=\left[\begin{array}{cc}
D_{1}^{-1} & O \\
\mathbf{1 1}^{\top} & D_{2}^{-1}
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{cc}
D_{1}^{-1} & O \\
\mathbf{1 1}^{\top} & D_{2}^{-1}
\end{array}\right]\left[\begin{array}{cc}
D_{1} & O \\
O & D_{2}
\end{array}\right]\left[\begin{array}{c}
p \\
q
\end{array}\right]=\left[\begin{array}{cc}
I & O \\
\mathbf{1 1}^{\top} D_{1} & I
\end{array}\right]\left[\begin{array}{c}
p \\
q
\end{array}\right] .
$$

It is easy to verify from the definition (2.29) that each row of the matrix $11^{\top} D_{1}$ is the $n_{1}{ }^{-}$ dimensional unit vector $(0, \ldots, 0,1)$ having 1 in the last entry. Therefore, $r=\left(p, p_{*} \mathbf{1}+q\right)$ as in (5.22).

Let

$$
\tilde{f}=f_{1} \oplus f_{2}, \quad \tilde{g}(r)=\tilde{f}(\tilde{D} r), \quad g_{1}(p)=f_{1}\left(D_{1} p\right), \quad g_{2}(q)=f_{2}\left(D_{2} q\right)
$$

Since $f_{1}$ and $f_{2}$ are multimodular by assumption, $g_{1}$ and $g_{2}$ are $\mathrm{L}^{\mathrm{h}}$-convex by Theorem 2.2 (only-if part). We prove the $\mathrm{L}^{\natural}$-convexity of $\tilde{g}$ by showing its discrete midpoint convexity:

$$
\begin{equation*}
\left.\tilde{g}(r)+\tilde{g}\left(r^{\prime}\right) \geq \tilde{g}\left(\left\lceil\frac{r+r^{\prime}}{2}\right\rceil\right)+\tilde{g}\left(\left\lvert\, \frac{r+r^{\prime}}{2}\right.\right\rfloor\right) \quad\left(r, r^{\prime} \in \mathbb{Z}^{n_{1}+n_{2}}\right) . \tag{5.23}
\end{equation*}
$$

The multimodularity of $\tilde{f}=f_{1} \oplus f_{2}$ follows from this by Theorem 2.2 (if part). It is noted that $\tilde{g} \neq g_{1} \oplus g_{2}$ in general.

On the left-hand side of (5.23) we have

$$
\begin{equation*}
\tilde{g}(r)=\tilde{f}(\tilde{D} r)=\tilde{f}(x, y)=f_{1}(x)+f_{2}(y)=g_{1}(p)+g_{2}(q), \tag{5.24}
\end{equation*}
$$

where $(x, y)^{\top}=\tilde{D} r, p=D_{1}^{-1} x$, and $q=D_{2}^{-1} y$. Similarly,

$$
\begin{equation*}
\tilde{g}\left(r^{\prime}\right)=\tilde{f}\left(\tilde{D} r^{\prime}\right)=\tilde{f}\left(x^{\prime}, y^{\prime}\right)=f_{1}\left(x^{\prime}\right)+f_{2}\left(y^{\prime}\right)=g_{1}\left(p^{\prime}\right)+g_{2}\left(q^{\prime}\right), \tag{5.25}
\end{equation*}
$$

where $\left(x^{\prime}, y^{\prime}\right)^{\top}=\tilde{D} r^{\prime}, p^{\prime}=D_{1}^{-1} x^{\prime}$, and $q^{\prime}=D_{2}^{-1} y^{\prime}$.
For the right-hand side of (5.23) we use $r=\left(p, p_{*} 1+q\right)$ and $r^{\prime}=\left(p^{\prime}, p_{*}^{\prime} 1+q^{\prime}\right)$ in (5.22) to see

$$
\begin{align*}
& {\left[\frac{r+r^{\prime}}{2}\right\rceil=\left(\left[\frac{p+p^{\prime}}{2}\right\rceil,\left\lfloor\left.\frac{p_{*}+p_{*}^{\prime}}{2} \mathbf{1}+\frac{q+q^{\prime}}{2} \right\rvert\,\right),\right.}  \tag{5.26}\\
& {\left[\frac{r+r^{\prime}}{2}\right]=\left(\left\lfloor\frac{p+p^{\prime}}{2}\right\rfloor,\left\lfloor\frac{p_{*}+p_{*}^{\prime}}{2} \mathbf{1}+\frac{q+q^{\prime}}{2}\right]\right) .} \tag{5.27}
\end{align*}
$$

We now apply Lemma 5.3
Suppose that $p_{*}+p_{*}^{\prime}$ is even. By Lemma 5.3 (with $a=p_{*}+p_{*}^{\prime}$ and $b=q_{i}+q_{i}^{\prime}$ for $i=1,2, \ldots, n_{2}$ ), we obtain

$$
\begin{aligned}
& {\left[\frac{r+r^{\prime}}{2} \left\lvert\,=\left(\left\lfloor\frac{p+p^{\prime}}{2} \left\lvert\,,\left[\frac{p_{*}+p_{*}^{\prime}}{2} \left\lvert\, \mathbf{1}+\left[\left.\frac{q+q^{\prime}}{2} \right\rvert\,\right)\right.,\right.\right.\right.\right.\right.\right.} \\
& \left.\left\lvert\, \frac{r+r^{\prime}}{2}\right.\right]=\left(\left\lfloor\frac{p+p^{\prime}}{2}\right],\left\lfloor\frac{p_{*}+p_{*}^{\prime}}{2}\right] \mathbf{1}+\left\lfloor\frac{q+q^{\prime}}{2}\right]\right) .
\end{aligned}
$$

These vectors are of the form $\left(\hat{p}, \hat{p}_{*} \mathbf{1}+\hat{q}\right)$ with

$$
(\hat{p}, \hat{q})=\left(\left\lceil\frac{p+p^{\prime}}{2}\right\rceil,\left\lceil\frac{q+q^{\prime}}{2}\right\rceil\right), \quad\left(\left\lfloor\frac{p+p^{\prime}}{2}\right\rfloor,\left\lfloor\frac{q+q^{\prime}}{2}\right\rfloor\right),
$$

respectively. Therefore,

$$
\begin{align*}
& \tilde{g}\left(\left[\frac{r+r^{\prime}}{2}\right\rceil\right)=g_{1}\left(\left[\frac{p+p^{\prime}}{2} \|\right)+g_{2}\left(\left[\frac{q+q^{\prime}}{2}\right\rceil\right),\right.  \tag{5.28}\\
& \left.\tilde{g}\left(\left\lvert\, \frac{r+r^{\prime}}{2}\right.\right\rfloor\right)=g_{1}\left(\left[\frac{p+p^{\prime}}{2}\right\rfloor\right)+g_{2}\left(\left\lfloor\frac{q+q^{\prime}}{2}\right\rfloor\right) . \tag{5.29}
\end{align*}
$$

By (5.24), (5.25), (5.28), (5.29), and the discrete midpoint convexity of $g_{1}$ and $g_{2}$, we obtain the discrete midpoint convexity of $\tilde{g}$ in (5.23).

Suppose that $p_{*}+p_{*}^{\prime}$ is odd in (5.26) and (5.27). By Lemma 5.3 (with $a=p_{*}+p_{*}^{\prime}$ and $b=q_{i}+q_{i}^{\prime}$ for $i=1,2, \ldots, n_{2}$ ), we obtain

$$
\begin{aligned}
& \left\lceil\frac{r+r^{\prime}}{2} \left\lvert\,=\left(\left\lfloor\frac{p+p^{\prime}}{2}\right\rceil,\left[\frac{p_{*}+p_{*}^{\prime}}{2} \left\lvert\, \mathbf{1}+\left\lfloor\frac{q+q^{\prime}}{2}\right]\right.\right),\right.\right.\right. \\
& \left.\left\lvert\, \frac{r+r^{\prime}}{2}\right.\right]=\left(\left\lfloor\frac{p+p^{\prime}}{2}\right],\left\lfloor\frac{p_{*}+p_{*}^{\prime}}{2}\right] \mathbf{1}+\left[\left.\frac{q+q^{\prime}}{2} \right\rvert\,\right) .\right.
\end{aligned}
$$

These vectors are of the form $\left(\hat{p}, \hat{p}_{*} \mathbf{1}+\hat{q}\right)$ with

$$
(\hat{p}, \hat{q})=\left(\left\lceil\frac{p+p^{\prime}}{2}\right\rceil,\left\lfloor\frac{q+q^{\prime}}{2}\right\rfloor\right), \quad\left(\left\lfloor\frac{p+p^{\prime}}{2}\right\rfloor,\left\lceil\frac{q+q^{\prime}}{2}\right\rceil\right),
$$

respectively. Therefore,

$$
\begin{align*}
& \tilde{g}\left(\left[\frac{r+r^{\prime}}{2}\right\rceil\right)=g_{1}\left(\left[\frac{p+p^{\prime}}{2}\right\rfloor\right)+g_{2}\left(\left\lfloor\frac{q+q^{\prime}}{2}\right\rfloor\right)  \tag{5.30}\\
& \left.\tilde{g}\left(\left\lvert\, \frac{r+r^{\prime}}{2}\right.\right\rfloor\right)=g_{1}\left(\left\lfloor\frac{p+p^{\prime}}{2}\right\rfloor\right)+g_{2}\left(\left[\frac{q+q^{\prime}}{2}\right\rfloor\right) \tag{5.31}
\end{align*}
$$

By (5.24), (5.25), (5.30), (5.31), and the discrete midpoint convexity of $g_{1}$ and $g_{2}$, we obtain the discrete midpoint convexity of $\tilde{g}$ in (5.23). This completes the proof of Proposition4.2,

### 5.3.2 Proof via polyhedral description

In this section we give an alternative proof of Proposition 3.2 for multimodular sets based on their polyhedral descriptions.

Let $S_{1} \subseteq \mathbb{Z}^{n_{1}}$ and $S_{2} \subseteq \mathbb{Z}^{n_{2}}$, and also $N_{1}=\left\{1,2, \ldots, n_{1}\right\}$ and $N_{2}=\left\{n_{1}+1, n_{1}+2, \ldots, n_{1}+n_{2}\right\}$. By the polyhedral description of multimodular sets (cf., Theorem 2.3 (only-if part)), $S_{1}$ and $S_{2}$ can be described as

$$
\begin{aligned}
& S_{1}=\left\{x \in \mathbb{Z}^{n_{1}} \mid a_{I}^{1} \leq x(I) \leq b_{I}^{1} \quad\left(I: \text { consecutive interval in } N_{1}\right)\right\}, \\
& S_{2}=\left\{y \in \mathbb{Z}^{n_{2}} \mid a_{J}^{2} \leq y(J) \leq b_{J}^{2}\left(J: \text { consecutive interval in } N_{2}\right)\right\}
\end{aligned}
$$

for some integers $a_{I}^{1}$ and $b_{I}^{1}$ indexed by consecutive intervals $I \subseteq N_{1}$, and $a_{J}^{2}$ and $b_{J}^{2}$ indexed by consecutive intervals $J \subseteq N_{2}$, where $a_{I}^{1}, a_{J}^{2} \in \mathbb{Z} \cup\{-\infty\}$ and $b_{I}^{1}, b_{J}^{2} \in \mathbb{Z} \cup\{+\infty\}$. For consecutive intervals $K \subseteq N_{1} \cup N_{2}$ define

$$
a_{K}=\left\{\begin{array}{ll}
a_{K}^{1} & \left(K \subseteq N_{1}\right), \\
a_{K}^{2} & \left(K \subseteq N_{2}\right), \\
-\infty & \text { (otherwise) },
\end{array} \quad b_{K}= \begin{cases}b_{K}^{1} & \left(K \subseteq N_{1}\right), \\
b_{K}^{2} & \left(K \subseteq N_{2}\right) \\
+\infty & \text { (otherwise) }\end{cases}\right.
$$

Then we have

$$
S_{1} \oplus S_{2}=\left\{z \in \mathbb{Z}^{n_{1}+n_{2}} \mid a_{K} \leq z(K) \leq b_{K}\left(K: \text { consecutive interval in } N_{1} \cup N_{2}\right)\right\}
$$

which shows, by Theorem 2.3(if part), that $S_{1} \oplus S_{2}$ is a multimodular set.

Remark 5.1. The above alternative proof of Proposition 3.2 for multimodular sets is shorter and simpler than the proof of Section 5.3 .1 based on discrete midpoint convexity. Furthermore, this gives an alternative proof of Proposition 4.2 for multimodular functions in the special case where $f_{1}$ are $f_{2}$ have bounded effective domains or they are convex-extensible. If $\operatorname{dom} f_{1}$ and $\operatorname{dom} f_{2}$ are bounded, then $\operatorname{dom}\left(f_{1} \oplus f_{2}\right)=\operatorname{dom} f_{1} \oplus \operatorname{dom} f_{2}$ is also bounded, and we may use Theorem 2.5 that characterizes a multimodular function in terms of its minimizers. Let $\tilde{f}=f_{1} \oplus f_{2}$ and $c=\left(c_{1}, c_{2}\right)$. Then we have

$$
\arg \min \tilde{f}[-c]=\left(\arg \min f_{1}\left[-c_{1}\right]\right) \oplus\left(\arg \min f_{2}\left[-c_{2}\right]\right) .
$$

Here, $\arg \min f_{1}\left[-c_{1}\right]$ and $\arg \min f_{2}\left[-c_{2}\right]$ are multimodular sets by Theorem 2.5 (only-if part), and their direct sum is also multimodular by Proposition 3.2. Therefore, $\tilde{f}$ is a multimodular function by Theorem 2.5 (if part).

### 5.4 Proof for the splitting of multimodular sets and functions

Here is a proof of Proposition 4.5 concerning the splitting of a multimodular function. Proposition 3.5 for a multimodular set follows from this as a special case for the indicator function of a set. The proof makes use of the reduction to $L^{\natural}$-convex functions, and it turns out that the splitting operation for a multimodular function corresponds to introducing a dummy variable to an $L^{4}$-convex function that does not affect the function value $3^{3}$

Let $f$ be a multimodular function and $g$ be an elementary splitting of $f$ defined by

$$
g\left(y_{1}, \ldots, y_{k-1}, y_{k}^{\prime}, y_{k}^{\prime \prime}, y_{k+1}, \ldots, y_{n}\right)=f\left(y_{1}, \ldots, y_{k-1}, y_{k}^{\prime}+y_{k}^{\prime \prime}, y_{k+1}, \ldots, y_{n}\right) .
$$

We can express this as

$$
g(y)=f(C y),
$$

where $y=\left(y_{1}, \ldots, y_{k-1}, y_{k}^{\prime}, y_{k}^{\prime \prime}, y_{k+1}, \ldots, y_{n}\right) \in \mathbb{Z}^{n+1}$ and $C=\left(C_{i j}\right)$ is an $n \times(n+1)$ matrix defined by

$$
C_{i j}= \begin{cases}1 & \text { if } 1 \leq i=j \leq k \text { or } k \leq i=j-1 \leq n,  \tag{5.32}\\ 0 & \text { otherwise } .\end{cases}
$$

The correspondence of the variables is given by

$$
\begin{equation*}
x=C y, \tag{5.33}
\end{equation*}
$$

where $x=\left(x_{1}, \ldots, x_{k-1}, x_{k}, x_{k+1}, \ldots, x_{n}\right) \in \mathbb{Z}^{n}$.
To show the multimodularity of $g$, we consider functions $\hat{f}$ and $\hat{g}$ defined by

$$
\hat{f}(p)=f\left(D_{n} p\right), \quad \hat{g}(q)=g\left(D_{n+1} q\right),
$$

where $D_{n}$ is the $n \times n$ matrix of the form of (2.29) and $D_{n+1}$ is the $(n+1) \times(n+1)$ matrix of the form of (2.29). The correspondences of the variables are given by

$$
\begin{equation*}
x=D_{n} p, \quad y=D_{n+1} q . \tag{5.34}
\end{equation*}
$$

It follows from (5.33) and (5.34) that

$$
p=D_{n}^{-1} x=D_{n}^{-1} C y=D_{n}^{-1} C D_{n+1} q .
$$

[^3]By straightforward calculation using the definitions (2.29), (2.30), and (5.32), we can obtain that the $(i, j)$ entry of $D_{n}^{-1} C D_{n+1}$ is given as

$$
\left(D_{n}^{-1} C D_{n+1}\right)_{i j}= \begin{cases}1 & \text { if } 1 \leq i=j \leq k-1 \text { or } k \leq i=j-1 \leq n,  \tag{5.35}\\ 0 & \text { otherwise } .\end{cases}
$$

Therefore, the correspondence of the variables $p=\left(p_{1}, \ldots, p_{k-1}, p_{k}, p_{k+1}, \ldots, p_{n}\right)$ and $q=$ $\left(q_{1}, \ldots, q_{k-1}, q_{k}^{\prime}, q_{k}^{\prime \prime}, q_{k+1}, \ldots, q_{n}\right)$ is given by

$$
\left(p_{1}, \ldots, p_{k-1}, p_{k}, p_{k+1}, \ldots, p_{n}\right)=\left(q_{1}, \ldots, q_{k-1}, q_{k}^{\prime \prime}, q_{k+1}, \ldots, q_{n}\right)
$$

This shows that $\hat{g}$ does not depend on $q_{k}^{\prime}$ and

$$
\begin{equation*}
\hat{g}\left(q_{1}, \ldots, q_{k-1}, q_{k}^{\prime}, q_{k}^{\prime \prime}, q_{k+1}, \ldots, q_{n}\right)=\hat{f}\left(q_{1}, \ldots, q_{k-1}, q_{k}^{\prime \prime}, q_{k+1}, \ldots, q_{n}\right), \tag{5.36}
\end{equation*}
$$

in which $\hat{f}$ is $\mathrm{L}^{\natural}$-convex. Therefore, $\hat{g}$ is $\mathrm{L}^{\natural}$-convex, which implies, by Theorem 2.2, that $g$ is multimodular.

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[^1]:    ${ }^{1}$ This concept ("jump M-convex function") is the same as "M-convex function on a jump system" in [20, 36].

[^2]:    ${ }^{2}$ In Part (4) for an L-convex function $f$, the boundedness of the effective domain is to be understood as the boundedness of dom $f$ intersected with a coordinate plane $\left\{x \mid x_{i}=0\right\}$ for some (or any) $i \in N$. Note that the effective domain of an L-convex function has the invariance in the direction of $\mathbf{1}$.

[^3]:    ${ }^{3}$ See (5.36) at the end of the proof, where the value of function $\hat{g}$ does not depend on the variable $q_{k}^{\prime}$.

