OFF-CENTER REFLECTIONS: CAUSTICS AND CHAOS

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1. Introduction

We study the properties of a particular one-parameter family of circle maps called *off-center reflections* defined in §2. This map, in its 2-dimensional version, is first introduced in an open problem by S. T. Yau, [Y, problem 21], who suggests a cross study of the dynamics and geometry. In this article, we attempt to explore the possible link between the dynamics of this family of circle maps and their *caustics*. Although our study has not much contents in differential geometry as Yau expected, it reveals some interesting phenomena. For example, we observe and partially prove that in a certain generic range of the parameter, the caustics have *exactly* 4 cusp points for odd iterations; whereas for even iterations, each caustic is a curve tangential to the circle at *exactly* four points. This may not be the best result that one could state about the dynamics and the geometry of the map; nevertheless, we still put it forward in the hope that our study may invite better understanding to the subject. The off-center reflection also bares several interesting analytic forms. It is a Blaschke product restricted to the circle. Moreover, it has an infinite series expression highlighting that it is a perturbation of rotation on the circle. Since the work of Arnold, [A2], a standard type of perturbations has attracted much interests in mathematics and physics communities, [BBJ, Di, Z]. This standard type is exactly a reduction of the series of the off-center reflection. This adds more flavor to our study.

This family of off-center reflections plays an interesting role in the space of circle maps. With the parameter r going from 0 to 1, it carries the initial antipodal map to the terminal doubling map, which provides a particularly nice way of deforming a simple dynamics to a

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chaotic one (see the asymptotic orbit diagram in §4.2). We expect that some notion of stability about the aforementioned cusp points and tangent points on the caustics might emerge from further study of this family of circle maps. We also hope to explore more systematically the dynamics of general circle maps through the method of symplectic and contact geometry in a forthcoming work.

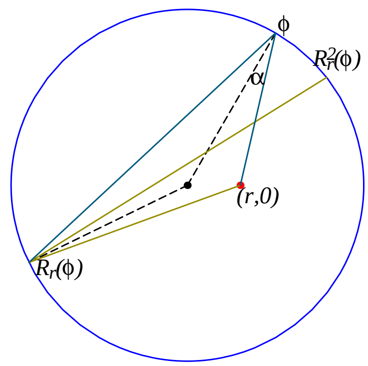
We will begin in §2 with the definition and some analytic properties of the map. Then the study is divided into three parts. In the first part (§3), our attention is given to the caustics of the map and its iterations. Our results concerning the caustics of odd iterations are more conclusive. Following a symplectic and contact geometry interpretation developed by Arnold [A1], we discover the "generating function" for the corresponding Lagrangian embedding. This in turns provides explicit formulae for the *orthotomics* of the caustics. Moreover, these orthotomics are always smooth simple closed curves (and hence the caustic of an off-center reflection has at least 4 cusp points following the classical 4 vertex theorem), and they are convex in a certain range of the parameter. The method fails for even iterations. Nevertheless, explicit computations still provide reasonable support for our prediction.

In the second part (§4), the main observations are presented together with graphical illustrations. There are also theoretical support for them. For example, we have partial result that the caustics is stable with $r \leq 1/3$. The tedious computations are separated into §4.2 for detailed reading.

In the third part (§5), we study the phenomenon of mode-locking behavior for this particular family of circle maps and the width of resonance zone is estimated. This is an attempt to understand the iterations of the map. This family extends a class of examples, which exhibits the same behavior, studied by Arnold and others. The mode-locking phenomenon of the off-center reflections and its "complex conjugates" are totally different. Moreover, $r = 1/3$ is the first value that this behavior undergoes a structural change. This probably is not simply a coincidence with the bifurcation values of cusps. There is room for investigation in this aspect.

2. Off-Center Reflection

An *off-center reflection* is a map $\mathbb{S}^1 \to \mathbb{S}^1$ defined as follows: Pick a point, say $(r, 0)$ in the interior of the unit disk D^2. For any $z \in \partial D^2 = \mathbb{S}^1$, emit a ray from $(r, 0)$ to z. The ray will be reflected at z with $\mathbb{S}^1$ as the curve of reflection and the reflected ray will hit another point $R_r(z)$ on $\mathbb{S}^1$. The map $z \mapsto R_r(z) : \mathbb{S}^1 \to \mathbb{S}^1$ is what we call an off-center reflection. The action of the map is shown in the figure where a point on $\mathbb{S}^1$ is represented by $\phi \mod 2\pi$ and α denotes the incident angle.



The iteration of this map is a little uncommon at first sight because it is different from the usual successive reflection in a curved mirror. However, maps similar to it have been a center of discussion in circle dynamics. Let us first establish two analytic expressions for the map $R_r : \mathbb{S}^1 \to \mathbb{S}^1$ where $0 \le r < 1$. By them, we may see R_r as a real function in terms of an infinite series as well as a complex function restricted to the circle. They are

$$R_r(\phi) = \phi + \pi - 2\alpha \qquad \mod 2\pi,$$
$$\alpha = \alpha(\phi) :\stackrel{\text{def}}{=\!=} \operatorname{Arg}(\cos\phi - r + \mathbf{i}\sin\phi) - \phi,$$

where Arg is the principal argument taking values in $(-\pi, \pi]$.

Lemma 2.1. *The angle of incident α has the following Fourier sine series,*

$$\alpha(\phi) = \operatorname{Arg}(\cos\phi - r + \mathbf{i}\sin\phi) - \phi = \sum_{k=1}^{\infty} \frac{r^k}{k}\sin(k\phi).$$

Thus,

$$R_r(\phi) = \phi + \pi - 2\sum_{k=1}^{\infty} \frac{r^k}{k}\sin(k\phi) \qquad \mod 2\pi. \tag{1}$$

Proof. It is clear that both α and $\partial\alpha/\partial r$ are odd functions in ϕ. So, they both have Fourier sine series expansions. Let $\alpha = \sum a_k(r)\sin(k\phi)$. Then, the coefficients of the series for $\partial\alpha/\partial r$ are given by

$$\begin{aligned} \frac{\partial a_k}{\partial r} &= \frac{2}{\pi}\int_0^{\pi} \frac{\sin\phi\sin(k\phi)}{1-2r\cos\phi+r^2}\,\mathrm{d}\phi \\ &= \frac{1}{\pi}\int_0^{\pi}\left(\frac{\cos(k-1)\phi}{1-2r\cos\phi+r^2} - \frac{\cos(k+1)\phi}{1-2r\cos\phi+r^2}\right)\mathrm{d}\phi \\ &= \frac{1}{\pi}\left(\frac{\pi r^{k-1}}{1-r^2} - \frac{\pi r^{k+1}}{1-r^2}\right) = r^{k-1}. \end{aligned}$$

Hence, those coefficients a_k for the series of α must be r^k/k. $\square$

In this paper, we often omit the mod 2π when it is clear in context. Formula (1) of R_r without the modulo 2π is exactly the lifting of R_r to a function from $\mathbb{R}$ to $\mathbb{R}$ taking 0 to π.

By playing with the argument of a complex number, we get another expression for the map R_r. This shows that the off-center reflection is a special case of the so-called Blaschke product. This map is not extendible to the hyperbolic disk.

Lemma 2.2. *For $|z| \le 1$, we have a complex function*

$$z \mapsto -z^2\frac{1-rz}{z-r}, \tag{2}$$

whose restriction to the boundary of the disk, $|z| = 1$, is the off-center reflection R_r.

Therefore, this function $R_r(\phi)$ is harmonic when (r, ϕ) are treated as polar coordinates as it is the argument of the analytic function $-z(1-z)^2$. We do not know whether this coupling between the parameters has additional physical or geometrical implications.

By changing a sign of the off-center reflection, we have another map $\overline{R}_r$ defined by

$$\overline{R}_r \ : \ \phi \mapsto \phi + \pi + 2\sum_{k=1}^{\infty}\frac{r^k}{k}\sin(k\phi) \qquad \mod 2\pi.$$

Geometrically, it is the mirror image of $R_r(\phi)$ reflected by the diameter joining ϕ to $\phi+\pi$. This map can be extended to the unit disk, namely,

$$z \mapsto -\frac{z-r}{1-rz}.$$

Therefore, it defines a map in $\mathrm{PSL}(2,\mathbb{R})$, the isometry group of the hyperbolic disk. The dynamics of R_r and this "conjugated" map $\overline{R}_r$ are completely different (see [H2] and §5.2).

For sufficiently small r, R_r behaves very similarly to R_0, i.e. the antipodal map. In fact, when $r < 1/3$, it is in the same component of R_0 in the group of orientation preserving diffeomorphisms of $\mathbb{S}^1$. However, $R_{1/3}$ is only a homeomorphism on $\mathbb{S}^1$ and R_r a degree 1 map when $1/3 < r < 1$. These can be easily concluded from the derivatives of R_r, which will also be useful later,

$$\begin{aligned}
R_r'(\phi) &= \frac{1-4r\cos\phi+3r^2}{1-2r\cos\phi+r^2},\\
R_r''(\phi) &= \frac{2r(1-r^2)\sin\phi}{(1-2r\cos\phi+r^2)^2},\\
R_r^{(3)}(\phi) &= \frac{2r(1-r^2)\left[(1+r^2)\cos\phi-2r(1+\sin^2\phi)\right]}{(1-2r\cos\phi+r^2)^3}.
\end{aligned}$$

In §4.2, we will give more information about the fixed point and other special points of R_r. More dynamical properties such as periodic cycles and whether they are attracting are discussed in [Au].

3. Caustics

3.1. **Caustic of Off-center Reflection.** In this section, we will discuss the cusp phenomenon of the caustic of R_r. Classical examples of caustics are the locus of focal points with respect to a point on a surface and the focal curve of a convex plane curve. Corresponding to these caustics, there are the famous Geometric Theorem (Conjecture) of Jacobi and Four-vertex Theorem. There are many interesting at-least-four results, see [A1, A5, T1, T2]. The caustic of off-center reflection provides another one. The conjugate locus of a point on a flat flying disc is, at degenerate situation, the caustic of the off-center reflection. It should be remarked that in the series of papers [BGG1, BG1, BG2, GK], the authors analyzed the singularities of the caustics produced by a point light source when it is reflected in a codimension 1 "mirror" in $\mathbb{R}^2$ and $\mathbb{R}^3$. Their emphasis though is on the "source genericity": whether the caustics could be made generic by moving the source. See also [BGG2].

For a circle map $f : \mathbb{S}^1 \to \mathbb{S}^1$, the family of lines joining ϕ to $f(\phi)$ is

$$\begin{aligned} F(\phi,x,y) &= (\sin f(\phi) - \sin\phi)(x - \cos\phi) - (\cos f(\phi) - \cos\phi)(y - \sin\phi) \\ &= (\sin f(\phi) - \sin\phi)x - (\cos f(\phi) - \cos\phi)y - \sin\left(f(\phi) - \phi\right). \end{aligned}$$

The *caustic* of the map f is defined to be the envelope of these lines. Thus, it is given by the equations $\frac{\partial F}{\partial \phi}(\phi,x,y) = 0 = F(\phi,x,y)$, that is,

$$\begin{pmatrix} \sin f(\phi) - \sin\phi & -\cos f(\phi) + \cos\phi \\ f'(\phi)\cos f(\phi) - \cos\phi & f'(\phi)\sin f(\phi) - \sin\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin(f(\phi) - \phi) \\ (f'(\phi) - 1)\cos(f(\phi) - \phi) \end{pmatrix}$$

Solving for x, y, we obtain a parameterization of the caustic

$$\begin{cases} x(\phi) = \dfrac{f'(\phi)\cos\phi + \cos f(\phi)}{1 + f'(\phi)} \\ y(\phi) = \dfrac{f'(\phi)\sin\phi + \sin f(\phi)}{1 + f'(\phi)}. \end{cases} \tag{3}$$

The tangent direction, which is degenerated at cusp points, of the caustic is given by

$$\begin{cases} x'(\phi) = \dfrac{f''(\phi)(\cos\phi - \cos f(\phi)) - f'(\phi)(1 + f'(\phi))(\sin\phi + \sin f(\phi))}{(1 + f'(\phi))^2} \\ y'(\phi) = \dfrac{f''(\phi)(\sin\phi - \sin f(\phi)) + f'(\phi)(1 + f'(\phi))(\cos\phi + \cos f(\phi))}{(1 + f'(\phi))^2}. \end{cases} \tag{4}$$

The caustic (3) of the off-center reflection may run to infinity since $1 + R_r'(\phi)$ may be equal to zero. In fact, it is so if and only if $r \geq 1/2$. It should be more appropriate to define the caustic as the envelop of the geodesic normal field on the sphere. After the stereographic projection, it does not matter whether the caustic is defined on the plane or the sphere as the local properties remain unchanged (Darboux Theorem of symplectic structure). As we will see, the local property of the caustic of R_r can be understood by direct computation.

Theorem 3.1. *For all $0 < r < 1$, there are exactly 4 cusp points on the caustic of R_r. Two of them correspond to the R_r-orbit, $\{0, \pi\}$, of period 2.*

Proof. The derivatives of x and y can be expressed completely in terms of r and ϕ, namely,

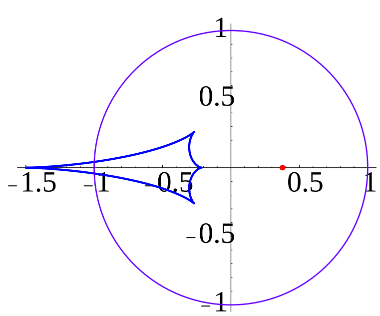
$$x'(\phi) = \frac{6r^2(-\cos\phi + r\cos(2\phi))(r-\cos\phi)\sin\phi}{(-1-2\,r^2+3\,r\,\cos\phi)^2}$$
$$y'(\phi) = \frac{6r^2(-1+2r\cos\phi)(r-\cos\phi)\sin^2\phi}{(-1-2\,r^2+3\,r\,\cos\phi)^2}.$$

The common solutions for $x'(\phi) = 0 = y'(\phi)$ are $\phi = 0, \pi$ and two values of ϕ with $\cos\phi = r$. Clearly, 0 and π are zeros of x' of first order and of y' of second order, thus, these are semicubical cusps. If $\cos\phi = r$, after further differentiation and evaluation at the point, one has

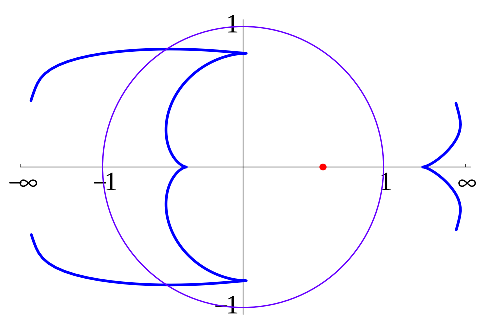
$$x''(\phi) = -12r^3; \qquad x'''(\phi) = \frac{12(5r^4+r^2)}{\sqrt{1-r^2}};$$
$$y''(\phi) = \frac{6r^2(2r^2-1)}{\sqrt{1-r^2}}; \qquad y'''(\phi) = \frac{-6r^3(10r^2-3)}{1-r^2}.$$

Thus, $x''y''' - x'''y'' = \frac{72r^4}{1-r^2} \neq 0$. Therefore, there are also semicubical cusps at those values of ϕ with $\cos\phi = r$. $\square$

Here are two pictures of the caustics of R_r, one for $r < 1/2$ and the other $r > 1/2$. Since the second one runs to infinity, it is drawn with "compressed" scale where a circle of radius > 1 represents the point of infinity and the caustics has a self-intersection there.



caustic of R_r with $r = 0.375$



caustic of R_r with $r \approx 1/\sqrt{2}$

3.2. **Symplectic and contact geometry interpretation.** The explicit computation in the previous section has its advantages and shortcomings. On the one hand, it gives a very exact count of the number of cusp points; on the other hand, it is too specific and also very complicated to apply especially when iterations are considered. In this section, we present the symplectic method which may work in general, though the result obtained is not as specific as before. The terminology of [A1] is followed.

Denote the coordinates of the unit cotangent bundle $ST^*(\mathbb{R}^2)$ by (p_x, p_y, x, y) where $(x, y) \in \mathbb{R}^2$ and $p_x^2 + p_y^2 = 1$. This bundle is a contact 3-manifold with the contact 1-form $p_x\, dx + p_y\, dy$, and the cotangent manifold $T^*(\mathbb{R}^2)$ is symplectic with the symplectic 2-form $d\,(p_x\, dx + p_y\, dy)$.

With our notation of ϕ and α in §2, the vector from ϕ to $R_r(\phi)$ is never zero and the unit vector in that direction (p_x, p_y) is given by

$$p_x = \cos(\phi + \pi - \alpha), \qquad p_y = \sin(\phi + \pi - \alpha).$$

Then

$$\begin{pmatrix} \phi \\ S \end{pmatrix} \mapsto \begin{pmatrix} p_x \\ p_y \\ x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\phi + \pi - \alpha) \\ \sin(\phi + \pi - \alpha) \\ \cos\phi + S\cos(\phi + \pi - \alpha) \\ \sin\phi + S\sin(\phi + \pi - \alpha) \end{pmatrix}$$

defines a map $L : \mathbb{S}^1 \times \mathbb{R}^1 \longrightarrow T^*(\mathbb{R}^2)$. This map may be thought of as a flow (in parameter S) of unit speed in the direction of the reflection lines, starting with the round circle $S = 0$. This is a case of what Arnold called "Legendrian collapsing" [A1].

Let $p : T^*(\mathbb{R}^2) \to \mathbb{R}^2$ be the canonical projection. The Jacobian of the map $p \circ L$ is

$$\begin{aligned} J(p \circ L) &= \det \begin{pmatrix} \cos(\phi + \pi - \alpha) & -\sin\phi - S(1 - \alpha')\sin(\phi + \pi - \alpha) \\ \sin(\phi + \pi - \alpha) & \cos\phi + S(1 - \alpha')\cos(\phi + \pi - \alpha) \end{pmatrix} \\ &= \cos(\phi + \pi - \alpha)\cos\phi + \sin(\phi + \pi - \alpha)\sin\phi + S(1 - \alpha') \\ &= -\cos\alpha + S(1 - \alpha') \end{aligned}$$

So the equation for the *critical curve* on $\mathbb{S}^1 \times \mathbb{R}^1$ is

$$S = \frac{\cos\alpha}{1 - \alpha'}.$$

Proposition 3.2. *The critical curve, when mapped to the (x, y)-plane, agrees with the caustic of R_r.*

Proof. Notice that since $R_r(\phi) = \phi + \pi - 2\alpha$, we have

$$R_r' = 1 - 2\alpha' \qquad \text{or} \qquad 1 + R_r' = 2(1 - \alpha').$$

Therefore,

$$\begin{aligned} x &= \cos\phi + \frac{\cos\alpha}{1-\alpha'}\cos(\phi+\pi-\alpha) \\ &= \frac{(1+R_r')\cos\phi + 2\cos\alpha\cos(\phi+\pi-\alpha)}{1+R_r'} \\ &= \frac{R_r'\cos\phi + \cos(\phi+\pi-2\alpha)}{1+R_r'}, \end{aligned}$$

which agrees with one of the equations (3). We may get the expression for y similarly. □

Furthermore, a straightforward calculation shows that

$$p_x\,dx + p_y\,dy = \sin\alpha\,d\phi + dS.$$

Thus, the image of L is a Lagrange cylinder in $T^*(\mathbb{R}^2)$. Notice further that $\alpha = \alpha(\phi)$ is an odd function of ϕ and therefore

$$\int_{\mathbb{S}^1} \sin\alpha\,d\phi = 0.$$

This implies that $p(L)$ is an exact Lagrange cylinder. Take a function $S(\phi)$ along the circle by

$$S(\phi) = -\int_0^\phi \sin\alpha\,d\phi.$$

It defines a section in the Lagrange cylinder $p(L)$. It is easy to see that $-S(\phi)$ is increasing for $0 \le \phi \le \pi$ and decreasing when $\pi \le \phi \le 2\pi$. Therefore, the curve C given by

$$x = \cos\phi + S(\phi)\cos(\phi+\pi-\alpha)$$

$$y = \sin\phi + S(\phi)\sin(\phi+\pi-\alpha)$$

is quite likely to be a convex plane curve. Let us show that this is the case when $r \le 1/2$.

First we note that C has a continuous normal field $(\cos(\phi+\pi-\alpha), \sin(\phi+\pi-\alpha))$. It is easy to compute

$${x'}^2 + {y'}^2 = 4\sin^2\alpha + (\cos\alpha - S(1-\alpha'))^2.$$

Therefore, ${x'}^2 + {y'}^2 = 0$ is possible only when $\phi = \pi$. But the number of zeros of ${x'}^2 + {y'}^2$ should be even (geometrically, because C is co-oriented). Thus ${x'}^2 + {y'}^2 > 0$ all the time and C is a smooth simple closed curve. We also point out that the curvature of C is

$$\kappa = \frac{1-\alpha'}{\sqrt{{x'}^2 + {y'}^2}}$$

which is non-negative when $r \leq 1/2$ and therefore C is convex. Now the family of reflection lines of R_r is identical to the family of normal lines of this convex curve C. Therefore, the caustic of R_r has at least 4 cusp points [A1].

The function $S(\phi)$ should be thought of as the *generating function* of the circle map $R_r(\phi)$. The curve C is related to the *orthotomic* of such a reflection. We will study such generating functions for general circle maps in a forthcoming work.

3.3. **Iterations of Reflections.** For an integer n, we denote the iteration of R_r by $R_r^n = R_r \circ R_r^{n-1} : \mathbb{S}^1 \to \mathbb{S}^1$. The equations (3) and (4) in §3.1, with $f = R_r^n$, give a parametrization of the caustic of R_r^n and its tangent.

The cusps on caustics of R_r^n are more intriging and complicated than that of R_r. There are fundamental differences between the caustics when n is odd or even. To see this difference, we may consider the trivial example that $r = 0$. For any odd n, R_0^n is the antipodal map and its caustic is a point (a degenerated curve with cusp). However, for even n, we have the identity map, so the caustic is defined by the family of tangents and it is the circle itself (a smooth curve). It is expected that this cusp versus "smooth" situation remains for r close to 0.

Theorem 3.3. *For sufficiently small $r > 0$, the caustic of R_r^{2m+1} has at least 4 cusp points.*

Proof. To some extent, the symplectic method in §3.2 may be adopted for R_r^{2m+1}. It can be proved by induction on m that

$$R_r^{2m+1}(\phi) = \phi + \pi - 2\tilde{\alpha}_m(\phi)$$

for some odd function $\tilde{\alpha}_m(\phi)$. For example,

$$\tilde{\alpha}_1(\phi) = \alpha(\phi) + \alpha(\phi + \pi - 2\alpha(\phi)) + \alpha(\phi - 2\alpha(\phi) - 2\alpha(\phi + \pi - 2\alpha(\phi))).$$

Therefore, we still have an exact Lagrangian cylinder and a sectional curve defined by

$$\tilde{S}(\phi) = -\int_0^\phi \sin \tilde{\alpha}_m \, d\phi.$$

For small r, it is a convex curve and hence there are at least 4 cusp points on the caustic. $\square$

Remark. The argument fails for even iterations of R_r, because the analog of $\tilde{\alpha}$ is not an odd function. Furthermore, it is not clear about how small the range of r should be. Yet, from experimental observation, there are at least four cusps for any $r > 0$ and there are exactly four for $0 < r < 1/3$. For more, please see the discussion after proposition 4.3.

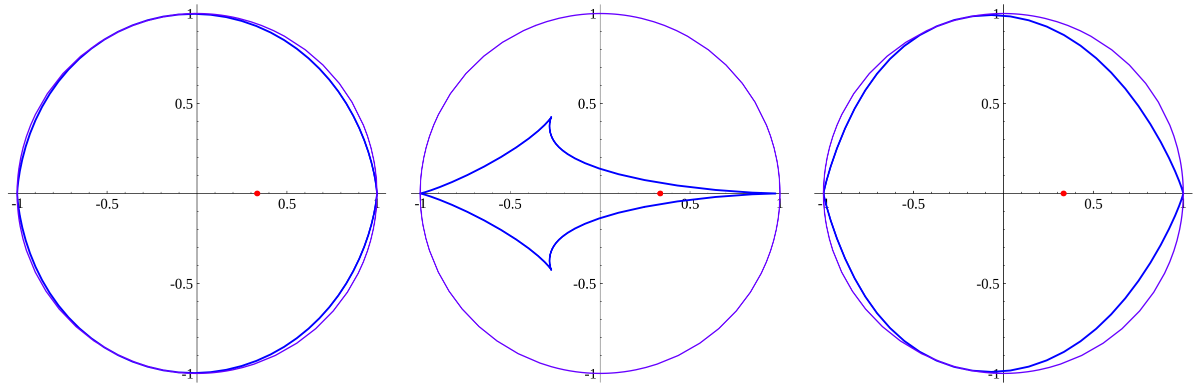
4. Experiments and Observations

To get more accurate information about the caustics of iterations, we have to rely on lengthy calculations. Our investigation is indeed partly theoretical and partly experimental. We will first describe some interesting properties with illustrations. The technical details of justification are left to the interested reader in §4.2.

4.1. **Observations.** We first put forward a conjecture about the exact picture of the caustic when R_r^n is still a diffeomorphism. Then we look at the bifurcation process of the structure of the caustics when r varies. Finally, we compare the caustics for different n. We will soon see that the 2-cycles of R_r play a special role (propositions 4.2 and 4.4). The 2-cycles are $\{0, \pi\}$ and $\{\pm\phi_c\}$ where $\phi_c \in (0, \pi)$ and $\cos\phi_c = \dfrac{1 - \sqrt{1 + 8r^2}}{4r}$. We will often refer to this notation.

Conjecture 4.1. *For $0 < r \leq 1/3$, the caustic of R_r^{2m+1} is a C^∞ curve with exactly four cusp singularities, with two of them occurring at $\phi = 0, \pi$. On the other hand, the caustic of R_r^{2m} is a differentiable curve; C^∞ everywhere except at exactly the four 2-periodic points of R_r, where the caustic is tangent to the unit circle.*

The conjecture is demonstrated by the following pictures, which are produced by programming in Mathematica. The purple (thin) curve is the unit circle and the blue (thick) curve is the caustic with the light source at the red dot, $(r, 0)$. Besides these pictures, we also have partial results that support our conjecture.



caption for $n = 2$, $r = 1/3$. caustic for $n = 3$, $r = 1/3$. caustic for $n = 4$, $r = 1/3$.

Proposition 4.2. *Any caustic curve of R_r^{2m} is tangential to the unit circle at $(\cos\phi, \sin\phi)$ at any point ϕ satisfying $R_r^{2m}(\phi) = \phi$. In particular, this includes the points 0, π, and $\pm\phi_c$. Moreover, if $r \le 1/3$, these are the only four points that the caustic meets the circle.*

Proof. By substitution of such ϕ into equations (3) with $f = R_r^n$, we have $x(\phi) = \cos\phi$ and $y(\phi) = \sin\phi$. Applying lemma 4.5 to such ϕ's, the assertion about the tangential property of the caustics of even iterations of R_r follows easily. Secondly, since $x^2 + y^2 = 1$, we have

$$(R_r^n)'(\phi)\cos\phi - \cos R_r^n(\phi) = 0$$

$$(R_r^n)'(\phi)\sin\phi - \sin R_r^n(\phi) = 0.$$

This leads to $R_r^n(\phi) = \phi$. If $r \le 1/3$, by lemma 4.8, n must be even and ϕ is one of the four 2-periodic points. $\square$

We have already seen from symplectic topology that caustics of odd iterations, R_r^{2m+1}, always have at least four cusps for sufficiently small $r > 0$. Moreover, for all R_r, two of the cusps occur at $\phi = 0, \pi$. Now, we may extend this result to odd iterations of R_r with isolated exceptional values of r. It turns out these exceptional values only occur in $r > 1/3$.

Proposition 4.3. *For $0 < r < 1/3$ and for generic $1/3 < r < 1$, the caustic of R_r^{2m+1} always has cusps at $0, \pi$.*

Proof. Note that for both $\phi_a = 0, \pi$, one has $R_r(\phi_a) = \phi_a + \pi \mod 2\pi$, thus we may apply lemma 4.5 to check whether there are cusps. Furthermore, $R_r^2(\phi_a) = \phi_a$ and $R_r''(\phi_a) = 0$, we can simplify the chain rules (lemma 4.6) and obtain $(R_r^{2m+1})''(\phi_a) = 0$ and

$$(R_r^{n+2})^{(3)}(\phi_a) = R_r'(R_r(\phi_a))(R_r^n)'(\phi_a)R_r^{(3)}(\phi_a) + \\ + R_r'(\phi_a)^3\left[(R_r^n)'(\phi_a)R_r^{(3)}(R_r(\phi_a)) + R_r'(R_r(\phi_a))^3(R_r^n)^{(3)}(\phi_a)\right].$$

Let us temporarily define, for positive integers n,

$$A_n = -(R_r^n)' + (R_r^n)'^3 + 2(R_r^n)^{(3)}.$$

So it suffices to check that $A_{2m+1}(\phi_a) \neq 0$. We will proceed by induction on m. First, by direct computation,

$$A_1(0) = \frac{24r^2}{(1-r)^2} > 0, \qquad A_1(\pi) = \frac{24r^2}{(1+r)^2} > 0.$$

Then, it can be shown that

$$\begin{aligned} A_{n+2}(\phi_a) &= \left(\frac{1-9r^2}{1-r^2}\right)^3 A_n(\phi_a) + \Big[2R_r^{(3)}(\phi_a+\pi)R_r'(\phi_a)^3 + 2R_r'(\phi_a+\pi)R_r^{(3)}(\phi_a) \\ &\qquad - R_r'(\phi_a)R_r'(\phi_a+\pi) + R_r'(\phi_a)^3R_r'(\phi_a+\pi)^3\Big](R_r^n)'(\phi_a) \\ &= \left(\frac{1-9r^2}{1-r^2}\right)^3 A_n(\phi_a) + A_2(\phi_a)\cdot(R_r^n)'(\phi_a). \end{aligned}$$

In particular,

$$\begin{aligned} A_{2m+1}(0) &= \left(\frac{1-9r^2}{1-r^2}\right)^3 A_{2m-1}(0) + A_2(0)\left(\frac{1-3r}{1-r}\right)^m\left(\frac{1+3r}{1+r}\right)^{m-1}; \\ A_{2m+1}(\pi) &= \left(\frac{1-9r^2}{1-r^2}\right)^3 A_{2m-1}(\pi) + A_2(\pi)\left(\frac{1-3r}{1-r}\right)^{m-1}\left(\frac{1+3r}{1+r}\right)^{m}; \end{aligned}$$

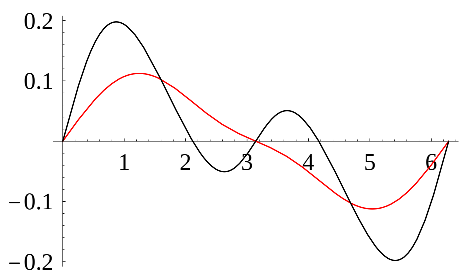
where

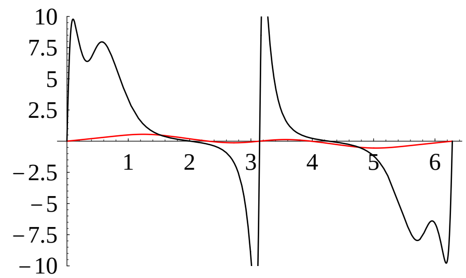
$$\begin{aligned} A_2(0) &= \frac{48r^2(1+r)}{(1-r^2)^3}(1-3r+13r^2-15r^3), \\ A_2(\pi) &= \frac{48r^2(1+r)}{(1-r^2)^3}(1+3r+13r^2+15r^3). \end{aligned}$$

It can be easily computed that $A_2(0) > 0$ for $0 < r < 1/3$ and $A_2(\pi) > 0$ for all $r > 0$. Thus, $A_{2m+1}(\phi_a) \geq \left(\frac{1-9r^2}{1-r^2}\right)^3 A_{2m-1}(\phi_a) > 0$.

For $r > 1/3$, A_{2m+1} may have zeros. We can only conclude from above that A_{2m+1} is a rational function in r with denominator being a power of $(1-r^2)$. Therefore, it has only isolated zeros and cusps at $0, \pi$ occur for $r > 1/3$ generically. $\square$

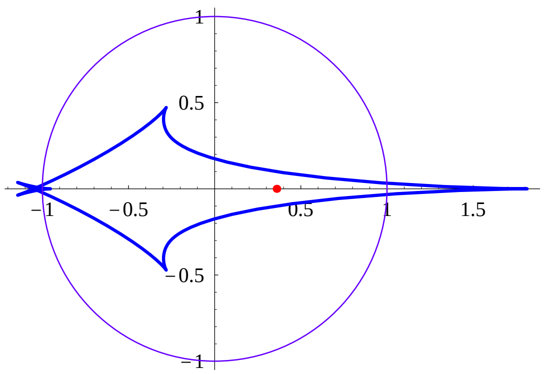
We observe that at cusp points, (4) gives a set of "homogeneous" equations which has zero determinant. Thus, except at a couple of ϕ's, it is sufficient to solve only one equation, say, $x'(\phi) = 0$. It is likely that this equation has exactly four solutions for $0 < r < 1/3$. However, it is still hard to solve explicitly especially for high iterations of R_r. Here are two pictures of the functions $(R_r^3)''/((R_r^3)'(1+(R_r^3)'))$ and $(\sin\phi + \sin R_r^3(\phi))/(\cos\phi - \cos R_r^3(\phi))$ for $r = 0.1$ and $r = 0.33$. Exactly four cusp solutions are demonstrated in each of them. We do not have a proof of this graphical fact. Perhaps it may be proved by detailed curve sketching argument and comparison of R_r^{2m-1} and R_r^{2m+1} using the known properties of R_r^2 given in lemma 4.8.



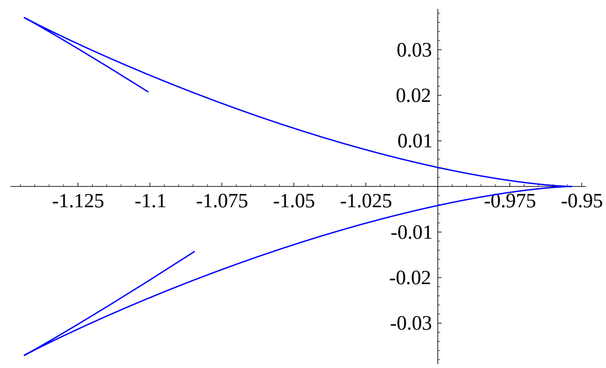


It is probably worthwhile to compare the situation here with the so-called Jacobi conjecture promoted by Arnold [A1, A5]. As mentioned in the beginning of §3.1, the caustics of R_r agrees with the locus of conjugate points of $(r, 0)$ on a flat flying disk. Although the loci of higher order conjugate points are not the same as the caustics of odd iterations of R_r, their common contact geometric nature indicates that two problems of whether there are exactly 4 cusps on the loci of higher order conjugate points and whether there are exactly 4 cusps of the caustics of odd iteration of R_r might be related. In both cases, since we do not have exact nice formulae for the loci of higher order conjugate points and the caustics of odd iterations of R_r, it would be very difficult to have an exact count of cusps. Of course, Conjecture 4.1 deals with a very special situation. From graphical evidence, it is tempting to think that a proof should not be out of reach with brute force calculation. Nevertheless, after spending much effort on this temptation, we think some more conceptual understanding of the caustics of iterations of R_r is needed in order to get an exact count of cusps.

Furthermore, in the proof above, we see that for $n \geq 3$, except $A_3(\pi)$, the quantity $A_n(0) = 0 = A_n(\pi)$ always at $r = 1/3$. There is a possible structural change on the caustic of R_r^n occurring at $r = 1/3$ for $n \geq 3$. We observe from experiment that, for the caustic of odd iterations, once $r > 1/3$, bifurcation of cusp may occur. Interestingly, from the computed pictures (only that of R_r^3 is shown), bifurcation only occur at the cusp corresponding to $\phi = 0$ but not others. Would the different properties between $A_2(0)$ and $A_2(\pi)$ be part of the reasons?



$n = 3$, $r = 0.36$.



its enlargement, $\phi = 0$.

On the other hand, bifurcation into cusps also occurs for even iterations of R_r at $r = 1/3$. We will discuss this by beginning with some Taylor expansions. Since R_r and its iterations are 2π-periodic odd functions, they have particular nice expansions at $\phi_a = 0, \pi$. This enables us to see the local properties of the caustics more clearly.

Let f be any even iteration of R_r, then $f(\phi_a) = \phi_a$. We write $\vartheta = \phi - \phi_a$ and $g(\vartheta) = f(\phi) - \phi_a$ and suppose it has an expansion

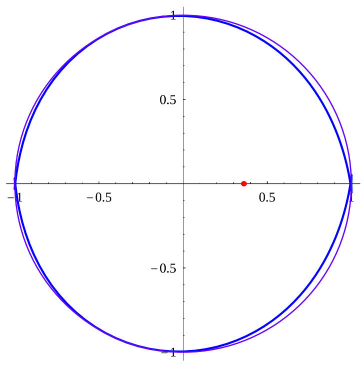
$$g(\vartheta) = \sum_{k=0} a_{2k+1} \vartheta^{2k+1}.$$

One can inductively work out the coefficients of the expansions of $1+f'(\phi) = 1+g'(\vartheta)$, $\cos f(\phi) = \pm \cos g(\vartheta)$, etc. If P_k, Q_k denote polynomials with $P_k(0, \dots, 0) = 0 = Q_k(0, \dots, 0)$, one has

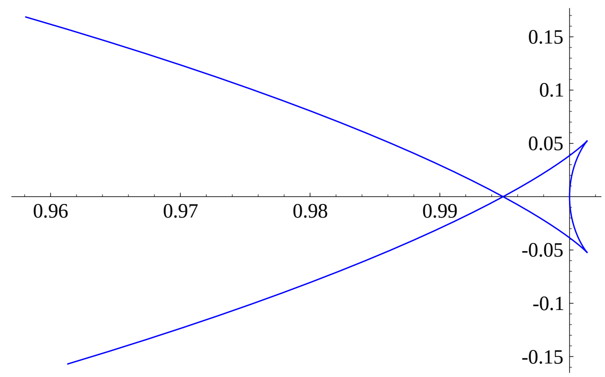
$$x(\phi) = \pm 1 \mp \frac{a_1}{2} \vartheta^2 + \sum_{k=1} \frac{P_k(a_1, \dots, a_{2k+1})}{(1+a_1)^{2k+1}} \vartheta^{2k};$$

$$y(\phi) = \pm \frac{2a_1}{1+a_1} \vartheta + \sum_{k=1} \frac{Q_k(a_1, \dots, a_{2k+1})}{(1+a_1)^{2k+1}} \vartheta^{2k+1}.$$

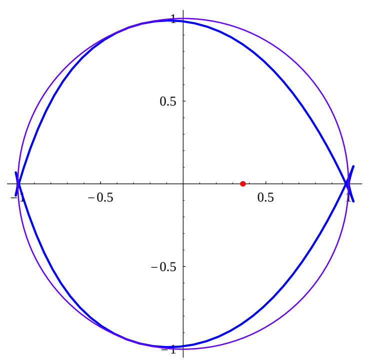
These expansions are helpful to understand the caustics of R_r^{2m} at $\phi_a = 0, \pi$. It would be convenient to look at the pictures before we go on.



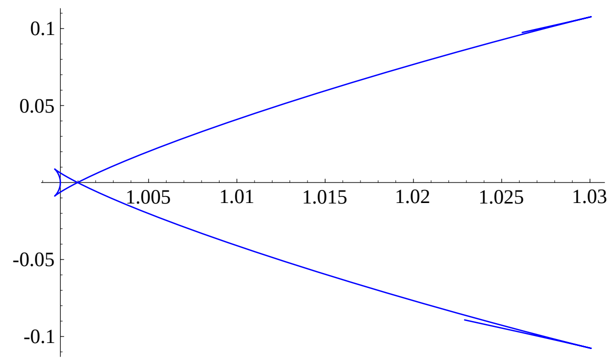
$n = 2,\ r = 0.36.$



its enlargement, $\phi = 0$.



$n = 4,\ r = 0.36.$



its enlargement, $\phi = 0$.

In the above pictures, cusps are born near $\phi = 0$ and π. From the enlargement, the caustic bifurcates into $2m$ cusps when r increases across $1/3$, where R_r changes from a diffeomorphism to a degree 1 map. In the expansion of R_r^{2m}, $a_1 = \left(\frac{1-9r^2}{1-r^2}\right)^m$. By this, we will see that $r = 1/3$ is the value that the caustic of R_r^{2m} changes at $\phi = 0, \pi$. In fact, the caustics of $R_{1/3}^2$ has the following Taylor expansions. At $\phi_a = 0$,

$$x(\phi) - x(0) = \frac{-27}{4}\vartheta^4 + \mathrm{O}(\vartheta^6)$$
$$y(\phi) - y(0) = 18\vartheta^3 + \mathrm{O}(\vartheta^5);$$

and at $\phi_a = \pi$,

$$x(\phi) - x(\pi) = \frac{243}{16}\vartheta^4 + \mathrm{O}(\vartheta^6)$$
$$y(\phi) - y(\pi) = \frac{-81}{2}\vartheta^3 + \mathrm{O}(\vartheta^5).$$

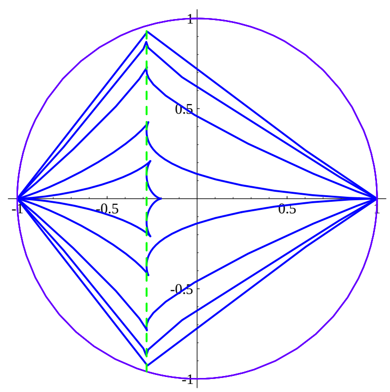
This shows that the caustic of R_r^2 undergoes a swallowtail bifurcation at $0, \pi$ when $r = 1/3$. We may further work out the expansion of $R_{1/3}^{2m}$ as the m-iteration of $R_{1/3}^2$. Using $a_1 = 0$ and $a_3 \neq 0$

for R_r^2, we have

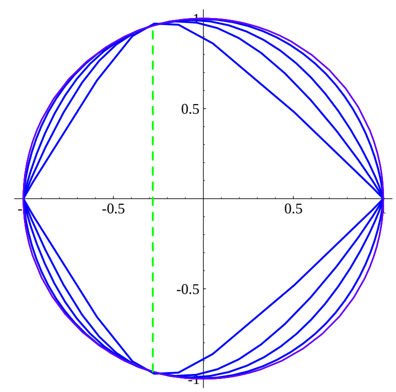
$$R_{1/3}^{2m}(\phi) = \phi_a + \vartheta^{3^m} U(\vartheta)$$

for some function U with $U(0) \neq 0$. The bifurcation of the caustics of R_r^{2m} at $\phi = 0, \pi$ should be of the type $(\vartheta^{3^m+1}, \vartheta^{3^m})$ when r passes $1/3$.

In the above, we consider the behavior of the caustics R_r^n with parameter r and n fixed. What happens to the caustics if r is fixed and n is allowed to vary? There is an interesting phenomenon for $r \leq 1/3$. Although there is fundamental difference when n is odd and even, this difference disappears as n goes to infinity. Here are the pictures showing how the caustics of R_r^{2m+1} and R_r^{2m} change–they tend to the same quadrilateral.



caustics of $R_{1/3}^{\text{odd}}$



caustics of $R_{1/3}^{\text{even}}$

The green (dashed) vertical line from ϕ_c to $-\phi_c$ in the picture of even or odd caustics is determined by r but not the number of iterations. When n is even, it is where the caustic is tangent to the circle. When n is odd, every caustic is tangent to this vertical line because it is the line joining ϕ_c and $R_r^{2m+1}(\phi_c) = -\phi_c$. The point of tangency occurs exactly at ϕ_c by definition.

Proposition 4.4. *For $0 < r \leq 1/3$, as $m \to \infty$, both the caustics of R_r^{2m+1} and R_r^{2m} approach the same quadrilateral defined by the four points 0, π, and $\pm\phi_c$, which are the only 2-periodic points of R_r.*

Proof. Let us first see that the caustics of R_r^{2m+1} at the four points tend to the circle as $m \to \infty$. These four ϕ's are the solution to $R_r^2(\phi) = \phi$. Thus, we have $(R_r^{2m+1})'(\phi) = R_r'(\phi)^{m+1} \cdot R_r'(R_r(\phi))^m$.

At $\phi = 0, \pi$, the coordinates of the caustic are

$$x(\phi) = \frac{\pm\left((R_r^n)'(\phi) - 1\right)}{1 + (R_r^n)'(\phi)}, \qquad y(\phi) = 0.$$

It is clear that $(R_r^{2m+1})'(0) = R_r'(0)^{m+1} R_r'(\pi)^m = \left(\frac{1-3r}{1-r}\right)^{m+1} \cdot \left(\frac{1+3r}{1+r}\right)^m$. Thus, $x(0) \to -1$ as $m \to \infty$. The situation at π is similar.

At the point ϕ_c with $R_r^n(\phi_c) = -\phi_c$, the coordinates of the caustic at this point are

$$x(\phi_c) = \cos\phi_c, \qquad y(\phi_c) = \frac{(-1 + (R_r^n)'(\phi_c))\sin\phi_c}{1 + (R_r^n)'(\phi_c)}.$$

Since R_r is odd and $R_r^2(\phi_c) = \phi_c$, it follows that $(R_r^n)'(\phi_c) = R_r'(\phi_c)^n$. Moreover, by $4r\cos\phi_c = 1 - \sqrt{1+8r^2}$, one may show that

$$\frac{(-1 + (R_r^n)'(\phi_c))}{1 + (R_r^n)'(\phi_c)} \to 1 \qquad \text{as } n \to \infty.$$

Again, these two cusps approach to the unit circle.

Secondly, from the lemmas 4.7 and 4.8, 0 and π are the attracting fixed points of R_r^2 while $\pm\phi_c$ are repelling. Moreover, the attracting basins for 0 and π are $(-\phi_c, \phi_c)$ and $(\phi_c, 2\pi - \phi_c)$ respectively. Thus, for any given neighborhood of 0, for sufficiently large m, for any neighboring $\phi_1, \phi_2 \in (-\phi_c, \phi_c)$, $R_r^{2m}(\phi_1)$ and $R_r^{2m}(\phi_2)$ lie in that neighborhood of 0. Hence, the intersection of the lines from ϕ_j to $R_r^{2m}(\phi_j)$ lies in a neighborhood of the quadrilateral. The proof for the cases at π and of odd iterations are similar. $\square$

4.2. **Technical Results.** In this section, a couple of technical results will be given. They are mostly done by direct computations and the methods may not be insightful. However, they may be the necessary evil for they will be used to justify our observations in §4.1. The first one deals with the existence of cusp at certain special "symmetric" positions.

Lemma 4.5. *Let f denote any iteration of R_r. On the caustic of f, the conditions for the occurrence of a semicubical cusp at ϕ are*

- $f'(\phi) = 0$ *and* $f''(\phi) \neq 0$ *if* $f(\phi) = \phi$*;*
- $f''(\phi) = 0$ *and* $-f'(\phi) + f'(\phi)^3 + 2f^{(3)}(\phi) \neq 0$ *if* $f(\phi) = \phi + \pi$*;*

Proof. This is proved by computing the derivatives of (3) and (4), then evaluate at the particular values ϕ_0 or ϕ_a. Then the result follows by verifying $x' = 0 = y'$ and $x''y''' - x'''y'' \neq 0$. □

Remarks. Although this lemma is stated for an iteration of R_r, it is actually true for any circle map. Likewise, many results in this section hold in a more general setting but we would like to focus on iterations of R_r. Furthermore, at a point ϕ with $f(\phi) = -\phi$, we always have $x'(\phi) = 0$. The conditions are

$$f'(\phi)(1 + f'(\phi)) \cos\phi + f''(\phi) \sin\phi = 0$$
$$f'(\phi)(2 + \cos(2\phi)) + 6f'(\phi)^2 \cos^2\phi + (1 + 2\cos(2\phi))f'(\phi)^3 - 2f^{(3)}(\phi) \sin^2\phi \neq 0.$$

Analogously, if $f(\phi) = \pi - \phi$, we have $y'(\phi) = 0$ and conditions

$$f'(\phi)(1 + f'(\phi)) \sin\phi - f''(\phi) \cos\phi = 0$$
$$f'(\phi)(2 - \cos(2\phi)) + 6f'(\phi)^2 \sin^2\phi + (1 - 2\cos(2\phi))f'(\phi)^3 - 2f^{(3)}(\phi) \cos^2\phi \neq 0.$$

The next one may be an exercise for calculus students.

Lemma 4.6. *The chain rules for R_r^n with $n = p + q$ are given by*

$$(R_r^n)'(\phi) = (R_r^p)'(R_r^q(\phi)) \cdot (R_r^q)'(\phi) = R_r'(\phi) \cdot R_r'(R_r(\phi)) \cdot \cdots \cdot R_r'(R_r^{n-1}(\phi))$$
$$(R_r^n)''(\phi) = (R_r^p)''(R_r^q(\phi)) \cdot (R_r^q)'(\phi)^2 + (R_r^p)'(R_r^q(\phi)) \cdot (R_r^q)''(\phi)$$
$$(R_r^n)^{(3)}(\phi) = (R_r^p)^{(3)}(R_r^q(\phi)) \cdot (R_r^q)'(\phi)^3 + (R_r^p)'(R_r^q(\phi)) \cdot (R_r^q)^{(3)}(\phi) +$$
$$+ 3(R_r^p)''(R_r^q(\phi)) \cdot (R_r^q)'(\phi) \cdot (R_r^q)''(\phi).$$

In determining the cusps on the caustic, some orbits in the iteration play a special role. We thus establish the following to handle that.

Lemma 4.7. *For $0 \leq r \leq 1/3$, R_r^{2m+1} has no fixed point and $R_r^{2m}(\phi) \neq \phi + \pi$.*

Proof. Firstly, one can obtain algebraically the four fixed points of R_r^2. The attracting ones are 0, π, while $\pm\phi_c$ are repelling. Then by simple calculus, we get the corresponding attracting basins and conclude that $R_r^{2m}(\phi)$ converges to 0 or $\pi \mod 2\pi$ monotonically. By the series expression (1) of $R_r(\phi)$, one can deduce the following estimate

$$|R_r(\phi) - \phi| \geq \pi - 2\left|\log(1-r)\right|.$$

Then, according to the convergence of $R_r^{2m}(\phi)$, the lemma follows. $\square$

Lemma 4.8. *For* $0 < r \leq 1/3$, *let* $\phi_c = \arccos\left(\frac{1-\sqrt{1+8r^2}}{4r}\right)$, *the solution sets to the following equations are given as,*

- $R_r^n(\phi) = \phi$ *has solutions if and only if* n *is even, which are* $0, \pi, \pm\phi_c$.
- $R_r^n(\phi) = -\phi$ *has solutions* $\pm\phi_c$ *if* n *is odd; and* $0, \pi$ *if* n *is even.*
- $R_r^n(\phi) = \phi + \pi$ *has solutions if and only if* n *is odd, two of them are* 0 *and* π.
- $R_r^n(\phi) = \pi - \phi$ *has solutions* $0, \pi$ *when* n *is odd and two solutions when* n *is even.*

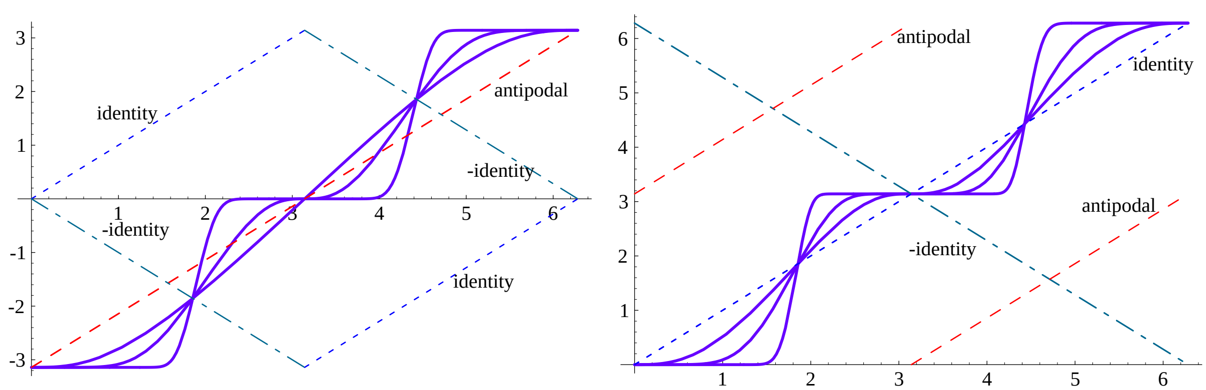
Proof. First of all, the preceding lemma already give us part of the conclusions. Furthermore, R_r^2 fixes the intervals $[-\pi, -\phi_c]$, $[-\phi_c, 0]$, $[0, \phi_c]$, and $[\phi_c, \pi]$ in a way that

$$R_r^2(\phi) < \phi \qquad \text{on } (-\pi, -\phi_c) \text{ or } (0, \phi_c),$$

$$R_r^2(\phi) > \phi \qquad \text{on } (-\phi_c, 0) \text{ or } (\phi_c, \pi).$$

Thus, induction process like $R_r^{2m}(\phi) < R_r^{2m-2}(\phi) < \phi$ on the corresponding intervals will give the only fixed points of R_r^{2m}. The other claims are done similarly. $\square$

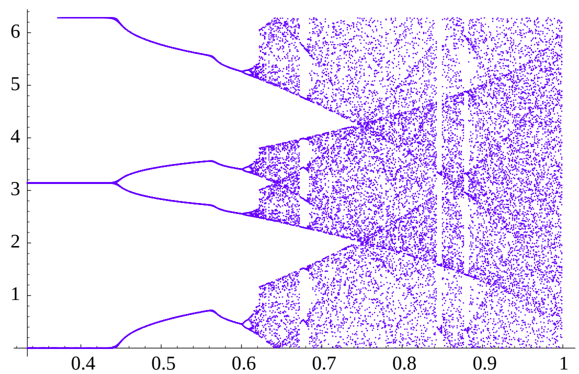
The above lemmas are illustrated by the following figures.



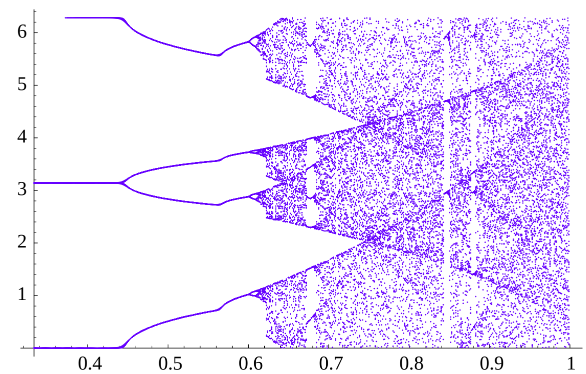
Plots of R_r^1, R_r^3 and R_r^7, $r = 1/3$.

Plots of R_r^2, R_r^6 and R_r^8, $r = 1/3$.

As an attempt to understand more about the iterations R_r^n, we computed the *asymptotic orbits* of the two critical points $\pm \arccos\left(\frac{1+3r^2}{4r}\right)$ for $1/3 \le r < 1$. The asymptotic orbit of ϕ is the set $\{\, R_r^n(\phi) \;:\; N_1 < n < N_2 \,\}$ for large N_1, N_2. The plot of asymptotic orbits against the parameter r is called a *bifurcation diagram*. It shows the attracting periodic cycles or chaotic behavior of a map according to the variation of the parameter. These diagrams have been further analyzed in [Au].



asymptotic orbits of $\arccos \frac{1+3r^2}{4r}$



asymptotic orbits of $-\arccos \frac{1+3r^2}{4r}$

5. Mode-locking

5.1. **Background.** The study of circle maps is closely related to the study of differential equations on torus (i.e., equations with double periodic coefficients). For any such differential equation, one may consider the Poincaré return map of the flow, which defines a map on a meridian circle of the torus. It turns out that the stability of the equation is reflected by this circle map.

Far back in 1959, in his Ph.D. project, Arnold investigated the circle map

$$\phi \mapsto \phi + a + \varepsilon \cos \phi$$

and obtained information on its resonance zone in the (a, ε)-plane, [A2, §12]. This gives rise to the famous picture of so-called Arnold tongues. Subsequently, there are numerous studies, by physicists and mathematicians, [BBJ, Di, FKP, JBB, K, P, Z], on the perturbation of a rotation

$$\phi \mapsto \phi + \Omega - \varepsilon \sin \phi, \qquad \varepsilon \in [0, 1).$$

The focus is on the phenomenon called mode-locking and the Devil's staircase. Arnold later gave a proof in [A4] of his observation for circle maps of the form

$$\phi \mapsto \phi + \Omega + \varepsilon(\text{trigonometric polynomial})$$

as well as analytic reduction of many circle maps, [A3, Ch. 3, §12]. The algebraic nature of the method is also apparent in the problem of particular differential equations. Arnold predicts that a general theorem exists for these equations and general circle maps.

In this section, we will provide further evidence towards Arnold's prediction by showing similar behavior in the off-center reflection. It should be remarked that the off-center reflection is not of the form studied by Arnold. Thus, it may be another small step towards the general theory.

We consider a two-parameter model of circle maps which arises from the off-center reflection map, namely, with parameters $r \in [0,1)$ and $\Omega \in (-\pi, \pi]$,

$$R_{r,\Omega}(\phi) = \phi + \Omega - 2\sum_{k=1}^{\infty} \frac{r^k}{k} \sin(k\phi).$$

Here we use r instead of ε to be consistent with previous sections. Note that unlike the models discussed above, r cannot be factored out. This map can be thought of as an imperfect off-center reflection on the circle where the reflected angle has a constant deviation from the incident angle. It is the original off-center reflection when $\Omega = \pi$. We may not get such a deviation by varying the metric of the circle; it is better understood in terms of symplectic geometry.

For $\phi_0 \in \mathbb{S}^1$, there is the *rotation number* $\omega(R_{r,\Omega}, \phi_0) = \lim_{n\to\infty} \dfrac{R^n_{r,\Omega}(\phi_0) - \phi_0}{n}$, where the right hand side is performed on a lifting of $R_{r,\Omega}$. It is independent of ϕ_0 if $R_{r,\Omega}$ is diffeomorphic. In such case, one simply denotes $\omega(R_{r,\Omega})$. If $R_{r,\Omega}$ is only a degree 1 map, one has a *rotation interval* instead. These notions are indeed defined for any circle map. Historically, attention has been centred around perturbations of rotations, $\phi \mapsto \phi + \Omega + u(\phi)$. It is natural to ask for the relation between Ω and ω. The physicists usually refer to Ω as internal frequency and ω as resonance frequency. When $\omega = \omega(\Omega)$ is a locally constant function, the situation is called *mode-locking*.

Herman has extensively studied the mode-locking property and obtained interesting results, [H1, H2]. These results are applicable to $R_{r,\Omega}$ because it satisfies the property $\mathbf{A}_0$ defined by Herman.

Theorem 5.1. *For all $\omega_0 \in 2\pi\mathbb{Q}$ and $0 < r \leq 1/3$, there is an interval $\mathcal{I} = \mathcal{I}_r$ of ω_0 such that for every $\Omega \in \mathcal{I}_r$, the diffeomorphism $R_{r,\Omega}$ has rotation number ω_0.*

The interval $\mathcal{I}$ is called *resonance interval* and its size depends on r (and of course ω_0). Its variance in terms of r defines a picture which looks like a tongue. We will discuss it later. Furthermore, from Herman's study, the off-center reflection model also demonstrates the well-known Devil's staircase.

Theorem 5.2. *For any $0 < r \leq 1/3$, the function $\Omega \mapsto \omega(R_{r,\Omega})$ is nondecreasing, locally constant at any rational number, and has a Cantor set of discontinuity.*

We have mentioned that if we alter a sign and form the "conjugate" family

$$\overline{R}_{r,\Omega}(\phi) = \phi + \Omega + 2\sum_{k=1}^{\infty} \frac{r^k}{k} \sin(k\phi),$$

the dynamics is completely different. Actually, $\overline{R}_{r,\Omega}$ can be extended to $e^{2\pi \mathbf{i}\Omega}\dfrac{z-r}{1-rz}$ on the hyperbolic disk, which defines a hyperbolic element in $\mathrm{PSL}(2,\mathbb{R})$. The mode-locking phenomenon does not occur, i.e., $\omega\left(\overline{R}_{r,\Omega}\right) = 2p\pi/q$ only if $\Omega = p/q$.

5.2. **Width of Resonance Zone.** In [A4], Arnold discusses the mode-locking situation of a rotation slightly perturbed by a trigonometric polynomial, $g(x)$,

$$f \ : \ x \mapsto x + \Omega + \varepsilon g(x).$$

The *resonance zone* is the set $\{\, (\Omega, \varepsilon) \ : \ \Omega \in \mathcal{I}_\varepsilon \,\}$. Arnold developed a formal calculation to estimate the width of the interval $\mathcal{I}_\varepsilon$ in terms of ε, which gives rises to a picture of the resonance zone. This formal calculation is related to the homological equation of analytical reduction, [A3]. If the rotation number is rational, the width of the resonance interval $\mathcal{I}_\varepsilon$ is bounded by a power of ε. The graphical plot of the resonance zone in the $\varepsilon\Omega$-plane form the so-called *Arnold's tongue*.

By a method similar to Arnold's, one may also estimate the width of $\mathcal{I}_r$ for the off-center reflections $R_{r,\Omega}$, $0 \le r \le 1/3$. We will show the different behaviors of $R = R_{r,\pi}$ and $\overline{R} = \overline{R}_{r,\pi}$ at the same time.

For simplicity of computation, let us first consider the resonance zone containing π. Writing $\Omega = \pi + a$, the second iterates of the maps are

$$R^2(x) = x + 2\pi + 2a - 2\sum_{k=1}^{\infty}\frac{r^k}{k}\sin(kx) - 2\sum_{k=1}^{\infty}\frac{r^k}{k}\sin k\left(x + \pi + a - 2\sum\frac{r^k}{k}\sin(kx)\right)$$
$$\overline{R}^2(x) = x + 2\pi + 2a + 2\sum_{k=1}^{\infty}\frac{r^k}{k}\sin(kx) + 2\sum_{k=1}^{\infty}\frac{r^k}{k}\sin k\left(x + \pi + a + 2\sum\frac{r^k}{k}\sin(kx)\right)$$

The equations of resonance are $R^2(x) = x + 2\pi$ and $\overline{R}^2(x) = x + 2\pi$. Let $v = a \mp 2\sum\frac{r^k}{k}\sin(kx)$, we have

$$0 = v \pm \sum_{k=1}^{\infty}\left[\frac{r^k}{k}\sin(kx) - \frac{(-r)^k}{k}\sin k(x+v)\right],$$

where $v = v_1 r + v_2 r^2 + v_3 r^3 + \cdots$. Note that the solutions of v's for R and $\overline{R}$ do not only differ by a sign. One can see this by the subtle combinations of the signs of the infinite series in their second iterates. Inductively, one may show that for $\overline{R}$, we have $v_k = \dfrac{2}{k}\sin(kx)$, while those for R are

$$v_1 = -2\sin(x),$$
$$v_2 = \sin(2x),$$
$$v_3 = 2\sin(x) - \frac{8}{3}\sin(3x),$$

The above result leads to a r-series for a and its maximum and minimum provide bounds for the resonance zone, namely,

$$a = \begin{cases} 2\sin(2x)r^2 + \left[2\sin(x) - \frac{7}{3}\sin(3x)\right]r^3 + \cdots, & \text{for the map } R, \\ 0 & \text{for the map } \overline{R}. \end{cases}$$

This calculation agrees with our previous remark that mode-locking (near $\omega = \pi$) does not occur for $\overline{R}$. Furthermore, we have

Theorem 5.3. *The width of $\mathcal{I}_r$ is bounded by Cr^2 for $\Omega = \pi$ and Cr for general Ω.*

The computation for the resonance zone at a general $\Omega = 2p\pi/q$ is more complicated. The equation to formally expand is $R^q_{r,a+2p\pi/q}(x) = x + 2p\pi$. The coefficients a_k of

$$a = a_1 r + a_2 r^2 + a_3 r^3 + \cdots$$

provide the estimates of $\mathcal{I}_r$. It turns out that the first term a_1 does not vanish, indeed,

$$qa_1 = 2\sum_{j=0}^{q-1} \sin\left(x + \frac{2jp\pi}{q}\right).$$

This may not be a sharp estimate, yet we can only conclude that the width of $\mathcal{I}_r$ is of order r in general.

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